Krylov Complexity of Fermionic and Bosonic Gaussian States

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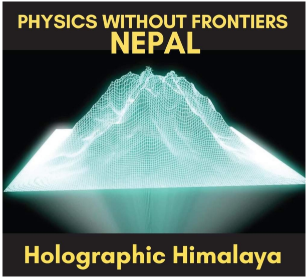


Gaussian Quantum States

- Gaussian states and their importance
- Coherent states
- Normalized coherent states
- Squeezed state
- Normalized squeezed state
- Squeezed coherent state
- Normalized squeezed coherent state

Gaussian states and their importance

- Gaussian state is defined as ground state of a Hamiltonian representing an ensemble of harmonic oscillators
- Derived from Gaussian function e^{-x^2}
- Gaussian state serve as a foundational basis for studying more complex systems and quantum states
- Gaussian states serve as simplified models which can be employed across various branches of physics

Coherent states

- Eigenstate of lowering operator $\hat{a}$ i.e.,

$$a |\alpha\rangle = \alpha |\alpha\rangle \, ; \alpha \in \mathrm{C} \tag{1}$$

- It closely resembles the motion of a classical harmonic oscillator so, called quasi-classical states.

Normalized coherent states

- $$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$
- Probability of finding n photons in $|\alpha\rangle$ obeys Poisson's distribution

$$P(n) = |\langle n|\alpha\rangle|^2 = \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!} \tag{2}$$

$P(n)$ vs n

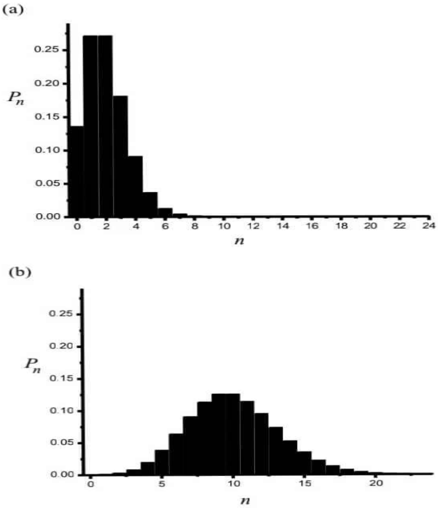


Figure: Coherent state photon number probability distributions for a) n=2 and b) n=10

Squeezed state

- For coherent state,

$$\Delta X \Delta P = \frac{\hbar}{2} \tag{3}$$

- reduce uncertainty in one of the quadratures and increase uncertainty in another canonical quadrature, we get squeezed state.For it,

$$\Delta X \Delta P \geq \frac{\hbar}{2} \tag{4}$$

- Squeezed state

$$|\psi_s\rangle = \hat{S}(\xi)\,|\psi\rangle\,;\text{where } \hat{S}(\xi) = e^{\frac{1}{2}\xi^* \hat{a}^2 - \frac{1}{2}\xi \hat{a}^{+^2}} \text{squeeze operator} \tag{5}$$

Normalized Squeezed state

- $$|\xi\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{m=0}^{\infty} (-1)^m \frac{\sqrt{(2m)!}\, e^{im\theta} \tanh^m r}{2^m m!} |2m\rangle \tag{6}$$
- Only even photon states have the solutions.
- Probability of obtaining $2m$ photons in $|\xi\rangle$ is

$$P_{2m} = |\langle 2m|\xi\rangle|^2 \tag{7}$$

$$= \frac{(2m)!(\tanh r)^{2m}}{2^{2m}(m!)^2 \cosh r} \tag{8}$$

- For $2m+1$ photons

$$P_{2m+1} = |\langle 2m+1|\xi\rangle||^2 = 0 \tag{9}$$

- The probability distribution for the squeezed vacuum state is oscillatory, vanishing for all odd photon numbers.

$P(n)$ vs n

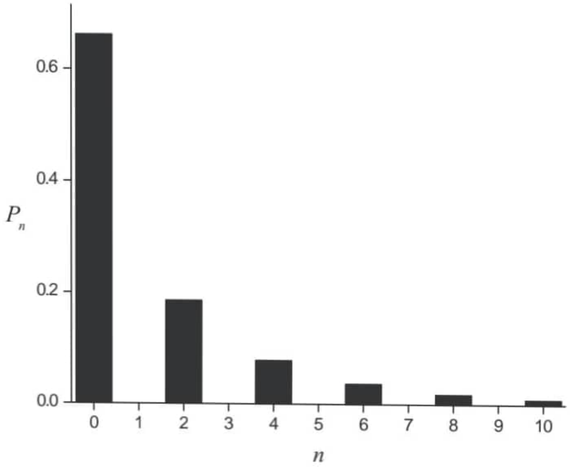


Figure: Histogram for photon number distribution for squeezed vacuum state

Squeezed coherent state (displaced squeeze state)

- Obtained by first acting with displacement operator $D(\alpha)$ on vacuum followed by squeeze operator $S(\xi)$ i.e.,

$$|\alpha, \xi\rangle = S(\xi)D(\alpha)|0\rangle \tag{10}$$

- It requires quadratic terms in a and $a^\dagger$ for generation.

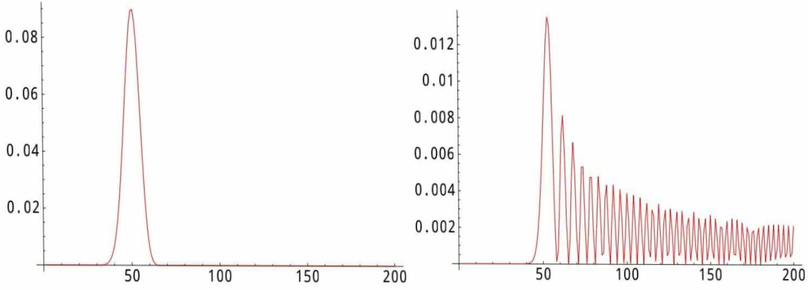
Normalized squeezed coherent

- $$|\alpha,\xi\rangle = \frac{1}{\sqrt{\cosh r}} e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}\alpha^{*2} e^{i\theta}\tanh(r)} * \sum_{n=0}^{\infty} \frac{[(1/2)e^{i\theta}\tanh(r)]^{\frac{n}{2}}}{\sqrt{n!}} H_n[\gamma(e^{i\theta}\sinh 2r)^{-\frac{1}{2}}]\,|n\rangle \tag{11}$$

- Probability of finding n photons in $|\alpha,\xi\rangle$ is given by

$$\begin{aligned} P(n) &= |\,\langle n|\alpha,\xi\rangle\,|^2 \\ &= \frac{[(1/2)\tanh r]^n}{n!\cosh r} e^{-|\alpha|^2 - \frac{1}{2}(\alpha^{*2}e^{i\theta} + \alpha^2 e^{-i\theta})\tanh(r)} \\ &\qquad |H_n[\gamma(e^{i\theta}\sinh(2r))^{-\frac{1}{2}}]|^2 \end{aligned} \tag{12}$$

$P(n)$ vs n



$|\alpha|^2 = 50, \theta = 0, r = 0.5$ $\quad$ $|\alpha|^2 = 50, \theta = 0, r = 4.0$

Figure: Squeezed coherent state photon number probability distributions:along vertical P(n) and horizontal n

Review of Krylov spread complexity

- The definition of Krylov spread complexity
- Using the Lanczos Algorithm to generate the Krylov basis
- Survival amplitude
- Using survival amplitude to compute Lanczos coefficients
- Compute the Krylov spread complexity

Krylov spread complexity

- Consider the evolution of a quantum state with a time independent Hamilton H:
$$|\psi(t)\rangle = e^{-iHt}\,|\psi(0)\rangle$$
- We define a **cost function** relative to a complete, orthonormal, ordered basis $\mathcal{B} = \{|B_n\rangle : n = 0, 1, 2, ...\}$ for the Hilbert space
$$C_{\mathcal{B}}(t) = \sum_n c_n |\langle\psi(t)|B_n\rangle|^2 = \sum_n c_n\, p_B(n, t)\,,$$
where c_n are positive increasing sequence of real numbers.
- Now the **spread complexity** of a quantum state is defined as
$$C(t) = \min_{\mathcal{B}} C_{\mathcal{B}}(t)$$

- Any complete **Krylov basis** $\{|K_n\rangle\}$ has a lower cost than any other basis (at least in the vicinity of t = 0).
- With $c_n = n$, the **Krylov complexity** is

$$C(t) = C_{\mathcal{K}}(t) = \sum_n n|\psi_n(t)|^2$$

where

$$|\psi(t)\rangle = \sum_n \psi_n(t)\,|K_n\rangle$$

Lanczos Algorithm

- Recursive application of the Gram-Schmidt orthogonalization to generate an orthonormal Krylov basis $\{|K_n\rangle\}$.
- The algorithm:

$$|A_{n+1}\rangle = (H - a_n |K_n\rangle) - b_n |K_{n-1}\rangle$$

$$\text{where} \qquad |K_n\rangle = b_n^{-1} |A_n\rangle , \qquad b_0 = 0, \qquad |K_0\rangle = |\psi(0)\rangle$$

.

- Here a_n and b_n are **Lanczos coefficients**

$$a_n = \langle K_n| H |K_n\rangle \ , \qquad b_n = \langle A_n|A_n\rangle^{\frac{1}{2}} \ .$$

- The quantum state $|\psi(t)\rangle$ in the Krylov basis is

$$|\psi(t)\rangle = \sum_n \psi_n(t)\,|K_n\rangle$$

- Using Schrodinger equation and the Lanczos algorithm, we get

$$i\partial_t\psi_n(t) = a_n\psi_n(t) + b_n\psi_{n-1}(t) + b_{n+1}\psi_{n+1}(t)$$

- Now we find the Lanczos coefficients using the **survival amplitude**.

Survival amplitude

- The **survival amplitude** is defined as

$$S(t) = \langle \psi(t) | \psi(0) \rangle = \langle \psi_0 | e^{iHt} | \psi_0 \rangle = \sum_n \mu_n \frac{t^n}{n!}$$

  where moments, $\mu_n = \langle \psi_0 | (iH)^n | \psi_0 \rangle$.
- Using a Markov chain representation, we get expressions for the moments in terms of the Lanczos coefficients such as

$$\mu_1 = ia_0 \,, \qquad \mu_2 = -a_0^2 - b_1^2 \qquad \text{and so on.}$$

- Finally, we can recursively find ψ_n using

$$i\partial_t \psi_n(t) = a_n \psi_n(t) + b_n \psi_{n-1}(t) + b_{n+1} \psi_{n+1}(t)$$

- Hence the **Krylov spread complexity** is computed as

$$C(t) = \sum_n n |\psi_n|^2$$

Review of the project

- Krylov complexity for coherent states
- Krylov complexity for single-mode squeezing and squeezed states
- Krylov complexity for displaced squeezing states

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Krylov complexity for Coherent States

- We found the Krylov basis coefficients to be:

$$\psi_n(t) = exp\left\{-\frac{1}{2}|z|^2\right\}\frac{|z|^n}{\sqrt{n!}}$$

- We get the expression for the complexity to be

$$C(t) = \alpha^2 t^2$$

when $z = i\alpha t$.

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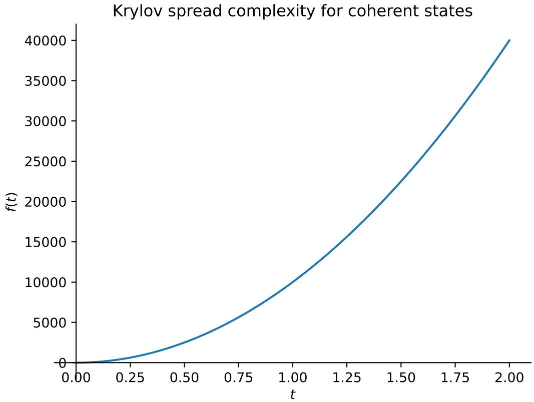

Figure: C(t) vs t for Coherent States

Krylov complexity for Single-mode Squeezing and squeezed states

- We found the Krylov basis coefficients to be:

$$|\psi_n(t)| = \frac{\sqrt{(2n-1)!!}tanh^n\eta t}{\sqrt{(2n)!!cosh\eta t}}$$

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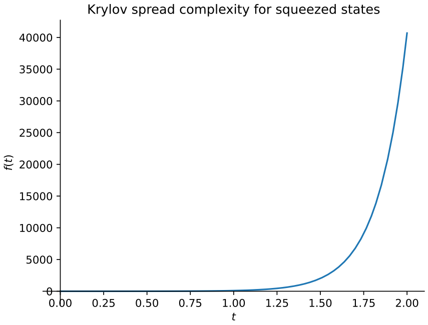

Figure: C(t) vs t for Squeezed States

Krylov complexity for Displaced Squeezing States

- We found the first three Krylov basis coefficients to be:
- $\psi_0(t) = \frac{1}{\sqrt{\cosh \eta t}} \exp\left[-\frac{1}{2}\alpha^2 t^2 + \frac{1}{2}\alpha^2 t^2 \tanh \eta t\right]$
- $$\psi_1(t) = -\frac{\sqrt{2}i\left(-\frac{\alpha^2 \eta t^2}{2} - \frac{\alpha^2 t \sinh(2\eta t)}{2} + \frac{\alpha^2 t \cosh(2\eta t)}{2} + \frac{\alpha^2 t}{2} + \frac{\eta \sinh(2\eta t)}{4}\right) e^{\frac{\alpha^2 t^2 (\tanh(\eta t) - 1)}{2}}}{\sqrt{2\alpha^2 + \eta^2} \cosh^{\frac{5}{2}}(\eta t)}$$
- $$\psi_2(t) = \frac{1}{A}\left(B - C - \frac{D1 + D2 + D3 + D4 + D5 \times \exp\left\{\frac{\alpha^2 t^2 \tanh(\eta t)}{2} - \frac{\alpha^2 t^2}{2}\right\}}{\sqrt{\alpha^2 + \frac{\eta^2}{2}}}\right)$$

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Krylov complexity for Displaced Squeezing States

where,

$$A = \sqrt{\frac{9\alpha^4\eta^2 + \left(-\alpha^2 - \frac{\eta^2}{2}\right)^3 + \left(\alpha^2 + \frac{\eta^2}{2}\right)\left(3\alpha^4 + 3\alpha^2\eta^2 + \frac{7\eta^4}{4}\right)}{\left(\alpha^2 + \frac{\eta^2}{2}\right)^2}}$$

$$B = 3\ \alpha^2\eta \left(-\frac{\eta e^{\frac{\alpha^2 t^2 \tanh(\eta t)}{2} - \frac{\alpha^2 t^2}{2}} \sinh(\eta t)}{2\cosh^{\frac{3}{2}}(\eta t)} + \frac{\left(\frac{\alpha^2\eta t^2 \cdot \left(1 - \tanh^2(\eta t)\right)}{2} + \alpha^2 t \tanh(\eta t) - \alpha^2 t\right) e^{\frac{\alpha^2 t^2 \tanh(\eta t)}{2} - \frac{\alpha^2 t^2}{2}}}{\sqrt{\cosh(\eta t)}}\right) \frac{}{\left(\alpha^2 + \frac{\eta^2}{2}\right)^{\frac{3}{2}}}$$

and so on...

Krylov complexity for Displaced Squeezing States

We can plot these $\psi_n(t)$ against t:

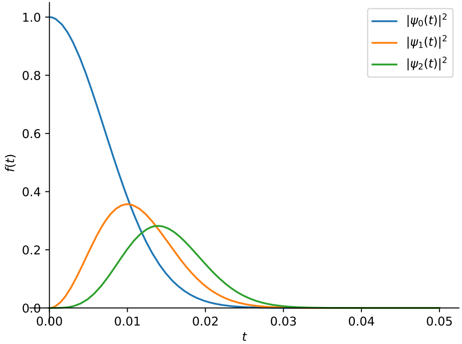

Figure: $\psi_0(t), \psi_1(t) and \psi_2(t)$ vs t

Krylov complexity for Displaced Squeezing States

Although we could not go beyond ψ_2 in our calculations, we can use these three wavefunctions or Krylov basis coefficients to plot complexity against time:

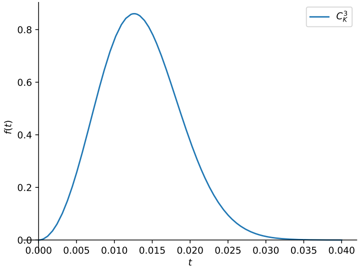

Figure: $C(t)$ vs t for Squeezed States

End of presentation

Thank you for your time!

Have Any Questions?