Remarks on Quasi-Equilibrium Theory

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ABSTRACT

A recent article by J. I. Yano has indicated that there is an inconsistency in the original formulation of the quasi-equilibrium theory of Arakawa and Schubert. He argues that this inconsistency results from a contradiction in the two asymptotic limits of the theory; that is, the fractional area covered by convection, and the ratio of the convective adjustment and large-scale timescales cannot simultaneously go to zero, $\sigma \rightarrow 0$ and $\tau_{ADJ}/\tau_{LS} \rightarrow 0$. Yano cites the heat engine theory proposed by Rennó and Ingersoll as ''formally establishing'' this contradiction. It is demonstrated in this paper that the quasi-equilibrium framework originally developed by Arakawa and Schubert is perfectly consistent with the heat engine theory for steady-state convection, that is, when the timescale associated with the large-scale forcing τ_{LS} approximates the effective adjustment timescale of the large-scale ensemble of convective clouds τ_{EFF}. Indeed, the quasi-equilibrium framework states that, on the large scale, the atmosphere is in quasi steady state.

1. Introduction

Quasi-equilibrium theory has provided the basis for many theoretical and modeling studies of large-scale atmospheric convection (e.g., Emanuel et al. 1994; Randall et al. 1997). Originally proposed by Arakawa (1969), applied to shallow nonprecipitating convection by Betts (1973) and to deep atmospheric convection by Betts (1974) and Arakawa and Schubert (1974, hereinafter AS), this theory essentially states that, over large areas, the production of instability by large-scale forcing (e.g., large-scale sensible and latent heat fluxes, radiative cooling, etc.) is balanced by its removal through cumulus convection. Although quasi-equilibrium theory has proved useful for both theoretical and modeling studies, Yano (1999) has recently pointed out that there is an inconsistency in the quasi-equilibrium theory put forth by AS. In this paper, we demonstrate that this inconsistency results solely from a confusion in the definition of the convective and effective adjustment timescales τ_{ADJ} and τ_{EFF}. Examining the quasi-equilibrium theory in terms of the heat engine framework for steady-state convecting atmospheres, in which the large-scale timescale τ_{LS} is the radiative relaxation timescale, we show that the inconsistency described by Yano disappears when the adjustment timescales are properly defined.

We turn first to a review of AS quasi-equilibrium theory, including its mathematical formulation and scaling analysis. This is followed by an examination of the Rennó and Ingersoll (1996, hereinafter RI) formulation of quasi equilibrium under the heat engine framework. Finally, we review Yano's critique of the quasi-equilibrium theory and demonstrate that the ''inconsistency'' in this theory results from using AS's ambiguous scaling arguments and not from the theory itself.

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2. The quasi-equilibrium theory

a. Arakawa–Schubert's mathematical model

The fundamental equation underlying quasi-equilibrium theory, Eq. (140) in AS,

$$\frac{dA}{dt} = \left(\frac{dA}{dt}\right)_C + \left(\frac{dA}{dt}\right)_{LS}, \qquad (1)$$

simply states that the time rate of change of the ''cloud work function'' A, which, for this discussion, can be thought of as convective available potential energy (CAPE), is a balance between the production of instability by large-scale forcing $(dA/dt)_{LS}$ and its removal by the ensemble of cumulus clouds at the subgrid cell scale $(dA/dt)_C$. In ''quasi equilibrium'' that is, in quasi steady state, dA/dt must be approximately zero. Arakawa and Schubert (1974) argue that the foundation upon which quasi-equilibrium theory lies is the separation of two timescales, that of the large-scale forcing τ_{LS}, and that of the convective adjustment τ_{ADJ}. This separation of timescales allows cumulus convection to quickly respond to changes in the large-scale forcing, maintaining quasi equilibrium. We show, in section 2b, that AS implicitly define the adjustment timescale τ_{ADJ} as the timescale for a convective updraft to travel from the surface to the top of the convective layer. Thus, their τ_{ADJ} is the timescale for *a local* convective adjustment, not the effective adjustment timescale τ_{EFF} (defined in section 3), of a large-scale ensemble of convective clouds.

On page 691 of their influential article AS state, ''The problem we are considering is a problem with two time scales: τ_{ADJ}, the adjustment time scale, and τ_{LS}, the time scale of the large-scale processes. Quasi equilibrium exists when $\tau_{ADJ} \ll \tau_{LS}$. When the adjustment process is filtered out, we obtain a sequence of quasi equilibria. In such a sequence, the large-scale forcing and the cumulus ensemble vary in time in a coupled way and, therefore, the timescale of the statistical properties of the ensemble is equal to the timescale of the large-scale processes, τ_{LS}.''

Mathematically, AS define the convective adjustment timescale τ_{ADJ}^{AS} (superscripts refer to timescale definitions appropriate to each author), and the large-scale timescale τ_{LS}^{AS} by their Eqs. (146) and (147), respectively:

$$\left|\left(\frac{dA}{dt}\right)_C\right| \sim \frac{A}{\tau_{ADJ}^{AS}}, \quad \text{and} \tag{2}$$

$$\left|\left(\frac{dA}{dt}\right)\right| \sim \frac{A}{\tau_{LS}^{AS}}. \tag{3}$$

Their adjustment timescale τ_{ADJ}^{AS} can be conceptualized as follows. Given the presence of CAPE (we are using CAPE interchangeably with the cloud work function of AS) and some triggering mechanism, cumulus activity will develop. If there is no large-scale forcing, the instability present in the large-scale ''grid box'' will be consumed by the cumulus ensemble thereby bringing the atmosphere to a neutral state. Here, τ_{ADJ}^{AS} is a measure of the time needed to reach this neutral state. In this sense, τ_{ADJ}^{AS} is really the adjustment timescale of the large-scale (grid box) ensemble of cumulus clouds (the timescale of the ''statistical properties of the ensemble'' mentioned in the quote above); that is, it is the effective adjustment timescale τ_{EFF} that brings the entire grid box to a neutral state. The large-scale timescale τ_{LS}^{AS} is defined by AS as being, not a relaxation timescale, but a timescale on which the large-scale forcing varies (an externally imposed timescale).

Arakawa and Schubert (1974) argue that quasi equilibrium exists when $\tau_{ADJ}^{AS} \ll \tau_{LS}^{AS}$. Yano (1999) follows AS employing these order of magnitude arguments to demonstrate that a contradiction exists in the asymptotic limit of quasi-equilibrium theory; namely, $\sigma \to 0$ and $\tau_{ADJ}^{AS}/\tau_{LS}^{AS} \to 0$ are inconsistent, where σ is the fractional area of the grid box covered by convective updrafts. We will show in section 3 that this inconsistency disappears when the convective adjustment and effective timescales are defined in a manner consistent with heat engine framework for steady-state convecting atmospheres. We turn first to AS's scaling arguments to reveal exactly how they define their convective adjustment and effective timescales.

b. Arakawa–Schubert's scaling analysis

By employing AS's order of magnitude analysis of the quasi-equilibrium theory, given on page 692 of their original article, we demonstrate that their effective convective adjustment timescale τ_{ADJ}^{AS} turns out to be the *local* convective adjustment timescale, that is, the time scale for a deep cumulus updraft to travel from the surface to its level of neutral buoyancy. We show in section 3 that it is the use of this local convective adjustment timescale τ_{ADJ}^{AS} over the entire grid box, instead of an effective adjustment timescale, *not* quasi-equilibrium theory per se, that leads to the inconsistency pointed out by Yano (1999).

Arakawa and Schubert (1974) state that $\tau_{ADJ}^{AS} \sim 10^3$–$10^4$ s [their Eq. (154)], assume a value of the vertical velocity $w \sim 1$–10 m s^{-1}, and of the depth of the convective layer $H \sim 10^4$ m. Thus, one must conclude that $\tau_{ADJ}^{AS} \sim H/w$ is a local convective adjustment timescale. It then follows that AS assume that rate of change of A by the large-scale ensemble of convective clouds is given by

$$\left(\frac{dA}{dt}\right)_C \sim \frac{A}{\tau_{ADJ}^{AS}}, \tag{4}$$

where $\tau_{ADJ}^{AS} \equiv H/w$. This equation is identical to Eq. (2). Therefore, this scaling argument implies that AS incorrectly assumes that the ensemble of convective elements in the grid box is adjusted on the same timescale as a single convective updraft (i.e., the timescale for local convective adjustment H/w). In the next section, we derive an effective adjustment timescale τ_{EFF} for the cumulus ensemble in the grid box. We demonstrate that this effective adjustment timescale is consistent with the asymptotic limits originally proposed by AS.

3. Heat engine theory and the steady-state convecting atmosphere

Another way to analyze the quasi-equilibrium theory is to examine the steady-state convecting atmosphere as described by the heat engine theory proposed by RI. As with AS, the basis for quasi equilibrium in a convecting atmosphere is that there is near equality between the production of CAPE by large-scale processes and its consumption by a large-scale ensemble of convective systems. That is, that over large scales the atmosphere is in quasi steady state. The amount of CAPE present in the quasi-steady-state convecting atmosphere is then a measure of the amount of mechanical dissipation of energy present. Yano (1999) uses estimates of the fractional area covered by convective drafts σ derived from

RI's heat engine theory to ''formally establish'' the ''contradiction'' in the asymptotic limits of AS's derivation. We demonstrate in this section that the effective adjustment timescale τ_{EFF} is estimated differently in RI from that of AS and the contradiction cited by Yano (1999) results from these different estimates.

In the heat engine theory for steady-state convection, RI assume that the large-scale atmosphere is in radiative–convective equilibrium. We can think of the steady-state convecting atmosphere in the following simple, qualitative way. Given a fixed value of the surface temperature, creation of instability ($A \sim$ CAPE) by radiative cooling of the atmosphere results in increased convective activity. This convective activity, realized through an ensemble of cumulus clouds, then forces large-scale subsidence that pushes the atmosphere away from its radiative equilibrium. Assuming that this perturbation to the atmosphere's radiative equilibrium is small, RI use the Newtonian cooling approximation to estimate the radiative timescale (the timescale for the atmosphere to radiatively relax back to the unadjusted state once convection has been ''turned off''). In steady state, this timescale must be equal to the effective adjustment timescale τ_{EFF}, that is, to the timescale over which the cumulus ensemble will bring the atmosphere from the unstable equilibrium to a neutral state if the large-scale forcing is turned off. Given this linear approach taken by RI (see sections 6 and 7 of their article), the large forcing term in Eq. (1) can be estimated as

$$\left|\left(\frac{dA}{dt}\right)_{\mathrm{LS}}\right| \sim \frac{g}{\Delta p}\eta F_{\mathrm{in}}, \tag{5}$$

where Δp is the thickness of the radiating layer (i.e., the troposphere), η is the thermodynamic efficiency of the convective heat engine, and F_{in} is the heat flux into the convective heat engine (i.e., sensible, latent, and radiative heat fluxes in to the near-surface air). Rennó and Ingersoll (1996) show in their Eq. (39) that this large-scale forcing is estimated as

$$\frac{g}{\Delta p}\eta F_{\mathrm{in}} \sim \frac{A}{\tau_{\mathrm{LS}}^{\mathrm{RI}}}, \tag{6}$$

where $\tau_{\mathrm{LS}}^{\mathrm{RI}} \sim \tau_R$ is the radiative relaxation time, and $A \sim$ CAPE is the total amount of work done by buoyancy forces. In quasi steady state, the magnitude of this large-scale term $(dA/dt)_{\mathrm{LS}}$ must be nearly equal to the magnitude of the cumulus ensemble term $(dA/dt)_C$. Rearranging terms from RI's Eqs. (34) and (39), we have

$$\frac{A}{\tau_{\mathrm{LS}}^{\mathrm{RI}}} \approx \frac{g}{\Delta p}\rho\sigma w A, \tag{7}$$

from which it follows

$$\frac{A}{\tau_{\mathrm{LS}}^{\mathrm{RI}}} \approx \frac{\sigma A}{\tau_{\mathrm{ADJ}}^{\mathrm{RI}}}, \tag{8}$$

where $\tau_{\mathrm{ADJ}}^{\mathrm{RI}} \approx H/w$ and $H \sim \Delta p/\rho g$. The effective adjustment timescale can be defined as $\tau_{\mathrm{EFF}} \equiv \tau_{\mathrm{ADJ}}^{\mathrm{RI}}/\sigma$, and under radiative–convective equilibrium conditions, Eq. (1) can be rewritten as

$$\frac{dA}{dt} \sim \frac{-A}{\tau_{\mathrm{EFF}}} + \frac{A}{\tau_{\mathrm{LS}}^{\mathrm{RI}}}. \tag{9}$$

Because in quasi-equilibrium conditions there is nearly balance between the terms on the rhs of Eq. (11), we have that

$$\tau_{\mathrm{LS}}^{\mathrm{RI}} \approx \tau_{\mathrm{EFF}} \equiv \frac{\tau_{\mathrm{ADJ}}^{\mathrm{RI}}}{\sigma}. \tag{10}$$

Thus, in quasi equilibrium the effective adjustment timescale is approximately equal to the large-scale timescale. Since, on the large scale, the fractional area covered by convective updrafts is much smaller than one ($\sigma \ll 1$), then $\tau_{\mathrm{EFF}} \gg \tau_{\mathrm{ADJ}}^{\mathrm{RI}} \approx \tau_{\mathrm{ADJ}}^{\mathrm{AS}}$.

4. Yano's critique

We now turn to Yano's critique of the quasi-equilibrium theory. Yano states that the two asymptotic limits under which the AS scheme is formulated, $\sigma \to 0$ and $\tau_{\mathrm{ADJ}}^{\mathrm{AS}}/\tau_{\mathrm{LS}}^{\mathrm{AS}} \to 0$, are in contradiction to each other and that ''the smallness of these two quantities is established only under a compromise.'' We argue that this inconsistency is due solely to AS's ambiguous definition of $\tau_{\mathrm{ADJ}}^{\mathrm{AS}}$. Following AS, Yano expresses the adjustment timescale [his Eq. (1)] in the same way as AS; that is, as

$$\tau_{\mathrm{ADJ}}^{\mathrm{AS}} \sim \left|\frac{A}{\left(\frac{dA}{dt}\right)_C}\right|. \tag{11}$$

This equation implies that the convective adjustment timescale is equal to the effective timescale; that is, $\tau_{\mathrm{EFF}} = \tau_{\mathrm{ADJ}}^{\mathrm{AS}}$. This result is inconsistent with AS's scaling analysis described in section 2b, which implies that $\tau_{\mathrm{ADJ}}^{\mathrm{AS}}$ is the timescale for local convective adjustment. Our expression for estimating the convective adjustment timescale [from Eqs. (1), (10), and (11)], in turn, is given by

$$\tau_{\mathrm{ADJ}}^{\mathrm{RI}} \sim \sigma\left|\frac{A}{\left(\frac{dA}{dt}\right)_C}\right|, \tag{12}$$

which shows that the convective adjustment timescale is equal to the product of the fractional area covered by convection with the effective timescale; that is, $\tau_{\mathrm{ADJ}}^{\mathrm{RI}} = \sigma\tau_{\mathrm{EFF}}$.

The contradiction that Yano points to in the AS scheme results directly from their failure to include σ in Eq. (11). If the equation for the magnitude of the convective timescale we derived in section 3 is used, then the contradiction in AS's quasi-equilibrium theory disappears as we demonstrate below.

It follows from Eq. (8) and the quasi-equilibrium assumption that

$$\tau_{\mathrm{ADJ}}^{\mathrm{RI}} \approx \sigma\tau_{\mathrm{LS}}^{\mathrm{RI}}, \tag{13}$$

where, again, $\tau_{\mathrm{LS}}^{\mathrm{RI}}$ depends only on the values of the large-scale forcing (e.g., sensible and latent heat fluxes, radiative cooling rate, etc.). Equation (13) shows that the convective timescale $\tau_{\mathrm{ADJ}}^{\mathrm{RI}}$ decreases with decreases in the fractional area covered by cumulus convection. This happens because, in steady state, the energy flux per convective draft increases as the fractional area covered by them decreases. Therefore, Yano's suggestion that τ_{ADJ} increases with decreasing fractional area covered by cumulus convection is incorrect in terms of the quasi-equilibrium idea defined by Eq. (9). Furthermore, it follows from Eq. (13) that

$$\lim_{\tau_{\mathrm{ADJ}}^{\mathrm{RI}}\to 0} \sigma \approx \lim_{\tau_{\mathrm{ADJ}}^{\mathrm{RI}}\to 0} \frac{\tau_{\mathrm{ADJ}}^{\mathrm{RI}}}{\tau_{\mathrm{LS}}^{\mathrm{RI}}} = 0, \tag{14}$$

demonstrating that the two asymptotic limits $\tau_{\mathrm{ADJ}}^{\mathrm{RI}}/\tau_{\mathrm{LS}}^{\mathrm{RI}} \to 0$ and $\sigma \to 0$ do not contradict each other when quasi equilibrium is expressed in terms of the radiative–convective equilibrium atmosphere.

5. Conclusions

The point of departure between the quasi-equilibrium theory expressed by AS and that by RI results from the estimation of the large-scale forcing term $(dA/dt)_{\mathrm{LS}}$ in Eq. (1). Arakawa and Schubert (1974) make no attempt at estimating the large-scale forcing term. Instead, they estimate the *total* time rate of change of the cloud work function (CAPE) over the grid cell [Eq. (3)] and, in doing so, introduce the characteristic large-scale timescale $\tau_{\mathrm{LS}}^{\mathrm{AS}}$. This timescale characterizes net changes in CAPE over the grid cell. Rennó and Ingersoll (1996), on the other hand, assume radiative–convective equilibrium, and that, in the linear regime of small perturbations to this equilibrium state, the large-scale term can be approximated by a Newtonian cooling rate, as in Eqs. (7)–(11). Following the arguments of RI, it is evident that the asymptotic limits of quasi-equilibrium theory do not lead to the contradiction in time–space scale separation mentioned by Yano. In this sense, RI's framework should not be cited as ''formally establishing'' this contradiction when, in fact, it is entirely consistent with the quasi-equilibrium time–space scale separation as originally proposed by AS.

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