Plan

- Day 1 : Bauer-Furuta invariant
- Day 2 : SW Floer stable homotopy type
- Day 3 : Real BF & Real SWF
- Day 4 : Computations & Applications
- Day 5 : Miyazawa's invariant & P^2-knot.

Day 1

what is the BF invariant?

Input $\qquad\qquad$ Output

$(X^4, \mathfrak{s}) \rightsquigarrow$ "$\operatorname{ind} \not{D} \longrightarrow H^+(X)$ up to suspension".

closed $\qquad$ continuous

oriented $\qquad$ S^1-equiv.

spin^c

smooth

4-manifold

Def.

Spin(n) = the double cover of $SO(n)$.

$$\mathrm{Spin}^c(n) := \frac{\mathrm{Spin}(n) \times U(1)}{\pm 1}$$

Def

oriented

A **spin structure** s on X^n is a principal $\mathrm{Spin}^c(n)$-bundle $\begin{matrix} P \\ \downarrow \\ X \end{matrix}$ with equipped

$$P \times_{\mathrm{Spin}(n)} SO \xrightarrow{\cong} Fr(X) \text{ as } SO(n)\text{-bundles.}$$

oriented frame bundle

$Spin^c$ str ... similar, use $Spin^c(n)$ in place of $Spin(n)$.

If $\exists$ spin str. on X,

$\{\text{spin structures}\}/\cong \; \curvearrowleft \; H^1(X;\mathbb{Z}/2)$

free
transitive

$\exists$ $spin^c$ str. on X,

$\{spin^c \text{ str. on } X\}/\cong \; \curvearrowleft \; H^2(X;\mathbb{Z})$

free
transitive

Output?

Thm 1 (Bauer–Furuta)

Given $(X^4, \mathfrak{s})$ closed spin^c smooth,

there is

$$f : (\mathbb{R}^m \oplus \mathbb{C}^n)^+ \longrightarrow (\mathbb{R}^{m'} \oplus \mathbb{C}^{n'})^+ \ : \ \text{based } S^1\text{-equiv. conti. map}$$

where

- $n - n' = \frac{1}{8}\big(\underbrace{c_1(\mathfrak{s})^2}_{\langle c_1(\mathfrak{s})^2, [X]\rangle} - \sigma(X)\big) \in \mathbb{Z}$

$$\left(\begin{array}{ccc} \{\text{spin}^c \text{ str. on } X\} & \longrightarrow & H^2(X;\mathbb{Z}) \\ \mathfrak{s} & \longmapsto & c_1(\mathfrak{s}) \end{array}\right)$$

bij if $\pi_1 X = 1$

- $m' - m = b^+(X)$.

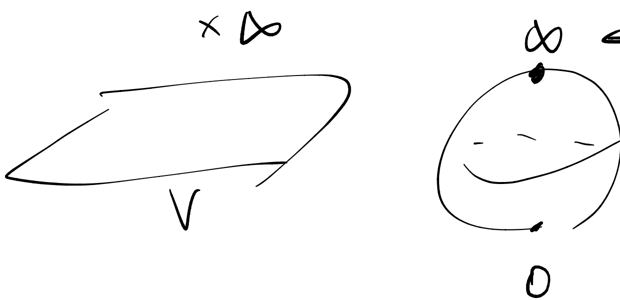
- $f^{S^1} : (\mathbb{R}^m)^+ \to (\mathbb{R}^{m'})$ comes from a linear injection $\mathbb{R}^m \hookrightarrow \mathbb{R}^{m'}$.

Remark.

Notation:

V : a finite dim vect. space

$V^+ = V \cup \{\infty\}$: compactification



regarded as the base point.

$$\begin{cases} S^1 \curvearrowright \mathbb{R} : \text{trivial} \\ S^1 \curvearrowright \mathbb{C} : \text{scalar multiplication.} \end{cases}$$

$S^1 = U(1) \subset \mathbb{C}$

- Spin case

$$\mathrm{Pin}(2) = U(1) \amalg jU(1) \subset \mathbb{C} \oplus j\mathbb{C} = \mathbb{H}.$$

$\mathrm{Pin}(2) \curvearrowright \tilde{\mathbb{R}}$: by

$$\begin{array}{ccc} \mathrm{Pin}(2) & \twoheadrightarrow & \mathbb{Z}/2 \curvearrowright \tilde{\mathbb{R}} \\ & & \| \\ \cup & & \{\pm 1\} \\ & & \cup \\ U(1) & \longrightarrow & 1 \\ jU(1) & \longrightarrow & -1 \end{array}$$

by the scalar multiplication

$Pin(2) \curvearrowright \mathbb{H}$: by $\mathbb{H}$-scalar multiplication

$\cap$

$\mathbb{H}$

<u>Thm 2</u> (Furuta)

$(X^4, \mathfrak{s})$: <u>spin</u> closed smooth

Then there is

$$\begin{cases} \bullet\ f : (\tilde{\mathbb{R}}^m \oplus \mathbb{H}^n)^+ \longrightarrow (\tilde{\mathbb{R}}^{m'} \oplus \mathbb{H}^{n'})^+ \\ \bullet\ n - n' = -\dfrac{\sigma(X)}{16} \\ \bullet\ m' - m = b^+(X) \end{cases}$$

: based $Pin(2)$-equiv. conti. map

• f^{S^1} comes from a linear inclusion.

Remark

+ Elkies

Thm 1 & $H^*_{S^1}$ $\Rightarrow$ Donaldson's diagonalization

Thm 2 & $K_{Pin(2)}$ $\Rightarrow$ Furuta's 10/8-inequality.

Rem $(X, \mathfrak{s}) \mapsto f$ gives an invariant by passing to the stable htpy class of f:

Stabilization: $V < \mathbb{R}^\infty \oplus \mathbb{C}^\infty$: fin. dim

$$\Sigma^V f : V^+ \wedge (\mathbb{R}^m \oplus \mathbb{C}^n)^+ \to V^+ \wedge (\mathbb{R}^{m'} \oplus \mathbb{C}^{n'})$$

You allow stabilizations and S^1-equiv. based htpy.

Spin case is similar: $V \subset \tilde{\mathbb{R}}^\infty \oplus \mathbb{H}^\infty$.

$S^1 \rightsquigarrow Pin(2)$.

$\rightsquigarrow$ The Bauer-Furuta invariant $[f]$. $\lrcorner$

- <u>Construction of f</u>

$$
\begin{array}{ccccc}
Spin(4) & = & Sp(1)\times Sp(1) & \ni & (q_+, q_-) \\
\downarrow{\scriptstyle \mathbb{Z}/2} & & \downarrow & & \downarrow \\
SO(4) & = & SO(\mathbb{H}) & \ni & (q \mapsto q_- q q_+^{-1})
\end{array}
$$

$$
Spin^c(4) = \frac{Sp(1)\times Sp(1)\times U(1)}{\pm 1}
\begin{array}{l}
\xrightarrow{\Delta_+} U(2) \\
\xrightarrow{\Delta_-} U(2) \\
\xrightarrow{\det} U(1)
\end{array}
$$

$$
\begin{array}{c} \Psi \\ g = [q_+, q_-, \lambda] \end{array}
$$

$$
\begin{cases}
\Delta_+(g)(\underset{\substack{\cap \\ \mathbb{H} = \mathbb{C}^2}}{q}) := q_+ q \lambda, \\
\Delta_-(g)(q) := q_- q \lambda,
\end{cases}
\qquad \det(g) := \lambda^2.
$$

$\mathfrak{s}$: a spin^c str. on X^4

$$\begin{array}{c} \mathfrak{s} \\ \updownarrow \\ P \\ \downarrow \mathrm{Spin}^c(4) \\ X \end{array} \quad \underset{\text{via } \Delta_+, \Delta_-, \det}{\overset{}{\underset{\text{associate}}{\rightsquigarrow}}} \quad \begin{array}{c} S^+ \\ \downarrow \mathbb{C}^2 \\ X \end{array}, \quad \begin{array}{c} S^- \\ \downarrow \mathbb{C}^2 \\ X \end{array}, \quad \begin{array}{c} L \\ \downarrow \mathbb{C} \\ X \end{array}$$

spinor bundles (S^+, S^-)

determinant line bdle (L)

$$c(\mathfrak{s}) := c_1(L)$$

The Seiberg-Witten equations:

$$\begin{cases} F_A^+ = \sigma(\Phi) \\ \not{D}_A \Phi = 0 \end{cases}$$

$(A, \Phi) \in \mathcal{A}(L) \times \Gamma(S^+)$ is the unknown functions.

$$\mathcal{A}(L) = \{U(1)\text{-connections on } L\}$$

Fix $A_0 \in \mathcal{A}(L)$, then

$$\begin{array}{ccc} \mathcal{A}(L) & \xleftrightarrow[1:1]{} & i\Omega^1 \\ \ni & & \ni \\ A & \longleftrightarrow & A - A_0 \end{array}$$

$$\Omega_g^+ = \{\omega \in \Omega^2 \mid *_g \omega = \omega\}$$

$$i\Omega^1 \oplus \Gamma(S^+) \cong \mathcal{A}(L) \times \Gamma(S^+) \longrightarrow i\Omega^+ \oplus \Gamma(S^-) \curvearrowleft \mathcal{G}$$

$$(A, \Phi) \longmapsto (F_A^+ - \sigma(\Phi), D_A \Phi)$$

$$\mathcal{G} = \mathrm{Map}(X, U(1))$$

$\xrightarrow[\text{restrict}]{}$

$$\mathrm{Ker}(d^* : i\Omega^1 \to i\Omega^0) \oplus \Gamma(S^+) \xrightarrow{F} i\Omega^+ \oplus \Gamma(S^-)$$

S^1 $\qquad$ S^1

Morally, "$F: \mathbb{R}^{\infty + \mathrm{ind}(F^{S})} \oplus \mathbb{C}^{\infty + \mathrm{ind}\, \not{D}} \to \mathbb{R}^{\infty} \oplus \mathbb{C}^{\infty}$".

Assume $b_1(X) = 0$ for simplicity.

Then $F^{-1}(0)$ is compact.

Do finite dimensional approximation:

Take $V' \subset i\Omega^{+} \oplus \Gamma(S^{-})$: large fin. dim subspace.

$F = l + c$, $l = (d^{+}, \not{D}_{A_0})$

$V := l^{-1}(V')$

$$V \xrightarrow[]{\overbrace{l + \pi_{V'} \circ c}^{=: f}} V' \quad (\pi_{V'} : L^2\text{- projection to } V'.)$$

Then, from the compactness of $F^{-1}(0)$,

it follows that f is proper:

$\rightsquigarrow$ $V^+ \xrightarrow{f} V'^+$ is induced.

$\circlearrowleft$ conti $\circlearrowright$

S^1 $\qquad\qquad$ S^1

• <u>Spin case</u>

$$\mathrm{Spin}(4) = Sp(1) \times Sp(1) \begin{array}{l} \xrightarrow{\Delta_+} Sp(1) \\ \xrightarrow[\Delta_-]{} Sp(1) \end{array}$$

$$\rightsquigarrow \quad \begin{array}{c} S^+ \\ \downarrow \mathbb{H} \\ X \end{array}, \quad \begin{array}{c} S^- \\ \downarrow \mathbb{H} \\ X \end{array} \qquad \begin{array}{c} L = X \times \mathbb{C} \\ \downarrow \\ X \end{array}$$

$$\rightsquigarrow \Gamma(S^\pm) \curvearrowleft \mathbb{H}$$

$$\begin{array}{c} \mathrm{Pin}(2) \\ \downarrow \\ \mathbb{Z}/2 \curvearrowright i\Omega^+ \\ \circlearrowleft \end{array}$$

scalar multiplication

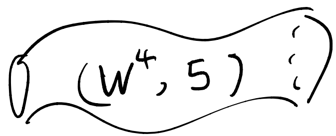
$A_0 :=$ trivial connection

$$\begin{array}{ccc} \mathcal{A}(L) & \longleftrightarrow & i\Omega^1 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ A & \longleftrightarrow & A - A_0 \end{array}$$

The SW eqs are Pin(2)-equivariant

Day 2 SW Floer stable homotopy type

Yesterday: $(X^4, \mathfrak{s})$ spin^c $\rightsquigarrow$ "$f : (\mathrm{ind}\, \not{D})^+ \to (H^+(X))^+$"



$(Y_0, \mathfrak{t}_0)$ $(Y_1, \mathfrak{t}_1)$

spin^c $\mathbb{Q}HS^3$

$\rightsquigarrow$ $f : (\mathrm{ind}\, \not{D})^+ \wedge SWF(Y_0) \longrightarrow (H^+(X))^+ \wedge SWF(Y_1)$.

S^1- / $Pin(2)$-equiv.

based continuous.

<u>relative BF invariant</u>

$(Y, \mathfrak{t}) \rightsquigarrow SWF(Y, \mathfrak{t})$: Seiberg-Witten Floer stable homotopy type.

$\mathbb{Q}HS^3$

spin^c

based finite CW cpx $\curvearrowleft S^1$ (or $\curvearrowleft Pin(2)$ if $\mathfrak{t}$: spin)

up to suspension / homotopy.

$\tilde{H}^*_{S^1}(SWF(Y, \mathfrak{t})) \cong \check{HM}^*(Y, \mathfrak{t})$: monopole Floer homology

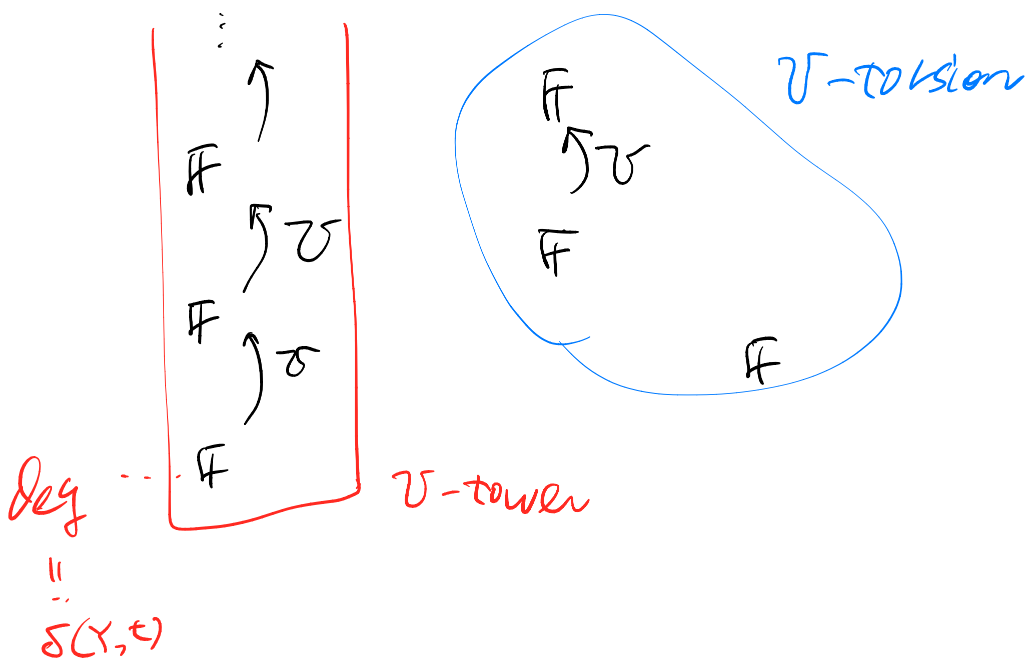
Lidman-Manolescu.

Frøyshov invariant

Y : $\mathbb{Q}HS^3$

deg $U = 2$

$$\check{HM}(Y,\mathfrak{t};\mathbb{F}_2) \curvearrowleft \mathbb{F}_2[U] = H^*_{S^1}(pt)$$



• Construction of SWF

$(Y, \mathfrak{t})$: spin^c $\mathbb{Q}HS^3$

$$\begin{array}{ccccc}
Spin(3) & = & Sp(1) & \ni & q \\
\downarrow & & \downarrow & & \downarrow \\
SO(3) & = & SO(\mathrm{Im}\,\mathbb{H}) & \ni & (q' \mapsto q q' q^{-1})
\end{array}$$

$$Spin^c(3) = \frac{Sp(1) \times U(1)}{\pm 1} \xrightarrow{\Delta} U(2), \quad \xrightarrow{\det} U(1)$$

$$\ni \quad g = [q, \lambda]$$

$$\Delta(g)(\underset{\substack{\in \\ \mathbb{H} = \mathbb{C}^2}}{q'}) := q q' \lambda, \quad \det(g) = \lambda^2.$$

$$t \longleftrightarrow \begin{array}{c} P \\ \downarrow \mathrm{Spin}^c(3) \\ Y \end{array}$$

$$\rightsquigarrow \begin{array}{c} S \\ \downarrow \mathbb{C}^2 \\ Y \end{array}, \quad \begin{array}{c} L \\ \downarrow \mathbb{C} \\ Y \end{array}$$

Pick $A_0 \in \mathcal{A}(L) = \{U(1)\text{-connections on } L\}$.

$$\begin{array}{ccc} \mathcal{A}(L) & \longleftrightarrow & i\Omega^1 \\ \Psi & & \cup \\ A & \longleftrightarrow & A - A_0 \end{array}$$

$$\mathcal{U} := \mathrm{Ker}(d^* : i\Omega^1 \to i\Omega^0) \oplus \Gamma(S)$$

(equipped with a (non-L^2) metric)

$CSD : \mathcal{V} \longrightarrow \mathbb{R}$: Chern-Simons-Dirac functional

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathbb{R} \\ \Psi & & \Psi \end{array}$$

$$(a, \Phi) \longmapsto \frac{1}{2} \int_Y \langle \Phi, \not{D}_{A_0+a} \Phi \rangle d\mathrm{vol} - \frac{1}{2} \int_Y a \wedge da$$

$$S^1 \curvearrowright \mathcal{V} = \underbrace{\ker d^*}_{\substack{\circlearrowleft \text{ triv.} \\ S^1}} \oplus \underbrace{\Gamma(S)}_{\substack{\circlearrowleft \text{ scalar multiplication} \\ S^1}}$$

CSD is S^1-invariant.

The gradient flow line equation for CSD is

$$-\frac{dx}{dt}(t) = (l + c)\, x(t)$$

$$(x : \mathbb{R} \longrightarrow V) \quad \begin{cases} l : V \circlearrowleft & : \text{linear} \\ c : V \circlearrowleft & : \text{non-linear} \end{cases}$$

fin. dim

$$V_\lambda^\mu := \bigoplus\Big(\underbrace{\text{eigenspaces}}_{\text{w.r.t. } l \,\circlearrowright\, V} \text{ with eigenvalues} \in (\lambda, \mu]\Big) \qquad \lambda \not\in \operatorname{Spec},\ \mu \not\in \operatorname{Spec}$$

$$V_\lambda^\mu \subset V$$

$p^{\mu}_{\lambda} : \mathcal{U} \longrightarrow V^{\mu}_{\lambda}$: the L^2-projection

Then $\ell + p^{\mu}_{\lambda} \circ c : V^{\mu}_{\lambda} \circlearrowleft$.

This gives a vector field:

$$\begin{array}{c} TV^{\mu}_{\lambda} \cong V^{\mu}_{\lambda} \times V^{\mu}_{\lambda} \\ \quad\quad\quad\quad \Psi \\ \quad\quad\quad\quad (x, (\ell + p^{\mu}_{\lambda} \circ c)(x)) \end{array}$$

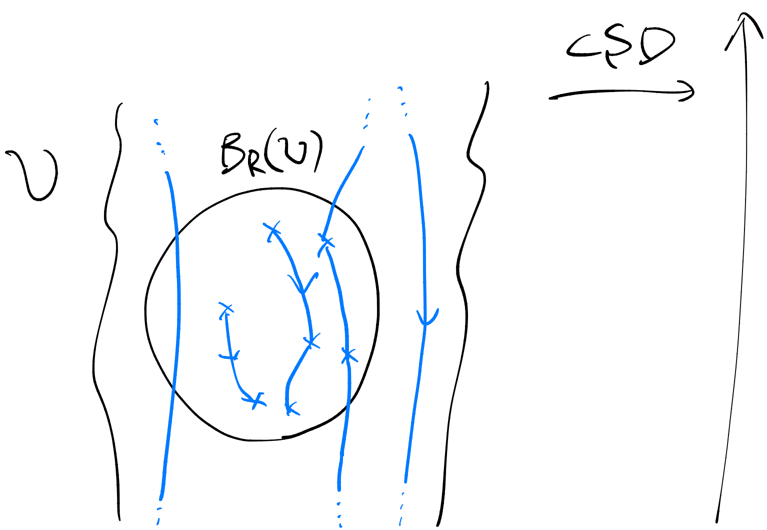
Consider the flow $\mathbb{R} \curvearrowright V^{\mu}_{\lambda}$ generated by this vector field.

after cut-off near the end.

Thm (Compactness)

$\exists R \gg 0$ s.t. $\{\text{crit pts of CSD}\} \subset B_R(v)$.

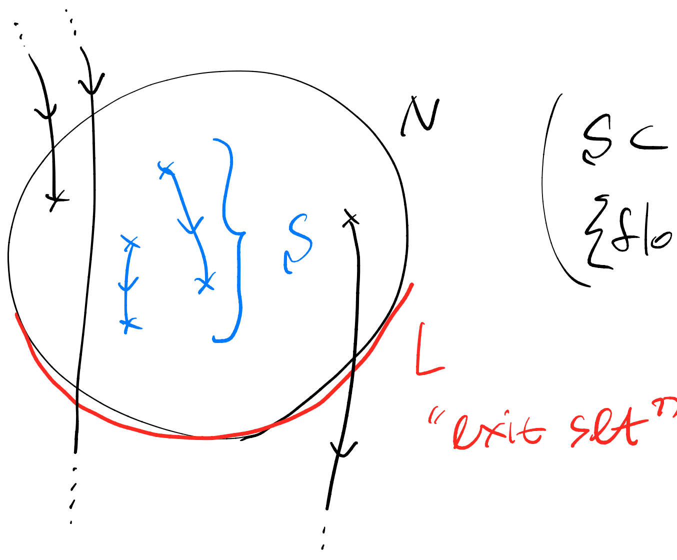
$\{\text{flow lines between crit pts of CSD}\} \subset$



Consider $S := \{\text{crit pts}\} \cup \{\text{flows between them}\} \subset B_R(V^\mu_\lambda)$

for the approximate flow $\mathbb{R} \curvearrowright V^\mu_\lambda$.

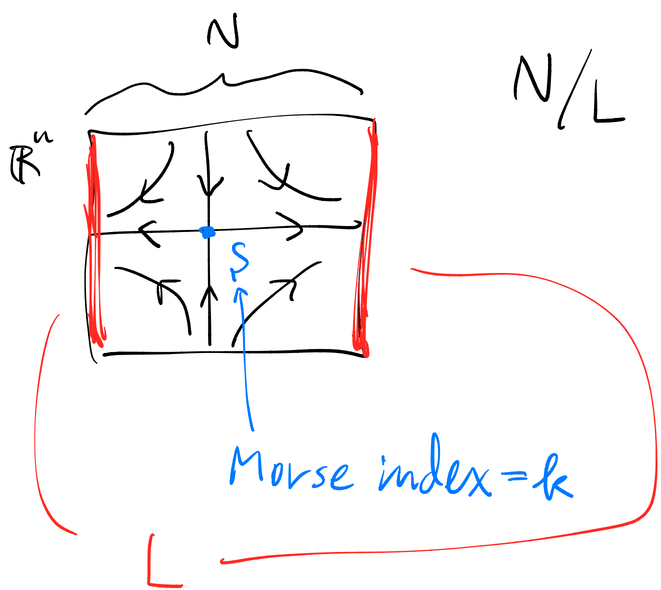
We can take the Conley index of S.

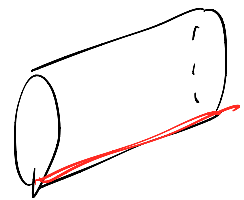
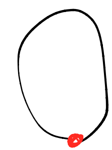
$V^\mu_\lambda \supset$ 

$\left(\begin{array}{l} S \subset \mathrm{Int}(N \setminus L), \\ \{\text{flows in } N \setminus L\} = S \end{array} \right)$

The Conley index of S is $I(S) := N/L \ni L/L$ base pt.

e.g. $V = \mathbb{R}^n$



$N/L =$  $\simeq$  S^k

$SWF(Y,t) := I(S)$ w/ some (de)suspension.

2 ambiguities:

$$\begin{cases} (1) \text{ Choice of } \lambda, \mu \\ (2) \text{ choice of a metric } g \text{ on } Y. \end{cases}$$

$$SWF(Y,t) = \underbrace{\Sigma^{-V^0_\lambda}}_{(1)} \underbrace{\Sigma^{n(Y,t,g)\mathbb{C}}}_{(2)} I(S).$$

(1): $\lambda < \lambda' < 0 < \mu' < \mu$

$$V_\lambda^\mu = V_{\lambda'}^{\mu'} \oplus V_\lambda^{\lambda'} \oplus V_{\mu'}^{\mu}$$

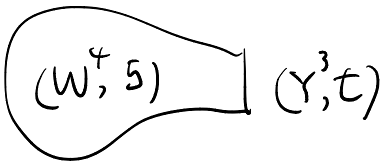
$$\underset{\substack{\text{Conley} \\ \text{index}}}{\rightsquigarrow} \quad I(V_\lambda^\mu) \simeq I(V_{\lambda'}^{\mu'}) \wedge (V_\lambda^{\lambda'})^+.$$

$\therefore$ "$\Sigma^{-V_\lambda^0} I(V_\lambda^\mu)$" is indep of λ, μ.

$$\uparrow \quad \underset{\text{top. spaces}}{\Sigma^{-V} Z \underset{\text{stably}}{\simeq} Z'} \iff Z \underset{\text{stably}}{\simeq} \Sigma^V Z'.$$

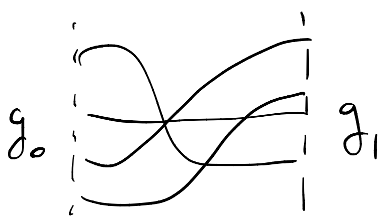
(2) : $n(Y,t,g) \in \mathbb{Q}$

$$\begin{gathered} n(Y,t,g) \\ \| \\ \mathrm{ind}_{\mathbb{C}} \not{D}_{(W,s)} - \frac{1}{4}\left(c_1(s)^2 - \sigma(X)\right) \end{gathered}$$



$$SF(\not{D}, g_0, g_1) = n(Y,t,g_0) - n(Y,t,g_1)$$

↑ spectral flow



Suspension by $n \in \mathbb{Q}$?

$$\Sigma^{n\mathbb{C}} Z \simeq \Sigma^{n'\mathbb{C}} Z'$$

$$\Leftrightarrow \quad n - n' \in \mathbb{Z}, \ \Sigma^{(n-n')\mathbb{C}} Z \underset{\text{stably}}{\simeq} Z'$$

Day 3 Real Bauer-Furuta / SWF stable homotopy type

Def $n = 4$ or 3

$(X^n, \mathfrak{s})$: spin 4-mfd $\left(\mathfrak{s} \leftrightarrow \begin{array}{c} P \\ \downarrow \mathrm{Spin}(n) \\ X \end{array} \right)$

$\iota : X \circlearrowleft$: smooth involution

ι is a <u>Real involution</u> w.r.t. $\mathfrak{s}$ if

$$\begin{cases} \iota^* \mathfrak{s} \cong \mathfrak{s} \quad \left(\Rightarrow \iota \text{ lifts to } \tilde{\iota} : \begin{array}{ccc} P & \to & P \\ \downarrow & \circlearrowleft & \downarrow \\ X & \to & X \end{array} \right) \\ \tilde{\iota} \text{ is order } 4 \end{cases}$$

Exercise $\tau^* 5 \cong 5 \Rightarrow \tilde{\tau}$ is of order $\underline{2}$ or $\underline{4}$,

Recall:

$$\begin{array}{l} P \\ \downarrow \mathbb{Z}/2 \\ Fr(X) \circlearrowleft \mathbb{Z}/2 = \langle \tau^* \rangle \end{array}$$

cf.

$$\begin{cases} \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \\ \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \end{cases}$$

Note. There are only two lifts: $\tilde{\tau}, -\tilde{\tau}$

Lem Exercise.

If $\iota \curvearrowright X^n$ ($n \le 4$) involution satisfies

$$\begin{cases} \iota^* \mathfrak{s} \cong \mathfrak{s} \\ X^\iota \neq \phi \quad \text{(fixed pt set)} \\ \operatorname{codim} X^\iota = 2 \end{cases} \Rightarrow \iota \text{ is Real w.r.t. } \mathfrak{s}.$$

Rem. There is also a free Real invol.

eg. $S^2 \times S^2 \circlearrowleft (\text{anti-podal})^{\times 2}$. (The quotient is not spin from Stipsicz's talk.)

- Induced involution on SW theory

Let ι be a Real involution on $(X, \mathfrak{s})$

$I : \Omega^*(X) \oplus \Gamma(S^\pm) \circlearrowleft$ In 3 dim case, just S.

(a, Φ)

$I(a, \Phi) = (-\iota^* a, \tilde{\iota}(\Phi) \cdot j)$ (Recall : $S^\pm \curvearrowleft \mathbb{H}$ in the spin case.)

$$\begin{cases} I^2 = id, \\ I \text{ is anti-complex linear on } \Gamma(S^\pm). \\ \text{i.e. } Ii = -iI \quad \leftsquigarrow \quad ij = -ji. \end{cases}$$

Exercise

$$SW : \operatorname{Ker} d^* \oplus \Gamma(S^+) \longrightarrow i\Omega^+ \oplus \Gamma(S^-)$$

with I acting on both sides ($I \circlearrowright$),

is I-equivariant.

$$\rightsquigarrow \quad SW^I : (\operatorname{Ker} d^* \oplus \Gamma(S^+))^I \longrightarrow (i\Omega^+ \oplus \Gamma(S^-))^I$$

$$\underset{\substack{\text{fin.}\\ \text{dim.}\\ \text{approx.}}}{\rightsquigarrow} \quad f^I : ((\operatorname{ind} \not{D})^I)^+ \longrightarrow ((H^+(X))^I)^+$$

Real Bauer-Furuta invariant.

$\underline{(\operatorname{ind} \not{D})^I :}$

$I \curvearrowright \operatorname{ind} \not{D}$: anti-complex linear

$\Rightarrow (\operatorname{ind} \not{D})^I = \frac{1}{2} \operatorname{ind} \not{D}$.

$\underline{\text{Lem}}$

V : complex vect. sp.

$\circlearrowleft$

I : anti-cpx linear invol

$\Rightarrow \dim V^I = \frac{1}{2} \dim V$.

$\underline{H^+(X)^I}$:

$I \curvearrowright H^+(X) \subset \Omega^+$ is given by $-2^* \curvearrowright H^+(X)$.

$$\begin{array}{l} \dim H^+(X)^I = \dim H^+(X)^{-2^*} = b^+(X) - \dim H^+(X)^{2^*} \\ \quad \| \\ b^+_R(X) \quad (\text{Real } b^+) \end{array}$$

- symmetry on SW^I.

$\Omega^*(X) \oplus \Gamma(S^\pm) \curvearrowleft Pin(2) \ni j$, $\curvearrowleft I$ don't commute, as $ij \neq ji$.

But $\mathbb{Z}/4 = \langle j \rangle$ and I commute.

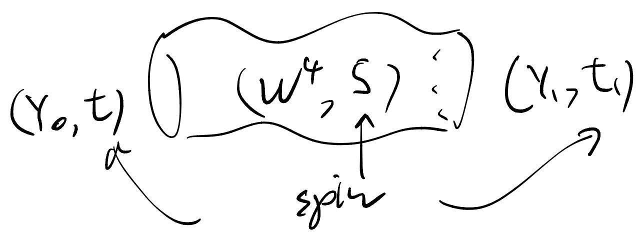
$\rightsquigarrow$ $SW^I : (\ker d^* \oplus \Gamma(S^+))^I \to (i\Omega^+ \oplus \Gamma(S^-))^I$ $\curvearrowleft \mathbb{Z}/4$

$\rightsquigarrow$ $f^I : ((\mathrm{ind}\, D)^I)^+ \longrightarrow ((H^+(X))^I)^+$: $\mathbb{Z}/4$-equiv. $\curvearrowleft \mathbb{Z}/4$

Similarly,

$$SWF(Y^3, \overset{\text{spin}}{\mathfrak{t}})^I \curvearrowleft \mathbb{Z}/4$$

Real Floer homotopy type



$$\rightsquigarrow \quad f^I : SWF(Y_0, \mathfrak{t}_0)^I \wedge \left((\operatorname{ind} D)^I\right)^+ \to SWF(Y_1, \mathfrak{t}_1)^I \wedge \left((H^+(X))^I\right)^+$$

: $\mathbb{Z}/4$-equiv.

◎ Non-spin case

$(X^n, \mathfrak{s})$: spin^c n-mfd
$$n = 4 \text{ or } 3$$

$\iota : X \circlearrowleft$: smooth involution

Assume $\iota^*\mathfrak{s} \cong \bar{\mathfrak{s}}$ $\qquad \left(\mathrm{Spin}^c(n) = \dfrac{\mathrm{Spin}(n) \times U(1)}{\pm 1},\ c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s}) \right)$

conjugate

$$\Downarrow$$

$$\begin{tikzcd}
I_\varphi : P \arrow[r, "c"] \arrow[drr, bend right] & \bar{P} \arrow[r, "\exists\varphi", "\cong"'] \arrow[dr] & \iota^* P \arrow[r, "\tilde{\iota}"] \arrow[d] & P \arrow[d] \\
& & X \arrow[r, "\iota"'] & X
\end{tikzcd}$$

$c : P \to \bar{P}$: complex conjugation

Lem

If $b_1(X) = 0$, $\exists\, u \in Map(X, U(1))$
s.t. $(u \cdot I\varphi)^2 = \pm id$.

Lem

If $X^2 \neq \phi$ and $\operatorname{codim} X^2 = 2$, $(u \cdot I\varphi)^2 = id$
for $\forall u$ w/ $(u \cdot I\varphi)^2 = \pm id$.

Put $I := u \cdot I\varphi$.

$I \curvearrowright \Gamma(S^\pm)$ is anti-complex linear.
(as $c : P \to \bar{P}$ is involved).

Lem

$\exists\ A_0 \in \mathcal{A}(L)$ s.t. $\begin{cases} \iota^* A_0 = A_0, \\ \iota^* F_{A_0} = -F_{A_0} \end{cases}$

use this

as the reference conn.

$$SW : \operatorname{Ker} d^* \oplus \Gamma(S^+) \longrightarrow i\Omega^+ \oplus \Gamma(S^-) \quad : I\text{-equiv.}$$

$$I \qquad\qquad I$$

$$I(a, \Phi) = (-\iota^* a, I(\Phi)).$$

The S^1-action & I don't commute

as I is anti-complex linear.

But $\mathbb{Z}/2 = \{\pm 1\} \subset S^1$ commutes with I.

$$\rightsquigarrow f^I : (\mathrm{ind}\, D^I)^+ \longrightarrow (H^+(X)^I)^+ : \mathbb{Z}/2\text{-equiv.}$$

$$\circlearrowleft \quad \circlearrowleft$$

$$\mathbb{Z}/2$$

Similarly, $\mathrm{SWF}(Y, t)^I \curvearrowleft \mathbb{Z}/2$.