joint w/ Daren Chen

based on arXiv: 2506.00824

Involutive Khovanov homology is interestingly trivial

Hongjian Yang (Stanford)

Involutive Heegaard Floer homology

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$\boxed{\mathbb{F} = \mathbb{F}_2}$

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Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, z)$

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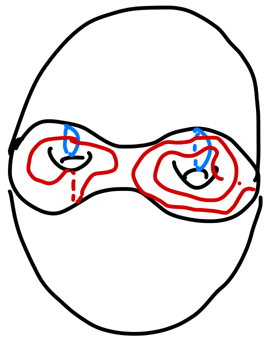
HF chain complex $\widehat{CF}(\mathcal{H})$



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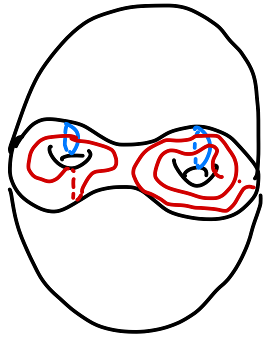

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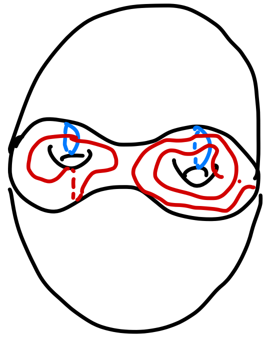

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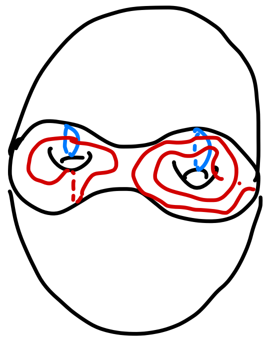

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Khovanov homology

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cube of resolutions + TQFT

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- Allow cut-and-glue techniques / invariants for tangles

  $\mathcal{F}(T_1 \circ T_2) = \mathcal{F}(T_1) \otimes \mathcal{F}(T_2)$ (Khovanov, Bar-Natan)

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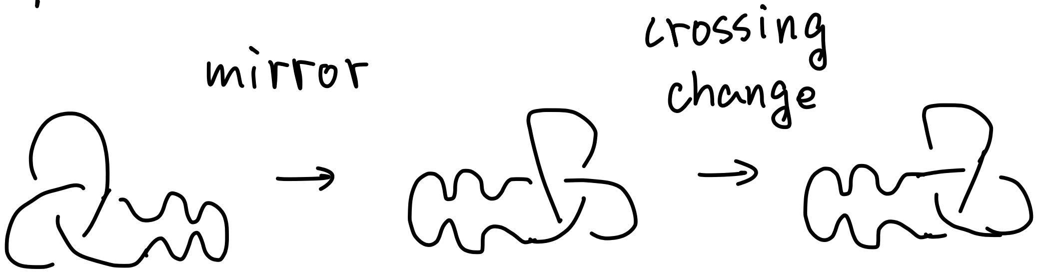

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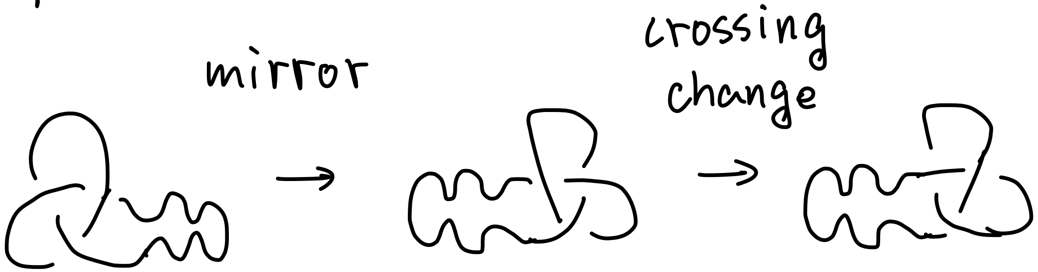

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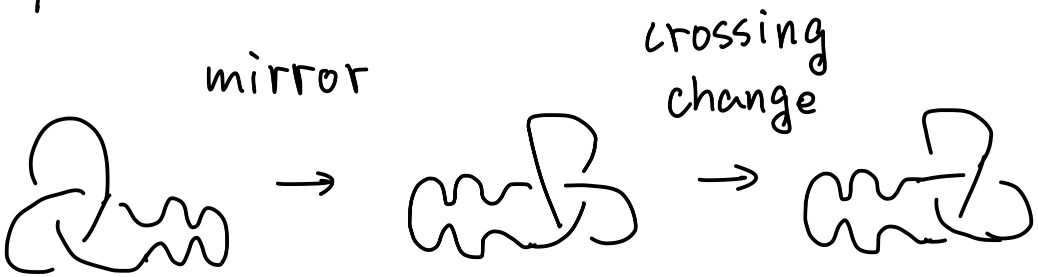

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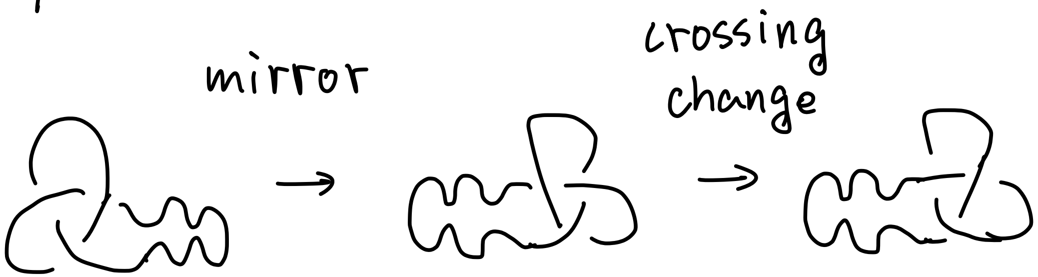

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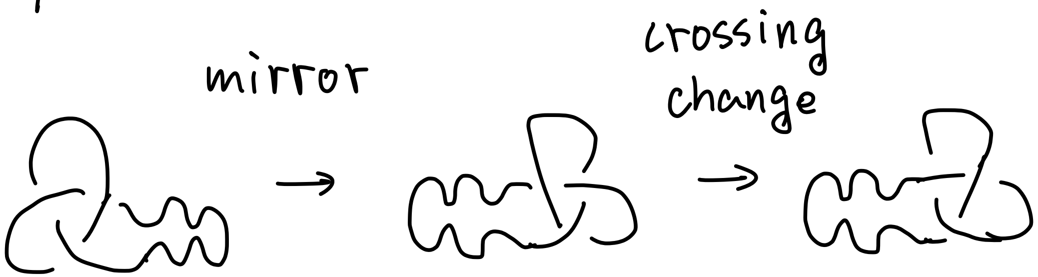

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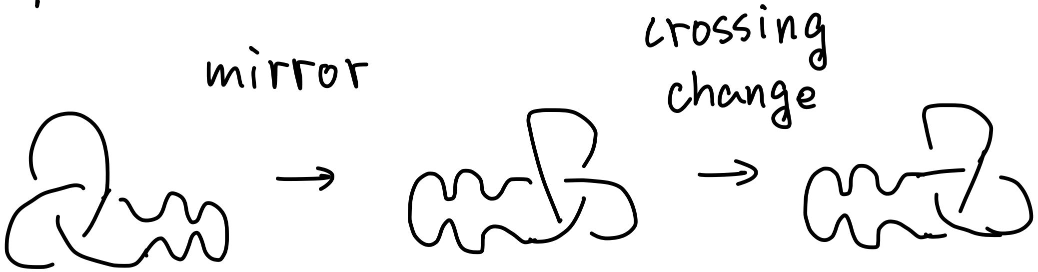

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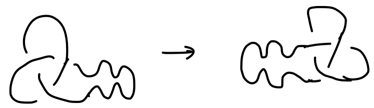
$$\Rightarrow \quad KhI(K;\mathbb{F}) \cong Kh(K;\mathbb{F}) \otimes_{\mathbb{F}} \mathbb{F}[Q]/Q^2 \qquad :($$

Computing involutive Kh



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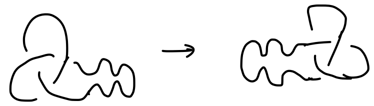

Observation: β k-strand braid Δ k-strand half twist

$$\begin{array}{|c|}\hline \Delta^{-1} \\ \hline \beta \\ \hline \Delta \\ \hline \end{array} \cong \beta^*$$

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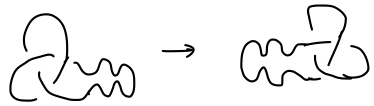

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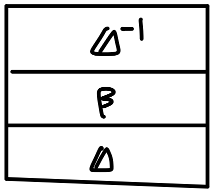

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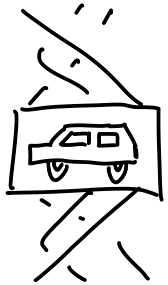

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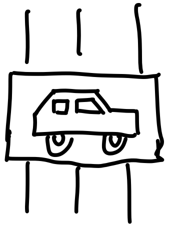

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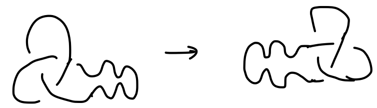

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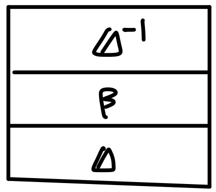

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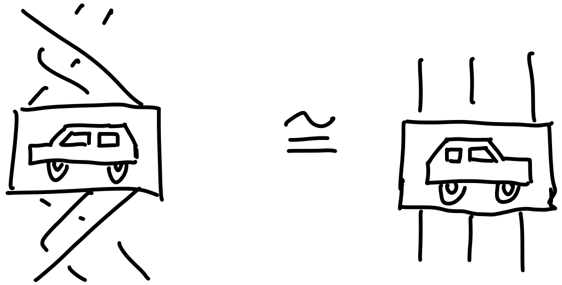
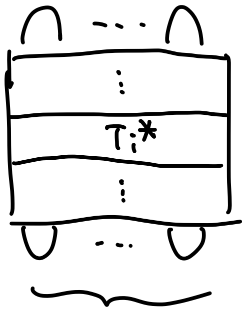


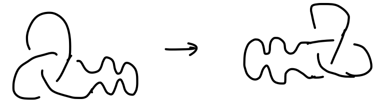
Now present K as a plat closure:

$D^* =$ 

$T_1 \sim T_n$: elementary braids

e.g. 

Computing involutive Kh



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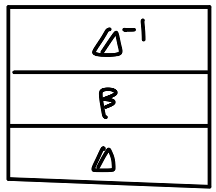

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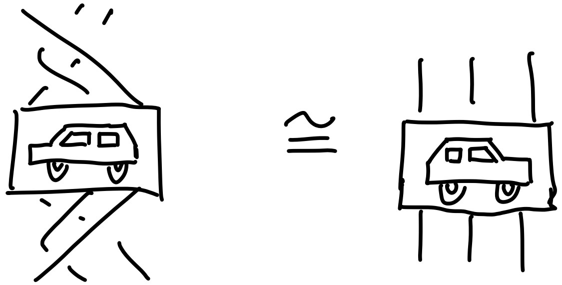

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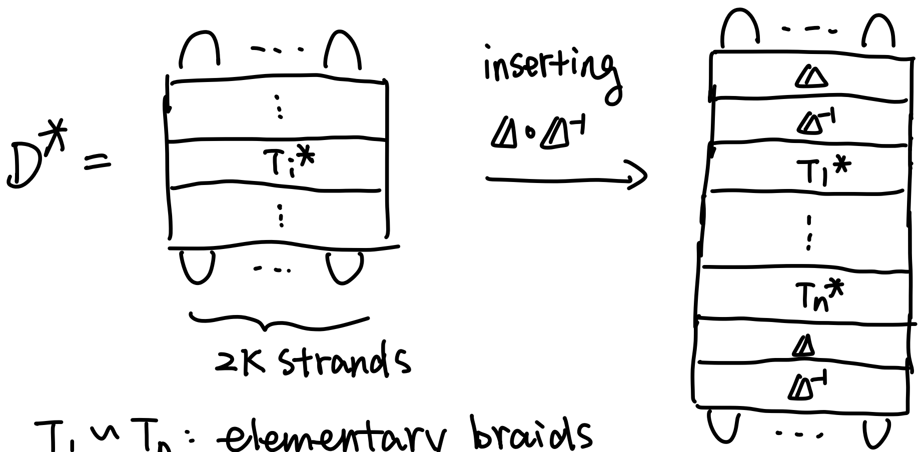

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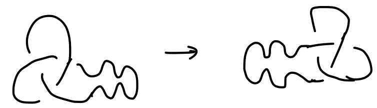

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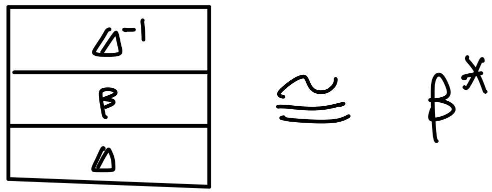

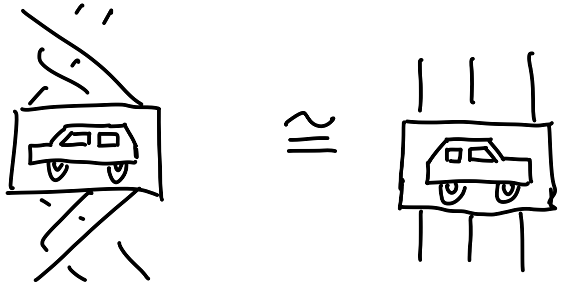

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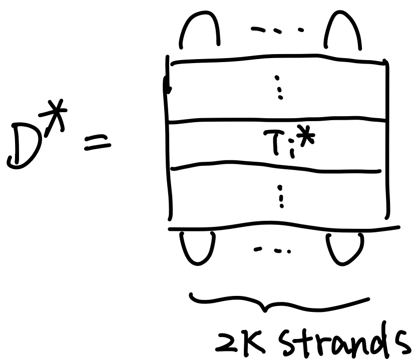

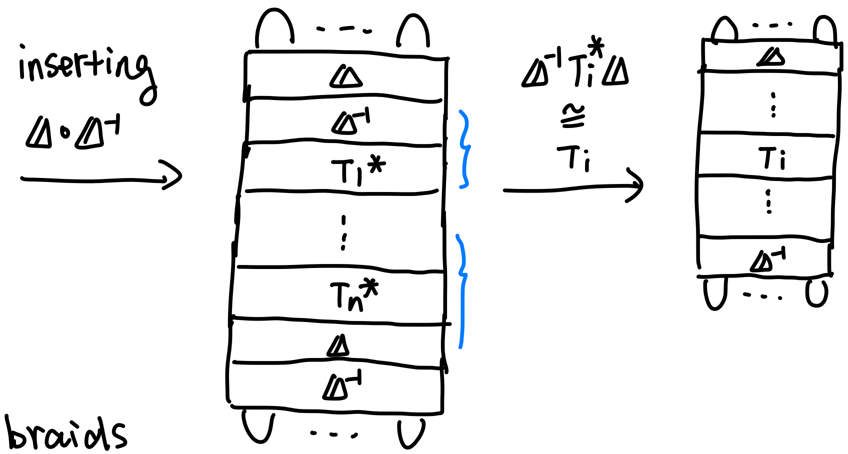

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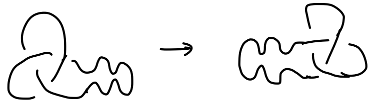

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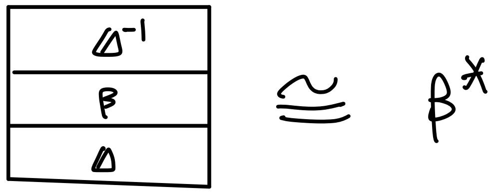

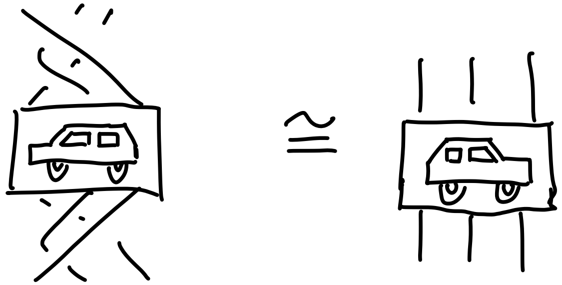

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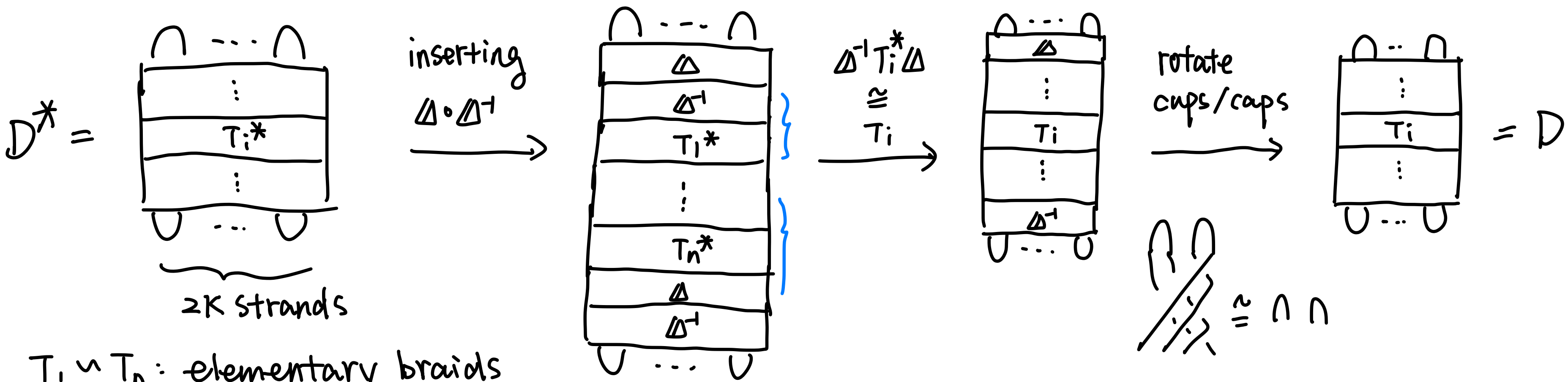

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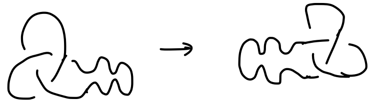

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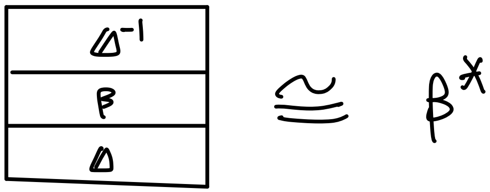

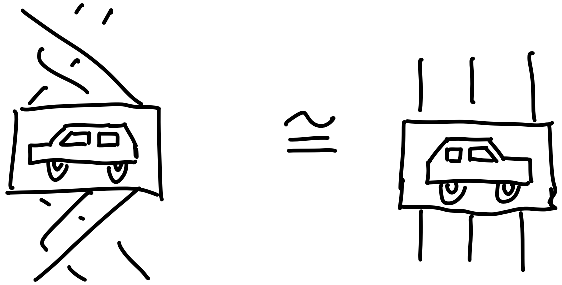

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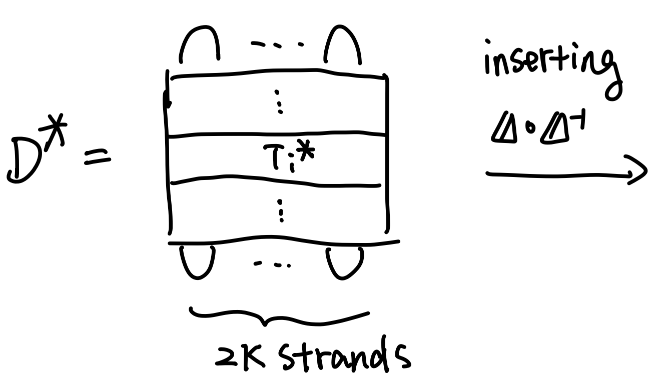

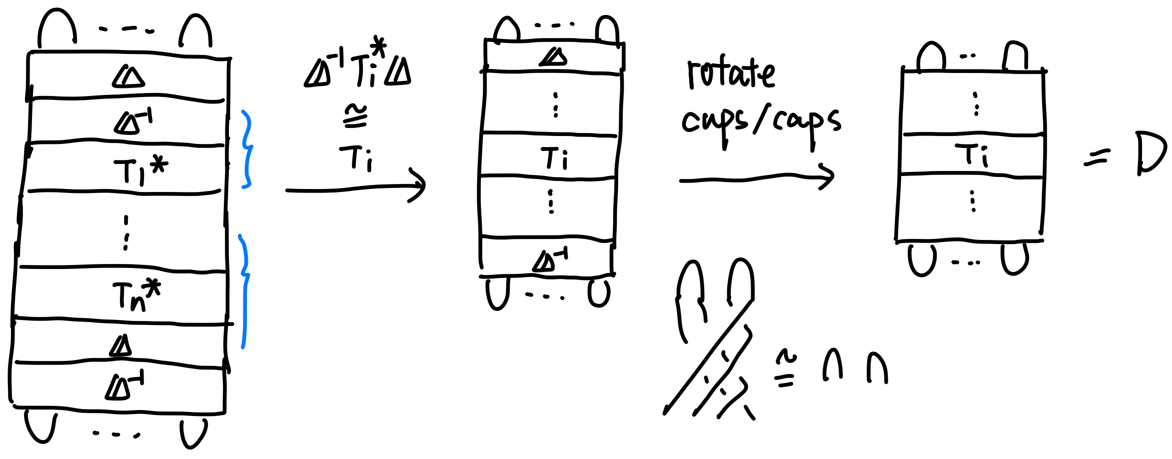

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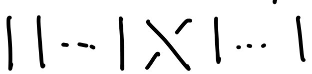
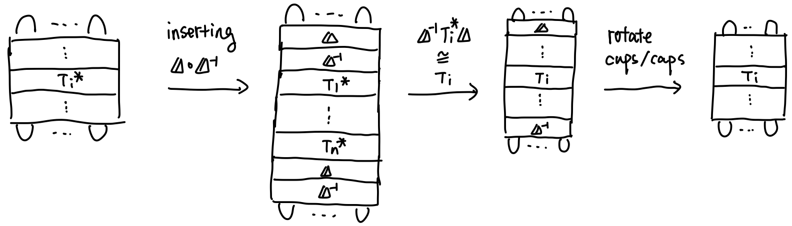
e.g. 

This gives a sequence of Reidemeister moves from D^* to D !

The local picture



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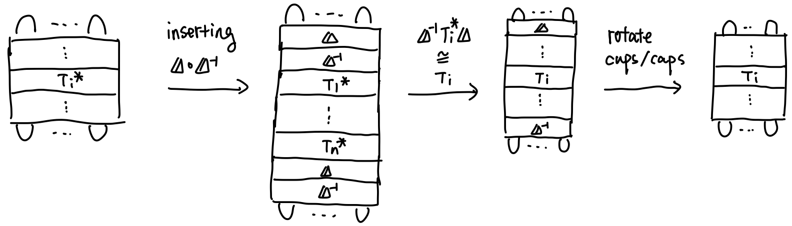

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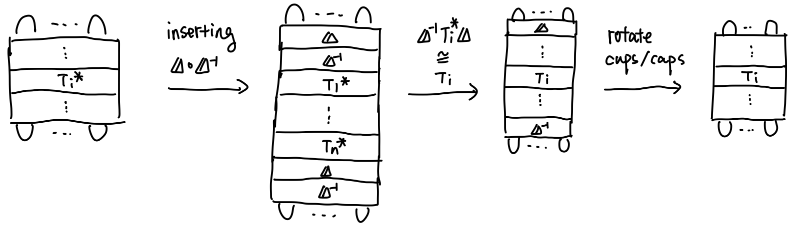

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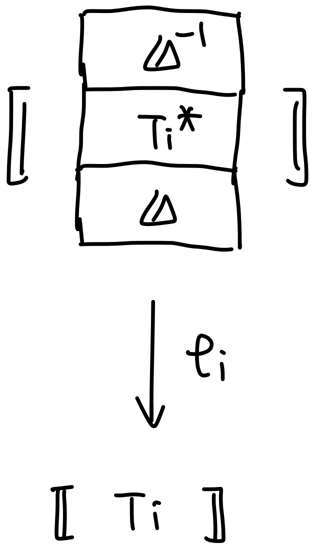

$[\![\,\cdot\,]\!]$: Bar-Natan's tangle invariant.

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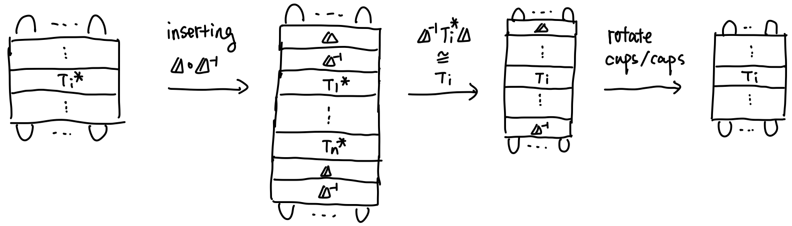

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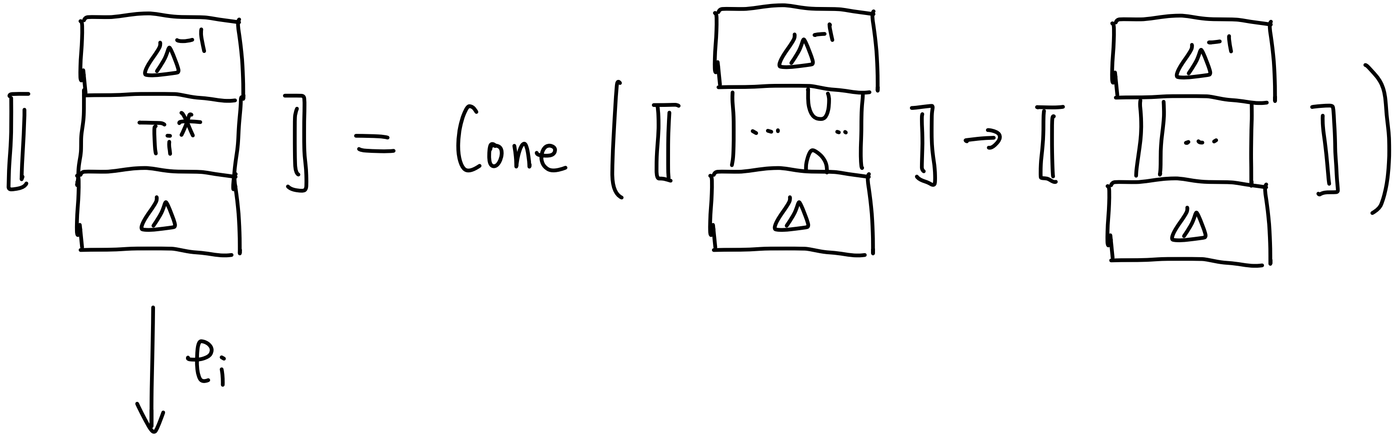
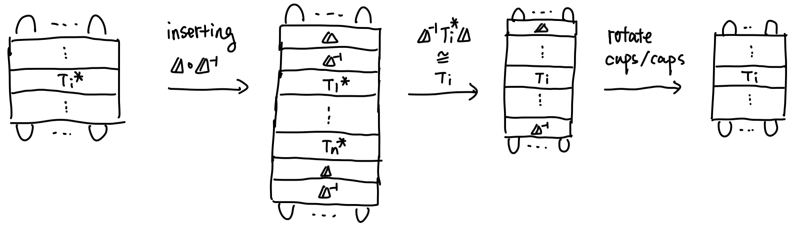


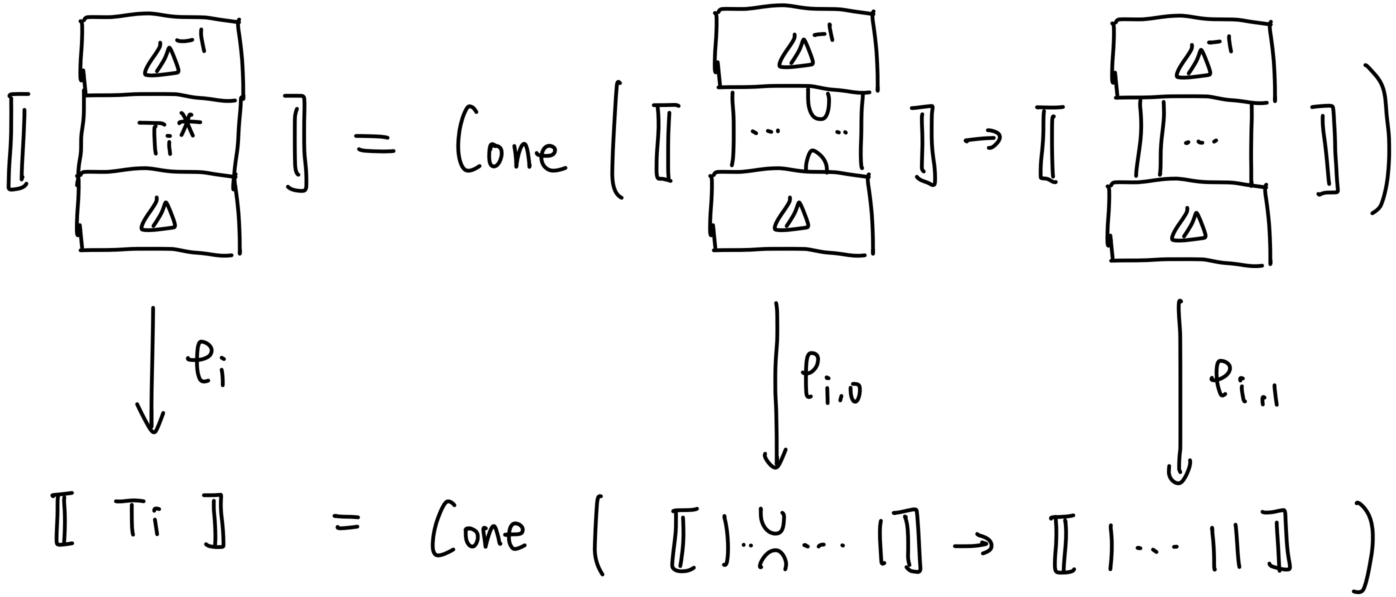
$$[\![\, T_i \,]\!] \quad = \quad \mathrm{Cone}\left([\![\, | \cdots \overset{\cup}{\cap} \cdots | \,]\!] \to [\![\, | \cdots | | \,]\!] \right)$$

$[\![\,\cdot\,]\!]$: Bar-Natan's tangle invariant.

The local picture



- The hardest part is to understand the second step.
- Locally:



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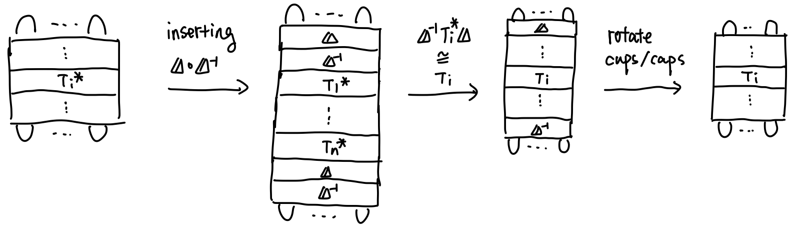

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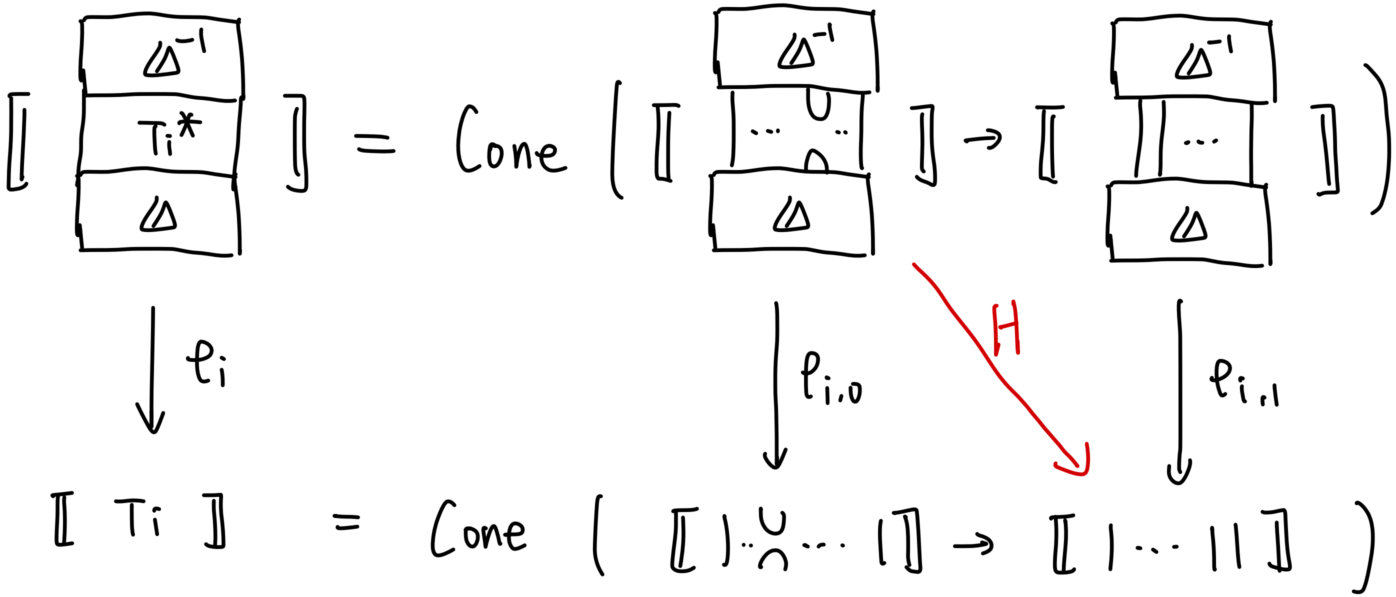

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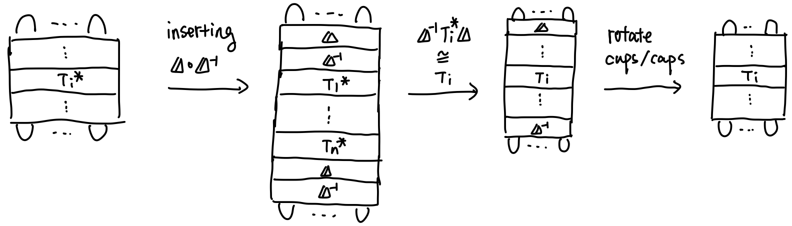

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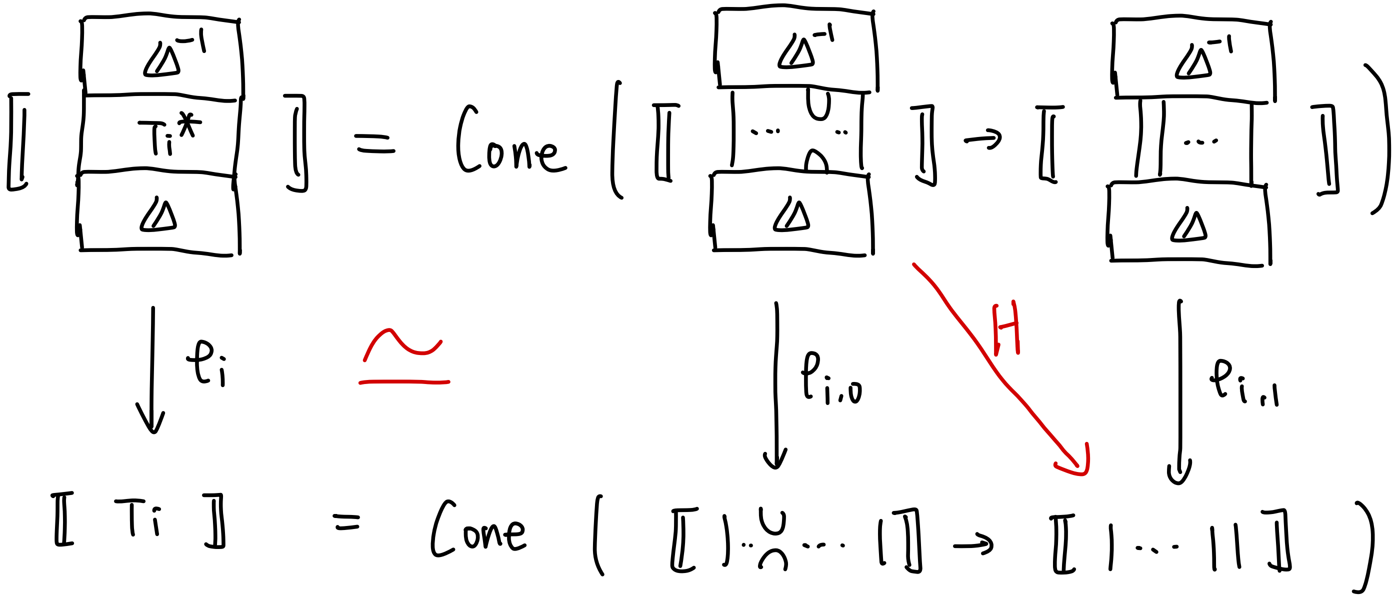
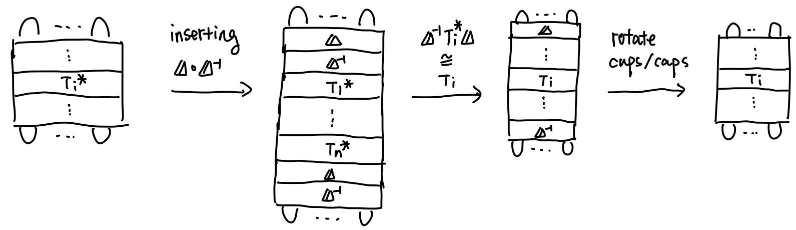


$[\![\cdot]\!]$: Bar-Natan's tangle invariant.

- Can use the right to compute the left!

cf. Rozansky. Willis.
Bar-Natan "Kh-simple"

The triviality



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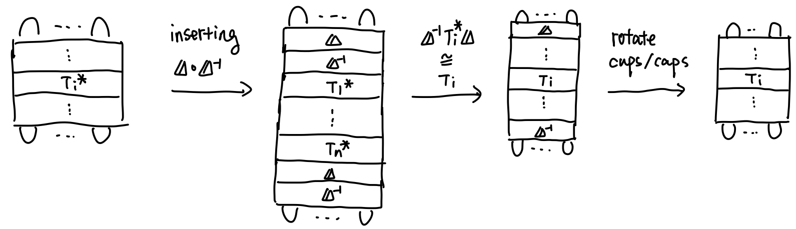
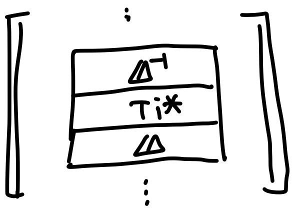
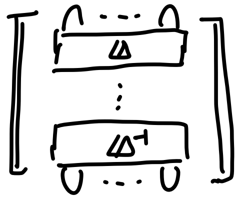
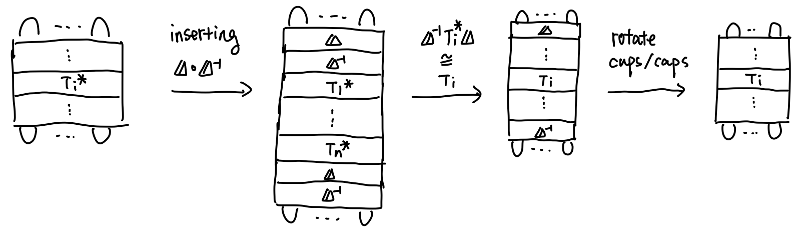


K can be computed by the composition

$[\![D]\!] \xrightarrow{\eta} [\![D^*]\!] \xrightarrow{\Omega}$ 

$\xrightarrow{\varphi}$ 
$\xrightarrow{\varphi_0} [\![D]\!]$

The triviality



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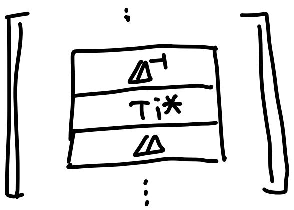
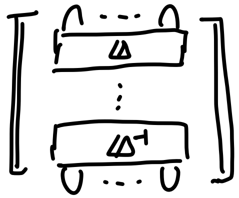
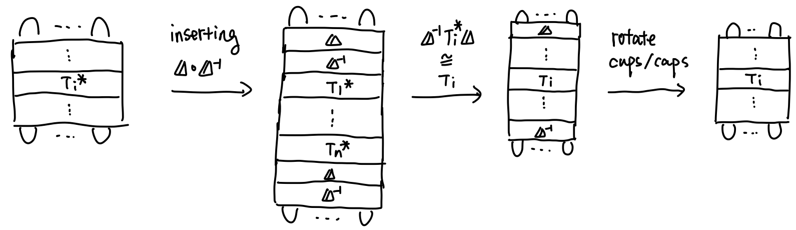
$[\![D]\!] \xrightarrow{\eta} [\![D^*]\!] \xrightarrow{\Omega}$ 

$\xrightarrow{\varphi}$ 

$\xrightarrow{\varphi_0} [\![D]\!]$

- Each complex appearing above has a $\{0,1\}^n$ / cubical grading from resolving crossings originally in D (i.e. the crossings in T_i).
- η, Ω, and φ_0 preserve the cubical grading; K preserves the homological grading.



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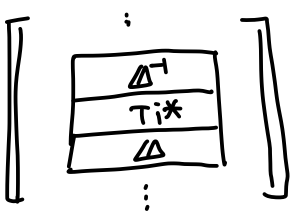
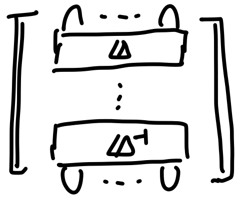
$[\![D]\!] \xrightarrow{\eta} [\![D^*]\!] \xrightarrow{\Omega}$  $\xrightarrow{\rho}$  $\xrightarrow{\rho_0} [\![D]\!]$

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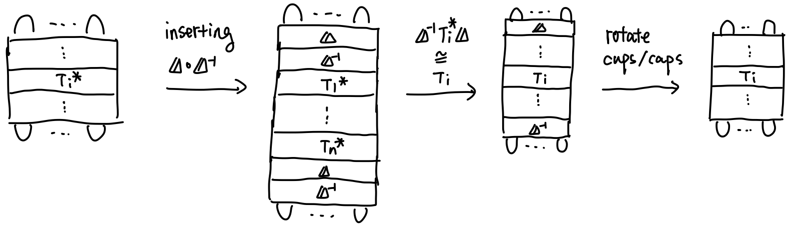
Claim. $\kappa \simeq \kappa'$ filtered w.r.t. the cubical grading.

cf. Alishahi-Truong-Zhang.

The triviality



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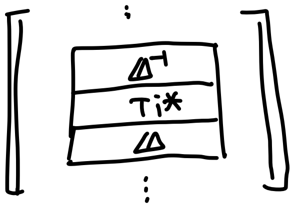
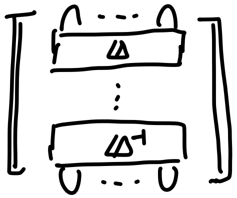
$[\![D]\!] \xrightarrow{\eta} [\![D^*]\!] \xrightarrow{\Omega}$  $\xrightarrow{\rho}$  $\xrightarrow{\rho_0} [\![D]\!]$

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$\Rightarrow$ κ' preserves (!) the cubical grading as $h = \sum$ cubical + constant.

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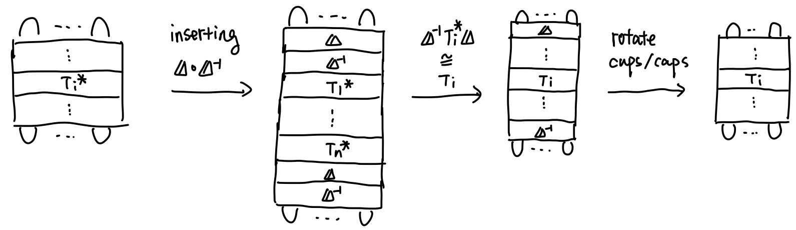
The triviality



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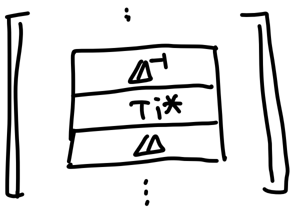
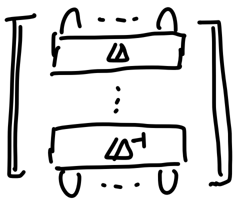
$[\![D]\!] \xrightarrow{\eta} [\![D^*]\!] \xrightarrow{\Omega}$ 

$\xrightarrow{\varphi}$ 

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$\Rightarrow$ $\kappa' = \mathbb{1}$ by computations for the unlinks.

cf. Alishahi-Truong-Zhang.

Strongly invertible knots

Strongly invertible knots

(K, τ) : strongly invertible

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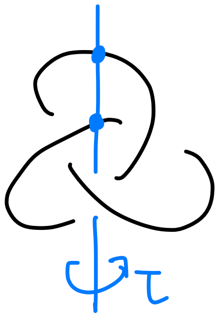
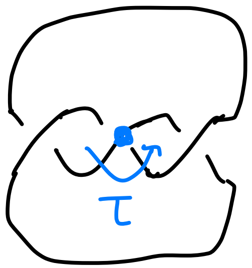
e.g



cf.
Lobb-Watson.
Sano

transvergent diagram



cf.
Lipshitz-Sarkar

intravergent diagram

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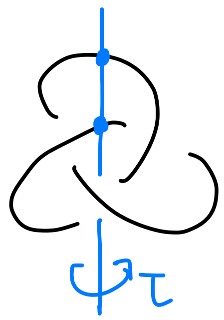

cf. Lobb-Watson. Sano

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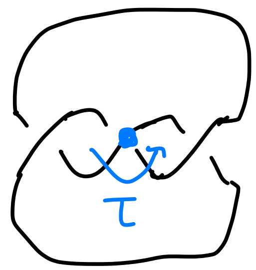

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intravergent diagram

They induce involutions on Khovanov chain complexes

$$\tau_t : CKh(D_t) \to CKh(D_t) \qquad \tau_i : CKh(D_i) \to CKh(D_i)$$

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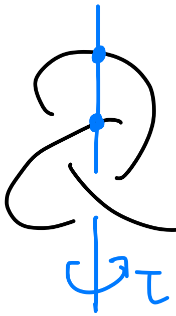

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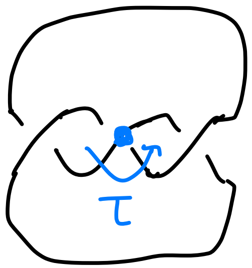

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Thm (Chen-Y.) $(\tau_t)_* = (\tau_i)_* : Kh(K) \to Kh(K)$ after fixing an iso.

$\varphi : Kh(D_i) \xrightarrow{\cong} Kh(D_t)$

induced by Reidemeister moves

Strongly invertible knots

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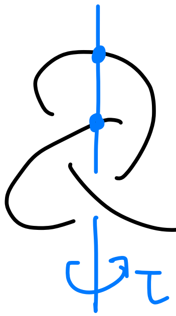

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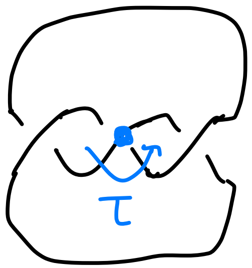

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· Corollary of $K \simeq \mathbb{I}$

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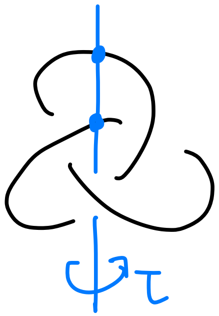

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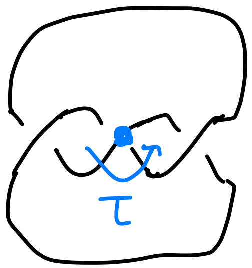

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- Corollary of $K \simeq \mathbb{I}$
- predicted by Witten's proposal.

Thank you for listening !