Matrix factorizations beyond type A

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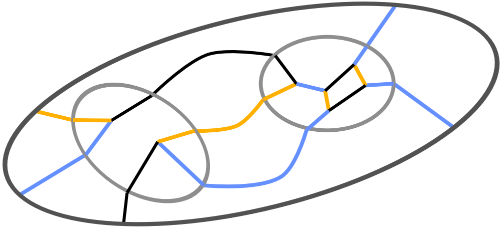
Contents

Goal: categorify spaces of invariant tensors

Fix $\mathfrak{g}$, set $\mathcal{V}$ of representations of $U_q(\mathfrak{g})$.

- For every sequence from $\mathcal{V}$, a linear, $\mathbb{Z}$-graded category whose Grothendieck group is the space of invariants in the corresponding tensor product
- Gluing functors categorifying index contractions

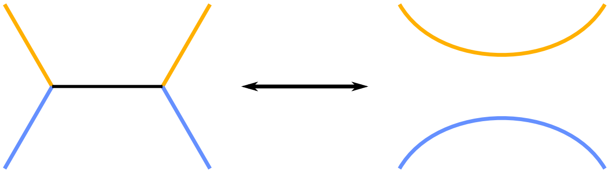


Approach via matrix factorizations

- Hope: accomplish this using (sub)categories of matrix factorizations, when $\mathfrak{g}$ is simply-laced and $\mathcal{V}$ is the set of minuscules.
  - Matrix factorization = chain complex, but $d^2 = w$.
  - w is a sum of terms from boundary points.
  - Gluing = fusion, tensor over glued vertices and forget those variables. Still finite if w nondegenerate.
- Done for $\mathfrak{g} = \mathfrak{sl}_n$: Khovanov-Rozansky 2004, Yonezawa 2011, Wu 2013, with some needed input on the decategorified side from Cautis-Kamnitzer-Morrison 2012
- Begun for $\mathfrak{g} = \mathfrak{so}_{2n}$: Khovanov-Rozansky 2007 constructed potentials, factorizations for vector representations, but decategorification and Reidemeister invariance conjectural.

Why matrix factorizations?

- Form a 2-category with very good duality properties (Carequeville-Murfet 2012)
  - Identities give arc factorizations
  - Adjunction maps give foam functoriality
- Easy to define Lee-type deformations, s-type invariants



- Two significant challenges
  - Fixing signs
  - Far from obvious that the decategorification is as expected

Recall: colored $\mathfrak{sl}(N)$ homology

$\mathfrak{sl}(N)$ homology: potentials and factorizations

- One potential for each minuscule representation = integer from 1 to N
- $w^{\wedge^k} = \frac{1}{N+1} p_{N+1}(e_1, \ldots, e_k$
  - $\partial w^{\wedge^k} / \partial e_i = \pm h_{N+1-i}$, so cohomology of unknot is $\mathcal{J}(w^{\wedge^k}) = H^* \mathcal{G}r(N, k)$
- A factorization whenever $k_1 + \cdots + k_j = N$
  - p_{N+1} is primitive in the Hopf algebra of symmetric functions
  - For $k = N$, the potential $w_{\wedge^N}$ is Morse $\rightsquigarrow$ Knörrer periodicity

$\mathfrak{sl}(N)$ homology: the categories

- To a specified boundary, assign the full subcategory of the homotopy category of matrix factorizations on all direct summands of fusions of these fundamental vertices.
- Direct sum decompositions of certain fusions $\rightarrow$ skein relations in the Grothendieck group
  - Exactly match sufficient skein relations for $\mathfrak{sl}_N$ invariant spaces
  - Reduce any closed diagram to a direct sum of empty diagrams, so in particular all Hom spaces in the category are in a single $\mathbb{Z}/2$ grading.

Potentials for Hermitian symmetric spaces

Hermitian Symmetric Spaces

- G/P for P a maximal parabolic corresponding to a minuscule node for G, G simply-laced.
- Equivalent to asking that, inside compact real G, P is the centralizer of some α with $\alpha^2 \in Z(G)$ and a central $U(1)$ in $Z(\alpha)$.
  - This $U(1)$ gives a complex structure
- $H^*(G/P)$ has same graded dimension as the corresponding representation.
- Classification:
  - Type A: $SU(N)/S(U(k) \times U(N-k))$
  - Type D: $SO(2N)/U(2N)$ (two versions), $SO(2N)/(SO(2N-2) \times SO(2))$
  - Type E: $E_6/(\mathrm{Spin}(10) \times U(1))$, $E_7/(E_6 \times U(1))$

Potentials

Observation: For X Hermitian symmetric, $H^*(X) = \mathcal{J}(w)$ for some potential w.

- Originally from physics, Lerche-Vafa-Warner 1989, see also Lerche-Warner 1991 and Gukov-Walcher 2005
- Idea – algebras of ground states in certain supersymmetric QFTs should have the form $\mathcal{J}(w)$, in particular Σ-models in Hermitian symmetric spaces.
- Type A potentials are the power sums $\frac{1}{N+1}p_{N+1}$, and the exceptional cases are computed in Lerche-Warner 1991.
  - $w^{E_6}(x, y) = xy^3 - 3x^5y^2 + 2x^9y - \frac{5}{13}x^{13}$

Type D Potentials

- $X = SO(2N)/U(N)$, the space of orthogonal, $\pm$-oriented complex structures carries a canonical $\mathbb{C}^N$ bundle E
- E has a section and $E \oplus \overline{E}$ is trivial, so

$$c(E) = Z(t) = 1 + \sum_{i=1}^{N-1} z_i t^i$$

  and $Z(t)Z(-t) = 1$.
- Alternatively, $Z^{\text{even}}(t) = \sqrt{1 + Z^{\text{odd}}(t)^2}$

Type D Potentials

- The potential $w^S = [t^{2N-1}]F(Z^{\mathrm{odd}}(t))$ has $\mathcal{J}(w^S) = H^*(X)$, for

$$F(t) = \int_0^t \sqrt{1+s^2}ds = \frac{1}{2}t\sqrt{1+t^2} + \frac{1}{2}\log(t+\sqrt{1+t^2})$$

- For the vector representation V,

$$w^V = xy^2 \pm \frac{1}{2n-1}x^{2n-1}$$

  - Used in Khovanov-Rozansky 2007's conjectural categorification of the Kauffman polynomial
  - x, y are Euler classes of canonical $\mathbb{R}^2$, $\mathbb{R}^{2N-2}$ bundles over real oriented Grassmannian.

Factorizations for flag varieties

Needed ingredients

Fundamental factorizations in type A correspond to

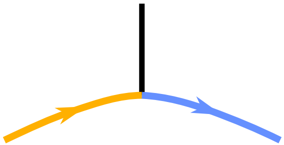
- $k_1 + \cdots + k_j = N$
- Tensor products of minuscules with one-dimensional invariant spaces
- Partial flag varieties

First goal: whenever A, B, C are minuscules with an invariant tensor in $A \otimes B \otimes C$, build a matrix factorization of $w^A + w^B + w^C$

- This can be done!

Trivalent intertwiners in type D

- There is a map $S^{\pm} \otimes V \to S^{\mp}$, so an invariant tensor in $S^{+} \otimes V \otimes (S^{\mp})^{\vee}$
- So, want a factorization of $w^{S}(z_i) + w^{V}(x, y) - w^{S}(z_i')$ to use as the fundamental vertex.
- Such a thing does exist, and its endomorphism ring is $H^{*}(SO(2N)/U(N))$.



Exceptional trivalent intertwiners

- The defining representation V of E_6 has an invariant tensor in $V \otimes V \otimes V$.
  - In fact, $w^{E_6}(x_1, y_1) + w^{E_6}(x_2, y_2) + w^{E_6}(x_3, y_3)$ lies in the ideal with the following generators, allowing for the construction:

$$x_1 + x_2 + x_3$$

$$y_1 + y_2 + y_3 - \frac{1}{2}(x_1^4 + x_2^4 + x_3^4)$$

$$x_1 y_1 + x_2 y_2 + x_3 y_3 - \frac{2}{5}(x_1^5 + x_2^5 + x_3^5)$$

- Dimension agrees with the cohomology of $E_6/(\mathrm{Spin}(8) \times U(1)^2)$, probably also agrees as a ring.

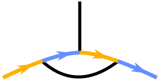
Gluings of the fundamental vertices

- Fusions of the vertex factorizations, respecting the edge colorings, give a matrix factorization assigned to any appropriately decorated graphs
  - Type D: Every vertex incident to one vector edge and two spinors
  - E_6: Edges oriented, every vertex a source or sink
- Main question: How do these fusions decompose under direct sums?
  - Special case: what is the vector space assigned to a closed graph?

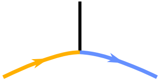
Conjecture: the K-theory of the subcategory generated by summands is isomorphic to the space of invariant tensors, intertwining fusion of factorization with index contraction.

Skein rules

Spinor Skeins

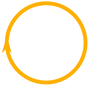


$$\cong [n-1]$$

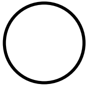


$$\cong [n]$$

$$\cong \frac{[2n-4]!!}{[n-2]!}$$

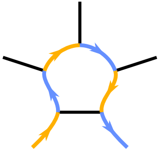
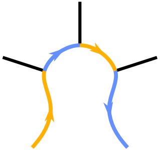
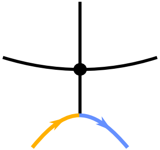


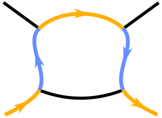
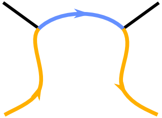
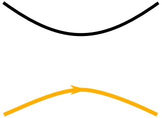
$$\cong \frac{[2n-2]!!}{[n-1]!}$$

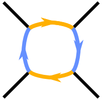
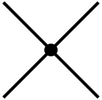


$$\cong \frac{[n][2n-2]}{[n-1]}$$

Spinor Skeins

 $\cong \; [n-3]$  $\oplus$ 

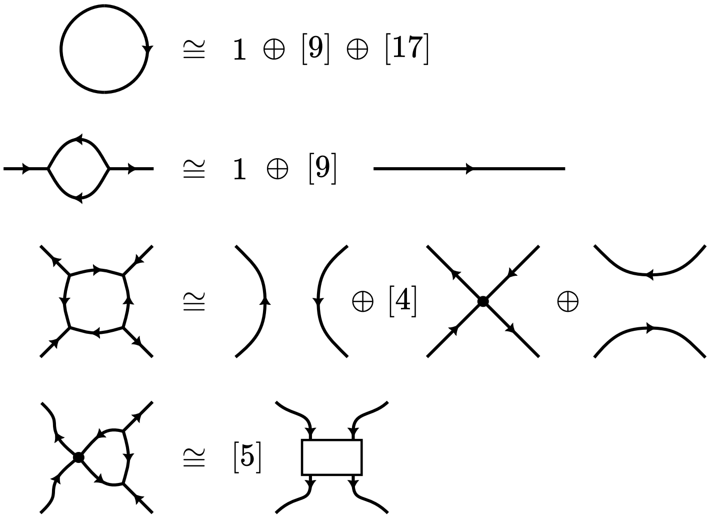
 $\cong \; [n-2]$  $\oplus$ 

 $\cong \; \dfrac{[2n-6]!!}{[n-3]!}$ 

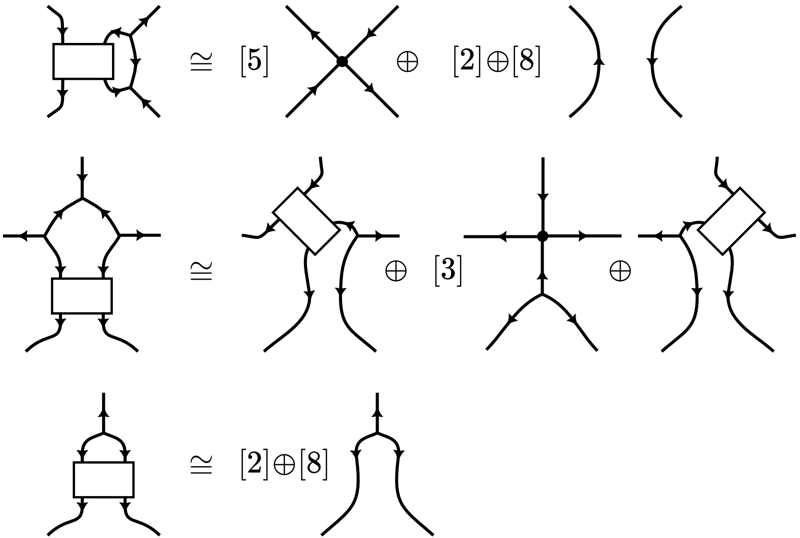
Spinor Skeins

- Decategorified versions of these relations hold in $\mathrm{Rep}(U_q(\mathfrak{so}_{2n}))$.
- Suffice to define and prove Reidemeister invariance for vector-vector and vector-spinor crossing complexes
- Not complete – missing relations among "ladder" graphs with four spinor boundary edges
    - When the correct relations are added, these suffice to compute any closed graph
    - Not clear how to prove categorified version
    - Also unknown if these relations are enough to calculate K-theory with general boundary

Conjectural E_6 Skeins



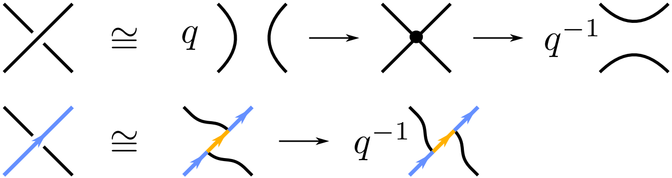
Conjectural E_6 Skeins



Conjectural E_6 Skeins

- These hold at the decategorified level, work in progress to check them for the categorification
- Suffice to define crossings, check Reidemeister moves
- Seems quite tricky to know if these relations can reduce any closed diagram
  - No other clear approach to computing dimensions of the spaces assigned to closed graphs, or to checking that they're concentrated in one $\mathbb{Z}/2$ degree

Crossing complexes



- Currently don't know how to categorify the spinor-spinor crossing
  - Should have $\sim N/2$ terms