Obstructing Cosmetic Crossing Changes

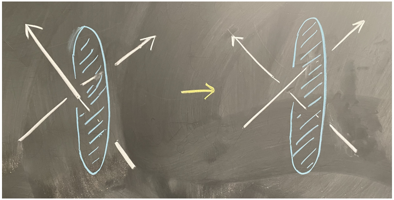
Joe Boninger (supported by NSF award 2202704)

Boston College

June 11, 2025

The Cosmetic Crossing Conjecture

$K \subset S^3$



A crossing change (and disk)

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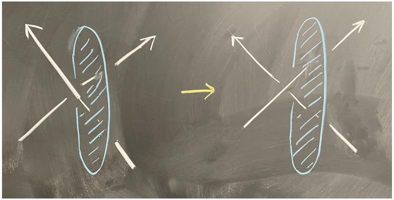

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A crossing change is cosmetic if it preserves K.

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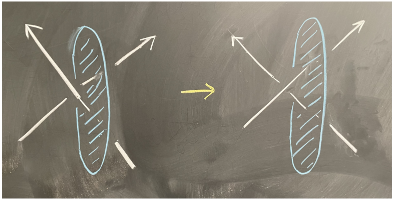

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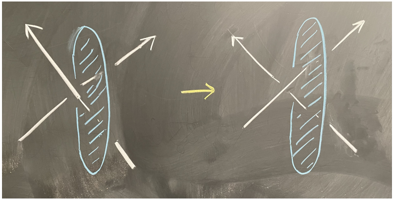

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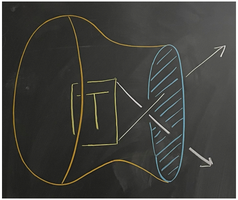

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Motivation and Background

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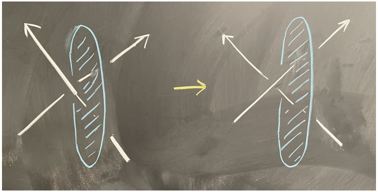

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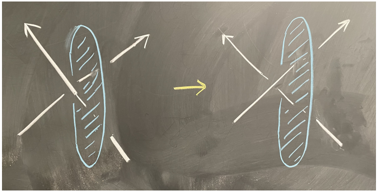

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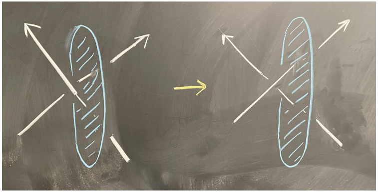

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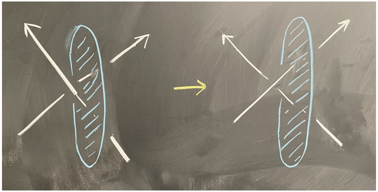

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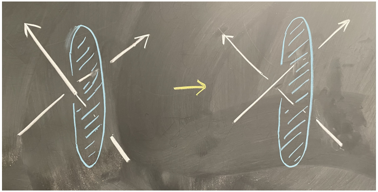

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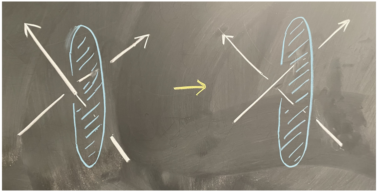
  - $+$ more (Balm, Friedl, Powell, Wang...)



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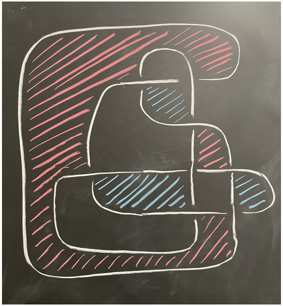
- $\uparrow$ Applies to all alternating knots.

Special Alternating Links

Definition

K is a special alternating link if an alternating diagram of K has an orientable checkerboard surface.



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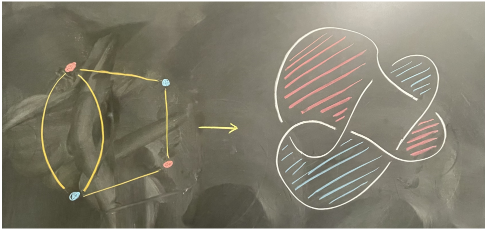

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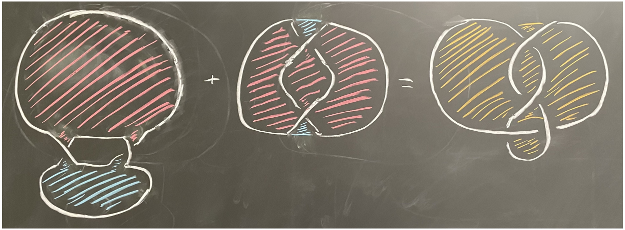
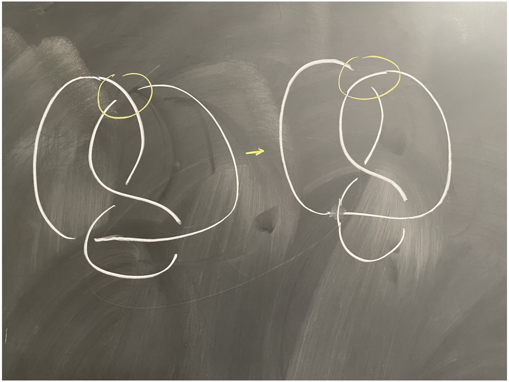
- SALs can be constructed from any bipartite planar graph.
- Every alternating knot is a Murasugi sum of SALs.



First Attempt

Proposition

Alternating link *diagrams* do not admit cosmetic crossings.



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1. Form the Tait graph G of D.

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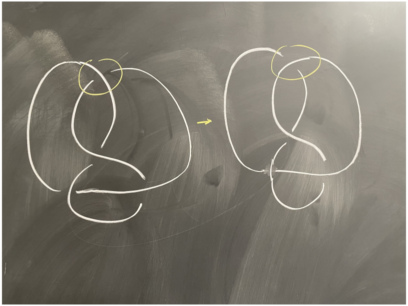

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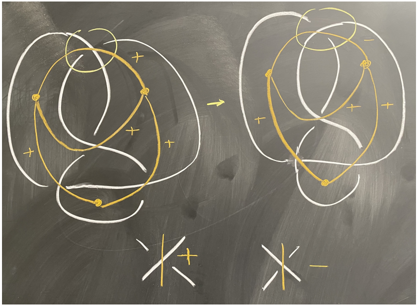

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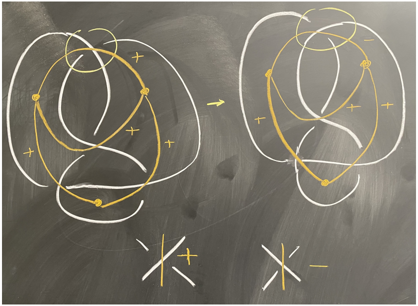

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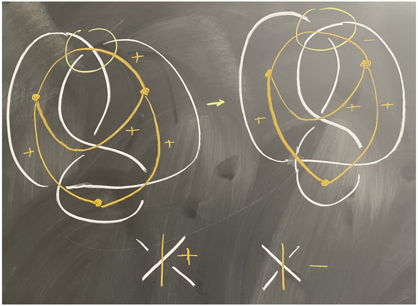

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4. For D alternating, changing a sign changes $\det(K)$.

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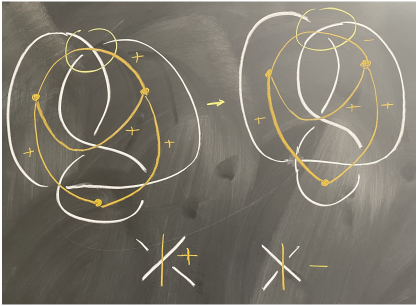

The Gordon-Litherland Form

For a surface $S \subset S^3$, $\partial S = K$, the Gordon-Litherland form $\mathcal{G}_S$ is a pairing $H_1(S)^2 \to \mathbb{Z}$ with $|\det(\mathcal{G}_S)| = \det(K)$.

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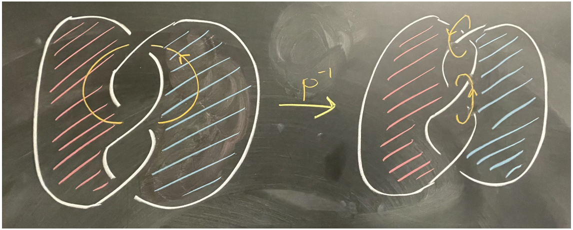

$$\mathcal{G}_S([\alpha],[\alpha]) = 2$$

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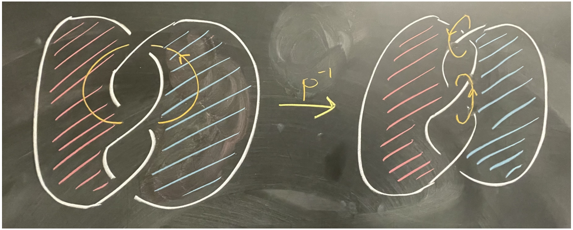

Proposition (Murasugi; Greene)

If D is alternating, the checkerboard surfaces of D have positive- and negative-definite GL forms.

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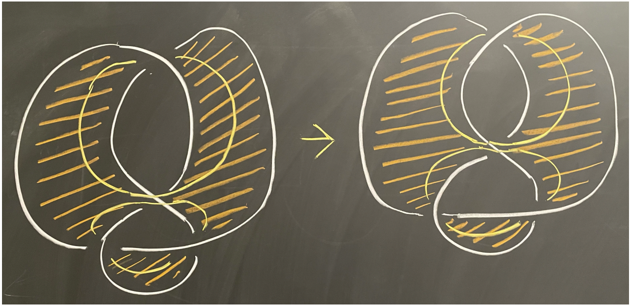

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The Gordon-Litherland form of S changes as follows:

$$A_+ = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & & A' \\ \vdots & & \end{bmatrix} \mapsto A_- = \begin{bmatrix} a_{11} - 2 & a_{12} & \cdots \\ a_{21} & & A' \\ \vdots & & \end{bmatrix}$$

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Since A_+ is definite, $\det(A') \neq 0$, so $|\det(A_+)| \neq |\det(A_-)|$. □

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Theorem 1 (B., 2023)

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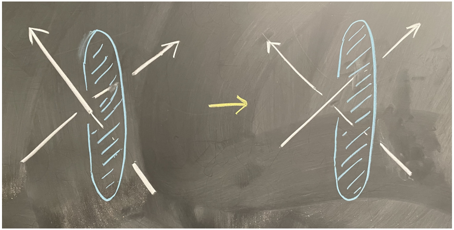

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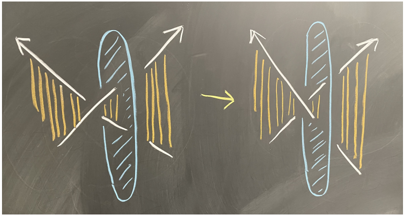

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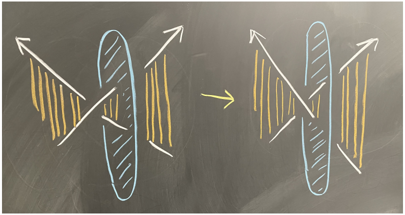
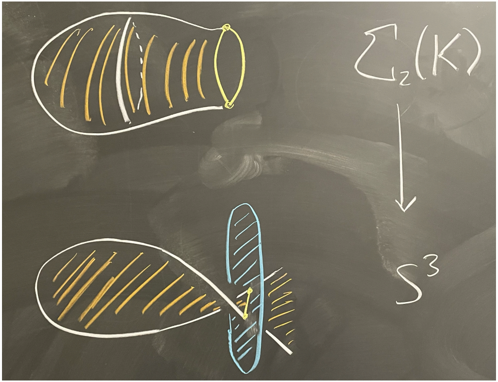


If U does not separate S, the previous argument works.

What if U Separates S?

...Then the lift of the crossing arc $U \cap S$ to $\Sigma(K)$ is nullhomologous.



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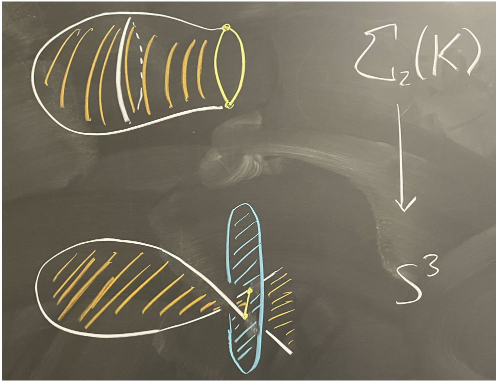
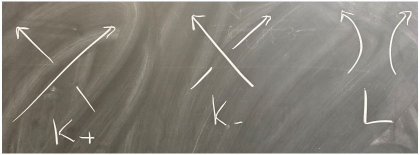


Lidman-Moore (+ Gainullin): if $\Sigma(K)$ is an L-space and crossing arc lift is nullhomologous, then U is nugatory.

What's Really Being Obstructed?



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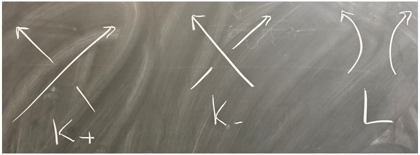

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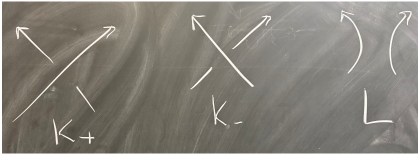

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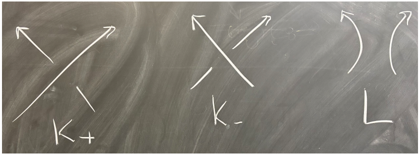

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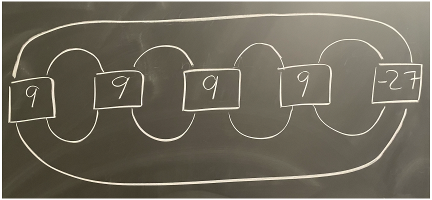
Additional Results

Corollary 1 (B., 2023)

If $L \subset S^3$ has $\Sigma(L)$ an L-space, and $H_1(\Sigma(L))$ has a minimal generating set of size $2g(L)$, then L does not admit cosmetic crossing changes.



$$K = P(9, 9, 9, 9, -27)$$

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Corollary 2 (B., 2024)

The cosmetic crossing conjecture holds for all pretzel knots $K = P(p_1, p_2, p_3, p_4, q)$ such that:

- $p_i \geq 1$ and $q > \min(p_1, \ldots, p_4)$.
- $p_i \equiv 1$ and $q \equiv 3 \bmod 4$.

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- The conjecture remains open for four knots with ≤ 10 crossings, and 18 alternating knots with 11 crossings.

Thanks for watching!