Non-sliceness of cables of figure-eight knot via real Seiberg-Witten theory

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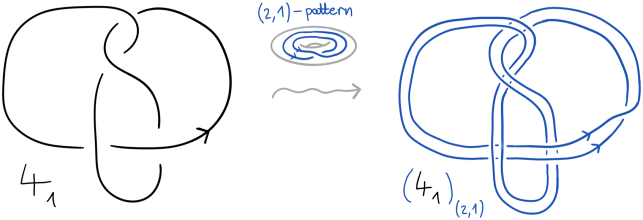
June 4, 2025

Today's result

In this talk, I announce the proof of the following theorem.

Theorem (K.–Park–Taniguchi 2025)

For any integer $n \neq 0$, the $(2n, 1)$-cable of the figure-eight knot is not smoothly slice.



But why is this important?

Smooth sliceness of knots

A knot K is smoothly slice if it bounds a smooth disk in B^4. How do we know if the given knot is smoothly slice?

It is incredibly hard when your knot might be smoothly slice.

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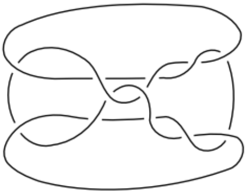
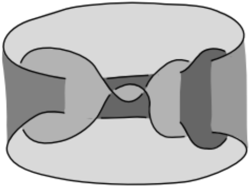




It is easier to think about ribbonness.

Slice–ribbon conjecture

Ribbon knots are smoothly slice. Are smoothly slice knots ribbon?

Slice–ribbon conjecture! (Fox 1962)

I don't think this conjecture is believable, but there has been no counterexamples anyway.

Potential counterexamples to slice–ribbon

Slice–ribbon might be false. How to find a counterexample?

Strategy: find a knot that is not ribbon but looks very slice.

Theorem (Miyazaki 1994)

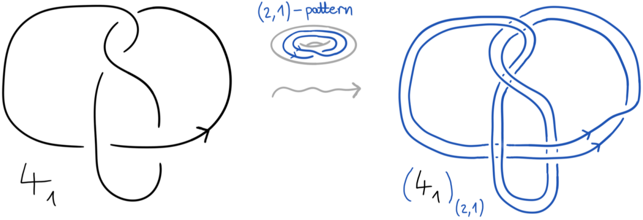
Let K be a knot that is negatively amphichiral, fibered, and has irreducible Alexander polynomial. Then for any integer $n > 0$, the cabled knot $K_{2n,1}$ is never ribbon.

$\rightarrow$ *Family of potential counterexamples!*

Potential counterexample: $(4_1)_{2,1}$

Simplest case ($n = 1$, $K = 4_1$): $(4_1)_{2,1}$.

$\Delta_{4_1}(t) = t - 3 + t^{-1}$ is irreducible.



Is $(4_1)_{2,1}$ slice? (Kawauchi 1980)

Cables of the figure-eight

Regarding smooth sliceness of even cables of figure-eight:

- $(4_1)_{2,1}$ is not smoothly slice and has infinite concordance order. (Dai–K.–Mallick–Park–Stoffregen 2022): *Involutive HF*
- $(4_1)_{4n+2,1}$ is not smoothly slice. (K.–Park–Taniguchi 2024): *Real Frøyshov invariants*
- $(4_1)_{2n,1}$ is not smoothly slice. (K.–Park–Taniguchi 2025): *Real 10/8 inequality*

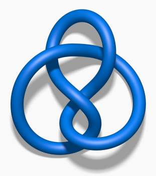
Note: odd cables of 4_1 were studied by Hom–K.–Park–Stoffregen in 2020.

Difficulty of showing the non–sliceness

The non–sliceness of $(4_1)_{2,1}$ had been open for 42 years. It is:

- algebraically slice;
- bounds nullhomologous disks in $\mathbb{C}P^2$ and $-\mathbb{C}P^2$;
- rationally slice. (4_1 is a *strongly negative amphichiral knot*)

*SNA*c*K*
à la Keegan Boyle



SNAK

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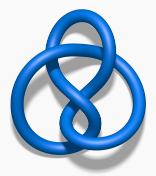

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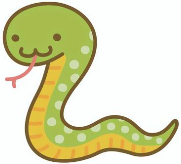


*SNAK*e

The $(2,1)$-cable case: A first try

Knot K $\rightarrow$ branched double cover $\Sigma(K)$.

Disk $D \subset B^4 \rightarrow \Sigma(D)$ is a smooth $\mathbb{Q}HB^4$. Thus:

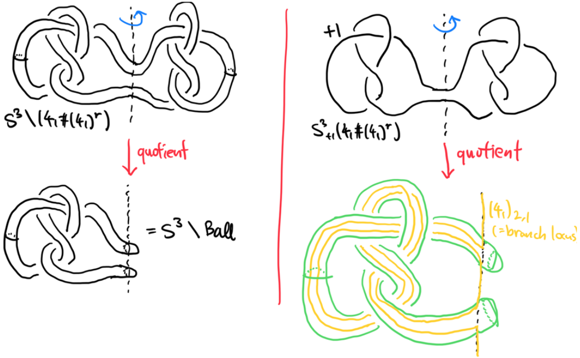
$\Sigma((4_1)_{2,1})$ does not bound a smooth $\mathbb{Q}HB^4$ W $\longrightarrow$ $(4_1)_{2,1}$ does not bound a smooth disk in B^4

But such a 4-manifold W exists.

First try fails

Montesinos trick: $\Sigma((4_1)_{2,1}) \simeq S^3_{+1}(4_1 \sharp (4_1)^r)$. $4_1 \sharp (4_1)^r$ is slice.

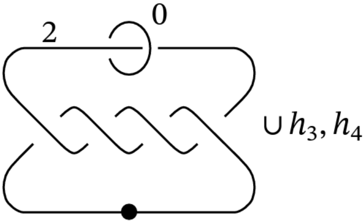
$\rightarrow \exists$ Contractible smooth 4–manifold bounding $\Sigma((4_1)_{2,1})$.



We need the *deck transformation* τ.

Second try also fails

For any smooth $\mathbb{Q}HB^4$ W bounding $\Sigma((4_1)_{2,1})$, τ does not extend smoothly to W $\longrightarrow$ $(4_1)_{2,1}$ is not slice



A smooth $\mathbb{Q}HB^4$ Z_0 containing a disk D bounding $(4_1)_{2,1}$.
$\Sigma(Z_0, D)$ is not *spin*!

Third times the charm

Obstruction: $\Sigma((4_1)_{2,1})$ does not bound a smooth $\mathbb{Q}HB^4$ W such that:

- τ extends to a smooth involution of W, and
- W carries an τ-invariant spin structure $\mathfrak{s}_0$.

In the case of $(2,1)$-cable, this obstruction was carried out using HF.

How the $(2,1)$-cable case was proven

Heegaard Floer homology:

$$\mathbb{Z}HS^3\ Y \mapsto \mathbb{Z}-\text{graded } \mathbb{F}_2[U]-\text{module } HF^-(Y).$$

Two symmetries:

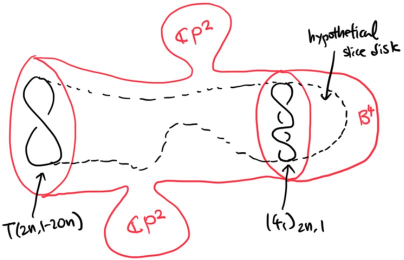
- Charge conjugation $\iota : HF^-(Y) \to HF^-(Y)$;
- $f \in \mathrm{Diff}^+(Y)/\text{isotopy} \mapsto f_* : HF^-(Y) \to HF^-(Y)$.

No "local" degree 0 element $x \in HF^-(\Sigma((4_1)_{2,1}))$ such that $\iota(x) = \tau_*(x) = x$ $\longrightarrow$ $(4_1)_{2,1}$ is not slice

But this works only in the $(2,1)$-cable case, due to a computability issue.

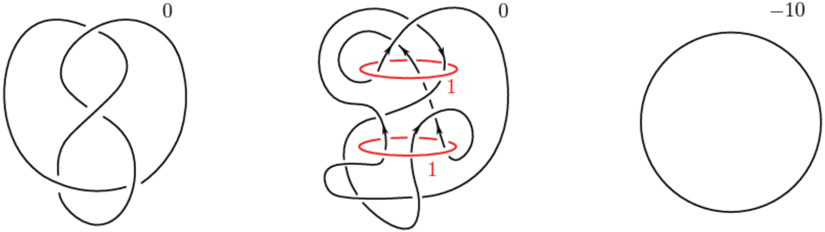
Observation of Aceto–Castro–Miller–Park–Stipsicz:

$\exists$ smooth annulus C_n from $(4_1)_{2n,1}$ to $T_{2n,1-20n}$ of class $(2n, 6n)$ in $2\mathbb{C}P^2$.

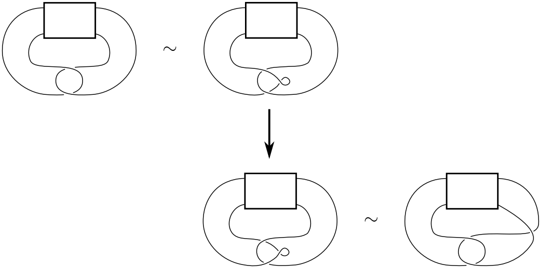


Constructing the annulus C_n

$C_n = (2n, 1)$–cable of this annulus from 4_1 to unknot.



This magical untwisting sequence came originally from Ballinger's paper on configurations of spheres in $n\mathbb{C}P^2$.



A concordance from twist knot K_n to K_{n-1} in $\mathbb{C}P^2$.

Linking number is 1 if n is even; 4 is n is odd.

Branched cyclic covers

$C_n \subset 2\mathbb{C}P^2$ bounds $(4_1)_{2n,1}$; homology class $(2n, 6n)$. Consider:

- $2n = 2^k \cdot m$, m odd;
- $W = \Sigma_{2^k}(C_n)$; (Instead of double cover, we take 2^kth cover!)
- τ is the deck transformation.

Then $\tau^2 = 1$ and W carries a unique τ-invariant spin structure $\mathfrak{s}_0$.

Perfect setting for applying real Seiberg–Witten theory!

Define $\kappa_R^{(k)}(K) := \kappa_R(\Sigma_{2^k}(K), \mathfrak{s}_0, \tau^{2^{k-1}}) \in \frac{1}{16}\mathbb{Z}$ for $K \subset S^3$.

The real 10/8 inequality

We apply the real 10/8 inequality of Konno–Miyazawa–Taniguchi.

Theorem (Konno–Miyazawa–Taniguchi)

Let K and K' be knots in S^3, and let X be an oriented smooth compact connected cobordism from S^3 to S^3 with $H_1(X;\mathbb{Z}) = 0$. Let S be an oriented, compact, connected, properly and smoothly embedded concordance in X from K to K' such that $\mathrm{PD}(w_2(X)) = [S]/2^k \pmod 2$. Then we have

$$-\frac{\sigma(\Sigma_{2^k}(S))}{16} + \frac{\alpha}{2} \leq b^+(\Sigma_{2^k}(S)) - b^+(\Sigma_{2^{k-1}}(S)) + \kappa_R^{(k)}(K') - \kappa_R^{(k)}(K).$$

In all cases, $\alpha = 0$ works. Moreover, if K is k-strongly spherical and $b^+(\Sigma_{2^k}(S)) > b^+(\Sigma_{2^{k-1}}(S))$, then $\alpha = 1$ works.

Higher sphericity of torus knots

What are k-strongly spherical knots?

Definition

For a positive integer k, we say that a knot K is *k-strongly spherical* if

$$SWF_R^{(k)}(K) := SWF_R(\Sigma_{2^k}(K), \tau^{2^{k-1}}) \simeq (\mathbb{C}^r)^+$$

as a $\mathbb{Z}_4$-equivariant stable homotopy type for some $r \in \mathbb{Q}$.

Note that, in the definition, $r = -\kappa_R^{(k)}(K)$.

Lemma (Higher sphericity of torus knots)

For any $k > 0$ and integers m, q, the torus knot $T_{2^k m,q}$ is k-strongly spherical.

Idea: use the correspondence of Mrowka–Ozsváth–Yu. Suppose:

- Y is a Seifert $\mathbb{Q}HS^3$ fibered over S^2;
- The canonical Spin^c structure $\mathfrak{s}_Y^{can}$ on Y is self-conjugate.

Then, with some nice choice of a metric on Y, the irreducible critical points of the CSD functional for $(Y, \mathfrak{s}_Y^{can})$ correspond to elements of

$$\bigcup_{\substack{0 \le \deg(E) < \frac{\deg\left(K_{\check{S}^2}\right)}{2} \\ [E]=[E_0] \in \mathrm{Pic}^t(Y/S^1)/\mathbb{Z}[L_Y]}} C_+(E) \sqcup C_-(E).$$

Goal: For $Y = \Sigma_{2^k}(T_{2^k m,q})$, the action of $j \circ \tau$ swaps $C_+(E)$ and $C_-(E)$.

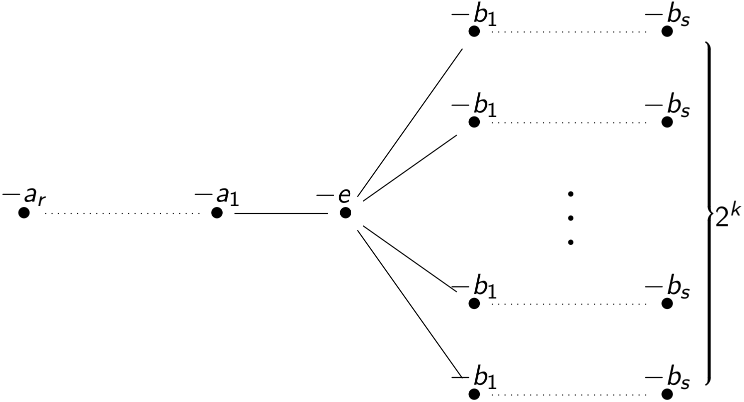
$\Rightarrow$ Fixed point part only comes from the reducible solution.

$\Rightarrow$ $SWF_R(Y, \tau)$ is a complex sphere.

Disclaimer: I am not a gauge theory expert.

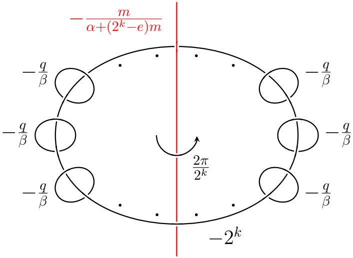
Proving this reduces to showing that the action of τ is holomorphic on the complex 1-orbifold Y/S^1.

To show that, we start from the almost-rational negative definite plumbing Γ with $W_\Gamma = \Sigma_{2^k}(T_{2^k m,q}) = \Sigma(2^k, 2^k m, q)$.

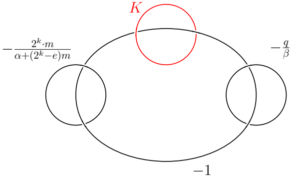


2^k identical legs $\Rightarrow$ 2^k-fold symmetry τ on W_Γ permuting them cyclically.

We turn it into a surgery diagram.



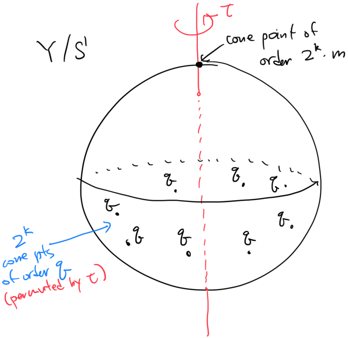
Action of τ on ∂W_Γ



Branching locus $K \subset \partial W_\Gamma/\tau$

$$\Rightarrow \partial W_\Gamma/\tau = S^3,\ K = T_{2^k m,q}.$$

The action of τ commutes with the Seifert S^1-action. Taking the S^1-quotient induces the following action.



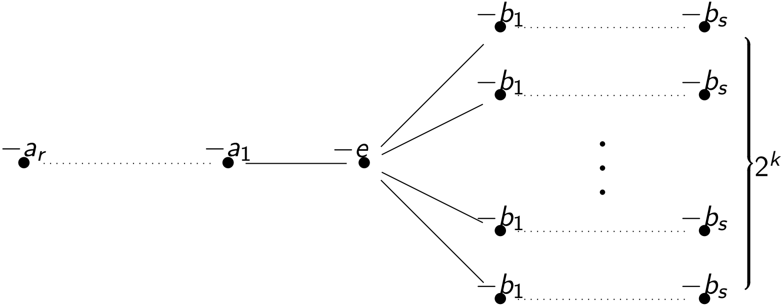
This action is "linear" and thus holomorphic.

$\Rightarrow$ *$T_{2^k m,q}$ is k-strongly spherical.*

Computing $\kappa_R^{(k)}(T_{2^k m,q})$

But the k-strong sphericity of $T_{2^k m,q}$ doesn't tell us about $\kappa_R^{(k)}(T_{2^k m,q})$.

For this, we follow the argument of K.–Park–Taniguchi 2024, which uses the Dai–Sasahira–Stoffregen lattice homotopy type.

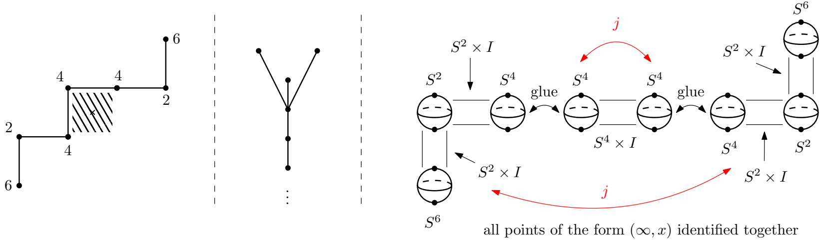


We start again from the $\mathbb{Z}_{2^k}$-fold symmetry τ on W_Γ.

2-fold symmetry $\tau^{2^{k-1}}$

$\Rightarrow \mathbb{Z}_2$-action $I = j \circ \tau^{2^{k-1}}$ on the lattice homotopy type $\mathcal{H}$.

The I-fixed locus of $\mathcal{H}$ is the sphere of dimension $\bar{\mu}(\Sigma(2^k, 2^k m, q), \mathfrak{s}_0)$.



(a) (b) (c)

The Dai–Sasahira–Stoffregen construction gives an $O(2)$-equivariant map

$$\mathcal{T} : \mathcal{H} \to SWF(\Sigma(2^k, 2^k m, q), \mathfrak{s}_0)$$

that is a nonequivariant homotopy equivalence.

Both $\mathcal{H}$ and SWF are finite $O(2)$-spectra

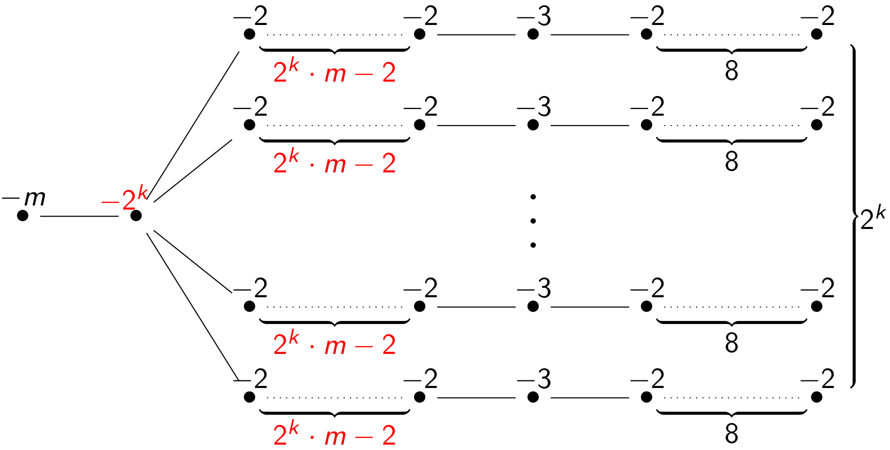
$\Rightarrow H^*(SWF(\Sigma(2^k, 2^k m, q), \mathfrak{s}_0)^{j\circ\tau^{2^{k-1}}}; \mathbb{Z}_2) \simeq \mathbb{Z}_2[-\bar{\mu}(\Sigma(2^k, 2^k m, q), \mathfrak{s}_0)]$

But $SWF(\Sigma(2^k, 2^k m, q), \mathfrak{s}_0)^{j\circ\tau^{2^{k-1}}}$ is spherical!

$$\kappa_R^{(k)}(T_{2^k m,q}) = -\frac{1}{2}\bar{\mu}(\Sigma(2^k, 2^k m, q), \mathfrak{s}_0).$$

Neumann-Siebenmann invariants are computed from plumbing graph.

Here is the plumbing graph for $\Sigma(2^k, 2^k m, 2^k \cdot 10m - 1)$.



The spherical Wu class w is supported on the central node and half of the $2^k \cdot m - 2$ consecutive (-2)-nodes.

$\Rightarrow w^2 = -2^k - 2^k(2^k m - 2) = 2^k - 2^{2k} \cdot m.$

There are $2^{2k} \cdot m + 7 \cdot 2^k + 2$ nodes, all negatively weighted.

$\Rightarrow \bar{\mu}(\Sigma(2^k, 2^k m, 2^k \cdot 10m - 1), \mathfrak{s}_0) = \frac{\sigma(\Gamma) - w^2}{8} = -2^k - \frac{1}{4}.$

$\Rightarrow \kappa_R^{(k)}(T_{2^k m, 2^k \cdot 10m - 1}) = 2^{k-1} + \frac{1}{8}.$

Amazing! So clean!

Computing the rest

Recall the real 10/8-inequality, adjusted to our setting:

$$-\frac{\sigma(\Sigma_{2^k}(C_n))}{16}+\frac{\alpha}{2} \leq b^+(\Sigma_{2^k}(C_n)) - b^+(\Sigma_{2^{k-1}}(C_n)) - \left(2^{k-1}+\frac{1}{8}\right),$$

where $\alpha = 1$ if $b^+(\Sigma_{2^k}(C_n)) > b^+(\Sigma_{2^{k-1}}(C_n))$.

Lemma (follows from G-signature theorem)

Let X be a closed 4-manifold and S be a properly embedded surface in $B^4\#X$. Let n be a prime power dividing $[S]$. Then we have:

$$b^+\big(\Sigma_n(S)\big) = n\, b^+(X) + (n-1)g(S) - \frac{n^2-1}{6n}[S]^2 + \frac{1}{2}\sigma^{(n)}(\partial S),$$

$$\sigma\big(\Sigma_n(S)\big) = n\,\sigma(X) - \frac{n^2-1}{3n}[S]^2 + \sigma^{(n)}(\partial S).$$

We only have to compute $\sigma^{(2^k)}(T_{2^k m, 2^k \cdot 10m-1})$ and $\sigma^{2^{k-1}}(T_{2^k m, 2^k \cdot 10m-1})$.

Lemma (Brute-force computation)

For any integers $p, m, n > 0$ that are not all equal to 1, we have

$$\sigma^{(p)}(T_{pm,\, pmn-1}) = 2(p-1) - \frac{p(p-1)(p+1)m^2 n}{3}.$$

Then we get:

$$b^+\big(\Sigma_{2^k}(C_n)\big) = 2^k + 1,\ b^+\big(\Sigma_{2^{k-1}}(C_n)\big) = 2^{k-1} + 1,\ \sigma\big(\Sigma_{2^k}(C_n)\big) = 2.$$

Wow! So clean!

Finalizing the proof

Now our real 10/8-inequality became:

$$-\frac{2}{16}+\frac{1}{2} \leq (2^k+1)-(2^{k-1}+1)-\left(2^{k-1}+\frac{1}{8}\right).$$

All nontrivial terms cancel out! Thus we get

$$\frac{1}{2} \leq 0.$$

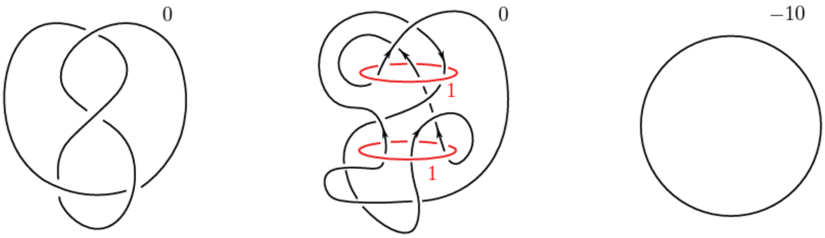
A contradiction. Therefore $(4_1)_{2^k m,1}$ is not smoothly slice.

A similar argument proves $\kappa_R^{(k)}(-c(4_1)_{2n,1}) \geq \frac{1}{2}$ for all $c, n > 0$.
$\Rightarrow$ $(4_1)_{2n,1}$ has infinite concordance order.

Question

Why is this miraculous cancellation happening?

Is it coming from some geometric property of the smooth concordance $C \subset (S^3 \times I)\#2\mathbb{C}P^2$ from 4_1 to unknot?



Some remaining questions about cables of 4_1

Every nontrivial cable of 4_1 has infinite concordance order.

But how about their smooth slice genus?

Question

Is there any $n > 0$ such that $g_4^{sm}((4_1)_{n,1}) \geq 2$?

Interestingly, $(4_1)_{2,1}$ bounds a genus 1 surface, so $g_4^{sm}((4_1)_{2,1}) = 1$.

Other questions

There are other related questions. I think they are very well known.

- Is any cable of 4_1 topologically slice?
- What about cables of other negative amphichiral knots (with irreducible Δ_K)? Might be related to geometric properties of C.
- Is $\{(4_1)_{2n,1} \mid n > 1\}$ linearly independent?

The end

Figure: A plumbing graph of $\Sigma(8, 8, 79)$, observed alive in real life.