

# ICTP Lectures on the Standard Model

## REFERENCES

### (\*) On the Particle Revolution

- R. Cahn and G. Goldhaber  
"The experimental foundation of particle physics"

### (\*) On the Standard Model

- M. Schwartz, "Quantum Field Theory and the Standard Model"
- J. Donoghue, E. Golowich, B.R. Holstein  
"Dynamics of the Standard Model"
- S. Weinberg, "The Quantum Theory of Fields" vol II

### (\*) On the Hierarchy Problem of the SM

- 't Hooft "Naturalness, chiral symmetry and spontaneous chiral symmetry breaking"  
Cargèse 1979 lectures
- Barbieri, Giudice Nucl. Phys. B 306 (1988) 63
- Susskind Phys. Rev. D 20 (1979) 2619
- Barbieri, Strumia "The LEP paradox" hep-ph/0007265
- Giudice "Naturally Speaking" arXiv:0801.2562

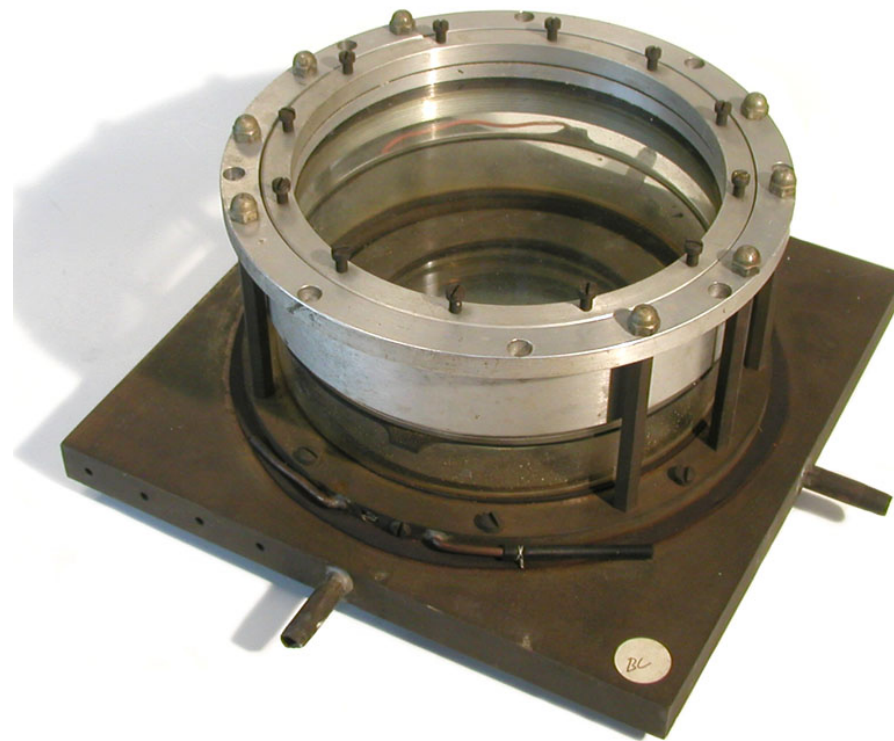


# First Particles from Cosmic Rays (1930-1940)

## Tools:

- Cloud chambers

*Invented by C.T.R. Wilson in 1911, filled with super-saturated vapour*



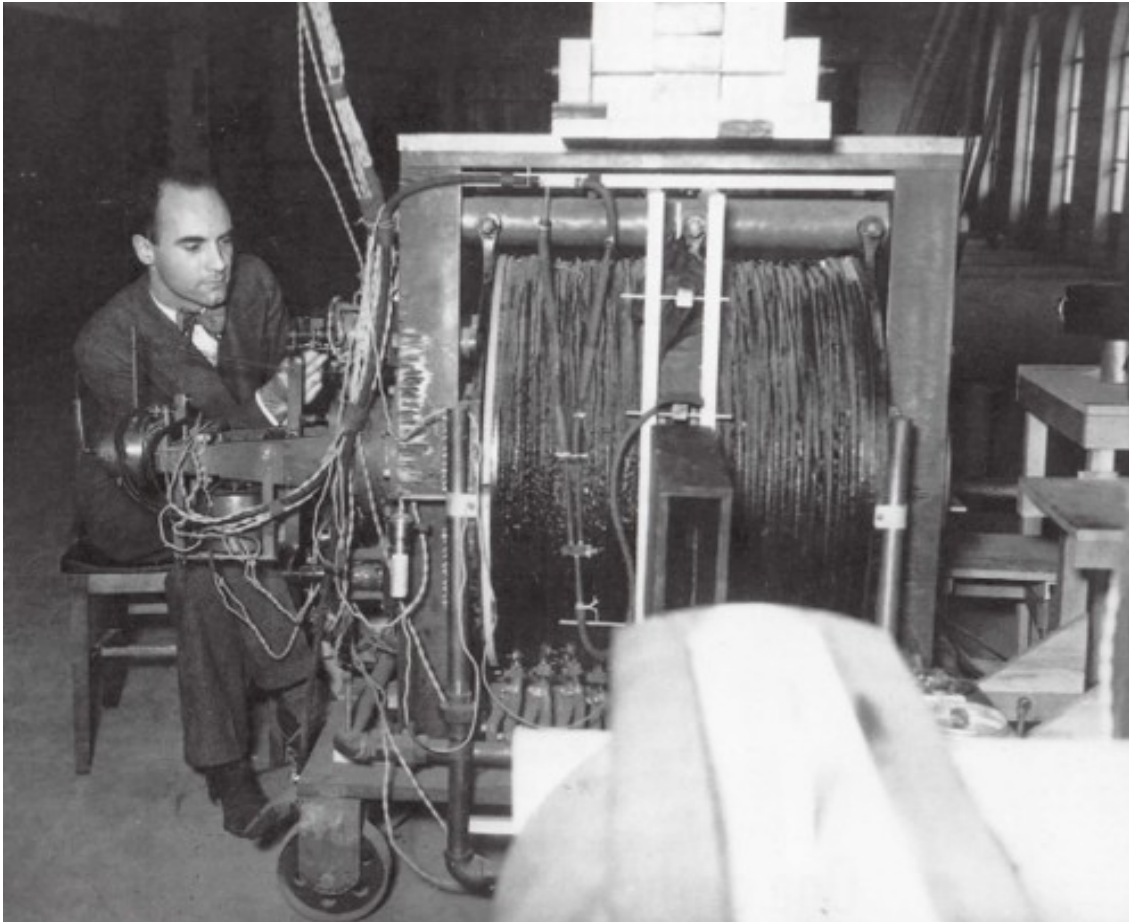
- Photographic emulsions



# The Positive Electron

CARL D. ANDERSON, *California Institute of Technology, Pasadena, California*  
(Received February 28, 1933)

Phys. Rev. 43 (1933) 491



Carl Anderson at the magnet cloud chamber  
Credits: Caltech Archives

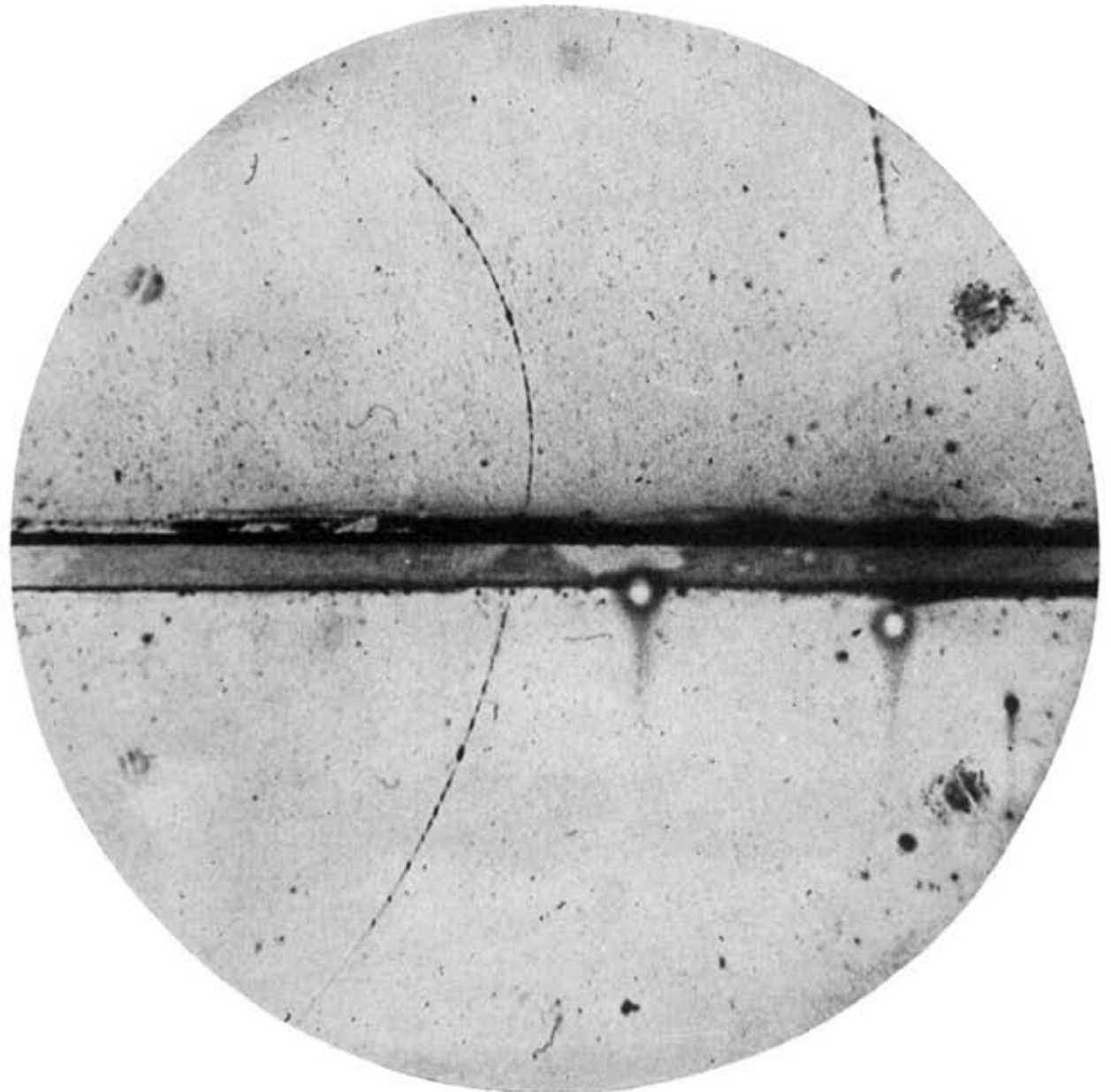


FIG. 1. A 63 million volt positron ( $H\rho = 2.1 \times 10^6$  gauss-cm) passing through a 6 mm lead plate and emerging as a 23 million volt positron ( $H\rho = 7.5 \times 10^4$  gauss-cm). The length of this latter path is at least ten times greater than the possible length of a proton path of this curvature.

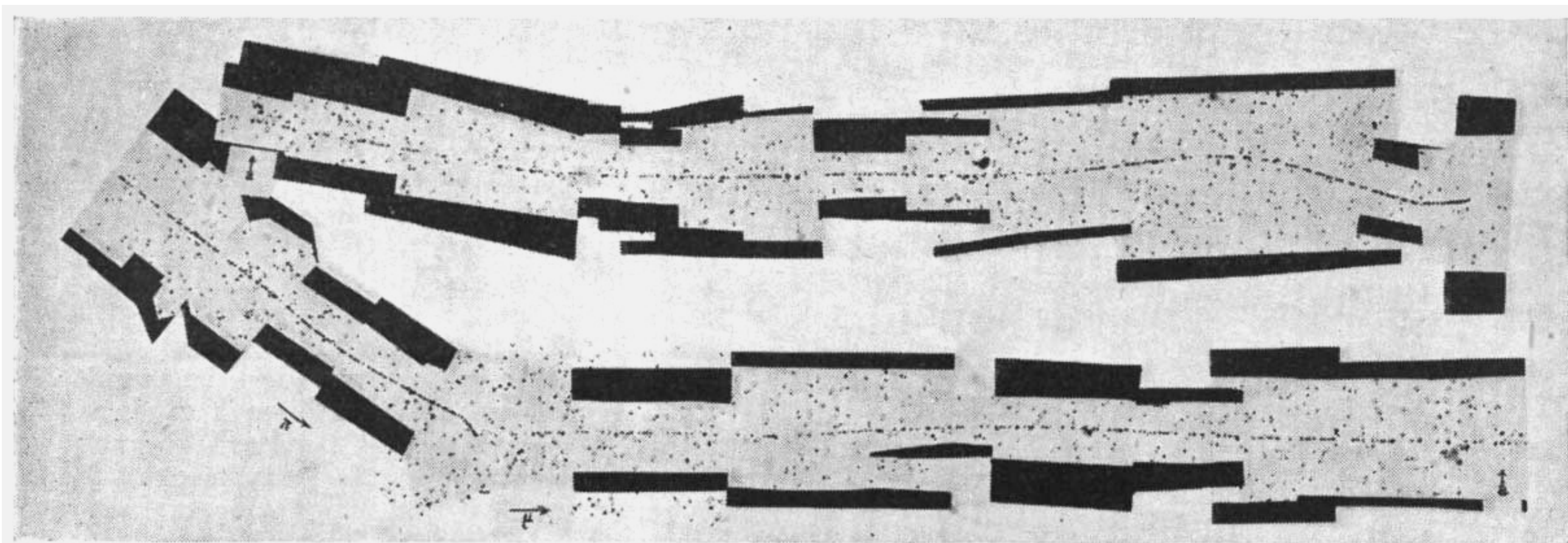


# OBSERVATIONS ON THE TRACKS OF SLOW MESONS IN PHOTO- GRAPHIC EMULSIONS\*

By C. M. G. LATTES, DR. G. P. S. OCCHIALINI  
and DR. C. F. POWELL

H. H. Wills Physical Laboratory, University of Bristol

Nature 160 (1947) 453



**Fig. 1.** OBSERVATION BY MRS. I. POWELL. COOKE  $\times 95$  ACHROMATIC OBJECTIVE; C2 ILFORD NUCLEAR RESEARCH EMULSION LOADED WITH BORON. THE TRACK OF THE  $\mu$ -MESON IS GIVEN IN TWO PARTS, THE POINT OF JUNCTION BEING INDICATED BY  $a$  AND AN ARROW

# Resonances at Colliders (1950-1960)

## Tools

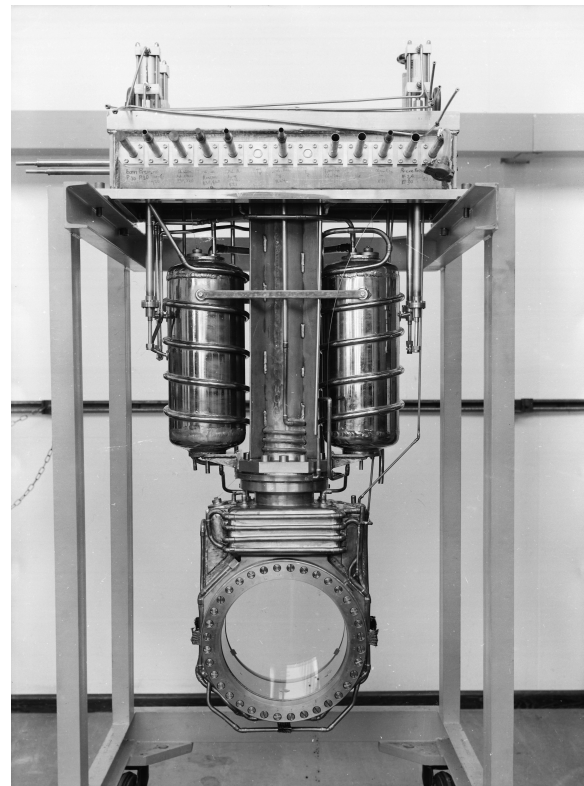
- (Synchro)cyclotrons and Synchrotrons



The 6 GeV proton synchrotron (Bevatron) at UC Berkeley in 1954

- Bubble Chambers

*Invented by Glaser in 1953, filled with a superheated liquid*



30-cm bubble chamber at CERN



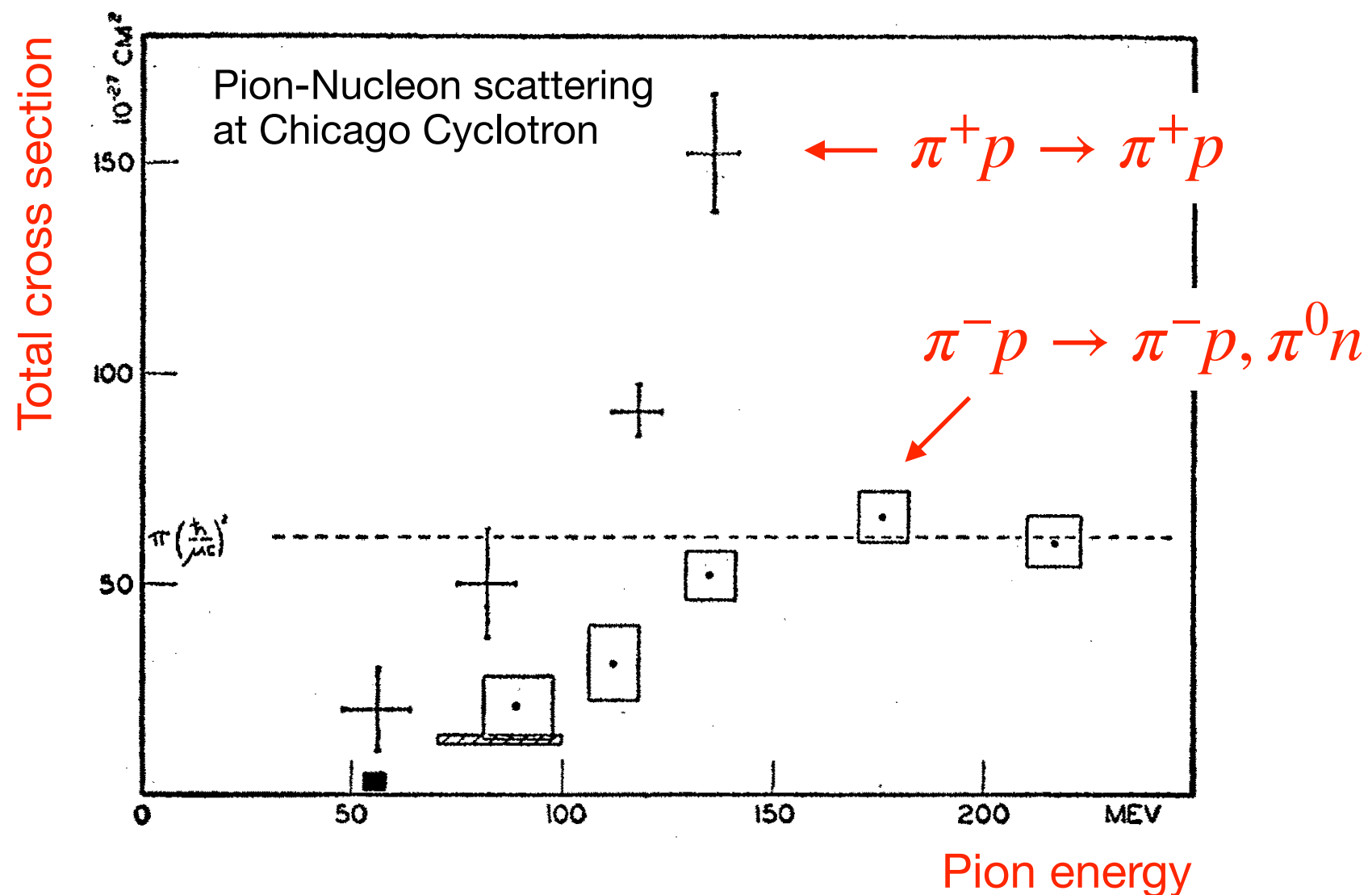
# The first resonance: $\Delta(1232)$ in $\pi N$ scattering

## Total Cross Sections of Positive Pions in Hydrogen\*

H. L. ANDERSON, E. FERMI, E. A. LONG,<sup>†</sup> AND D. E. NAGLE

*Institute for Nuclear Studies, University of Chicago,  
Chicago, Illinois*

Phys. Rev. 85 (1952) 936



# $\Sigma(1385)$ resonance in $K^-p \rightarrow \Lambda\pi^+\pi^-$ scattering

## RESONANCE IN THE $\Lambda\pi$ SYSTEM\*

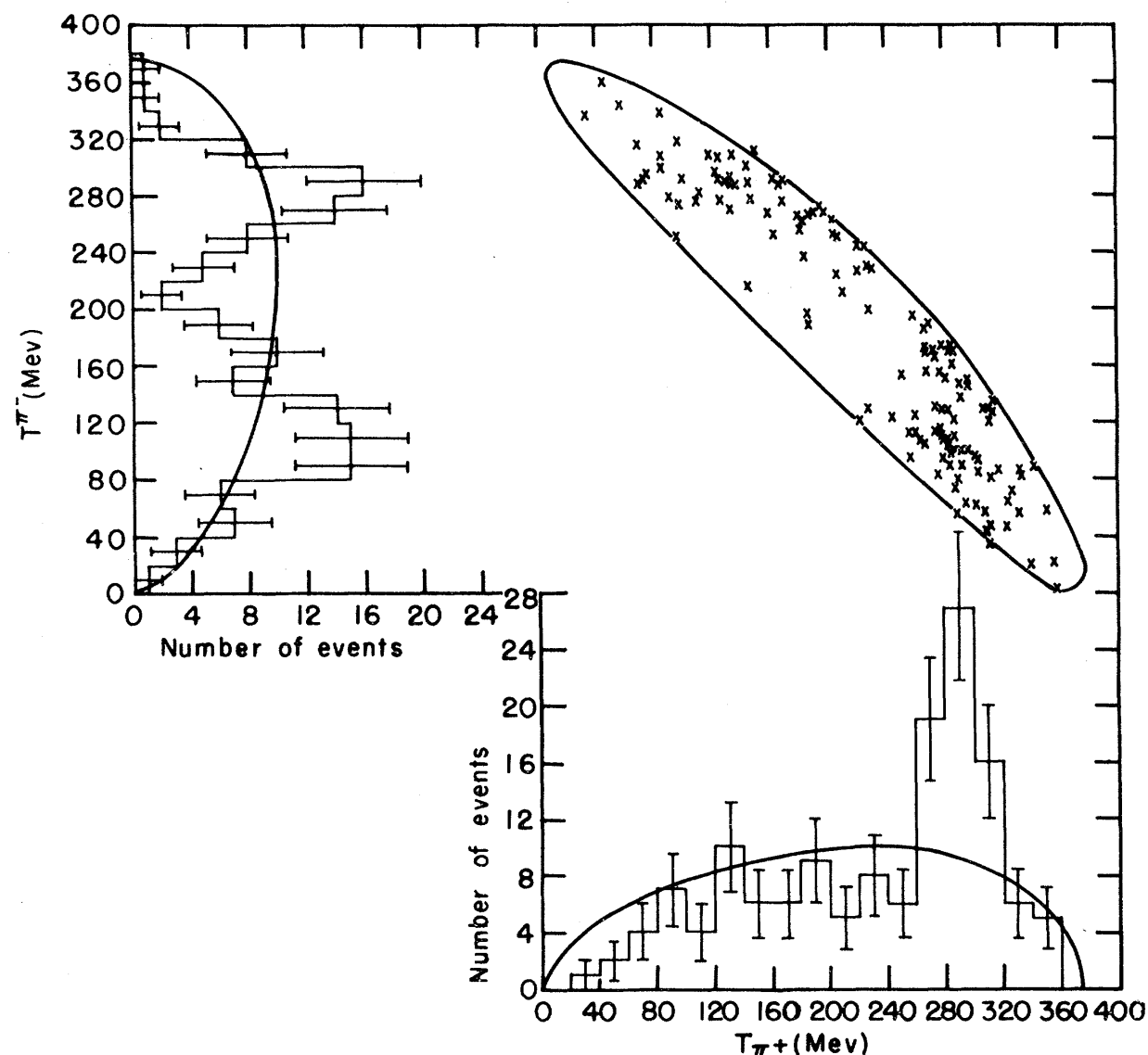
Margaret Alston, Luis W. Alvarez, Philippe Eberhard,<sup>†</sup> Myron L. Good,<sup>‡</sup>

William Graziano, Harold K. Ticho,<sup>||</sup> and Stanley G. Wojcicki

Lawrence Radiation Laboratory and Department of Physics, University of California, Berkeley, California

Phys. Rev. Lett. 5 (1960) 520

FIG. 1. Energy distribution of the two pions from the reaction  $K^- + p \rightarrow \Lambda + \pi^+ + \pi^-$ . Each event is plotted only once on the Dalitz plot, which should be uniformly populated if phase space dominated the reaction. The two energy histograms are merely one-dimensional projections of the two-dimensional plot, and each event is represented once on each histogram. The solid lines superimposed over the histograms are the phase-space curves.



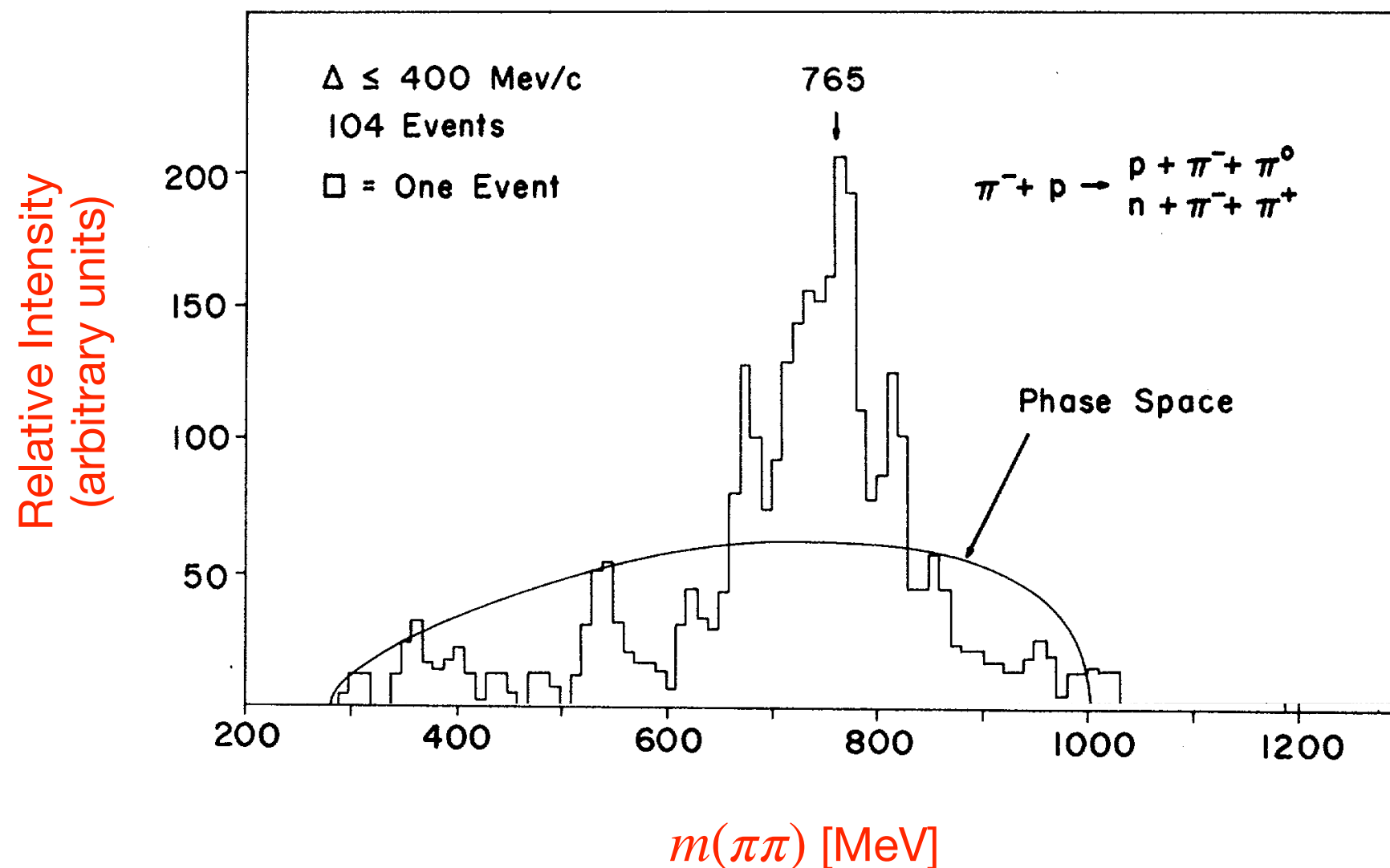
# The $\rho$ (predicted by Nambu and by Frazer, Fulco)

EVIDENCE FOR A  $\pi$ - $\pi$  RESONANCE IN THE  $I=1, J=1$  STATE\*

A. R. Erwin, R. March, W. D. Walker, and E. West

Brookhaven National Laboratory, Upton, New York and University of Wisconsin, Madison, Wisconsin

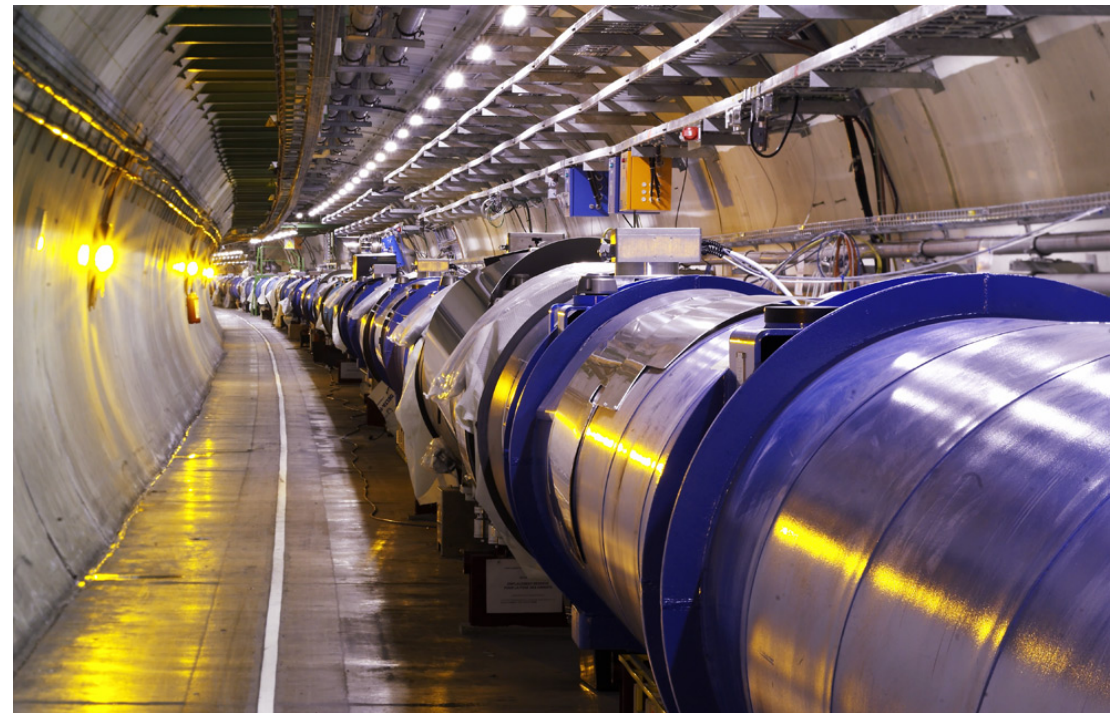
Phys. Rev. Lett. 6 (1961) 628



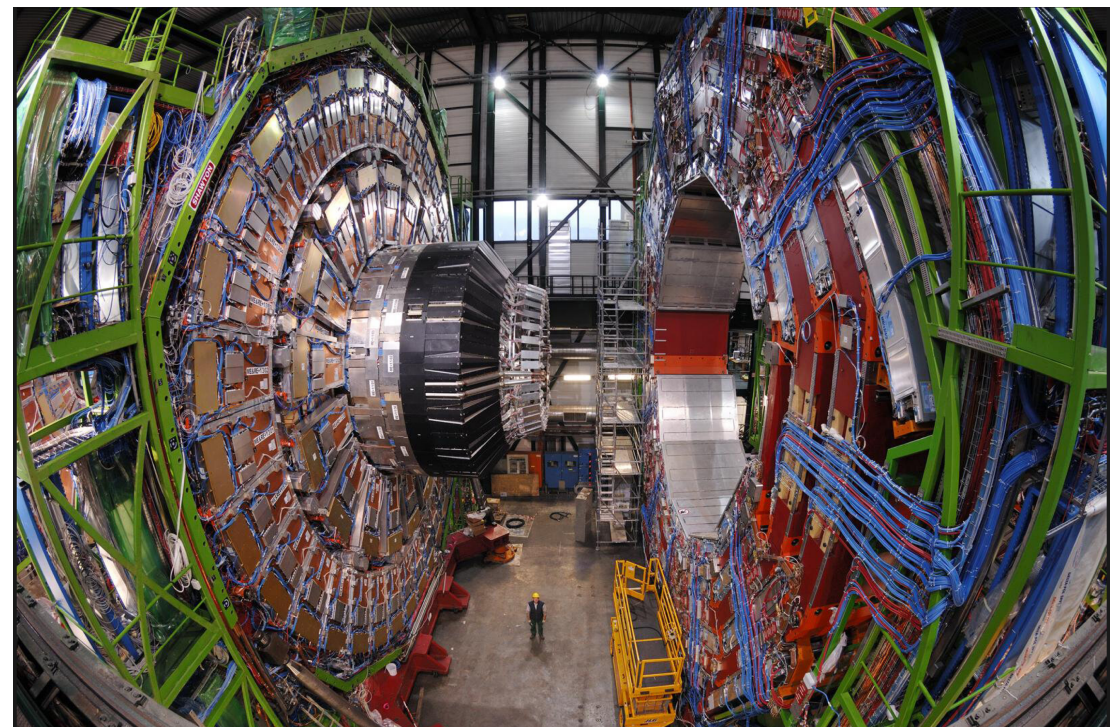
# The Last Discoveries (1995-2012)

## Tools:

- High-Energy Colliders
- General-purpose Detectors



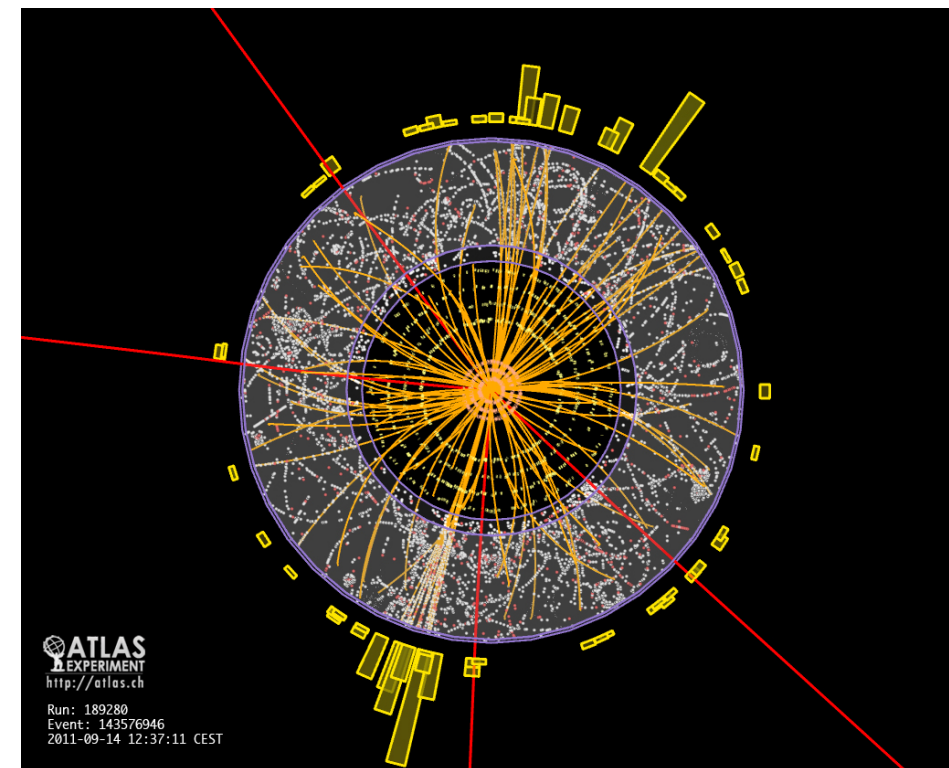
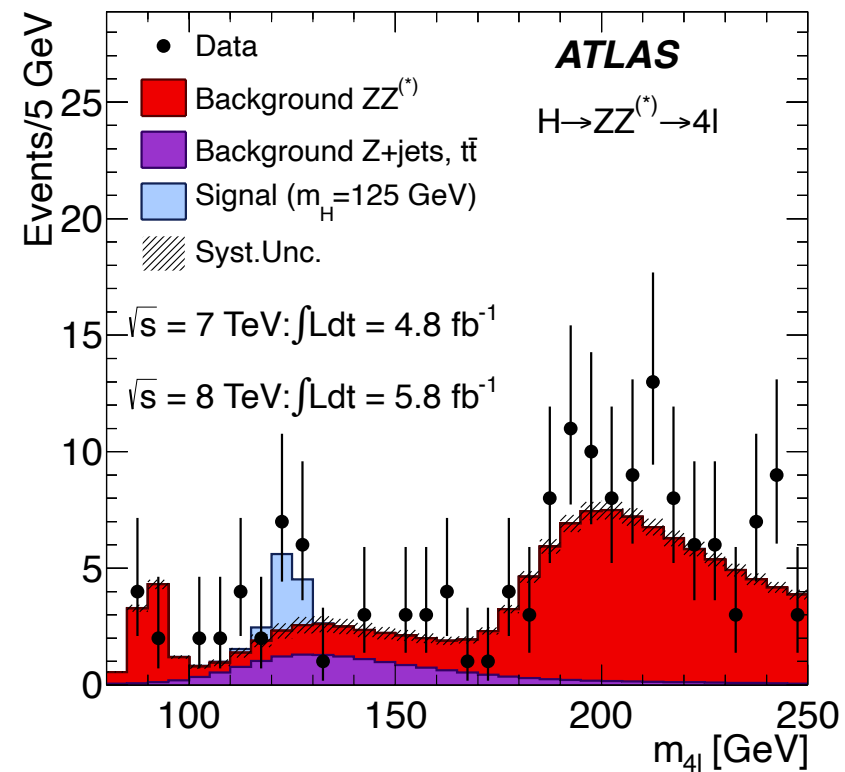
LHC tunnel. Credits: CERN



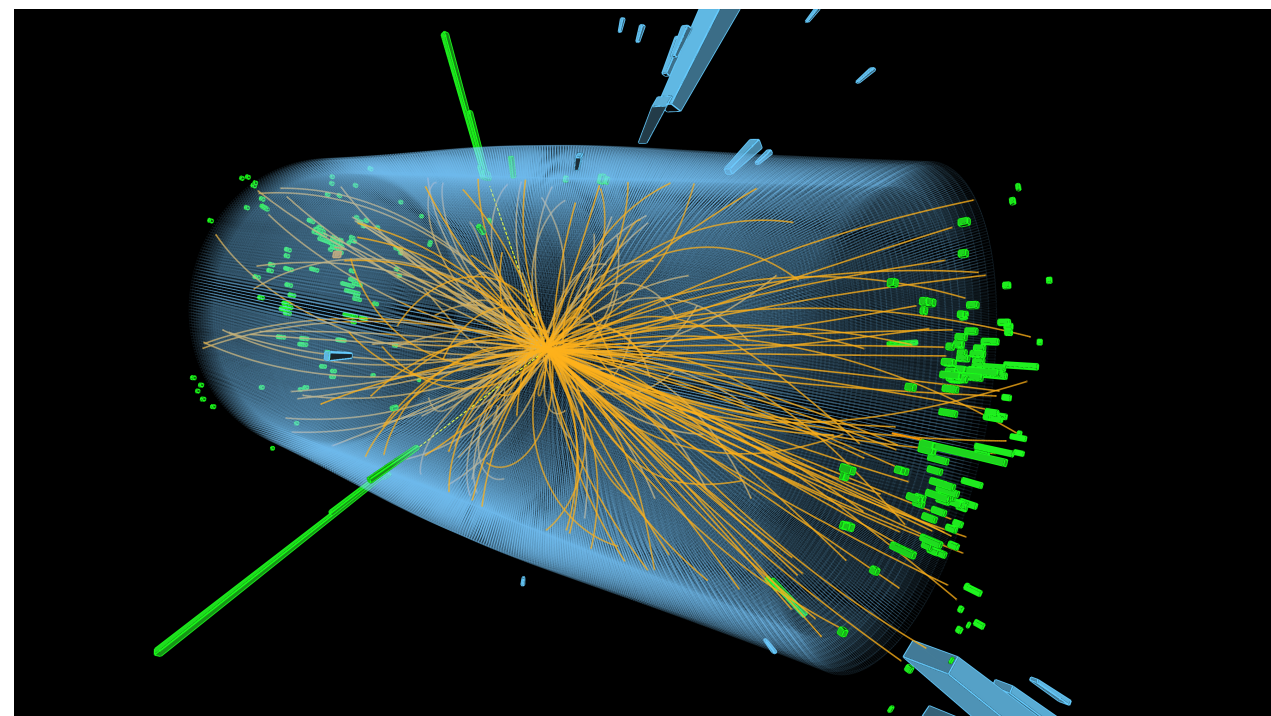
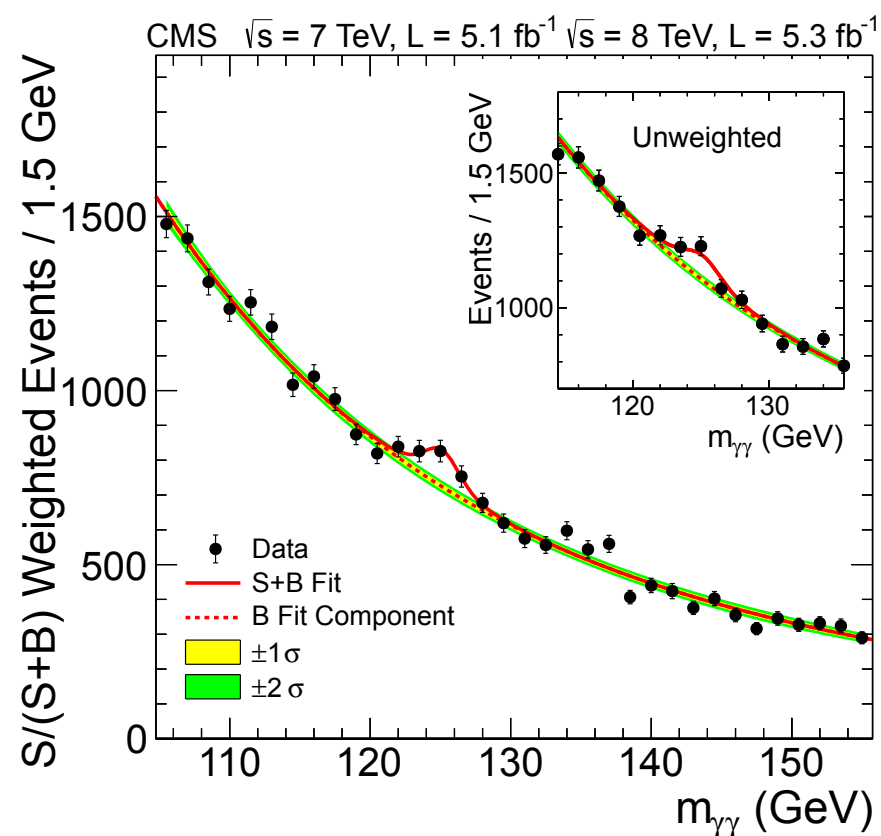
CMS detector. Credits: CERN/Maximilien Brice



# Born on the 4th of July (2012): the Higgs boson



Higgs to 4-muon event at ATLAS. Credits: CERN



Higgs to 2 photons event at CMS. Credits: CERN

(1)

These lectures give an introduction to the Standard Model (SM) of particle physics, our current theory of fundamental interactions. They assume a previous knowledge of Quantum Field Theory (QFT) and its application to Quantum Electrodynamics (QED).

The SM was built to describe the properties and interactions of the zoo of subatomic particles discovered starting from the early 1930's. Evidence of the existence of such particles come into two different forms:

- (1) "Particles" were observed as tracks in cloud (and later bubble) chambers or photographic emulsions, and were coming from cosmic rays or produced by accelerators (synchrocyclotrons, synchrotrons). Information on the energy and charge of the particles was coming from the length of the track and its curvature in a magnetic field.

These techniques led to the discovery of the positron  $e^+$  (Anderson 1933, using a cloud chamber), the muon  $\mu$  (Anderson and Neddermeyer 1937, using a cloud chamber), the pion  $\pi^\pm$  (Perkins 1947, using photographic emulsions) and various other particles.

(2) "Resonances" were discovered from the study of scattering processes and manifest themselves as peaks in the invariant mass of two final particles or as high density of events in Dalitz plots. They are too short lived ( $\tau < 10^{-10} \text{ s}$ ) to be seen as tracks in cloud chambers or emulsions.

The first resonance to be discovered was the  $\Delta(1232)$ , observed first by Anderson, Fermi, Long and Neele in 1932 in  $\pi N \rightarrow \pi N$  scattering. It was soon followed by the discovery of many others: the  $\Sigma(1385)$  from  $K^+ \bar{p} \rightarrow \Lambda \pi^+ \pi^-$  (L. Alvarez et al. 1960),  $K^*(890)$  from  $K^+ \bar{p} \rightarrow \bar{K}^0 \pi^+ \bar{p}$  (L. Alvarez et al. 1961),  $\rho$  from  $\pi N \rightarrow \pi N$  (A.R. Erwin et al. 1961),  $\omega$  from  $p \bar{p} \rightarrow \pi^+ \pi^- \pi^+ \pi^- \pi^0$  (Majlić et al. 1961), etc.

Now we know that resonances are just particles that decay via strong interactions and have a very short lifetime.

In general, therefore, by particle one denotes a point-like object characterized by its mass, spin, 4-momentum and charge (other quantum numbers like parity can also apply). In modern

Jargon:

Particles are unitary irreps of the Poincaré group and of any other global symmetry group (like electromagnetism) in the theory.



(3)

The SM must describe and organize all these particles in a clear logical scheme, and explain as well their interactions. These are (neglecting the very feeble gravitational interaction, irrelevant at microscopic distances):

- electromagnetism (long range)
- weak force (short range,  $r \sim 10^{-3} \text{ fm}$ )
- strong force (short range,  $r \sim \text{fm}$ )

Quantum field theory explains <sup>these</sup> interactions in terms of exchange of force carriers - hence particles - as the consequence of gauge invariance. Particles emerge as quantum excitations of local fields.

The SM is thus a GAUGE THEORY, with  $SU(3) \times SU(2)_{EW} \times U(1)_Y$  gauge group, built to describe electromagnetic, weak and strong interactions. The field content is chosen so as to reproduce the spectrum of all known particles. These can be classified into two broad classes:

LEPTONS ( $e^\pm, \mu^\pm, \tau^\pm, \nu$ ) : particles without strong interactions

HADRONS ( $p, n, \pi, \rho, \Delta, \dots$ ) : particles with strong interactions

to which one must add the

4

FORCE CARRIERS ( $\gamma, W^\pm, Z^0$ )

and the

HIGGS BOSON ( $h$ )

All these particles can be described by the following set of fields

LEPTON fields :  $e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau$

QUARK fields :  $u, d, s, c, b, t$

Gauge fields :  $B_\mu$  (hypercharge),  $W_\mu^{1,2,3}$ ,  $G_\mu^{1,\dots,8}$  (gluons)

HIGGS field :  $\phi^\pm, \phi^0, h$

by means of a Lagrangian density  $\mathcal{L}(\varphi, \partial_\mu \varphi)$  - here  $\varphi$  is a collective symbol for all the fields - invariant under the group of local (gauge) transformations  $SU(3)_c \times SU(2)_{EW} \times U(1)_Y$ . The Lagrangian density, or more precisely the action, is also required to be invariant under the Poincaré group of global transformations (translations + Lorentz transformations).

Fields are properly organized according to their quantum numbers under the  $SU(3)_c \times SU(2)_{EW} \times U(1)_Y$  group and the  $SO(3,1)^+$  group of proper and orthochronous Lorentz transformations (or, more precisely of its covering group  $SL(2, \mathbb{C})$ ):

| Field                                                       | $SO(3,1)^+$ | $SU(3)_c$ | $SU(2)_{EW}$ | $U(1)_Y$       |                |
|-------------------------------------------------------------|-------------|-----------|--------------|----------------|----------------|
| $Q_L^i = \begin{pmatrix} u^i \\ d^i \end{pmatrix}_L$        | $(2,1)$     | $\square$ | $\square$    | $\frac{1}{6}$  | } quarks       |
| $u_R^i$                                                     | $(1,2)$     | $\square$ | 1            | $\frac{2}{3}$  |                |
| $d_R^i$                                                     | $(1,2)$     | $\square$ | 1            | $-\frac{1}{3}$ |                |
| $l_L^i = \begin{pmatrix} \nu^i \\ e^i \end{pmatrix}_L$      | $(2,1)$     | 1         | $\square$    | $-\frac{1}{2}$ | } leptons      |
| $e_R^i$                                                     | $(1,2)$     | 1         | $\square$    | -1             |                |
| $B_\mu$                                                     | $(2,2)$     | 1         | 1            | 0              | } gauge fields |
| $W_\mu^a$                                                   | $(2,2)$     | 1         | Adj          | 0              |                |
| $G_\mu^\alpha$                                              | $(2,2)$     | Adj       | 1            | 0              |                |
| $H = \begin{pmatrix} \phi^\pm \\ h + i\phi^0 \end{pmatrix}$ | $(1,1)$     | 1         | $\square$    | $\frac{1}{2}$  | } Higgs field  |

The index  $i=1,2,3$  runs over three families of fields with identical quantum numbers.

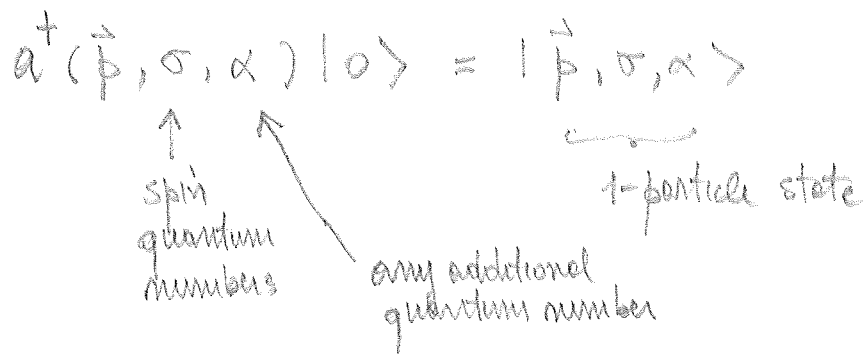


What spectrum of particles does the SM predict?

(6)

Answering this question is not simple, and in fact requires knowing the dynamics of the theory. In a QFT particles are excited (interpolated) from the vacuum by field operators. Any operator can indeed be written as a linear superposition of products of creation and annihilation operators (for a proof of this theorem, see Weinberg "The Quantum Theory of Fields", volume I chapter 4). As this name says, creation operators create particles from the vacuum,

$$a^\dagger(\vec{p}, \sigma, \alpha) |0\rangle = |\vec{p}, \sigma, \alpha\rangle$$



↑  
spin  
quantum  
numbers

↑  
any additional  
quantum number

1-particle state

hence field operators will do the same. Clearly, the interpolating operator of a particle will have to have its same quantum numbers, and these are determined by how the operator transforms under global symmetry transformations.

It is important to notice that the same particle (i.e. 1-particle state) will have in general several interpolating operators, all sharing the same quantum numbers:

$$\langle 0 | O_{\sigma, \alpha} | \vec{p}, \sigma, \alpha \rangle \neq 0$$

Which operators can we choose to interpolate the particles experimentally discovered?

Notice that the spectrum of particles will also include some that are interpreted here as bound states of other, more fundamental, ones. For example, the hydrogen atom is interpreted as a bound state of one electron and one proton; more in general all atoms and molecules are described as composite systems of more fundamental particles. Field operators will interpolate these bound states as well, and, as a matter of fact, the boundary between what we call "fundamental" particles and their bound states is less obvious than one would naively think.

In a weakly-coupled QFT, one can use the fields appearing in the Lagrangian to interpolate the physical states. In other words, the spectrum of physical states can be read off from the fields in the Lagrangian, i.e. from their quantum numbers. In fact, this procedure does not capture all the particles in the spectrum, since it leaves out in general bound states made of them. As an example, in QED positronium is not interpolated by the electron field, but it can be interpolated (at least for ortho-positronium with  $S=1$ ) by the electromagnetic current  $j_\mu = \bar{\psi} \gamma_\mu \psi$ .

In a gauge field theory, the procedure of interpolating physical states by means of the fundamental fields in the Lagrangian makes sense only after fixing the gauge. The point is that physical states obviously are gauge invariant, while the Lagrangian fields are not in general. QED is a well known example of such situation. Among all possible choices of gauges, the unitary gauge is the one which makes this procedure most transparent. In other gauges, not all the states

interpolated by the fundamental fields are physical (consider for example would-be Nambu-Goldstone bosons in a  $\Sigma$ -gauge for a spontaneously-broken gauge symmetry).

Summarizing, the spectrum of a weakly-coupled gauge theory can be determined by

- i) working in the unitary gauge and interpolating fundamental states by means of fundamental fields (i.e. those in the Lagrangian)
- ii) interpolating bound states by means of composite operators. The presence of these states can be inferred by using perturbation theory, since the theory is weakly coupled.

The prototype of theory in which this procedure can be applied is QED.

Things are not so simple in a strongly-coupled QFT, where the physical spectrum of particles can be very much different from the one naively corresponding to fundamental fields.

This means that determining the spectrum of a QFT, in general, requires knowing the evolution of (gauge) couplings with the energy and, more in particular, the phase of the theory in the far infrared. Such knowledge can only be obtained by either solving analytically the model (this is possible for a few  $1+1$  dimensional systems) or making use of non-perturbative numerical methods.

Although a detailed reconstruction of the spectrum can only be achieved by means of these approaches, the quantum numbers of the particles possibly appearing in the spectrum can be obtained by inspecting all gauge-invariant (composite) operators that can interpolate them. ⑨

The use of gauge-invariant operators is particularly useful because it removes any ambiguity due to the gauge redundancy, therefore isolating the physical degrees of freedom.

It is in fact inevitable if no gauge fixing is performed (this is a possibility, for example, if the theory is put on a lattice discretization, working at finite volume): in that case physical states must be interpolated by gauge-invariant operators.

Finally, physical states can be interpolated by gauge-invariant operators even if the theory is weakly coupled. We will comment on this later.

Let us now return to the Standard Model: how can we determine its particle spectrum?



We said that determining the spectrum of physical states (particles) requires knowing the phase of the theory, and in particular whether it is weakly or strongly coupled for  $E \rightarrow 0$ . It is remarkable that, working at the 1-loop level, the behavior of the gauge couplings with the energy can be determined without knowing the details of the Lagrangian, but simply based on the gauge quantum numbers of fields.

Gauge couplings run with the energy scale  $\mu$  according to the renormalization group (RG) equation

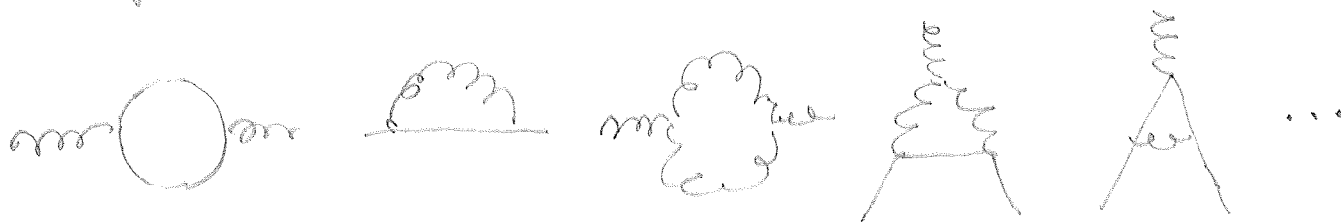
$$\mu \frac{d}{d\mu} g = \beta(g)$$

Here we assume to define the renormalized coupling  $g(\mu)$  by means of Dimensional Regularization (DR) with a Minimal Subtraction scheme (MS or  $\overline{\text{MS}}$ ). This is particularly convenient since the  $\beta$ -function turns out to be a function of  $g(\mu)$  alone.

For small values of  $g(\mu)$ ,  $\beta(g)$  can be Taylor expanded:

$$\beta(g) = \frac{\beta_0}{16\pi^2} g^3 + \frac{\beta_1}{(16\pi^2)^2} g^5 + \dots$$

The coefficient  $\beta_0$  can be computed in terms of 1-loop diagrams



Its value is given by a simple general formula.

(11)

For a non-abelian  $SU(N)$  gauge theory with  $n_f$  Weyl fermions and  $n_s$  (complex) scalar fields in the fundamental representation:

$$\beta_0^{SU(N)} = -\frac{11}{3} C_2(\text{Adj}) + \frac{2}{3} T(\square) n_f + \frac{1}{3} T(\square) n_s$$

where  $C_2(\text{Adj}) = N$  is the quadratic Casimir in the adjoint representation ( $\sum_a (T^a(r) T^a(r))_{ii} = C_2(r) \delta_{ii}$ ) and  $T(\square)$  is the Dynkin index in the fundamental representation ( $\text{Tr}(T^a(r) T^b(r)) = T(r) \delta^{ab}$ ); I will normalize the  $SU(N)$  generators  $T^a$  such that  $T(\square) = 1/2$ .

For an abelian  $U(1)$  gauge theory with fermion charges  $y_v^F$  and scalar charges  $y_v^S$ , one has (again, for Weyl fermions)

$$\beta_0^{U(1)} = \frac{2}{3} \sum_i (y_v^F)^2 + \frac{1}{3} \sum_i (y_v^S)^2$$

Let us apply these formulas to compute  $\beta_0^{(1)}$ ,  $\beta_0^{(2)}$ ,  $\beta_0^{(3)}$  of  $U(1)_Y$ ,  $SU(2)_{EW}$ ,  $SU(3)_C$  respectively.

(12)

$$\beta_0^{(1)} = \frac{2}{3} \left\{ N_c \times \left[ \overbrace{\left(\frac{1}{6}\right)^2}^{q_L} \times 2 + \overbrace{\left(\frac{2}{3}\right)^2}^{u_R} + \overbrace{\left(-\frac{1}{3}\right)^2}^{d_R} \right] + 2 \times \overbrace{\left(-\frac{1}{2}\right)^2}^{l_L} + \overbrace{(-1)^2}^{e_R} \right\} \cdot N_f$$

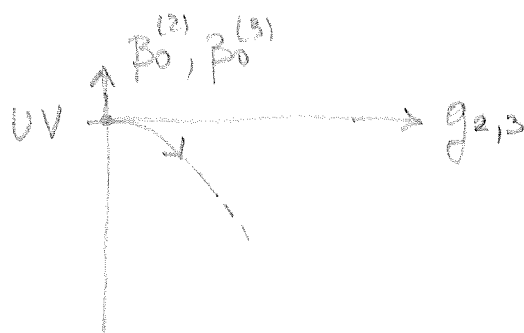
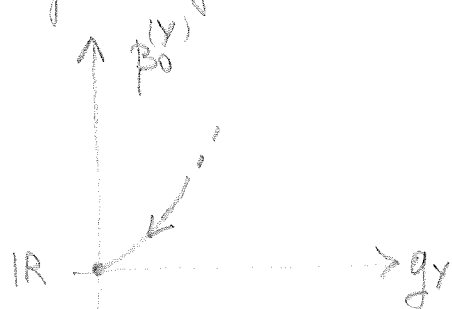
$$+ \frac{1}{3} \underbrace{\left(\frac{1}{2}\right)^2 \times 2}_H = \frac{41}{6}$$

$$\beta_0^{(2)} = -\frac{11}{3} \cdot 2 + \frac{2}{3} \cdot \frac{1}{2} \cdot N_f \left( \overbrace{N_c+1}^{q_L} \overbrace{1}^{l_L} \overbrace{1}^H \right) + \frac{1}{3} \times \frac{1}{2} = -\frac{19}{6}$$

$$\beta_0^{(3)} = -\frac{11}{3} \cdot 3 + \frac{2}{3} \cdot \frac{1}{2} \cdot N_f \left( \overbrace{2}^{q_L} \overbrace{1}^{u_R} \overbrace{1}^{d_R} \right) = -7$$

where  $N_c=3$  is the number of colors and  $N_f=3$  is the number of families.

We see that  $\beta_0^{(1)} > 0$  while  $\beta_0^{(2)}, \beta_0^{(3)} < 0$ . This means that if we look at values of the couplings sufficiently small, for which the 1-loop truncation of  $\beta(g)$  is a reasonable approximation, then the  $\beta$ -functions have the following behavior:



We can thus deduce that

(13)

(i) the hypercharge coupling diminishes at lower energies and increases at higher energies.

(ii)  $SU(2)_{EW}$  and  $SU(3)_c$  are asymptotically free but their couplings grow at lower energies.

Point (i) is particularly important since it tells us that the SM becomes strongly coupled in the far UV. Let us stick to the 1-loop formula for  $\beta(g)$  and see what it predicts at high energies. We can solve the RG equation and find

$$\mu \frac{d}{d\mu} g = \frac{\beta_0}{16\pi^2} g^3 \quad \Leftrightarrow \quad \mu \frac{d}{d\mu} \frac{1}{g^2} = -\frac{\beta_0}{8\pi^2}$$

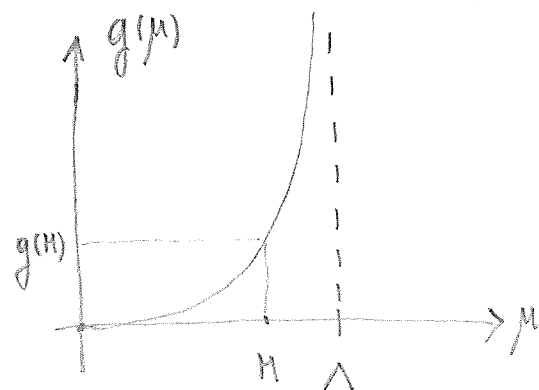
$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(M)} - \frac{\beta_0}{8\pi^2} \log \frac{\mu}{M}$$

$$g^2(\mu) = \frac{g^2(M)}{1 - \frac{\beta_0}{8\pi^2} g^2(M) \log \frac{\mu}{M}}$$

Here  $M$  is a reference energy scale where the value  $g(M)$  of the coupling is assumed to be known. If we take  $\mu > M$ , then we see that for  $\beta_0 > 0$  the denominator becomes zero at a finite value of  $\mu \equiv \Lambda$  (Landau pole):

$$1 - \frac{\beta_0}{8\pi^2} g^2(\mu) \log \frac{\Lambda}{\mu} = 0$$

$$\Lambda = \mu \exp \left( \frac{8\pi^2}{\beta_0} \frac{1}{g^2(\mu)} \right)$$



If we apply this formula to the hypercharge coupling and take

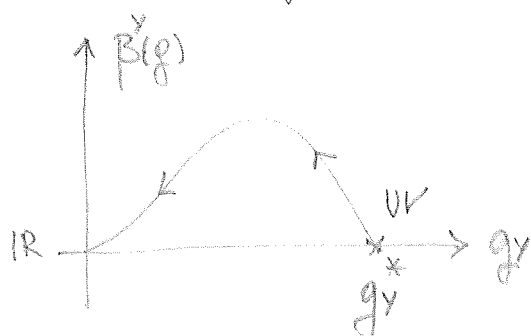
$$\mu = m_Z = 91 \text{ GeV}$$

$$\alpha_Y(m_Z) \equiv \frac{g_Y^2(m_Z)}{4\pi} = 0.0102$$

$$\beta_0^{(Y)} = \frac{41}{6}$$

we find  $\Lambda_Y \approx 10^{41} \text{ GeV}$ . This is an astronomically large energy scale, much above the Planck scale, yet the fact that hypercharge blows up at a finite energy (although very large) means that our theory must be replaced with a more fundamental one at  $E \lesssim \Lambda_Y$ .

One might object that this conclusion has been drawn by using the 1-loop approximate formula, which does not hold at large values of the couplings. Indeed, there is the possibility that at large values of  $g_Y$  the  $\beta$ -function has a UV fixed point:





In that case,  $g_Y$  would tend asymptotically to the fixed point value  $g_Y^*$  for  $\mu \rightarrow \infty$ . This scenario is called Asymptotic Safety. Although it might seem an attractive solution, in practice establishing whether it is realised in the SM requires being able to perform calculations in the strongly-coupled regime of the theory. Such techniques do not exist at the moment (lattice discretisation is not a viable option for a chiral theory like the SM).

Furthermore, when the hypercharge coupling becomes strong, one should also check that all the other couplings remain asymptotically free or tend to non-trivial UV fixed points. Finally even before approaching the energies at which  $g_Y$  becomes strong, gravity becomes non perturbative.

A realisation of Asymptotic Safety within the SM must then also include a UV completion of gravity. This seems a formidable task and in the following we will assume the more conservative scenario where the SM is replaced by a new theory before it becomes strongly-coupled. In other words, we will consider the SM as an Effective QFT.

We have discussed the implications of the RG evolution of the hypercharge coupling. But what about the  $SU(2)_{EW}$  and  $SU(3)_c$  couplings?

Let us start analyzing  $g_2$ : We found that its value increases at lower energies. We know that  $SU(2)_{EW}$  is responsible for the weak interactions, which are short ranged. Starting from the early days of the SM construction, the short-ranged nature of weak interaction was interpreted as the consequence of a massive force carrier. It later became clear (Higgs, Brout & Englert, Hagen, Guralnik & Kibble) that massive gauge bosons can be obtained by means of the so-called "Higgs mechanism", in which the  $SU(2)_{EW}$  gauge group is "spontaneously broken".

The crucial point about the Higgs mechanism is that it stops the RG running of  $g_2$  while it is still small, so that the theory remains weakly coupled. This justifies the use of perturbation theory and will allow us to read the spectrum (and the interactions) of particles from the Lagrangian in the unitary gauge. Let us therefore start by writing down the Lagrangian of the SM and see how  $SU(2)_{EW} \otimes U(1)_Y$  is spontaneously broken.

Since, as we said, the SM should be regarded as an EFT, (17)  
its Lagrangian must be written as a sum of terms

$$\mathcal{L}_{SM} = \mathcal{L}^{\text{ren}} + \mathcal{L}^{(5)} + \mathcal{L}^{(6)} + \dots \equiv \mathcal{L}^{\text{ren}} + \mathcal{L}^{\text{non-ren}}$$

where  $\mathcal{L}^{(m)}$  contains operators with dimension  $D=m$ . Therefore,  
 $\mathcal{L}^{\text{non-ren}}$  contains all non-renormalizable operators (those with  
dimension  $D > 4$ ):

$$\mathcal{L}^{\text{non-ren}} = \sum_i c_i \mathcal{O}_i \quad \text{where } [c_i] = 4 - [\mathcal{O}_i]$$

$\uparrow$  effective coefficients       $\nwarrow$  local operators made of SM fields

Being an EFT, the SM is valid up to a cutoff scale  $\Lambda$ , at which new degrees of freedom will appear and the theory is incorporated in a larger mathematical scheme. The effective operators  $\mathcal{O}_i$  are thus obtained by integrating out the new physics degrees of freedom at the scale  $\Lambda$ ; on dimensional ground, one therefore expects that

$$c_i \sim \left( \frac{1}{\Lambda} \right)^{[\mathcal{O}_i] - 4}$$

This means that the contribution of operator  $\mathcal{O}_i$  to some observable measured at an energy  $E \ll \Lambda$  is suppressed by a factor  $(E/\Lambda)^{[\mathcal{O}_i] - 4} \ll 1$ .

This is the reason why operators with  $[O_1] > 4$  are also called IRRELEVANT.

(18)

Since we are now interested in the properties of the vacuum (i.e. in the phase of the theory), we can neglect  $\mathcal{L}^{\text{non-ren}}$  and focus on the renormalizable Lagrangian. The latter can be written as follows:

$$\mathcal{L}^{\text{ren}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_\theta + \mathcal{L}_{\text{Yuk}} - V_{\text{scalar}}$$

$$\mathcal{L}_{\text{kin}} = \sum_{\psi \in \{q, u, d, l, e\}} \sum_{j=1}^3 \bar{\psi}^{(j)} i \not{D} \psi^{(j)} + |D_\mu H|^2$$

$$- \frac{1}{4g_Y^2} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4g_2^2} W_{\mu\nu}^i W^{\mu\nu, i} - \frac{1}{4g_3^2} G_{\mu\nu}^a G^{\mu\nu, a}$$

$$(\psi = q_L, u_R, d_R, l_L, e_R)$$

$$\mathcal{L}_\theta = \frac{\theta_1}{32\pi^2} B_{\mu\nu} \tilde{B}^{\mu\nu} + \frac{\theta_2}{32\pi^2} W_{\mu\nu}^i \tilde{W}^{\mu\nu, i} + \frac{\theta_3}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{\mu\nu, a}$$

$$(\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma})$$

$$\mathcal{L}_{\text{Yuk}} = - \sum_{j,k}^{1,2,3} \left[ (Y_u)_{jk} \bar{q}_L^{(j)} H^c u_R^{(k)} + (Y_d)_{jk} \bar{q}_L^{(j)} H d_R^{(k)} \right. \\ \left. + (Y_e)_{jk} \bar{l}_L^{(j)} H e_R^{(k)} + \text{h.c.} \right]$$

$$(H^c \equiv i\sigma^2 H^*)$$

$$V_{\text{scalar}} = \mu^2 H^\dagger H + \lambda (H^\dagger H)^2$$

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This Lagrangian has been written by including all possible terms that are compatible with the requests of i) invariance under  $SU(3)_C \times SU(2)_{EW} \times U(1)_Y$  transformations ; ii) invariance under  $SO^+(3,1)$  transformations. It is clear that it includes many unphysical parameters that can be removed by field redefinitions. We will come back to this later.

At this point we assume that the electroweak interactions remain weak at low energy and determine the properties of the electroweak vacuum by using perturbation theory. In practice, we look for configurations of the Higgs field  $H(x)$  that minimize the classical Hamiltonian. The kinetic energy is minimized by constant field configurations, and among those we look for configurations that minimize the potential  $V_{\text{scalar}}$ . These configurations are solutions of the classical equations of motion.

Let us start by adopting the following parametrization of the Higgs field :

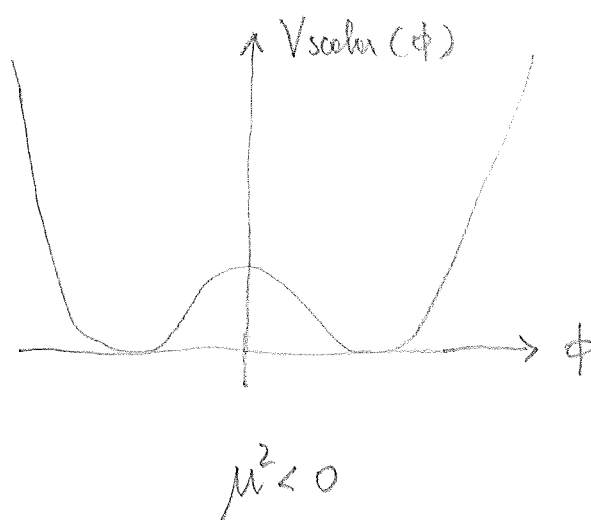
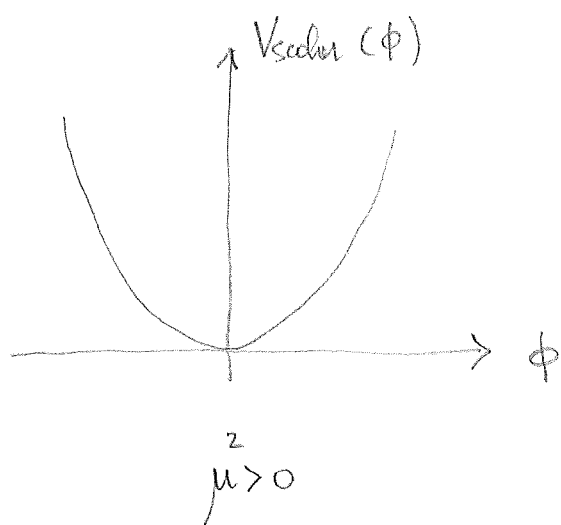
$$H(x) = e^{i\varphi^a(x)T^a} \begin{pmatrix} 0 \\ \frac{\phi(x)}{\sqrt{2}} \end{pmatrix} \quad a = 1, 2, 3$$

where  $T^a$  are the  $SU(2)_{EW}$  generators,  $T^a = \sigma^a/2$ .

The potential  $V_{\text{scalar}}$  only depends on  $\phi(x)$  and can be rewritten as

$$V_{\text{scalar}} = \frac{\mu^2}{2} \phi^2(x) + \frac{\lambda}{4} \phi^4(x)$$

We now consider how  $V_{\text{scalar}}$  varies as a function of a constant background value  $\phi(x) = \phi$ ; the behavior of  $V_{\text{scalar}}$  depends on the sign of  $\mu^2$ :



For  $\mu^2 < 0$  there are minima at

$$0 = \frac{dV_{\text{scalar}}}{d\phi} = \mu^2 \phi + \lambda \phi^3 \rightarrow$$

$$\phi^2 = \frac{-\mu^2}{\lambda} \equiv v^2$$

$$\frac{d^2 V_{\text{scalar}}}{d\phi^2} = \mu^2 + 3\lambda \phi^2 \rightarrow$$

$$\left. \frac{d^2 V_{\text{scalar}}}{d\phi^2} \right|_{\phi=v} = -2\mu^2 > 0$$

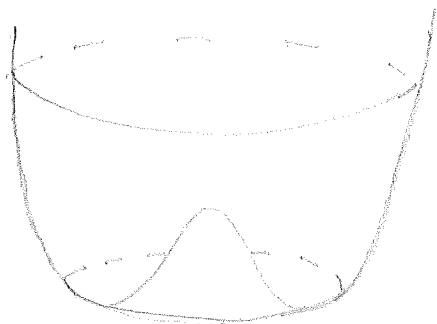
The potential has a valley of minima parametrized by

$$\langle H(x) \rangle = e^{i T^a \theta^a} \begin{pmatrix} 0 \\ v \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$



where  $\theta^a = 1, 2, 3$  are arbitrary phases. The valley of minima lies at the bottom of the potential

(21)



We can take any of these classical minima as the starting point to construct the Fock space of our quantum field theory. Since under  $SU(2)_{EW} \times U(1)_Y$  the Higgs field transforms as

$$SU(2)_{EW} : \quad H(x) \rightarrow \exp(i T^a \alpha^a) H(x)$$

$$U(1)_Y : \quad H(x) \rightarrow \exp(i \frac{Y}{2}) H(x)$$

the vev  $\langle H \rangle$  breaks spontaneously the global  $SU(2)_{EW} \times U(1)_Y$  invariance down to a  $U(1)_{em}$  subgroup generated by the generator  $Q$ :

$$Q = a_0 Y + a_i T^i \quad a_0, a_i \in \mathbb{R}$$

$$Q \cdot \langle H \rangle = 0$$

If we choose  $\theta^a = 0$  in the expression of  $\langle H \rangle$ , then we have:

$$\begin{bmatrix} \frac{a_0}{2} + \frac{a_3}{2} & \frac{a_1 - i a_2}{2} \\ \frac{a_1}{2} + i \frac{a_2}{2} & \frac{a_0}{2} - \frac{a_3}{2} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{a_1 - i a_2}{2} \\ \frac{a_0 - a_3}{2} \end{pmatrix} \equiv 0$$

$$\rightarrow \begin{cases} a_1 = a_2 = 0 \\ a_0 = a_3 \end{cases}$$

 $\rightarrow$ 

$$Q = T_3 + Y$$

The spontaneous breaking  $SU(2)_{EW} \times U(1)_Y \rightarrow U(1)_{em}$  implies namely 3 Nambu-Goldstone bosons (NGBs).

These can be identified with the fields  $\chi_{(x)}^{i=1,2,3}$  that parametrize massless excitations along the valley of minima in the classical potential.

The massive excitation in the radial direction, instead, is identified with the Higgs boson. We can set  $\phi(x) = v + h(x)$ ,  $\varphi^a(x) = \frac{\chi^a(x)}{v}$  and write:

$$H(x) = e^{i T^a \chi^a(x) / v} \begin{pmatrix} 0 \\ \frac{v + h(x)}{\sqrt{2}} \end{pmatrix}$$

We recall that this analysis makes sense only as long as perturbation theory is valid and a gauge fixing is performed. In fact, the presence of the gauge fixing invalidates the Goldstone theorem: depending on the choice of the gauge the NGBs can be either absent or unphysical.

In the so-called unitary gauge, the would-be NGB one is absent from the Lagrangian. This is obtained by performing a gauge transformation such that

$$H(x) \rightarrow e^{iT^a \alpha^a(x)} H(x) \equiv \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix}$$

In practice, one sets  $\alpha^a(x) = -X^a(x)/v$  to remove the NGB from the field. In general, if present the NGB have a mixing term with the gauge fields of the form  $g v \partial_\mu X A^\mu$ , which thus requires a diagonalization. Gauge fixings where this mixing is removed (like the unitary gauge) are particularly convenient since the kinetic term is already diagonal. In the SM Lagrangian, the mixing term arises from the kinetic term of the Higgs field:

$$\begin{aligned} D_\mu H &= \partial_\mu H - i \frac{g_Y}{2} B_\mu H - i g_2 W_\mu^a \frac{\sigma^a}{2} H \\ &= e^{iX^a \sigma^a / 2v} \left[ i \frac{\sigma^a}{2v} \partial_\mu X^a \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\partial_\mu h}{v} \end{pmatrix} - i \frac{g_Y}{2} B_\mu \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} \right. \\ &\quad \left. - i \frac{g_2}{2} W_\mu^a \sigma^a \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} \right] \end{aligned}$$

Now we define  $\sigma^\pm \equiv \frac{\sigma_1 \pm i \sigma_2}{2}$ ;  $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$   $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$W_\mu^\pm \equiv \frac{W_\mu^1 \mp i W_\mu^2}{\sqrt{2}}$$

$$\chi^\pm \equiv \frac{\chi^1 \mp i \chi^2}{\sqrt{2}}$$

(24)

hence

$$\begin{cases} W_\mu^1 = \frac{W_\mu^+ + W_\mu^-}{\sqrt{2}} \\ W_\mu^2 = i \frac{W_\mu^+ - W_\mu^-}{\sqrt{2}} \end{cases}$$

$$\begin{cases} \chi^1 = \frac{\chi^+ + \chi^-}{\sqrt{2}} \\ \chi^2 = i \frac{\chi^+ - \chi^-}{\sqrt{2}} \end{cases}$$

$$\frac{1}{2} (\sigma^1 W_\mu^1 + \sigma^2 W_\mu^2) = \frac{1}{\sqrt{2}} (\sigma^+ W_\mu^+ + \sigma^- W_\mu^-)$$

$$\frac{1}{2} (\sigma^1 \chi^1 + \sigma^2 \chi^2) = \frac{1}{\sqrt{2}} (\sigma^+ \chi^+ + \sigma^- \chi^-)$$

We can thus write

$$D_\mu H = e^{i\chi^a \sigma^a / 2v} \left[ \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} - \frac{i}{2} \left( g_Y B_\mu - g_Z W_\mu^3 - \frac{\partial_\mu \chi^3}{v} \right) \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} - \frac{i}{\sqrt{2}} \left( g_Z W_\mu^+ - \frac{\partial_\mu \chi^+}{v} \right) \begin{pmatrix} \frac{v+h}{\sqrt{2}} \\ 0 \end{pmatrix} \right]$$

from which it follows :

$$|D_\mu H|^2 = \frac{1}{2}(\partial_\mu h)^2 + \frac{(\nu+h)^2}{8} \left( g_Y B_\mu - g_2 W_\mu^3 - \frac{\partial_\mu \chi^3}{\nu} \right)^2 \\ + \frac{(\nu+h)^2}{4} \left( g_2 W_\mu^+ - \frac{\partial_\mu \chi^+}{\nu} \right) \left( g_2 W_\mu^- - \frac{\partial_\mu \chi^-}{\nu} \right)$$

We see that

$$|D_\mu H|^2 \supset -\frac{\nu}{4} \partial_\mu \chi^3 (g_Y B_\mu - g_2 W_\mu^3) \\ + \frac{\nu g_2}{4} (\partial_\mu \chi^+ W_\mu^- + \text{h.c.})$$

In the unitary gauge  $\chi^a(x) = 0$ , hence

$$|D_\mu H|^2 = \frac{1}{2}(\partial_\mu h)^2 + \frac{(\nu+h)^2}{8} (g_Y B_\mu - g_2 W_\mu^3)^2 \\ + \frac{(\nu+h)^2}{4} g_2^2 W_\mu^+ W_\mu^-$$

It is at this point convenient to make a field redefinition (a so-called Weinberg rotation):

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix}$$

with  $\sin\theta_W \equiv \frac{g_Y}{\sqrt{g_Y^2 + g_2^2}}$   $\cos\theta_W \equiv \frac{g_2}{\sqrt{g_Y^2 + g_2^2}}$

This gives (in the unitary gauge) :

(26)

$$|D_\mu H|^2 = \frac{1}{2} (\partial_\mu h)^2 + \left( \frac{g_2^2 v^2}{4} \right) \left( 1 + \frac{h(x)}{v} \right)^2 W_\mu^+ W_\mu^- \\ + \frac{1}{2} \left( v^2 \frac{(g_1^2 + g_2^2)}{4} \right) \left( 1 + \frac{h(x)}{v} \right)^2 Z_\mu Z_\mu$$

From this expression it is easy to read off the masses of  $W^\pm$  and  $Z$  :

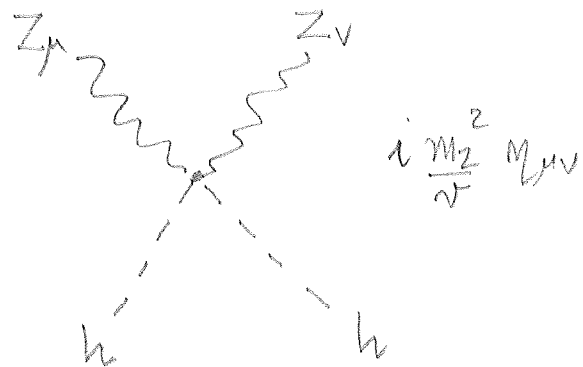
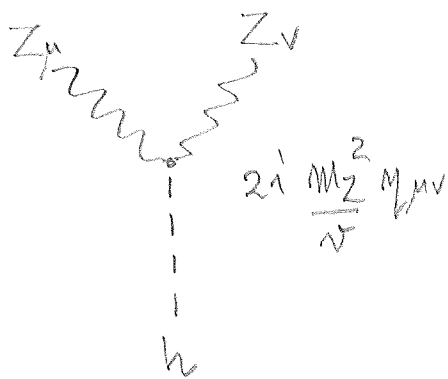
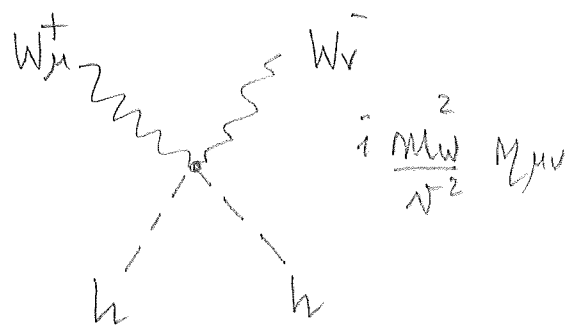
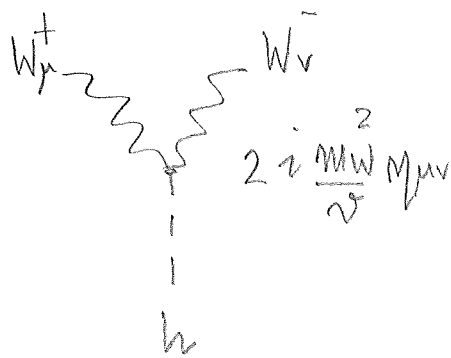
$$m_W^2 = \frac{g_2^2 v^2}{4}$$

$$m_Z^2 = \frac{v^2 (g_1^2 + g_2^2)}{4}$$

while the photon ( $A_\mu$ ) remains massless. We conclude that three (out of the four initial ones) gauge fields become massive and mediate short-range forces (the range of the interactions being  $\sim 1/m_W$  and  $\sim 1/m_Z$  respectively). The fourth gauge field remains massless and is identified with the photon. Its gauge invariance is still exact in the unitary gauge. The scale of symmetry breaking  $SU(2)_W \times U(1)_Y \rightarrow U(1)_{em}$ ,  $v$ , is easily determined by means of experimental data (muon decay). Before doing that, however, it is useful to finish deriving the spectrum and the couplings of the theory.



From the kinetic term of the Higgs field, in the unitary gauge, (27)  
 we can immediately obtain the interaction vertices between  
 the field  $h(x)$  (the Higgs boson) and the EW gauge fields:



The couplings of the gauge bosons to fermions can be derived by looking  
 at the fermions' covariant derivative. For a generic representation

$$\begin{aligned}
 D_\mu &= \partial_\mu - i g_Y B_\mu Y - i g_2 W_\mu^3 T^3 - i g_2 (W_\mu^1 T^1 + W_\mu^2 T^2) - i g_3 g_\mu^a \frac{\lambda^a}{2} \\
 &= \partial_\mu - i g_Y (\cos\theta_W A_\mu - \sin\theta_W Z_\mu) - i g_2 T_3 (\sin\theta_W A_\mu + \cos\theta_W Z_\mu) \\
 &\quad - i \frac{g_2}{\sqrt{2}} (T^+ W_\mu^+ + T^- W_\mu^-) - i g_3 g_\mu^a \frac{\lambda^a}{2} \\
 &= \partial_\mu - i A_\mu (Y g_Y \cos\theta_W + g_2 T_3 \sin\theta_W) - i Z_\mu (-Y \sin\theta_W g_Y + T_3 \cos\theta_W g_2) \\
 &\quad - i \frac{g_2}{\sqrt{2}} (T^+ W_\mu^+ + T^- W_\mu^-) - i g_3 g_\mu^a \frac{\lambda^a}{2}
 \end{aligned}$$

$$e \equiv \frac{g_Y g_2}{\sqrt{g_Y^2 + g_2^2}} ; \quad \frac{1}{e^2} = \frac{1}{g_Y^2} + \frac{1}{g_2^2} ; \quad \tan \theta_W = \frac{g_Y}{g_2}$$

We finally obtain :

$$D_\mu = \partial_\mu - ieQ A_\mu - i \frac{e}{\sin \theta_W \cos \theta_W} Z_\mu (T_3 - Q \sin^2 \theta_W) - i \frac{g_2}{\sqrt{2}} (T^+ W_\mu^+ + T^- W_\mu^-) - ig_3 g_\mu^a \frac{\lambda^a}{2}$$

where, we recall, the EM generator is  $Q = T_3 + Y$ .

We are thus led to identify  $e$  with the electromagnetic coupling, while  $Q$  gives the EM charges in terms of the  $SU(2)_{EW} \times U(1)_Y$  quantum numbers

|         | $T_3$          | $Y$            | $Q$            |
|---------|----------------|----------------|----------------|
| $u_L$   | $+\frac{1}{2}$ | $\frac{1}{6}$  | $\frac{2}{3}$  |
| $d_L$   | $-\frac{1}{2}$ | $\frac{1}{6}$  | $-\frac{1}{3}$ |
| $u_R$   | $0$            | $\frac{2}{3}$  | $\frac{2}{3}$  |
| $d_R$   | $0$            | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| $\nu_L$ | $+\frac{1}{2}$ | $-\frac{1}{2}$ | $0$            |
| $e_L$   | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$           |
| $e_R$   | $0$            | $-1$           | $-1$           |

This table shows one important property of our theory:

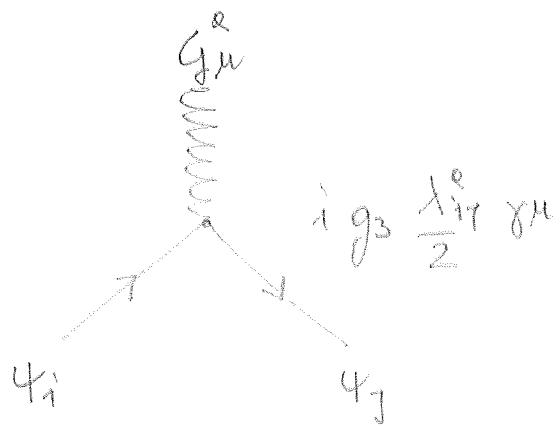
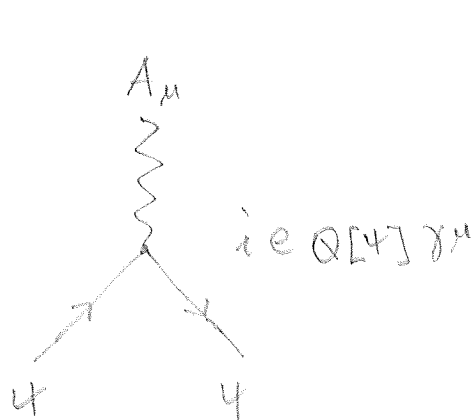
(29)

For each non-vanishing value of the EM charge, there exist both left- and right-handed fields.

In other words, the spectrum of EM charges is vector-like.

This crucial property of QED is reproduced in the SM by means of a non-trivial interplay of  $SU(2)_{EW}$  and  $U(1)_Y$  quantum numbers. Spelled in yet another way:  $U(1)_{em}$  is a vectorial subgroup of  $SU(2)_{EW} \times U(1)_Y$ .

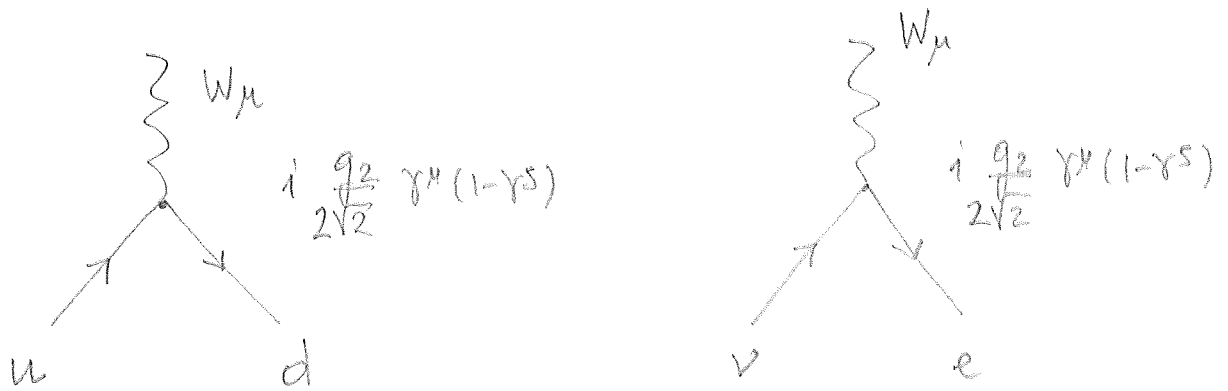
In fact, it is easy to check that the whole  $SU(3)_c \times U(1)_{em}$  is a vectorial subgroup of  $SU(3)_c \times SU(2)_{EW} \times U(1)_Y$ : for each rep of  $SU(3)_c \times U(1)_{em}$  there are fields with both left and right chiralities. This implies that the interactions that we have derived between fermions and  $SU(3)_c \times U(1)_{em}$  are precisely those postulated in QCD and QED:



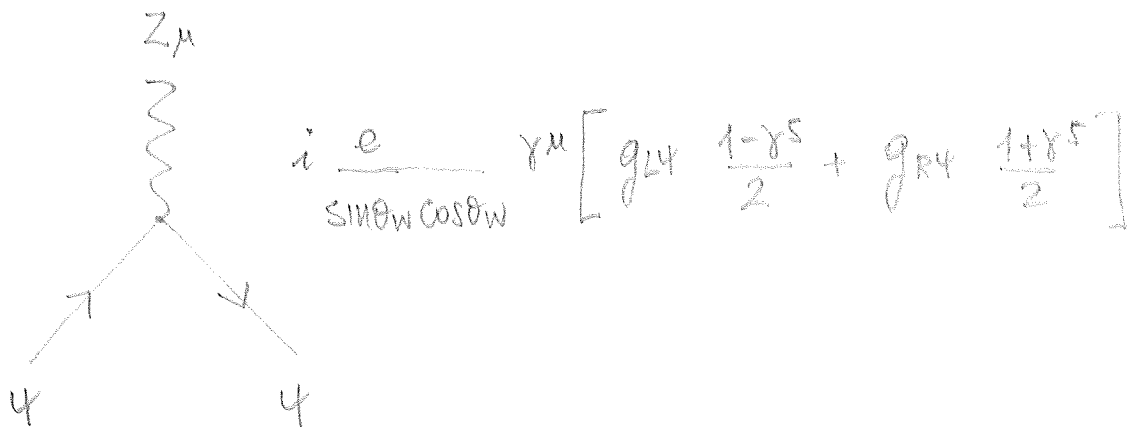
These interaction vertices are proportional to  $\gamma_\mu$  and for this reason they are called vectorial. The interactions with the  $W_\mu^\pm, Z_\mu$  bosons are instead chiral, i.e. left- and right-handed fields have different coupling.

Since only left-handed fields transform under  $SU(2)_W$ , the interaction with  $W_\mu^\pm$  is proportional to  $\gamma^\mu(1-\gamma^5)$ :

(30)



The interaction with  $Z_\mu$  instead is of the form:



$$\text{where } \begin{cases} g_L^\psi \equiv T_3[\psi] - \sin^2\theta_W Q[\psi] \\ g_R^\psi \equiv -\sin^2\theta_W Q[\psi] \end{cases}$$

Finally, the kinetic Lagrangian for fermions can be written as

$$\mathcal{L}_{\text{kin}}^{(f)} = \bar{u} i \not{\partial} u + \bar{d} i \not{\partial} d + \bar{e} i \not{\partial} e + \bar{\nu} i \not{\partial} \nu \\ + e A_\mu \left[ \frac{2}{3} \bar{u} \gamma^\mu u - \frac{1}{3} \bar{d} \gamma^\mu d - \bar{e} \gamma^\mu e \right]$$

$$+ g_3 g_\mu^a \left( \bar{u} \gamma^\mu \frac{\lambda^a}{2} u + \bar{d} \gamma^\mu \frac{\lambda^a}{2} d \right)$$

$$+ \frac{g_2}{2\sqrt{2}} W_\mu^+ \left[ \bar{u} \gamma^\mu (1-\gamma^5) d + \bar{\nu} \gamma^\mu (1-\gamma^5) e \right] + h.c.$$

$$+ \frac{e}{\cos\theta_W \sin\theta_W} Z_\mu \left\{ \bar{u} \gamma^\mu \left[ \frac{1-\gamma^5}{2} g_{Lu} + \frac{1+\gamma^5}{2} g_{Ru} \right] u \right. \\ \left. + \bar{d} \gamma^\mu \left[ \frac{1-\gamma^5}{2} g_{Ld} + \frac{1+\gamma^5}{2} g_{Rd} \right] d \right. \\ \left. + \bar{e} \gamma^\mu \left[ \frac{1-\gamma^5}{2} g_{Le} + \frac{1+\gamma^5}{2} g_{Re} \right] e \right. \\ \left. + \bar{\nu} \gamma^\mu \frac{1-\gamma^5}{2} g_{Lv} \nu \right\}$$

$$g_{Lu} = \frac{1}{2} - \frac{2}{3} \sin^2\theta_W$$

$$g_{Ld} = -\frac{1}{2} + \frac{1}{3} \sin^2\theta_W$$

$$g_{Ru} = -\frac{2}{3} \sin^2\theta_W$$

$$g_{Rd} = \frac{1}{3} \sin^2\theta_W$$

$$g_{Le} = -\frac{1}{2} + \sin^2\theta_W$$

$$g_{Lv} = \frac{1}{2}$$

$$g_{Re} = \sin^2\theta_W$$

Here we omitted for simplicity the family indices, since all terms of the Lagrangian are (so far) diagonal in flavor space.

The field basis we have been working in is not, however, the mass eigenstate basis. To see this, let us analyze the Yukawa terms.

In the unitary gauge,

32

$$H(x) = \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix} \quad H^c(x) \equiv i\sigma^2 H^*(x) = \begin{pmatrix} \frac{v+h(x)}{\sqrt{2}} \\ 0 \end{pmatrix}$$

the Yukawa terms include mass terms for quarks and charged leptons:

$$\begin{aligned} \mathcal{L}_{\text{Yuk}} &= -\bar{q}_L^i (\gamma_\mu)^{ij} H^c U_R^j - \bar{q}_L^i (\gamma_\mu)^{ij} H d_R^j - \bar{l}_L^i (\gamma_\mu)^{ij} H e_R^j + \text{h.c.} \\ &= -\left(\frac{v+h}{\sqrt{2}}\right) \left[ \bar{u}_L^i \gamma_\mu^{ij} U_R^j + \bar{d}_L^i \gamma_\mu^{ij} d_R^j + \bar{e}_L^i \gamma_\mu^{ij} e_R^j + \text{h.c.} \right] \end{aligned}$$

The matrices  $Y_u, Y_d, Y_e$  are generic  $3 \times 3$  complex matrices that can be diagonalized by means of two unitary transformations. We thus perform the following field redefinition:

$$\begin{aligned} u_R^i &\rightarrow (U_R)_{ij} u_R^j & u_L^i &\rightarrow (U_L)_{ij} u_L^j \\ d_R^i &\rightarrow (D_R)_{ij} d_R^j & d_L^i &\rightarrow (D_L)_{ij} d_L^j \\ e_L^i &\rightarrow (E_L)_{ij} e_L^j & U_R, U_L, D_L, D_R, E_L, E_R &\in SU(3) \\ e_R^i &\rightarrow (E_R)_{ij} e_R^j \end{aligned}$$

such that

$$\begin{aligned} Y_u &\rightarrow U_L^\dagger Y_u U_R \equiv Y_u^{\text{diag}} \\ Y_d &\rightarrow D_L^\dagger Y_d D_R \equiv Y_d^{\text{diag}} \\ Y_e &\rightarrow E_L^\dagger Y_e E_R \equiv Y_e^{\text{diag}} \end{aligned}$$

It is possible to make the diagonal entries of  $Y_{u,d,e}$  real by the following further field redefinition acting only on right-handed fields:

(33)

$$\begin{cases} u_R^i \rightarrow e^{i\phi_u^i} u_R^i \\ d_R^i \rightarrow e^{i\phi_d^i} d_R^i \\ e_R^i \rightarrow e^{i\phi_e^i} e_R^i \end{cases}$$

The Yukawa terms finally read:

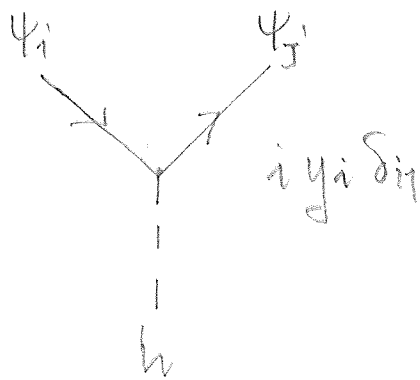
$$\mathcal{L}_{\text{Yuk}} = - \left( 1 + \frac{h(x)}{v} \right) \left\{ \bar{u}^i \begin{pmatrix} m_u & & \\ & m_c & \\ & & m_t \end{pmatrix}_{ij} u^j + \bar{d}^i \begin{pmatrix} m_d & & \\ & m_s & \\ & & m_b \end{pmatrix}_{ij} d^j + \bar{e}^i \begin{pmatrix} m_e & & \\ & m_\mu & \\ & & m_\tau \end{pmatrix}_{ij} e^j \right\}$$

where

$$m_i \equiv \frac{v}{\sqrt{2}} Y_{ii}^{\text{diag}}$$

is the mass of each fermion. Notice that neutrinos do not get mass in this way. The Lagrangian  $\mathcal{L}_{\text{Yuk}}$  also contains Yukawa interactions between fermions and the Higgs boson:





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$$y_i \equiv \frac{m_i}{v}$$

Notice that these interactions are diagonal in flavor space; this is a property of the SM that does not hold when we include non-renormalizable operators.

What is the impact of our field redefinitions on the remaining terms of the Lagrangian? It is easy to see that, since  $U_{LR}, D_{LR}, E_{LR}$  are unitary ( $SU(3)$ ) transformations, the only terms that are affected are  $W_\mu$  interactions:

$$\bar{u}_L^i \gamma^\mu W_\mu^+ d_L^j \rightarrow \bar{u}_L^i \gamma^\mu W_\mu^+ (U_L^\dagger D_L)_{ij} d_L^j$$

$$\bar{\nu}_L^i \gamma^\mu W_\mu^+ e_L^j \rightarrow \bar{\nu}_L^i \gamma^\mu W_\mu^+ E_L^{ij} e_L^j$$

The phase redefinitions of right-handed fields, in particular, do not modify any other term in the Lagrangian (Notice: some of these transformations are in fact anomalous under the gauge group; this means that at the quantum level they will imply a redefinition of the  $\Theta$  terms).

We still have the freedom to redefine the neutrino fields and bring back the lepton interaction with  $W_\mu$  to a flavor-diagonal form:

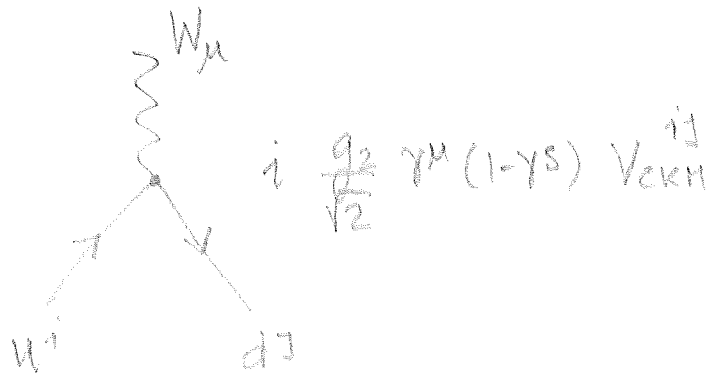
$$V_L^i \rightarrow (E_L)_{ij} V_L^j$$

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We cannot, on the other hand, remove the matrix that appears in the quark vertex with  $W_\mu$ , since this would change the Yukawa terms. We conclude that quark weak interactions with  $W_\mu$  (charge-current interactions) mix different flavors through the  $SU(3)$  matrix

$$V_{CKM} \equiv V_L^\dagger D_L$$

This is the so-called 'Cobbi-Kobayashi-Maskawa' matrix. We thus update the expression of the  $\bar{u} W_\mu d$  vertex to



Notice that Neutral-current interactions (those involving  $Z_\mu$ ) do not mix different flavors. In other words, there are no Flavor-Changing Neutral Currents (FCNC) at tree level in the SM renormalizable Lagrangian. This is not true anymore when non-renormalizable operators are included.

Finally, there is no flavor mixing in the lepton sector at the level of the renormalisable Lagrangian.

(36)

How many physical parameters does  $V_{CKM}$  contain? It is a  $3 \times 3$  special unitary matrix and as such it has 8 real parameters. In comparison, a  $3 \times 3$  real orthogonal matrix has 3 real parameters, i.e. three angles. We can thus identify the 8 parameters of  $V_{CKM}$  as 3 angles plus 5 phases.

Not all these 8 parameters are however physical. We can indeed perform vectorlike (i.e. the same for left and right fields) phase redefinitions of the fields :

$$u_L^i \rightarrow e^{i\theta_i} u_L^i$$

$$d_L^i \rightarrow e^{i\tilde{\theta}_i} d_L^i$$

$$u_R^i \rightarrow e^{i\theta_i} u_R^i$$

$$d_R^i \rightarrow e^{i\tilde{\theta}_i} d_R^i$$

These do not change the mass terms but can remove some of the phases in  $V_{CKM}$ . How many? Let us count. We have  $3+3=6$  phase redefinition in total at disposal. An overall phase (baryon number), however, does not modify  $V_{CKM}$ . An opposite phase transformation of up and down quarks would instead introduce an overall complex phase in  $V_{CKM}$ , and should not be performed. The remaining 4 relative phases can be used to remove 4 unphysical parameters in  $V_{CKM}$ .

One is left with

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3 angles + 1 phase (physical)

The standard parametrization of  $V_{CKM}$  in terms of these angles and phase (the one recommended by PDG) is the following:

$$V_{CKM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where  $\theta_{12}, \theta_{13}, \theta_{23}$  are three Euler angles and  $\delta$  is a CP-violating phase.

Finally, let us analyze the scalar potential. In the unitary gauge

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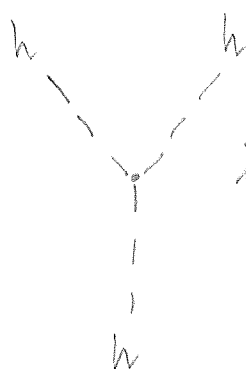
$$\begin{aligned}
 V_{\text{scalar}} &= \frac{\mu^2}{2} (v+h)^2 + \frac{\lambda}{4} (v+h)^4 \\
 &= \frac{\mu^2}{2} v^2 + \mu^2 v h(x) + \frac{\mu^2}{2} h(x)^2 + \frac{\lambda}{4} (v^4 + h^4 + 4h^3 v + 4h v^3 + 6h^2 v^2) \\
 &= \left( \frac{\lambda v^4}{4} + \frac{\mu^2 v^2}{2} \right) + h (\lambda v^3 + \mu^2 v) + \frac{h^2}{2} (\mu^2 + 3\lambda v^2) \\
 &\quad + \lambda v h^3 + \frac{\lambda}{4} h^4
 \end{aligned}$$

$$V_{\text{scalar}} = \frac{m_h^2}{2} h(x)^2 + \lambda v h(x)^3 + \frac{\lambda}{4} h(x)^4 + \text{constant}$$

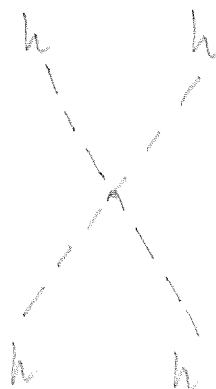
where we used that  $\lambda v^3 + \mu^2 v = 0$  and defined the mass of the Higgs boson to be

$$m_h^2 = 2\lambda v^2 = -2\mu^2$$

We thus predict a massive spin-0 boson with the following self-interactions:



$$i\lambda v = i \frac{m_h^2}{2v}$$

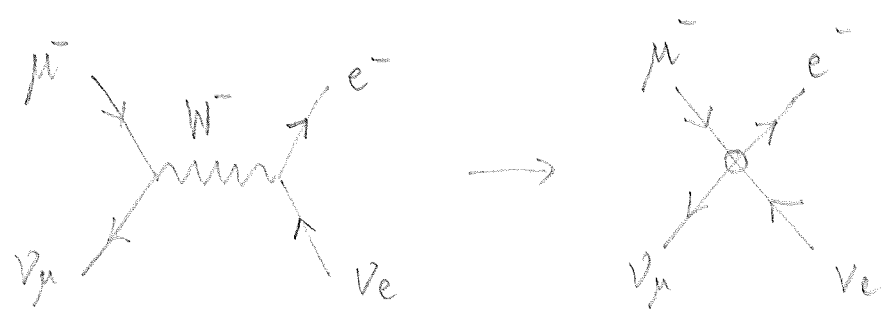


$$i \frac{\lambda}{4} = i \frac{m_h^2}{8v^2}$$

This completes our derivation of masses and couplings using perturbation theory. We have found that for  $\mu^2 < 0$  three of the gauge bosons acquire a mass, so that the force they mediate is short-ranged. At energies below this scale,  $E \ll M_W, M_Z$ , the dynamics is described by an  $SU(3)_c \times U(1)_{em}$  vectorlike effective theory. The corresponding Lagrangian is that of QCD and QED plus higher-dimensional operators:

$$\mathcal{L}_{E \ll M_W, M_Z} = \underbrace{\mathcal{L}_{QCD+QED}}_{\substack{\uparrow \\ \text{renormalizable} \\ \text{Lagrangian with} \\ \text{QCD + QED} \\ \text{interactions}}} + \underbrace{\mathcal{L}^{non-ren}}_{\substack{\uparrow \\ \text{non-renormalizable} \\ \text{operators break P, CP} \\ \text{and flavor symmetries}}}$$

Here  $\mathcal{L}^{non-ren}$  includes four-fermion operators obtained by integrating out  $W_\mu$  and  $Z_\mu$  plus others generated at loop level. As an example consider the operator



$$- \frac{G_F}{\sqrt{2}} \left[ \bar{\mu} \gamma^\mu (1 - \gamma^5) \nu_\mu \right] \left[ \bar{\nu}_e \gamma_\mu (1 - \gamma^5) e \right]$$

The Fermi constant  $G_F$  can be derived by matching the expressions of the Feynman amplitudes in the SM and in the low-energy theory:

$$M_{SM} = J_\mu^{(\mu)} J_\nu^{(e)} \left( \frac{1}{2!} 2 \right) \left( \frac{g}{2\sqrt{2}} \right)^2 \frac{-i}{p^2 - m_W^2} \left( \eta^{\mu\nu} - \frac{p^\mu p^\nu}{m_W^2} \right)$$

$$M_{\text{eff}} = -i \frac{G_F}{\sqrt{2}} J_\mu^{(\mu)} J_\nu^{(e)}$$

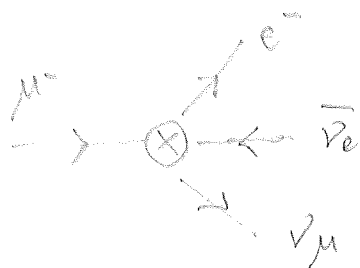
Considering that the term  $p^\mu p^\nu$  in the  $W$  propagator gives zero when contracted with conserved currents, and that

$$\frac{1}{p^2 - m_W^2} \simeq -\frac{1}{m_W^2} \left( 1 + O\left(\frac{p^2}{m_W^2}\right) \right) \quad \text{for } p^2 \ll m_W^2$$

one finds

$$\boxed{\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} = \frac{1}{2v^2}}$$

By applying the Fermi low-energy theory to muon decay



one finds that

$$\Sigma\mu = 2.2 \times 10^{-6} \text{ s} \Rightarrow G_F = 1.16 \times 10^{-5} \text{ GeV}^{-2}$$

This in turn fixes the value of the EW scale

$$V = 246 \text{ GeV}$$

The experimentally measured values of the W and Z masses are

$$m_W = 80.4 \text{ GeV}$$

$$m_Z = 91.2 \text{ GeV}$$

from which one infers the value of the EW couplings at the same scale :

$$g_Y(m_Z) \simeq 0.35$$

$$g_Z(m_W) \simeq 0.65$$

These are perturbative values, confirming the validity of our perturbative analysis.

Notice that the QED + QED renormalizable Lagrangian preserves C and P (with the exception of the  $\theta_3$  term, which breaks P but is constrained to be very small by data) and contains no flavor violation. All these effects come from the non-renormalizable operators in  $\mathcal{L}^{\text{non-ren}}$ .



This is a good point to pause and have a critical look of our analysis: our derivation of the spectrum and couplings relies on perturbation theory and required fixing the gauge. We have found that the Higgs field develops a vev and this in turn suggests that the  $SU(2)_E \times U(1)_Y$  symmetry is spontaneously broken.

But how can a gauge symmetry be broken, given that it is not actually a symmetry but merely a redundancy in the description of our theory, introduced to be able to work with local fields?

Furthermore: physical states, including the vacuum, are certainly gauge invariant, so how can the vacuum break the gauge symmetry?

A serious drawback of the perturbative approach is that it requires a gauge fixing, something that obscures the gauge invariance of physical quantities. It is therefore worth trying a different strategy. Wilson showed that it is possible (and in fact even desirable!) to define gauge theories on the lattice at finite volume without performing any gauge fixing. The continuum theory is obtained (actually defined) by taking the limit of vanishing lattice spacing, assuming that such limit exists. On the lattice, gauge fixing is not necessary since the gauge redundancy amounts to a finite multiplicative factor in the functional integral, which can always be removed by a proper normalization.

It is quite intuitive that if one tries to compute the vacuum expectation value of any gauge dependent operator in this theory, one obtains a vanishing result, given that the action in the functional integral is

$$\langle A^g \rangle = \langle A \rangle \Rightarrow \langle A \rangle = 0 \text{ if } A \neq A^g$$

This result can be proven rigorously and goes under the name of Elitzur's Theorem (Elitzur PRD 12 (1975) 3978). We must conclude, therefore, that the Higgs vev found in the perturbative approach is a gauge artifact and that there is no such phenomenon as spontaneous breaking of a gauge symmetry. The jargon commonly used in textbooks can be misleading.

Can one characterize the spectrum without sacrificing manifest gauge invariance? The answer is yes: one can resort to gauge-invariant interpolating operators, as first pointed out in these papers:

- Banks and Robinson NPB 160 (1979) 349
- 't Hooft NATO Sci. Ser. B59 (1980) 117 (Compere 1979 lectures)
- Frölich, Morchio, Strocchi Phys. Lett. 97B (1980) 249  
NPB 190 (1981) 553

Frölich, Morchio and Strocchi, in particular, showed that there is a one-to-one correspondence between gauge-invariant operators and those used in the perturbative approach. Such correspondence is established with reference to a critical orbit which minimizes the scalar potential ( $\langle H^\dagger H \rangle = v^2/2$ ), such that, when computed on a reference point on the orbit, the gauge invariant operator coincides with the perturbative one.

The gauge invariant operators are most easily constructed when the residual gauge group on the critical orbit (i.e. the subgroup which is unbroken in a generic point on the orbit) is the identity. In that case the gauge-invariant operators are also local.

Let us discuss this case by considering a modified version of the SM where the hypercharge factor  $U(1)_Y$  is not gauged. For simplicity, we will also keep only leptons among the fermions (in this way the theory has an odd number of  $SU(2)_{EW}$  doublets, hence a Witten anomaly, but we will ignore this problem for the sake of the argument).

In the perturbative approach the Higgs vev breaks spontaneously  $SU(2)_{EW} \times U(1)_Y$  down to  $U(1)_{em}$ , where the latter remains as a global unbroken symmetry and its quantum numbers characterize the physical states. The latter are interpolated by the following fields:

| field       | <u><math>U(1)_{em}</math> charge</u> |
|-------------|--------------------------------------|
| $e_L, e_R$  | -1                                   |
| $\nu_L$     | 0                                    |
| $h$         | 0                                    |
| $W_\mu^\pm$ | $\pm 1$                              |
| $W_\mu^3$   | 0                                    |

In the gauge-invariant (non-perturbative) approach there is no spontaneous breaking and one makes use of the global  $U(1)_Y$  symmetry to characterize the composite operators, hence the physical states.

For each perturbative field, one can construct a corresponding gauge-invariant operator. The latter coincides with the perturbative field, up to higher-order terms, when expanding around

$$H_0 \equiv \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

and its hypercharge is equal to the EM charge of the perturbative field. The list of gauge-invariant interpolating operators is the following:

| <u>Gauge-invariant operator</u>                            | <u>U(1)<sub>Y</sub> charge</u> | <u>Corresponding perturbative operator</u> |
|------------------------------------------------------------|--------------------------------|--------------------------------------------|
| $H^\dagger L_L$                                            | -1                             | $e_L$                                      |
| $e_R$                                                      | -1                             | $e_R$                                      |
| $\varepsilon^{\alpha\beta} H^\alpha L_L^\beta$             | 0                              | $\nu_L$                                    |
| $H^\dagger H$                                              | 0                              | $h$                                        |
| $H^\dagger D_\mu H$                                        | 0                              | $W_\mu^3$                                  |
| $\varepsilon^{\alpha\beta} H^\alpha (D_\mu H)^\beta$       | +1                             | $W_\mu^+$                                  |
| $\varepsilon^{\alpha\beta} (H^c)^\alpha (D_\mu H^c)^\beta$ | -1                             | $W_\mu^-$                                  |

This shows that a gauge-invariant characterization of the spectrum is possible, and coincides with that obtained using perturbation theory if the theory is weakly coupled. In the limit in which the theory is strongly coupled,

on the other hand, only the non-perturbative approach is viable. To see how the theory can become strongly coupled, let us decrease the value of  $v$  (as defined by the critical orbit  $\langle H^\dagger H \rangle = v^2/2$ ). This can be obtained by decreasing  $(-\mu^2)$  in the Lagrangian. When  $v$  decreases, the value of the gauge coupling  $g_2$  evaluated at that scale will increase ( $\beta_0^{(2)} < 0$ !). When  $v$  becomes comparable to the  $SU(2)_{EW}$  dynamical scale the theory is strongly coupled. Notice that this conclusion holds in our toy SM model with no quarks: in the full SM,  $SU(2)_{EW}$  is spontaneously broken by the quark condensate at energies  $E \sim 1 \text{ GeV}$ , at which  $g_2$  is still perturbative.

Finally, we want to mention how to construct interpolating gauge invariant fields in the case in which the residual group on the critical orbit is not the identity. In practice, we can focus on the case in which the residual group is a  $U(1)$  factor, like in the SM. In that case, physical states can be charged under the  $U(1)$ . While neutral states are still interpolated by gauge-invariant local operators, as discussed previously, charged states are interpolated by non-local (yet still gauge-invariant) operators. The construction of these non-local operators parallels what is done in QED for the electron. A possible interpolating operator for the electron is ( $q = -1$  is the charge of the electron)

$$W_q(\infty, x) \psi(x) \quad \text{where} \quad W_q(y, x) \equiv \exp \left[ i q \int_{\gamma(x \rightarrow y)} dx^\mu A_\mu(x) \right]$$

Here  $\gamma(x \rightarrow y)$  is a curve starting at  $x^\mu$  and ending at  $y^\mu$ . Under gauge transformations, the Wilson line  $W_q(y, x)$  transforms as

$$W_g(y, x) \rightarrow \exp \left[ i g \int_{x(x \rightarrow y)} dx'^\mu (A_\mu(x') + \partial_\mu \alpha) \right]$$

$$= e^{i g \alpha(y)} W_g(y, x) e^{-i g \alpha(x)}$$

whence  $\psi(x) \rightarrow e^{i g \alpha(x)} \psi(x)$ . Since  $\lim_{x \rightarrow \infty} \alpha(x) = 0$  for small gauge transformations, it follows that the operator  $W_g(\infty, x) \psi(x)$  is gauge invariant (i.e. invariant under small gauge transformations). On the other hand, under global  $U(1)$  transformations  $A_\mu$  does not transform, while  $\psi(x) \rightarrow e^{i g \alpha} \psi(x)$ . It follows that  $W_g(x, \infty) \psi(x)$  has (global)  $U(1)$  charge  $q$ .

The strategy in the presence of a non-trivial critical gauge orbit is thus the following (Fröhlich, Morchio and Strocchi): dress matter fields with the would-be NGB of the coset  $G/G_0$ , where  $G$  is the full gauge group and  $G_0 = U(1)$  is the residual subgroup. The local fields obtained in this way are representations of the local  $U(1)$ . With these, construct non-local strings that are gauge invariant but have non-vanishing  $U(1)$  charge.

We have seen that the  $SU(2)_{EW} \times U(1)_Y$  model gives a correct characterization of weak interactions ( $\beta$  decay), and that one recovers the familiar QED phenomenology at low energies. There is one aspect of the electroweak spectrum which is unsatisfactory, however: neutrinos are predicted to be massless. This is in conflict with data on neutrino oscillations, which require neutrinos to have a finite (though very small) mass. How can we reconcile our model with such experimental evidence?

The conclusion that neutrinos are massless followed by our analysis of the renormalizable part of the Lagrangian. Can a non-vanishing neutrino mass arise from higher-dimensional operators? The answer is yes, fortunately. There is only one  $SU(3)_C \times SU(2)_{EW} \times U(1)_Y$  invariant (local) operator with dimension  $D=5$ , which turns out to give the neutrino a mass upon electroweak symmetry breaking (notice: we are again using the sloppy perturbative language!). The operator is called the Weinberg operator and has the following form (S. Weinberg, IRL 43 (1979) 1566):

$$O_W = C_{ij} (\ell_i^{a,\alpha} H^b \epsilon_{ab}) (\ell_j^{a',\alpha'} H^{b'} \epsilon_{a'b'}) \epsilon_{\alpha\alpha'}$$

Here

|                     |                          |
|---------------------|--------------------------|
| $a, b, \alpha', b'$ | are $SU(2)_{EW}$ indices |
| $\alpha, \beta$     | are Lorentz indices      |
| $i, j$              | are family indices       |

It is easy to convince oneself that  $O_W$  is gauge invariant, (49) since each  $(lH)$  factor is, as well as Lorentz invariant.

The flavor matrix  $C_{ij}$  must be symmetric, due to Fermi statistics of  $l$  and Bose statistics of  $H$ . Indeed, if we exchange

$$(l_i^{\alpha, \alpha} H^b \epsilon_{ab}) \leftrightarrow (l_j^{\alpha', \alpha'} H^{b'} \epsilon_{ab'})$$

we get a  $(-1)$  factor due to Fermi statistics and exchange  $i \leftrightarrow j$  and  $\alpha \leftrightarrow \alpha'$ . Since  $\epsilon_{\alpha\alpha'}$  is antisymmetric under the last exchange and  $O$  must be overall symmetric, we conclude that  $C_{ij} = C_{ji}$ . When expanding around the Higgs vev, we find

$$O_W = \frac{v^2}{2} C_{ij} \nu_{L,i}^{\alpha} \nu_{L,j}^{\alpha'} \epsilon_{\alpha\alpha'} + \dots$$

where the dots indicate interaction terms. We thus obtain a Majorana mass term for left-handed neutrinos:

$$(M_M)_{ij} = \frac{v^2}{2} C_{ij}$$

Notice that  $C_{ij}$  has dimension  $[C_{ij}] = -1$ , hence  $M_M$  has dimension 1. The presence of a mass matrix for neutrino leads to mixing in the charge-current interactions. To see this, let us diagonalize the mass matrix, similarly to what we have done for the quark and charged lepton sectors. We start from the basis where



$m_u, m_d, m_e$  (mass matrices for up-type, down-type quarks and charged leptons) are diagonal and where the  $\bar{\nu}_L W e_L$  interaction is universal in family space. In that basis, the Majorana neutrino mass matrix  $M_N$  is a generic complex, symmetric  $3 \times 3$  matrix:

$$\mathcal{L}_{\nu \text{ mass}} = - \bar{\nu}_L^i (M_N)_{ij} \nu_L^j + \text{h.c.}$$

We can diagonalize it by redefining the neutrino fields by means of an  $SU(3)$  transformation:

$$\nu_L^i \rightarrow N_{ij} \nu_L^j \quad N \in SU(3)$$

such that

$$N^T M_N N = M_N^{\text{diag}}$$

Proof: Any complex matrix can be diagonalized by means of two unitary transformations (singular value decomposition):

$$M = A M_{\text{diag}} B \quad A, B \in SU(3)$$

If  $M$  is symmetric, then

$$A M_{\text{diag}} B = M = M^T = B^T M_{\text{diag}} A^T \Rightarrow \boxed{A = B^T}$$

Therefore:  $M = B^T M_{\text{diag}} B$ .

The matrix  $B$  can be constructed from the (normalised) eigenvectors of the Hermitian matrix  $M^\dagger M$ :

(51)

$$M^\dagger M = B^\dagger M \text{diag}^* B^* B^T M \text{diag} B = B^\dagger (M \text{diag}^* M \text{diag}) B$$

The diagonal entries of  $M \text{diag}^* M \text{diag}$  will be complex in general. They can be made real by redefining the phases of the neutrino fields:

$$\nu_L^i \rightarrow e^{i\theta_i} \nu_L^i$$

In such mass eigenstate basis, the interaction term with  $W_\mu$  has the form

$$\frac{g_2}{\sqrt{2}} \bar{\nu}_L \gamma^\mu W_\mu^+ U_{\text{PMNS}} e_L + \text{h.c.}$$

where  $U_{\text{PMNS}} \equiv \text{diag}(e^{i\theta_i}) \cdot N^\dagger$ .

The Pontecorvo-Maki-Nakagawa-Sakata matrix  $U_{\text{PMNS}}$  is the analog of the Cabibbo-Kobayashi-Maskawa matrix, since it encodes flavor mixing in charged lepton interactions. To identify the physical parameters in  $U_{\text{PMNS}}$ , we first notice that it is unitary but its determinant is not 1:  $U_{\text{PMNS}} \in U(3)$ . Therefore, it contains 9 parameters that we can identify as

$$3 \text{ angles} + 6 \text{ phases}$$

Not all these parameters are physical, however. Indeed, we can perform the field redefinitions

(52)

$$e_L^i \rightarrow e^{i\phi_i} e_L^i$$

$$e_R^i \rightarrow e^{i\phi_i} e_R^i$$

which leave the charged lepton matrix unchanged, to reabsorb three phases. Notice that we cannot change the neutrino phases otherwise the Majorana mass term gets modified. We conclude that the physical parameters in  $U_{\text{PMNS}}$  are

3 angles + 3 phases

Compared to the CKM matrix, there are two additional phases. The presence of these phases is a direct consequence of the Majorana nature of the neutrino mass term. A possible parametrization of  $U_{\text{PMNS}}$  is the following:

$$U_{\text{PMNS}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta} \\ 0 & c_{13} & 0 \\ -s_{13} e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\eta_1} & 0 & 0 \\ 0 & e^{i\eta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where  $\delta$  is the Dirac phase, while  $\eta_1, \eta_2$  are the Majorana phases.

For a review on neutrino masses and mixings, see the 2024 PDG Review "Neutrino masses, mixings and oscillations" by H.C. Gonzalez-Garcia and H. Yokoyama.

Let us continue our discussion about the spectrum of particles (and resonances) predicted by the SM by considering the quark sector. Quarks interact through  $SU(3)_c$  interactions and we have seen that these get stronger at lower energies. The solution of the RG equation at 1-loop

$$g^2(\mu) = \frac{g^2(H)}{1 - \frac{\beta_0}{8\pi^2} g^2(H) \log \frac{\mu}{H}}$$

implies that, for  $\beta_0 < 0$ , the coupling grows logarithmically when the energy decreases. We have seen that  $\beta_0^{(3)} = -7$  if one includes all six quarks in the running. In order to determine the evolution of  $g^3(\mu)$  at low energies it is important at this point to specify the value of the quark masses. This is because when the energy  $\mu$  becomes smaller than the mass of a given quark, that quark effectively stops contributing to the running. In the spirit of effective field theories, it must be integrated out of the theory. Now, we anticipate that a correct physical spectrum is obtained by fixing the quark masses to these values:

$$m_t \sim 173 \text{ GeV}$$

$$m_s \sim 100 \text{ MeV}$$

$$m_b \sim 4 \text{ GeV}$$

$$m_d \sim 5 \text{ MeV}$$

$$m_c \sim 1.3 \text{ GeV}$$

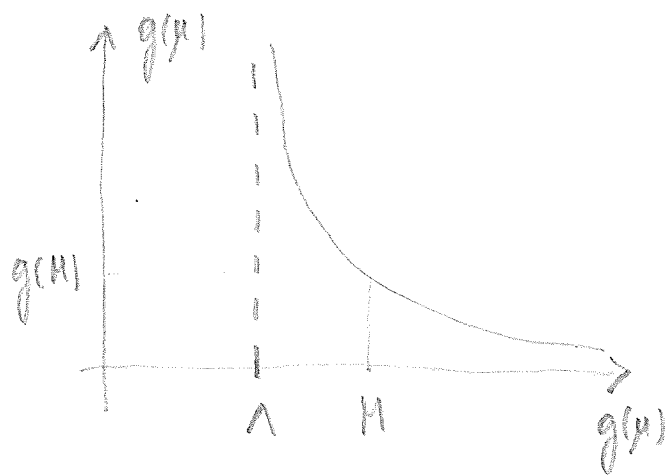
$$m_u \sim 2 \text{ MeV}$$

We don't need to be precise here, and we skip all details about the fact that quark masses are running quantities whose value thus depend on the scale and the renormalization scheme. We only need to know that top, bottom and charm are heavier than the others, and that up, down, strange have a mass in the range a few-100 MeV.

If we apply the RG running keeping only the dynamical quarks at each scale, we find that  $g_3(\mu)$  blows up at  $\mu \approx$  few hundred MeV. This means that the lightest three quarks (u, d, s) remain dynamical and contribute to the  $\beta$ -function for the whole RG evolution. The existence of a scale at which the coupling blows up is a simple prediction of the 1-loop RG equation: for  $\beta_0 < 0$  and  $\mu < M$ ,  $g(\mu)$  blows up at  $\mu = \Lambda$ , where

$$1 - \frac{\beta_0}{8\pi^2} g^2(M) \log \frac{\Lambda}{M} = 0$$

$$\Lambda = M \exp \left( - \frac{8\pi^2}{|\beta_0|} \frac{1}{g^2(M)} \right)$$



To compute the value of  $\Lambda$  using this formula we have to plug the value of  $\beta_0$  in a theory with 3 flavors

$$\beta_0^{(3)} = -\frac{11}{3} N_c + \frac{4}{3} \cdot \frac{1}{2} \cdot 3 = -9$$

and to specify the value of  $g_3$  at some scale  $M$  as extracted from

experimental data. In this way one obtains a value

$$\Lambda_{QCD} \approx 200-300 \text{ MeV}$$

depending on the renormalization scheme. The value in fact also depends on whether we work at 1-loop level or if we include higher-loop corrections to the  $\beta$ -function. In fact, the analytical solution of the RG equation can never be trusted when the coupling becomes too large, since perturbation theory breaks down. The message here is that (differently from the EW gauge coupling  $g_2$ ),  $g_3$  grows at low energy and becomes non perturbative at scales of order 200-300 MeV.

Since the quark sector becomes non perturbative at low energies, we cannot read the spectrum from the Lagrangian as we made for the EW sector. While a quantitative study requires some non-perturbative approach like lattice gauge theories, we can already obtain some qualitative information of the spectrum by simply considering interpolating operators. Since we have a simple non-abelian gauge group,  $SU(3)_c$ , and no abelian  $U(1)$  factors, the interpolating operators will be local operators.

The physical states excited by any operator are characterized by its quantum numbers, so we ask: what are the relevant quantum numbers? To answer this question, let us consider the QCD Lagrangian:

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4g_s^2} G_{\mu\nu}^a G^{\mu\nu,a} + \frac{\theta_3}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{\mu\nu,a} \\ + \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R - (\bar{q}_L m_q q_R + \text{h.c.})$$

Here  $q$  must be interpreted as a vector in flavor space. For example, if we consider only the three light quarks:

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad \text{hence} \quad m_q = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

What are the symmetries of the QCD Lagrangian? Besides the obvious Lorentz invariance, if we neglect the quark masses (this approximation is justified by the fact that  $m_{u,d,s} \ll \Lambda_{\text{QCD}}$ ) then  $\mathcal{L}_{\text{QCD}}$  is invariant under separate unitary rotations of left and right quarks:

$$q_L^i \rightarrow (V_L)^{ij} q_L^j \quad V_L \in U(3)_L$$

$$q_R^i \rightarrow (V_R)^{ij} q_R^j \quad V_R \in U(3)_R$$

In other words, the Lagrangian is invariant under

$$SU(3)_L \times U(1)_L \times SU(3)_R \times U(1)_R$$

(Here we do not pay attention to the global difference that there exist between  $U(n)$  and  $SU(n) \times U(1)$  and focus on the algebra - one can easily show that  $U(n) = SU(n) \times U(1) / \mathbb{Z}_n$ )

(57)

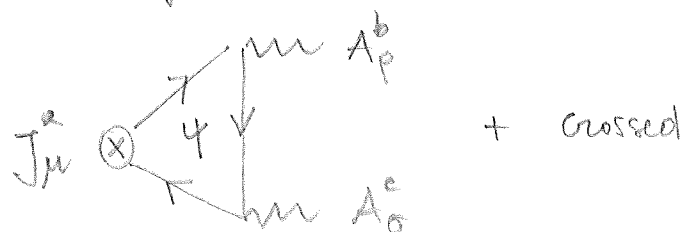
At this point we need to pay attention to the fact that not all symmetries of the classical Lagrangian survive at the quantum level: some of them are anomalous, i.e. are simply not symmetries of the quantum field theory. Anomalies are signalled by the non-conservation of a Noether current in the background (i.e. in presence) of gauge fields. Let  $J_\mu^a$  be a Noether current associated with a generator  $T^a$ ; then, in the background of gauge fields  $A_\mu$

$$\langle \partial_\mu J^{\mu a}(x) \rangle_{A_\mu} = - \frac{g^2}{16\pi^2} D^{abc} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^b F_{\rho\sigma}^c$$

$$D^{abc} \equiv \frac{1}{2} \text{Tr} [T^a \{T^b, T^c\}]$$

Here  $T^a, T^b, T^c$  are generators in the fermion representations and we are considering a collection of left-handed Weyl fermions  $\psi$ . The contribution of right-handed fermions is just equal and opposite.

The anomaly can be derived from a perturbative calculation, in which case the (only) relevant Feynman diagrams are





The anomaly arises as the impossibility to introduce a regulator while preserving both the global and the gauge symmetry. An alternative approach due to Fujikawa makes use of the functional integral. In that case the anomaly arises from the non-invariance of the fermion measure under a global transformation

$$d\psi d\bar{\psi} \rightarrow d\psi d\bar{\psi} e^{i\alpha^a \int d^4x A^a(x)}$$

$$A^a(x) = -\frac{1}{16\pi^2} D^{abc} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^b F_{\rho\sigma}^c$$

If we apply these formulas to QCD, we see that the  $U(1)_A$  subgroup of  $SU(3)_L \times SU(3)_R \times U(1)_L \times U(1)_R$  is anomalous in the gluon background. Here  $U(1)_L \times U(1)_R = U(1)_V \times U(1)_A$  where

$$U(1)_V: q_L^i \rightarrow e^{i\alpha} q_L^i, \quad q_R^i \rightarrow e^{i\alpha} q_R^i$$

$$U(1)_A: q_L^i \rightarrow e^{-i\alpha} q_L^i, \quad q_R^i \rightarrow e^{i\alpha} q_R^i$$

In this case the generators  $T^b, T^c$  are just Gell-Mann generators of  $SU(3)_c$  ( $T^b = \lambda^b/2$ ), and  $T^a$  is just the identity. By summing over both left and right-handed quarks, using the fact that they have opposite axial charges, we find

$$\langle \partial_\mu J_A^\mu \rangle = \frac{g_3^2}{16\pi^2} m_f^2 \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}^a G_{\rho\sigma}^a$$

where  $n_f = 3$  is the number of flavors.

(59)

We conclude that, in absence of quark masses, QCD is invariant under the global symmetry

$$G = SU(3)_L \times SU(3)_R \times U(1)_B$$

where we have normalized the generator of the  $U(1)$  factor such that each quark has  $U(1)_B$  charge  $1/3$ :  $U(1)_B$  is called baryon number.

In addition to the continuous (approximate) symmetry group  $G$ , the QCD Lagrangian is (approximately) invariant under two discrete symmetries: parity ( $P$ ) and charge conjugation ( $C$ ). While  $C$  is exact, parity is explicitly broken by the  $\theta$ -term. The bound on the electric dipole moment of the neutron sets a quite stringent limit  $\theta_3 \lesssim 10^{-10}$ , which means that the violation of parity (hence of  $CP$ ) is very tiny in the QCD sector, if not absent at all. Therefore,  $P$  and  $C$  can also be used, together with the Lorentz group, to characterize the physical states in the spectrum.

The symmetry group  $G$ , on the other hand, can be only partly used to characterize the spectrum. Indeed, the QCD vacuum is invariant only under the vectorial subgroup

$$G_V = SU(3)_V \times U(1)_B$$

where left- and right-quarks transform in the same way under  $SU(3)_V$ .

(60)

The 8 generators of  $G$  that are not contained in  $G_v$  are spontaneously broken. By the Goldstone theorem, for each broken generator there exists a Nambu-Goldstone boson (NGB). These would be exactly massless spin-0 particles if the symmetry  $G$  were exact. Quark masses break explicitly  $G$  and lift the NGB masses to non-zero values. Since  $m_q \ll \Lambda_{\text{QCD}}$ , the NGB are expected to be much lighter than the other QCD states. The first to realize this (even before the formulation of the Goldstone theorem) was Yoichiro Nambu, who identified pions as particles that would be massless in the limit of unbroken  $G$  (Y. Nambu PRL 4 (1960) 380).

The local order parameter associated with the spontaneous breaking  $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$  is the quark condensate  $\langle \bar{q}_L q_R \rangle$ .

Interestingly, Vafa and Witten proved that, in a vectorlike gauge theory like QCD, the vectorial subgroup  $SU(3)_V \times U(1)_B$  cannot be spontaneously broken (Vafa, Witten NPB 234 (1984) 173).

A proof of the fact that  $SU(3)_L \times SU(3)_R$  must be spontaneously broken to  $SU(3)_V$ , instead, can be formulated by making use of 't Hooft anomaly matching (see S. Weinberg, "The Quantum Theory of Fields" Vol 2, section 22.5; 't Hooft, NATO Sci. Ser. B59 (1980) 135; L. Ciambelli et al. PRD 110 (2024) 114035).

As a consequence of spontaneous symmetry breaking, only the linearly-realized subgroup  $SU(3)_V \times U(1)_B$  can be used to characterize physical states.

(61)

After discussing the (approximate) symmetries of QCD, we are now ready to characterize the spectrum of physical states by analyzing gauge-invariant local operators. All these operators will be  $SU(3)_c$  singlets.

A first question we may ask is the following: are there massless states? In other words: is the spectrum of the theory gapped? This apparently innocent question poses a formidable challenge: proving the existence of a mass gap in Yang-Mills gauge theory is one of the Millennium Prize Problems of the Clay Mathematics Institute (Jaffe and Witten, "Quantum Yang Mills Theory")! The existence of a mass gap is in fact required to explain why the nuclear force is short-ranged (massless particles would mediate long-range interactions). We will thus assume the existence of a mass gap in the following.

Notice that 't Hooft anomaly matching can be used to rule the absence of a mass gap in QCD with massless quarks. In that case one can deduce the existence of some massless state in the spectrum. Under the assumption of confinement, such massless states turn out to be the NGBs of chiral symmetry breaking. We have already used this fact when anticipating the presence of light pseudo-NGB in the spectrum due to the spontaneous breaking  $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$ .

Implicit in the discussion we had so far is that the theory leads to a spectrum of particles. Notice that this is not necessarily the case for a QFT: conformal field theories are an obvious counterexample where physical states are not particle-like. Without entering into the details of the possible phases of a theory like QCD, we will assume that at low energy it will develop color confinement. This means that the spectrum consists of particles (hadrons) interpolated from the vacuum by color-singlet local operators. This assumption seems compatible, as far as we know, with the number of colors and quark flavors in the theory. One should on the other hand keep in mind that a proof of confinement is still missing.

After clarifying the assumptions behind our analysis, we start considering the available interpolating operators starting from the lowest-dimensional ones (these are expected to excite the lightest particles in the spectrum).

The first obvious possibility is MESON operators made of one quark and one anti-quark field:

$$\bar{q}(x) \Gamma q(x)$$

Here  $\Gamma$  denotes a matrix in Lorentz and  $SU(3)$  flavor space. Operators of this kind have zero baryon number and transform under  $SU(3)_V$  like singlets or adjoint representations ( $3 \times \bar{3} = 1 + 8$ ).

Since we are anticipating massive particles, we can use the  $SO(3)$  rotational subgroup of the Lorentz group to characterize their spin. Each quark field transforms as a spin  $-1/2$  spinor under  $SO(3)$ , hence mesons can have spin 0 or 1 ( $1/2 \times 1/2 = 0 + 1$ ). For example,  $(\bar{q} \gamma^5 q)$  can excite a  $J^P = 0^-$  singlet of  $SU(3)_v$ , while  $(\bar{q} \gamma^\mu \lambda^a q)$  can excite a  $J^P = 1^+$  adjoint of  $SU(3)_v$ .

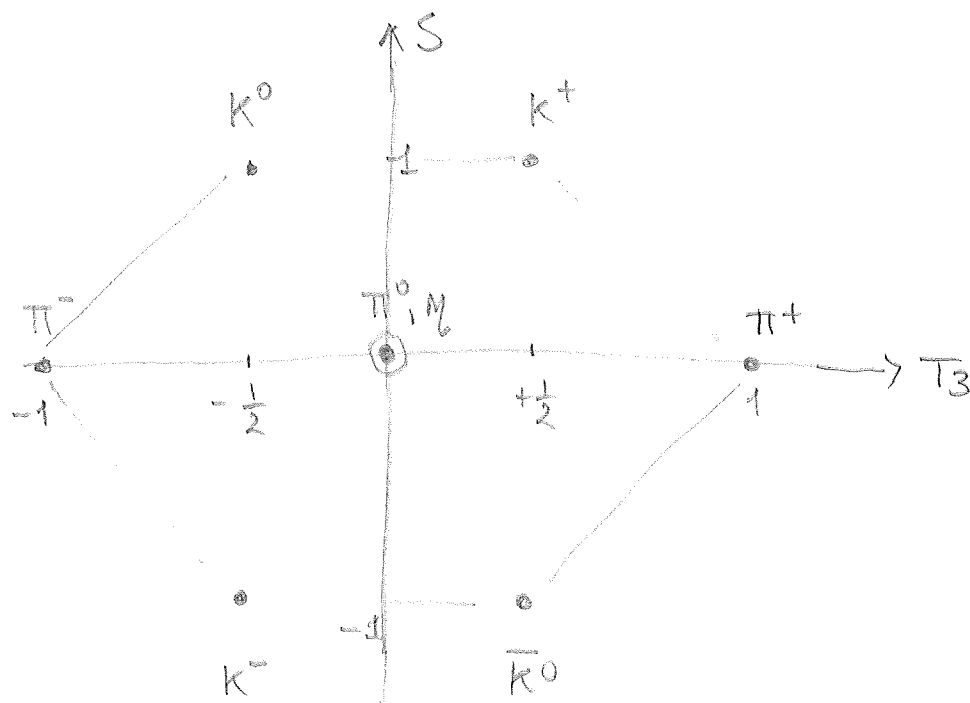
The hierarchy of masses in the spectrum cannot be in general deduced simply based on symmetry arguments, it must be extracted by solving the non-perturbative dynamics. As we already said, a notable exception is NGBs, which are expected to be the lightest states in the spectrum. By the Goldstone theorem, they are interpolated by the broken axial currents  $J_A^{\mu a} = \bar{q} \gamma^\mu \gamma^5 \lambda^a q$ , which transform as an octet of  $SU(3)_v$ . Starting with Nambu, these states have been identified with  $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$  and have mass of order

$$m_\pi \sim 135-140 \text{ MeV}$$

$$m_K \sim 500 \text{ MeV}$$

$$m_\eta \sim 500 \text{ MeV}$$

They can be characterized by their isospin ( $T_3$ ) and strangeness ( $S$ ) quantum numbers - these are the two Cartan generators of the  $SU(3)_v$  algebra. The octet is thus represented as follows in the plane  $(T_3, S)$ :

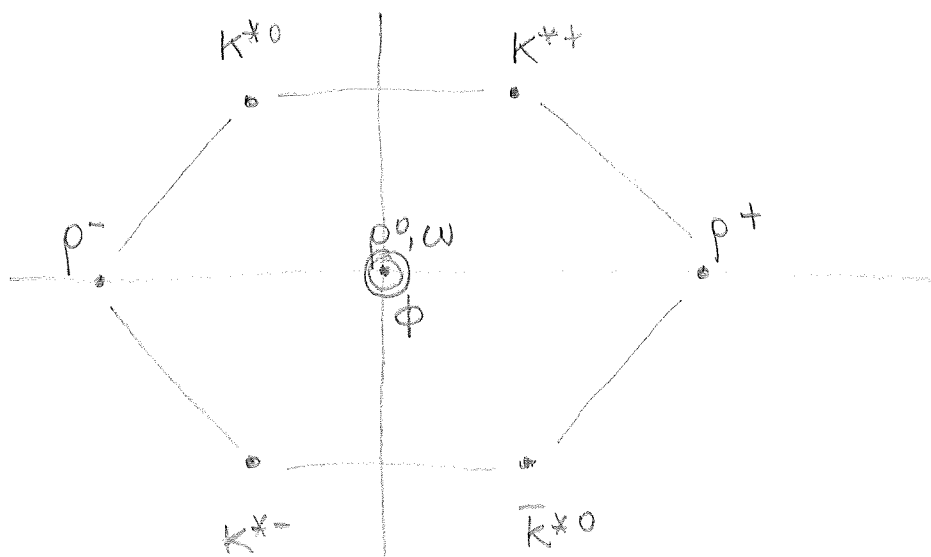


We said that the NGBs are interpolated from the vacuum by the broken axial currents, that is the matrix element

$$\langle 0 | J_A^{\mu a}(x) | \chi^b(p) \rangle = i f \delta_{ab} p^\mu e^{i p \cdot x}$$

is non-vanishing (it can be used as a definition of the decay constant  $f$  - here  $\chi^b$  denotes a generic NGB). Notice however that another possible interpolating operator is the pseudo-scalar density  $(\bar{q} \gamma^5 \lambda^a q)$ , since it has the same quantum numbers under  $SU(3)_V$  and  $P$  (although  $J_A^{\mu a}$  and  $(\bar{q} \gamma^5 \lambda^a q)$  do have different transformation rules under  $SU(3)_L \times SU(3)_R$ !). This example clearly illustrates that a given physical state can (and in fact will) have more than one interpolating operator.

The next lightest mesons that can be identified in the observed spectrum are vector mesons interpolated by vector currents  $J_V^{\mu a} = \bar{q} \gamma^\mu \lambda^a q$  and  $J_V^\mu = \bar{q} \gamma^\mu q$ , respectively transforming as an octet and a singlet of  $SU(3)_V$ . They are represented as follows in the plane  $(T_3, S)$ :



$SU(3)$ -breaking effects lead to a mixing between the  $T=0$  isosinglet component of the  $SU(3)$  octet and the  $SU(3)$  singlet. The physical states  $\omega, \phi$  are thus superpositions of singlet and octet.



Another set of states is made of those excited by BARYON operators of the form

(66)

$$q^i(x) q^j(x) q^k(x) \epsilon^{ijk} \leftarrow \text{color indices}$$

They have baryon number 1 (each quark has baryon number  $+1/3$ ). Their transformation rules under  $SU(3)_c$  and the rotation group  $SO(3)$  are restricted by Fermi statistics. The baryonic operator must indeed be invariant under the exchange of any quark pair. Under such exchange one gets a  $(-1)$  factor due to Fermionic nature of the quark field plus another  $(-1)$  factor from the exchange of color indices. Therefore, the operator must be overall symmetric under the combined exchange of flavor and spin indices. In other words, baryons must have identical Young tableaux under  $SU(2)$  spin and  $SU(3)$  flavor. Possible  $SU(2)$  spin tableaux are

$$\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array}$$

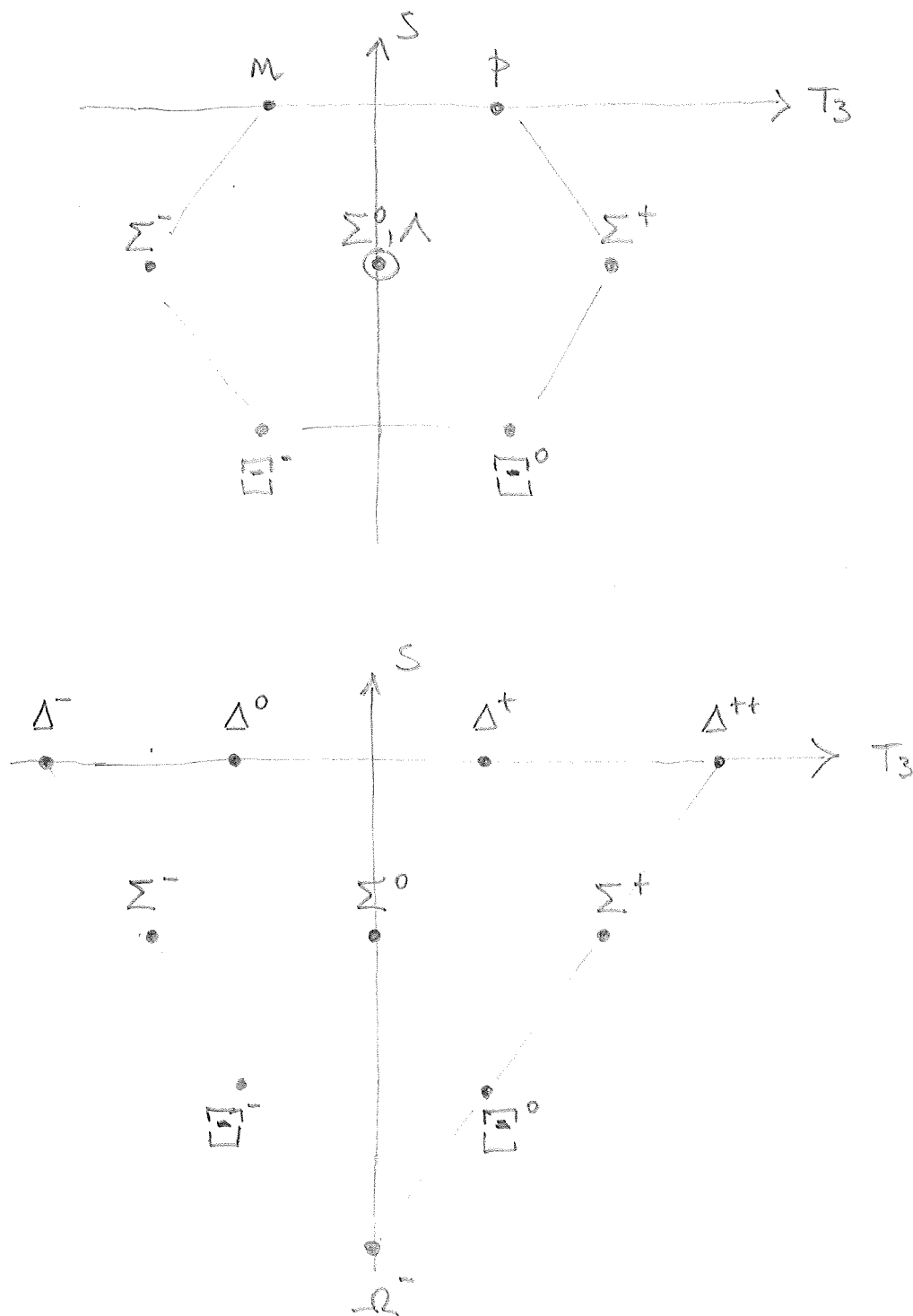
$$S = 3/2 \quad S = 1/2$$

which, when interpreted as  $SU(3)$  tableaux, correspond respectively to a 10 and an 8 of  $SU(3)$ . We thus expect

$$\begin{array}{ll} \text{spin} = 1/2 & \text{octets of } SU(3) \\ \text{spin} = 3/2 & \text{decuplets of } SU(3) \end{array}$$

The experimentally observed baryons do have these quantum numbers. The lightest octet and decuplet are represented as follows in the  $(T_3, S)$  plane:

(67)



The  $\Delta$ , in particular, was the first resonance to be discovered: it was found by Fermi and collaborators in their study of nucleon-pion scattering in 1952.

While  $\bar{q}q$  and  $qqq$  operators are exciting the lowest-lying states in the spectrum, other more exotic states are also predicted based on other interpolating operators. These include operators of the kind

$q\bar{q}q\bar{q}$  (tetraquarks)

$qqqq\bar{q}$  (pentaquarks)

$gg, ggg, \dots$  (glueballs)

While there seems to be robust evidence for the existence of tetraquark and pentaquark states, the experimental support for glueballs is much weaker. For more details see the following 2024 PDG reviews :

- Amaler, Gude, DeGand "Quark Model"
- Karlmer, Skwarwicki "Pentaquarks"
- Gutsche, Hanhart, Mitchell "Heavy non- $q\bar{q}$  Mesons"

Our characterization of the hadronic spectrum has been necessarily qualitative and based only on the quantum numbers of the states. We already mentioned that lattice gauge theories can give a more quantitative information on the QCD spectrum by means of numerical simulations.

An introduction to the relevant literature on this subject is found on the "Quark Model" 2024 PDG review by Amsler, Goeke and DeGrand.

Here we just mention the way in which masses of hadrons are computed on the lattice: one starts from the two-point Green function of some interpolating operator  $O(\vec{x}, t)$ , averaged over position, and insert a complete set of intermediate states:

$$\begin{aligned} \int d^3x \langle 0 | O(\vec{x}, t) O(\vec{0}, 0) | 0 \rangle &= \sum_m \int d^3x \langle 0 | e^{-i\vec{P} \cdot \vec{x}} O(\vec{0}, 0) e^{i\vec{P} \cdot \vec{x}} | m \rangle \\ &\quad \cdot \langle m | O(\vec{0}, 0) | 0 \rangle \\ &= \sum_m e^{iM_m t} |\langle 0 | O(0) | m \rangle|^2 \end{aligned}$$

By performing a Wick rotation to Euclidean space ( $t = i\tau$ ):

$$\int d^3x \langle 0 | O(x) O(0) | 0 \rangle = \sum_m e^{-M_m \tau} |\langle 0 | O(0) | m \rangle|^2$$

For  $\tau \rightarrow \infty$ , the correlator is dominated by the lowest-lying states, and its computation on the lattice allows one to extract the values of the masses  $M_m$ .

Hadrons can be thought as relativistic, non-perturbative bound states of quarks and gluons. Despite the strong coupling, one can try to model the bound state by means of an effective potential among quark constituents. This strategy leads to quark models for mesons and baryons. Such models do not have a rigorous justification from QCD, and in this sense are just phenomenological approaches, but they are rather successful to characterize the spectrum of at least the lowest-lying modes. They should thus be regarded as a useful tool at our disposal.

For example,  $q\bar{q}$  mesons are described as bound states of one constituent quark and one constituent antiquark. Their parity is predicted to be  $(-1)^{l+1}$ , where  $l$  is the orbital angular momentum. Since the total spin is  $S=0,1$ , one predicts  $J^P = 0^-, 1^-$  states with  $l=0$ , and  $J^P = 0^+, 1^+, 2^+$  states with  $l=1$ .

Notice that both  $0^-, 1^-$  and  $0^+, 1^+$  are interpolated by  $(\bar{q}\Gamma q)$  operators, while  $2^+$  states require interpolating operators with a covariant derivative, of the form  $(\bar{q}\Gamma D q)$ .

For a discussion about quark models see Donoghue, Golowich and Holstein, and the "Quark Model" 2024 PDG review.

So far we have focused on bound states made of light quarks. Heavy charm and bottom quarks can be straightforwardly included in the discussion and their presence leads to a spectrum of heavy hadron. Quarkonium made of a  $Q\bar{Q}$  heavy pair, for example, entail a rich phenomenology that can be studied rather successfully by means of quark models.

The case of the top quark, on the other hand, is rather special since it decays before having the time to hadronize (i.e. to form relativistic bound states).

The time of hadronization can be naively estimated to be

$$\tau_{\text{had}} \approx \frac{1}{\Lambda_{\text{QCD}}} \sim 3-5 \text{ GeV}^{-1}$$

The decay width of the top quark into bottom plus a W boson is estimated to be



$$\Gamma_t \sim \frac{1}{8\pi} \left( g_2 \frac{m_t}{m_W} \right)^2 m_t \sim \frac{G_F m_t^3}{8\pi} \sim 2 \text{ GeV}$$

Notice that the factor  $(m_t/m_W)$  in the decay amplitude comes from the longitudinal polarization of the W. The experimentally measured value is

$$\Gamma_t = 1.42^{+0.19}_{-0.15} \text{ GeV}, \text{ in excellent agreement with our naive estimate.}$$

It follows a top lifetime

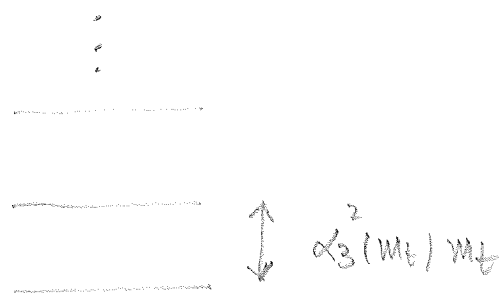
$$\tau_{\text{top}} \sim 1 \text{ GeV}^{-1}$$

Since  $\Sigma_{\text{top}} < \Sigma_{\text{had}}$ , we conclude that the top quark decays before hadronizing.

(72)

One might ask if there exist perturbative, non-relativistic bound states made of top quarks. Toponium, made of  $t\bar{t}$ , would be one of such states. If the top quark were stable, toponium would have a spectrum of energy levels very much similar to the hydrogen atom or, rather, to positronium. Toponium is a perturbative bound state since the strength of the color interaction at distances of order  $1/m_t$  is  $\alpha_3(m_t) \sim 0.1 \ll 1$ . The binding energy of toponium, hence the splitting between two consecutive levels, is of order

$$E_B \sim \alpha_3^2(m_t) m_t \sim 2 \text{ GeV}$$



Now, since the top decay width  $\Gamma_{\text{top}}$  is of order  $E_B$  (this is a coincidence of otherwise unrelated scales!), this means that, due to the decay of the top quark, the energy levels of toponium get smeared by an  $O(1)$  amount. In other words, the decay of the top quark turns the energy spectrum of toponium into a continuum hardly distinguishable in  $t\bar{t}$  scattering.

Another way to reach the same conclusion is by comparing the typical size for a constituent top to circle one orbit

$$\tau_{\text{orbit}} = \frac{r_B}{v} \sim \frac{1}{\alpha_3^2(m_t) m_t} \sim 0.5 \text{ fm}^{-1}$$

$$v \sim \alpha_3(m_t) \quad \text{orbit velocity}$$

$$r_B \sim \frac{1}{\alpha_3(m_t) m_t} \quad \text{Bohr radius}$$

with the top lifetime. Since  $\tau_{\text{orbit}} \sim \tau_{\text{top}}$ , toponium has barely the time to form before it decays.





$$O_i^{\text{exp}} = O_i'(\{O_k^{\text{input}}\})$$

(2)

should be satisfied without strong cancellations among different terms. One way to check for the existence of strong cancellations (fine-tuning) is to vary the value of the input observables (around their experimental value) and see how the observable  $O_i$  varies. In practice, we say that the experimental value of  $O_i$  is reproduced naturally in our theory if

$$\left| \frac{O_k^{\text{input}}}{O_i} \frac{\partial O_i}{\partial O_k^{\text{input}}} \right|_{O_i = O_i^{\text{exp}}} < \Delta \quad \forall k$$

where  $\Delta$  is some chosen amount.

Notice that the prediction  $O_i = O_i'(\{O_k^{\text{input}}\})$  is a relation among observables and as such does not depend on the way in which we perform the calculation, i.e. on the renormalization scale or scheme. In practice, if we perform our calculation in perturbation theory by truncating the perturbative series to a finite order, this introduces some spurious dependence on the RG scale and scheme.

Depending on the theory under study and on the set of chosen observables, deriving the relations between input and output observables might be cumbersome.

A possible, equivalent strategy to test naturalness is by monitoring the sensitivity of the predicted observable on the Lagrangian parameters (Guidice, Barbieri NPB 306 (1988) 63). Specifically, one varies the Lagrangian parameters  $\{a_i\}$  and tests how large is the variation of the predicted observable:

$$\left| \frac{\partial a_k}{\partial O_i} \frac{\partial O_i}{\partial a_k} \right|_{O_i = O_i^{\text{exp}}} < \Delta \quad \forall k$$

In this way, each Lagrangian parameter  $a_k$  is a proxy for a corresponding input observable  $O_k^{\text{input}}$ .

This is the so-called Barbieri-Guidice criterion, originally formulated to test naturalness of supersymmetric theories.

Here we are mostly interested in the naturalness of the mass spectrum of a given field theory, and in particular of the SM mass spectrum.

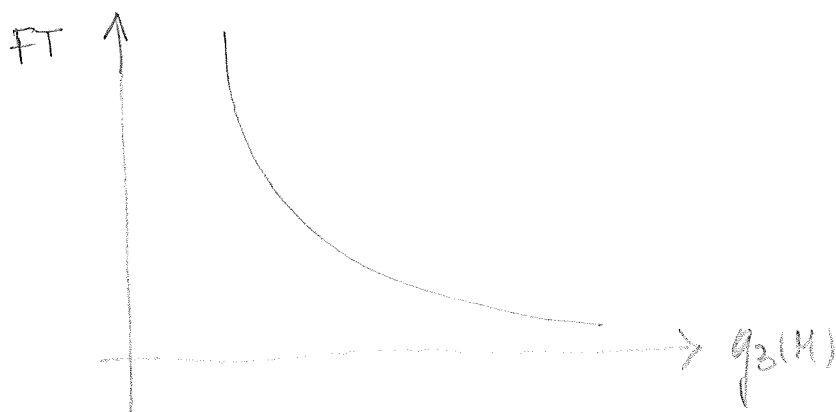
It is instructive to see that the value of the QCD scale, in terms of which all hadronic masses are fixed, is natural compared to a much higher value of the SM cutoff scale, like for example the

Planck scale. To see this, let us apply the Barbieri-Giudice (4) criterion. We know that

$$\Lambda_{\text{QCD}} = M \exp \left( - \frac{8\pi^2}{|\beta_0^{(3)}|} \frac{1}{g_3^2(H)} \right)$$

where  $M$  is a very high-energy scale (possibly the Planck scale) at which  $g_3(H)$  is perturbative. Hence, at sufficiently high energies,  $\Lambda_{\text{QCD}}/M$  depends on just one Lagrangian parameter:  $g_3$ . By varying with respect to  $g_3(H)$  we find

$$FT \equiv \left| \frac{g_3(H)}{\Lambda_{\text{QCD}}} \frac{d\Lambda_{\text{QCD}}}{dg_3(H)} \right| = \frac{16\pi^2}{|\beta_0^{(3)}|} \frac{1}{g_3^2(H)}$$



Since  $\Lambda_{\text{QCD}} = M \exp(-FT/2)$ , we can obtain  $\Lambda_{\text{QCD}} \ll M$  by choosing  $FT \approx \frac{16\pi^2}{|\beta_0^{(3)}|} \frac{1}{g_3^2(H)}$  somewhat larger than 1 thanks to the exponential suppression. For example, the 19 orders of magnitude between

$M_{Pl} \approx 10^{18} \text{ GeV}$  and  $\Lambda_{QCD} \approx 200 \text{ MeV}$  are reproduced for (5)

$$FT = 2 \log \frac{M_{Pl}}{\Lambda_{QCD}} \simeq 86, \text{ which corresponds to fixing the}$$

value of  $g_3(M_{Pl})$  with a precision of order 1%.

While this might suggest the need for fine-tuning, we will see that reproducing our EW scale of order  $100 \text{ GeV}$  implies a much larger tuning. In general, dimensional transmutation is a way to generate scales that are exponentially suppressed compared to a reference scale.

We chose to estimate the fine tuning  $FT$  by varying  $\Lambda_{QCD}$  with respect to  $g_3(M)$ , a proxy for some input observable. It is possible and easy to repeat our calculation by expressing  $\Lambda_{QCD}$  in terms of such observable. For example, we can choose as input a scattering cross section at the scale  $M$  (since QCD is perturbative at scales  $M \gg \Lambda_{QCD}$ , it is not important which scattering process we consider, for example we can consider  $q\bar{q} \rightarrow q\bar{q}$ ). We expect

$$\sigma(\sqrt{s}=M) \simeq A \cdot \frac{g_3^4(M)}{8\pi} \frac{1}{M^2}$$

where  $A$  is some constant. Therefore

$$\Lambda_{QCD} = M \cdot \exp\left(-\frac{1}{16\pi^2} \sqrt{\frac{8A}{\sigma \cdot M^2}}\right)$$

so that

(6)

$$FT = \left| \frac{\sigma}{\Lambda_{\text{QCD}}} \frac{d\Lambda_{\text{QCD}}}{d\sigma} \right| = \frac{1}{2} \left( \frac{1}{|\beta_0|} \sqrt{\frac{8A}{5 \cdot M^2}} \right)$$

In this way we find that  $\Lambda_{\text{QCD}} = M \exp(-2FT)$ . As before,  $\Lambda_{\text{QCD}}/M$  depends exponentially on  $FT$ , though now the coefficient in the exponent is different. Our new estimate suggests that  $\Lambda_{\text{QCD}} = 200 \text{ MeV}$  implies a fine tuning of order  $FT \simeq 22\%$  if  $M = M_{\text{Pl}}$ .

Let us now consider QED and ask about the naturalness of the electron mass. We treat QED as an EFT since the electromagnetic coupling has a Landau pole at high energies and we know that the UV completion is the SM.

Formulated in this way, the theory has (at least) three input parameters:  $e$ ,  $m$  (the electron mass term) and  $\Lambda$  (the cutoff scale).

We can fix  $e$  and  $m$  by measuring the properties of positronium. The difference between the first and second energy levels of positronium,  $\Delta E(ee)$  and the para-positronium decay width to two photons,  $\Gamma(ee)$ , scale as

$$\Delta E(ee) \sim \alpha^2 m$$

$$\Gamma(ee) \sim \alpha^5 m$$

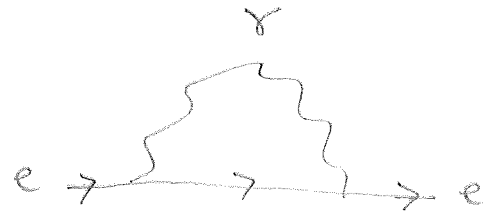
Hence we can extract  $\alpha$  and  $m$  from  $\Gamma(ee)$  and  $\Delta E(ee)$ :

$$\alpha \sim \left( \frac{\Gamma(ee)}{\Delta E(ee)} \right)^{1/3} \quad m \sim \Delta E(ee) \left( \frac{\Delta E(ee)}{\Gamma(ee)} \right)^{2/3}$$

(7)

The physical electron mass, at 1-loop, has an expression in terms of the input parameters of the form

$$m_e \sim m \left( 1 + \frac{\alpha}{\pi} \log \Lambda/m \right)$$



where we have omitted for simplicity  $O(1)$  coefficients. Noticed that we omitted, for simplicity, loop corrections to  $\Delta E(ee)$  and  $\Gamma(ee)$ , it is sufficient for our purposes to retain them only in the expression of  $m_{phys}$ .

As long as  $\alpha$  is small ( $\alpha \sim 1/137$ ), the theory explains in a simple way the hierarchy of scales

$$m_e \gg \Delta E(ee) \gg \Gamma(ee)$$

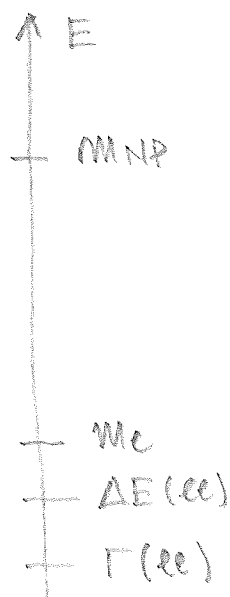
This is in fact a small hierarchy that follows from the smallness of the electromagnetic coupling. Now the question is whether a cutoff scale much larger than  $m_{phys}$  is also natural in our theory.

In the spirit of EFTs, we can interpret  $\Lambda$  as a proxy for the mass scale of the new particles appearing in the UV completed theory:

$$\Lambda \sim m_{NP}.$$

We thus ask if a hierarchy  $M_{NP} \gg m_e$  is natural:

(8)



To answer this question, let us write  $m_e$  in terms of the three input observables  $\Delta E(ee)$ ,  $\Gamma(ee)$ ,  $M_{NP}$  and vary with respect to  $M_{NP}$ :

$$m_e \sim \frac{\Delta E(ee)}{\left(\frac{\Gamma(ee)}{\Delta E(ee)}\right)^{2/3}} \left[ 1 + \left(\frac{\Gamma(ee)}{\Delta E(ee)}\right)^{1/3} \log \left( \frac{M_{NP}}{\Delta E(ee)} \left(\frac{\Gamma(ee)}{\Delta E(ee)}\right)^{2/3} \right) \right]$$

$$FT = \left| \frac{M_{NP}}{m_e} \frac{\partial m_e}{\partial M_{NP}} \right| \sim \left(\frac{\Gamma(ee)}{\Delta E(ee)}\right)^{1/3} \frac{1}{m_e} \frac{\Delta E(ee)}{\left(\frac{\Gamma(ee)}{\Delta E(ee)}\right)^{2/3}}$$

$$\sim \frac{\alpha}{4\pi} \frac{1}{1 + \alpha \log 1/m}$$

We see that the tuning is always small,  $FT \approx \alpha/4\pi \ll 1$ . This means that a hierarchical spectrum with  $M_{NP} \gg m_e$  is natural, i.e. it implies no strong cancellations.



(9)

The main reason why the fine-tuning turned out to be small, is that the 1-loop correction to the electron mass term was only logarithmically, and not linearly, divergent:  $\delta m \propto m$ . Indeed, this implies that if  $m$  is small (compared to 1) then also  $\delta m$  will be small. In other words, the loop corrections do not destabilize the hierarchy.

But why is the 1-loop correction  $\delta m$  not linearly divergent? The answer to this question involves a symmetry argument.

As first pointed out by 't Hooft in his 1979 Cargèse lectures ("Naturalness, chiral symmetry and spontaneous symmetry breaking"), in the limit  $m \rightarrow 0$  QED acquires a larger global symmetry. Indeed, for  $m=0$  the QED Lagrangian is invariant under an axial phase transformation

$$U(1)_A: \quad \psi(x) \rightarrow e^{i\alpha \gamma^5} \psi(x)$$

The mass term breaks  $U(1)_A$  since

$$m \bar{\psi}_L \psi_R + \text{h.c.} \rightarrow m e^{2i\alpha} \bar{\psi}_L \psi_R + \text{h.c.}$$

This suggests, however, that we can assign a (fictitious) transformation rule to  $m$  as if it was a field rather than a parameter:

$$m \rightarrow e^{-2i\alpha} m$$

In this way, the mass term becomes formally  $U(1)_A$  invariant. (10)  
A parameter which is assigned a transformation rule under some symmetry is called a SPURION of that symmetry. The spurion is just a trick to encode the way in which the symmetry gets broken.

Since now  $U(1)_A$  has been formally restored, any equation must be  $U(1)_A$  covariant. In particular, radiative corrections of the form  $\delta m \propto \Lambda$  are not allowed since this relation is not covariant under  $U(1)_A$ :  $\Lambda$  is just a constant, while  $\delta m$  must transform like  $m$ . On the other hand,  $\delta m \propto m$  is allowed because  $U(1)_A$  covariant. We thus conclude that  $m$  is multiplicatively renormalized because it is the only spurion of  $U(1)_A$ .

Following 't Hooft, we can generalize what we have learned from the QED example by saying that: a small value of some predicted observable  $O_i$  is natural if the theory acquires a larger global symmetry in the limit  $O_i \rightarrow 0$ . The electron mass is just an example of this kind of situation.

It is instructive to reanalyze the naturalness of the electron mass by choosing input observables at some high energy  $M$  (at which the theory is still perturbative). A possible choice is of course  $M = M_{NP}$ . For this discussion will be sufficient to consider tree-level expressions of the input observables. For the better, we choose two differential (unpolarized)

cross sections for  $e^+e^- \rightarrow \gamma\gamma$  at energies  $\sqrt{s} = M_{NP}$  and  $\sqrt{s} = M_{NP}/2$  respectively. We will fix the scattering angle to  $\theta = \bar{\theta}$ . At leading order (tree level) in perturbation theory

(11)

$$\frac{d\sigma}{d\cos\theta}(e^+e^- \rightarrow \gamma\gamma) = \frac{2\pi\alpha^2}{s} f(\theta, m^2/s)$$

where  $f(\theta, m^2/s)$  is some adimensional function of  $\theta$  and of the ratio  $m^2/s$ . For  $m^2 \ll s$ , in particular, one finds

$$f(\theta, \frac{m^2}{s}) \simeq \frac{1+\cos^2\theta}{\sin^2\theta} + \frac{4m^2}{s} \left[ \frac{3}{\sin^2\theta} - \frac{1}{2} \frac{(1+\cos^2\theta)^2}{\sin^4\theta} \right] + O(m^4)$$

Our inputs are thus

$$\sigma_1 \equiv \frac{d\sigma}{d\cos\theta}(e^+e^- \rightarrow \gamma\gamma) \Big|_{\substack{\theta = \bar{\theta} \\ \sqrt{s} = M_{NP}}} = \frac{2\pi\alpha^2}{M_{NP}^2} f(\bar{\theta}, \frac{m^2}{M_{NP}^2})$$

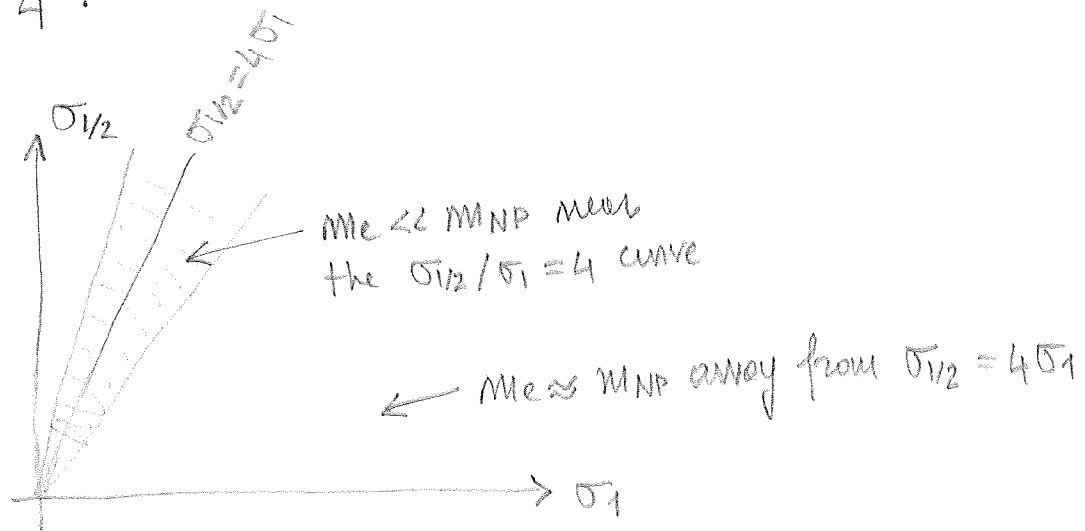
$$\sigma_{1/2} \equiv \frac{d\sigma}{d\cos\theta}(e^+e^- \rightarrow \gamma\gamma) \Big|_{\substack{\theta = \bar{\theta} \\ \sqrt{s} = \frac{M_{NP}}{2}}} = \frac{8\pi\alpha^2}{M_{NP}^2} f(\bar{\theta}, \frac{4m^2}{M_{NP}^2})$$

From these we extract  $\alpha$  and  $m$  and predict the electron mass using the tree-level relation  $m_e = m$ . In this way,  $m_e$  is obtained by solving the equation

$$\frac{\sigma_{1/2}}{\sigma_1} = 4 \frac{f(\bar{\theta}, \frac{4m_e^2}{M_{NP}^2})}{f(\bar{\theta}, \frac{m_e^2}{M_{NP}^2})}$$

For generic values of  $\sigma_1$  and  $\sigma_{1/2}$  of order  $1/M_{NP}^2$ , the electron mass predicted by this formula is of order  $M_{NP}$ . To obtain  $m_e \ll M_{NP}$  one must tune the values of  $\sigma_{1/2}$  and  $\sigma_1$  near the special ratio

$$\sigma_{1/2} / \sigma_1 = 4 :$$



One is thus led to ask : is the choice of input observables  $\sigma_{1/2} \simeq 4\sigma_1$  a tuning ? The answer is that it is not a tuning since for  $\sigma_{1/2} \rightarrow 4\sigma_1$  the theory recovers a global  $U(1)_A$  symmetry.

For  $m=0$  the electron becomes massless and chirality corresponds to helicity. We can thus use a spurious argument to characterize the transformation rule of the scattering amplitude under  $U(1)_A$  :

$$|e_L^\pm\rangle \rightarrow e^{-i\alpha} |e_L^\pm\rangle$$

$$|e_R^\pm\rangle \rightarrow e^{+i\alpha} |e_R^\pm\rangle$$

$$|\bar{e}_L e_R^+\rangle \rightarrow |\bar{e}_L e_R^+\rangle$$

$$|\bar{e}_R e_L^+\rangle \rightarrow |\bar{e}_R e_L^+\rangle$$

$$|\bar{e}_L e_L^+\rangle \rightarrow e^{-2i\alpha} |\bar{e}_L e_L^+\rangle$$

$$|\bar{e}_R e_R^+\rangle \rightarrow e^{+2i\alpha} |\bar{e}_R e_R^+\rangle$$

$$A(\bar{e}_L e_R^+ / \bar{e}_R e_L^+ \rightarrow \gamma\gamma) \rightarrow m^0$$

$$A(\bar{e}_L e_L^+ \rightarrow \gamma\gamma) \sim m$$

$$A(\bar{e}_R e_R^+ \rightarrow \gamma\gamma) \sim m^*$$

Here we have used the fact that the S-matrix operator is U(1)<sub>A</sub> invariant and used the spinor transformation rules of  $m$ .

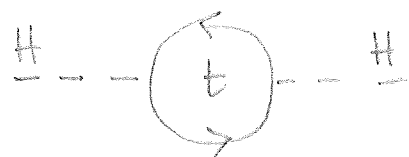
We thus conclude that (for  $m$  small) the  $O(m^0)$  contribution to  $f(\theta, m^2/s)$  comes from the  $\bar{e}_L e_R^+$ ,  $\bar{e}_R e_L^+$  channels, while the  $O(m^2)$  comes from  $\bar{e}_L e_L^+$ ,  $\bar{e}_R e_R^+$ . In particular, only

$\bar{e}_L e_R^+$ ,  $\bar{e}_R e_L^+$  contribute for  $m=0$ .

The prediction  $\sigma_{1/2} = 4\sigma_1$  can be seen as a consequence or a hallmark of U(1)<sub>A</sub> invariance.

One thus concludes, following 't Hooft, that the hierarchy  $m_e \ll m_W$  is natural in QED.

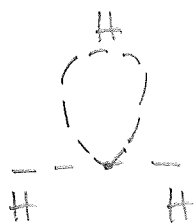
Let us now turn to consider the naturalness of the electroweak scale in the SM EFT. The electroweak spectrum of masses is fixed in terms of the Higgs mass term, which is quadratically divergent under radiative corrections: (14)



$$\delta\mu_{\text{top}}^2 \simeq -\frac{N_c}{8\pi^2} y_t^2 \Lambda^2$$



$$\delta\mu_{\text{gauge}}^2 \simeq \frac{3}{4} (3g_2^2 + g_Y^2) \frac{\Lambda^2}{16\pi^2}$$



$$\delta\mu_{\text{Higgs}}^2 \simeq \frac{3}{8\pi^2} \lambda \Lambda^2$$

Notice that, since we are focusing on the quadratically divergent contribution to  $\delta\mu^2$ , we can perform the calculation using  $SU(2)_{\text{EW}} \times U(1)_Y$  fields, i.e. ignoring EWSB. Notice also that the cutoff  $\Lambda$  can be different in different diagrams (one can for example choose a hard cutoff in the loop integrals).

The physical Higgs boson mass will thus have the following expression (up to  $O(1)$  coefficients):

$$m_H^2 \simeq -2\mu^2 + \frac{N_c}{4\pi^2} y_t^2 \Lambda^2 - \frac{3}{2} (3g_2^2 + g_Y^2) \frac{\Lambda^2}{16\pi^2} - \frac{3}{4\pi^2} \lambda \Lambda^2$$

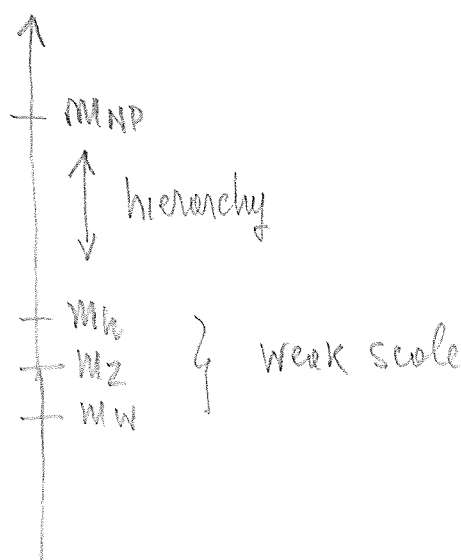
We can now

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- (i) replace  $\Lambda$  with the mass of new particles,  $M_{NP}$ , appearing in the UV complete theory
- (ii) fix  $\mu^2$ ,  $y_t$ ,  $g_2$ ,  $g_Y$ ,  $\lambda$  by means of a corresponding set of input observables

and ask about how natural is a hierarchical spectrum with

$$M_{NP} \gg M_H, M_W, M_Z :$$



We see that, differently from the QED case, now the dependence on  $M_{NP}$  is quadratic, and thus expect that  $M_{NP} \gg M_H$  will require some unnatural tuning of parameters to be realized. Let us quantify the tuning by means of the Barbieri-Giudice criterion:

$$FT = \left| \frac{\frac{M_{NP}^2}{M_H^2}}{\frac{\partial M_H^2}{\partial M_{NP}^2}} \right| \simeq \frac{N_c}{4\pi^2} y_t^2 \frac{M_{NP}^2}{M_H^2} \simeq \left( \frac{M_{NP}}{456 \text{ GeV}} \right)^2$$

Here we have retained only the 1-loop top contribution because it is the most sizable one.

(16)

This estimate shows that in a natural UV completion of the SM, new states cannot be much heavier than  $500\text{ GeV} - 1\text{ TeV}$ . These considerations motivated the construction of the LHC, whose goal was (and still is) discovering the new states.



## — Previous application of Naturalness arguments —

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It is useful and instructive to discuss two cases in which naturalness arguments as formulated above lead to a correct estimate of new particles: i) the GIM mechanism and ii) the electromagnetic correction to the pion mass.

Let us start with the GIM mechanism (for an historical review see Maiani arXiv: 1303.6154).

Prior to 1974, only three quarks were known ( $u, d, s$ ) and their weak interactions were described by Fermi's theory (an  $SU(2) \times U(1)$  gauge theory had been formulated by Glashow and then by Weinberg and Salam but was applicable only to leptons):

$$\mathcal{L}_F = \frac{G_F}{\sqrt{2}} (J_\mu^{\text{had}} + J_\mu^{\text{lep}}) (J_\mu^{\text{H}} + J_\mu^{\text{lep}})$$

$$J_\mu^{\text{lep}} = \bar{\nu}_e \gamma^\mu (1 - \gamma^5) e + \bar{\nu}_\mu \gamma^\mu (1 - \gamma^5) \mu$$

$$J_\mu^{\text{had}} = \bar{u} \gamma^\mu (1 - \gamma^5) (\cos \theta_c d + \sin \theta_c s) \quad (\theta_c = \text{Cabibbo's angle})$$

Fermi's theory with three quarks predicted that the  $\Delta S = 2$  amplitude responsible for  $K\bar{K}$  oscillations diverges quadratically:

$$\propto G_F^2 \Lambda^2 \sin^2 \theta_c \cos^2 \theta_c$$

The real part of the  $\Delta S=2$  amplitude determines the mass splitting between  $K_L^0$  and  $K_S^0$ . By using the experimental value

$$\Delta M_K \equiv M_{K_L^0} - M_{K_S^0} = 3.5 \times 10^{-12} \text{ MeV}$$

and using a naturalness argument, it was possible to estimate the value at which new physics should appear:  $\Lambda \approx 3 \text{ GeV}$ . Similar estimates could be obtained from other FCNC processes, in particular from  $\Delta S=1$  kaon decays such as  $BR(K_L^0 \rightarrow \mu^+ \mu^-) \sim 7 \times 10^{-9}$ .

It was known at the time that Fermi's 4-fermion interaction could be obtained through the virtual exchange of a massive W boson, so that

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W^2}$$

While the value of  $G_F$  was well known, the W mass was not (its prediction in the  $SU(2) \times U(1)$  model requires knowing  $\sin \theta_W$ , whose measurement was made only in 1973 by the Gargamelle experiment at CERN).

Identifying  $\Lambda$  with  $m_W$ , however, was leading to a puzzling conclusion

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$$m_W = \Lambda = 3 g_W \Rightarrow g^2 = 4\sqrt{2} m_W^2 G_F \simeq 6 \times 10^{-4}$$

This was puzzling because the  $SU(2) \times U(1)$  gauge theory predicted

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2}$$

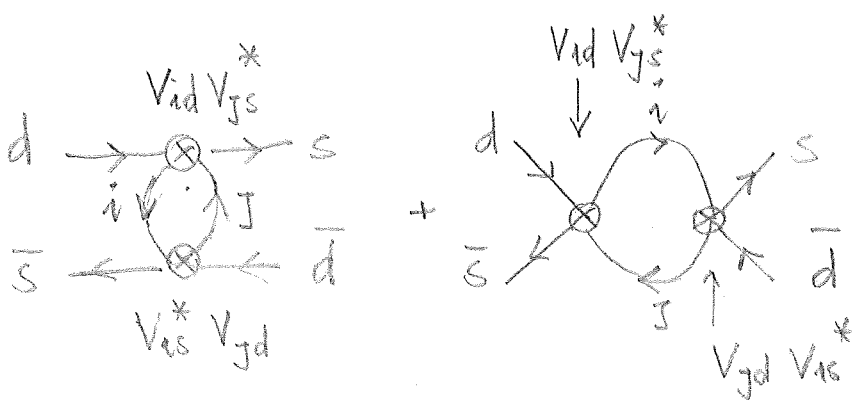
whence the expectation  $g, g' \gtrsim e \simeq 0.3$ .

Glashow, Iliopoulos and Maiani (PRD 2(1970) 1285) proposed a simple solution: the cutoff  $\Lambda$  should be identified with the mass of a fourth quark, the charm, whose weak interactions are described by Fermi's theory in terms of the hadronic current

$$J_{\text{had}}^\mu = \bar{c} \gamma^\mu (1 - \gamma^5) (-\sin\theta_c d + \cos\theta_c s)$$

$$\begin{aligned} J_{\text{had}}^\mu &= (\bar{u} \ \bar{c}) \gamma^\mu \begin{bmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{bmatrix} \begin{pmatrix} d \\ s \end{pmatrix} \\ &= (\bar{u} \ \bar{c}) \gamma^\mu V \begin{pmatrix} d \\ s \end{pmatrix} \end{aligned}$$

The matrix  $V$  is orthogonal and real, hence unitary. This implies that the quadratic divergence in the  $K\bar{K}$  amplitude is exactly canceled by the charm contribution:



$$i, j = u, c$$

$$A \propto G_F^2 \sum_{i,j=\{u,c\}} V_{id} V_{is}^* V_{jd} V_{js}^* F(m_i, m_j)$$

where the loop function  $F(m_i, m_j)$  has the form

$$F(m_i, m_j) = \Lambda^2 + m_i m_j \log \Lambda f(m_i/m_j) + \text{finite terms}$$

with  $f(x)$  some (dimensionless) function.

The unitarity of the matrix  $V_{ij}$  ensures that the quadratically divergent term, which does not depend on the quarks' masses, cancels out. This is the GIM mechanism. To see how it works, let's define

$$\lambda_u \equiv V_{ud} V_{us}^*$$

$$\lambda_c \equiv V_{cd} V_{cs}^*$$

so that  $\lambda_u + \lambda_c = 0$  (unitarity).

Then

$$A \propto G_F^2 \left[ \lambda_c^2 F(m_c, m_c) + 2\lambda_u \lambda_c F(m_c, m_u) + \lambda_u^2 F(m_u, m_u) \right]$$

The coefficient of the quadratically divergent term does not depend on the quarks' masses and is thus proportional to

$$\lambda_c^2 + 2\lambda_u \lambda_c + \lambda_u^2 = (\lambda_u + \lambda_c)^2 = 0$$

where the last equality follows from unitarity of  $V_{ij}$ .

For the same reason, in the limit of equal up and charm masses the amplitude vanishes:

$$\lim_{m_u \rightarrow m_c} A \propto G_F F(m_c, m_c) (\lambda_c^2 + 2\lambda_u \lambda_c + \lambda_u^2) = 0$$

More in general, any flavor-violating process vanishes in the limit  $m_u = m_c$  (in a theory with two generations). This is because for  $m_u = m_c$  the theory has an unbroken  $U(1)_1 \times U(1)_2$  global symmetry, where  $U(1)_i$  is the conserved number of family  $i$ . To see this, one can perform a flavor rotation

$$U_L^i \rightarrow V_{ij} U_L^j \quad U_R^i \rightarrow V_{ij} U_R^j$$

that diagonalizes the  $W$  interaction term and leaves the mass term of up-type quarks unchanged:

$$\bar{U}_L^i \gamma^\mu W_\mu \delta_{ij} d_L^j - \bar{U}_L^i \delta_{ij} m_u U_R^j - \bar{d}_L^i \delta_{ij} m_{di} d_R^j + h.c.$$

In this basis it is evident that the Lagrangian is invariant under

$$U(1)_1 : \begin{cases} U_{L,R} \rightarrow U_{L,R} e^{i\varphi_1} \\ d_{L,R} \rightarrow d_{L,R} e^{i\varphi_1} \end{cases} \quad U(1)_2 : \begin{cases} c_{L,R} \rightarrow c_{L,R} e^{i\varphi_2} \\ s_{L,R} \rightarrow s_{L,R} e^{i\varphi_2} \end{cases}$$

Thanks to the GIM mechanism, the quadratic divergence cancels out and the leading term of the amplitude is proportional to the quarks' mass squared. Working at leading order in  $m_c$  and setting  $m_u = 0$  for simplicity, one has

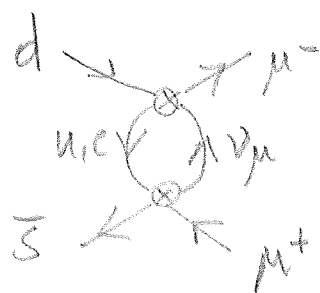
$$A \propto G_F^2 (3\lambda_c^2 + 2\lambda_u \lambda_c) m_c^2 = \sin^2 \theta_c \cos^2 \theta_c G_F^2 m_c^2$$

We see that :

- i) the quadratic divergence  $\Lambda^2$  has been replaced by  $m_c^2$ , hence  $m_c$  acts effectively as a cutoff for the  $K\bar{K}$  amplitude
- ii) the  $\log \Lambda$  divergence also cancels out.

The cancellation of the  $\log \Lambda$  term does not follow from the GIM mechanism, but is accidental for the  $K\bar{K}$  amplitude. If one considers another FCNC amplitude, the coefficient of the  $\log \Lambda$  term does not vanish in general.

For example, the amplitude for  $K_L^0 \rightarrow \mu^+ \mu^-$  goes as



$$A \propto G_F^2 (\lambda_c F(m_c) + \lambda_u F(m_u))$$

$$K_L^0 \rightarrow \mu^+ \mu^-$$

where we have neglected the neutrino mass.

The loop function goes as

(23)

$$F(m) \simeq \Lambda^2 - m^2 \log \Lambda^2 + \text{finite terms}$$

so that (using  $\lambda_u = -\lambda_c = \sin\theta_c \cos\theta_c$ ):

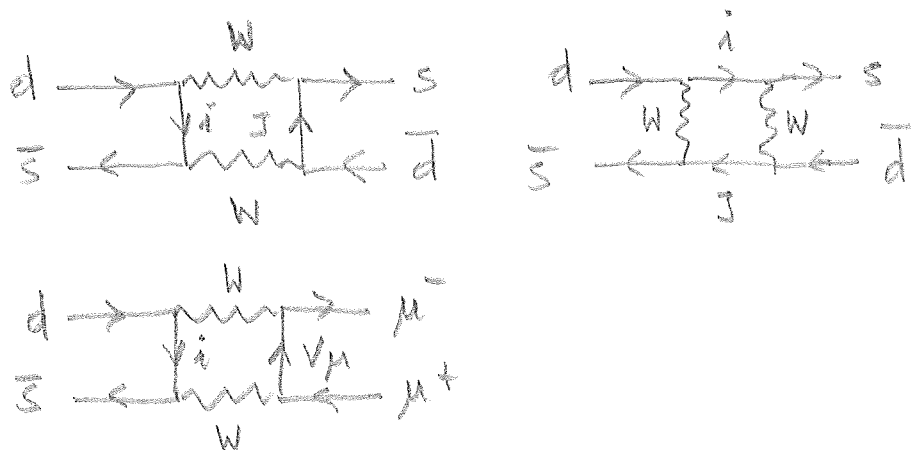
$$A(K_L^0 \rightarrow \mu^+ \mu^-) \propto G_F^2 \sin\theta_c \cos\theta_c (m_c^2 - m_u^2) \log \Lambda + \text{finite terms.}$$

In this case there's a residual log divergence, although the GIM mechanism ensures that the quadratic divergence gets cut off by the charm mass.

Comparison with the experimental values of  $\Delta M_K$  and  $BR(K_L^0 \rightarrow \mu^+ \mu^-)$  allows one to determine the value of the charm mass:  $m_c \approx 1.3 \text{ GeV}$ . This prediction was later confirmed in 1974 with the discovery of the  $J/\psi$ .

Notice that the contribution of the quarks, as a consequence of the GIM mechanism, is non-decoupling. Therefore, one could ask why the presence of the top quark did not jeopardize the GIM prediction. The answer is that the contribution of the top quark to the real part of  $A(K\bar{K})$  is negligible due to its GIM suppression. Hence the GIM prediction of  $\Delta M_K$  is accurate. The top, on the other hand, gives an important contribution to the imaginary part of the amplitude; including the top quark is thus crucial to predict CP processes.

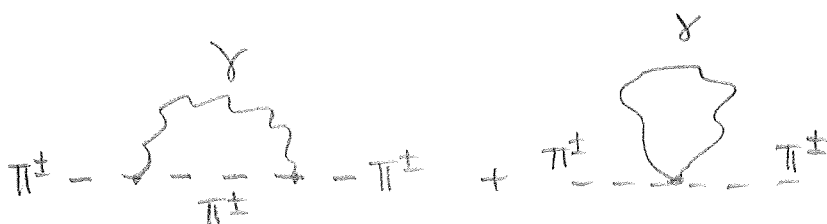
Notice that in the SM the amplitudes of  $K \leftrightarrow \bar{K}$  and  $K_L^0 \rightarrow \mu^+ \mu^-$  are both finite; the corresponding diagrams are



plus those involving the would-be NGB (in a  $\Sigma$  gauge).

It thus follows that the logarithmic divergence in  $K_L^0 \rightarrow \mu^+ \mu^-$  is cut off by the  $W$  mass:  $\log \Lambda \rightarrow \log m_W$ .

As another historical application of naturalness arguments, let us now consider the electromagnetic correction to the pion mass. The virtual exchange of the photon gives a shift to the mass of the charged pion compared to that of the neutral pion:

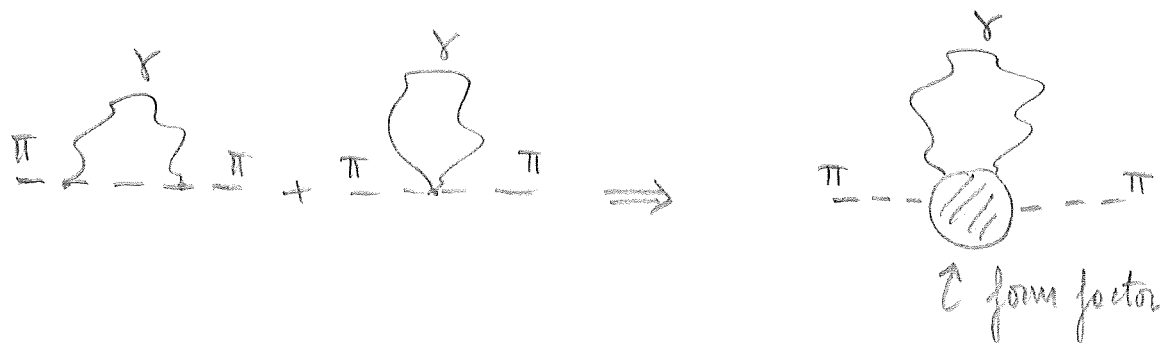


$$\Delta m_\pi^2 \propto \frac{e^2}{16\pi^2} \Lambda^2$$



In the low-energy theory of pions, the diagrams leading to the mass splitting are quadratically divergent. On the other hand, we now know that the pion is a  $q\bar{q}$  bound state, and that  $\Delta M_\pi^2$  is finite in QCD.

Hence, finiteness of  $\Delta M_\pi^2$  is a consequence of pion compositeness in this case (as well as of the QCD UV behavior):



By using quark-hadron duality, it is possible to describe the form factor involved in the loop in terms of the exchange of an infinite tower of resonances. By assuming that the quadratic divergence is mainly canceled by the lowest resonance (Vector Meson Dominance), it is possible to estimate the mass of the latter from the experimental value  $M_{\pi^+} - M_{\pi^0} \equiv \Delta M \approx 4.6 \text{ MeV}$ .

$$\Delta M_\pi^2 \approx 2 \Delta M \cdot M_{\pi^0} \approx \frac{\alpha}{4\pi} \cdot \Lambda^2 \rightarrow \Lambda \approx 1.5 \text{ GeV}$$

A more refined estimate performed using Weinberg's sum rules and PCAC (PRL 18 (1967) 759) gives  $\Lambda \approx 717 \text{ MeV}$ , which is in reasonable agreement with the  $\rho$  mass,  $M_\rho = 770 \text{ MeV}$ , the  $\rho$  being the lightest resonance contributing to the form factor.

Naturalness of the EW scale is the main motivation to expect New Physics (at least those particles which cut off the quadratic divergence in the Higgs mass term) to be around  $\approx 0.5 - 1 \text{ TeV}$ . Precision Tests at LEP and SLC, as well as searches for NP at the LHC, on the other hand, have excluded the existence of new particles such light. This is the so-called Little Hierarchy Problem: the lack of evidence for NP so far implies a minimum amount of tuning on simple UV completions of the SM at the level of a few percent.

Very strong, though indirect, constraints on NP come from the EW precision tests made at LEP and SLC. By estimating the coefficients of dim-6 operators in the SM effective Lagrangian to be  $\pm/\Lambda^2$  (this generically assumes  $O(1)$  couplings), EWPT set a lower bound  $\Lambda \gtrsim 3 - 10 \text{ TeV}^*$  (this was dubbed the LEP Paradox in Barbieri, Strumia hep-ph/0007265).

While such bounds severely constrain strongly-coupled UV completions of the SM (Technicolor and Composite Higgs models), they are easily satisfied by (weakly-coupled) SUSY, where corrections to the EW precision observables are generically small.

\* see for example: J. de Blas et al. arXiv: 1710.05402

Supersymmetric models are, on the other hand, significantly constrained by NP searches performed at the LHC. For example, LHC has excluded a gluino with mass smaller than  $\simeq 2 \text{ TeV}$  and stops lighter than  $1.2 - 1.4 \text{ TeV}$  (27)

[see: ATL-PHYS-PUB-2022-013 (March 2022, Moriond)].

These bounds imply sub-percent tuning in the Higgs mass term (notice that in the case of SUSY, the tuning is exacerbated by the presence of the residual logarithmic divergence:

$$\delta m_h^2 \simeq \frac{N_c}{2\pi^2} \left( \frac{m_t}{v} \right)^2 m_{\tilde{t}}^2 \log \frac{\Lambda^2}{m_{\tilde{t}}^2}$$