Lattice Gauge Theories – An Introduction

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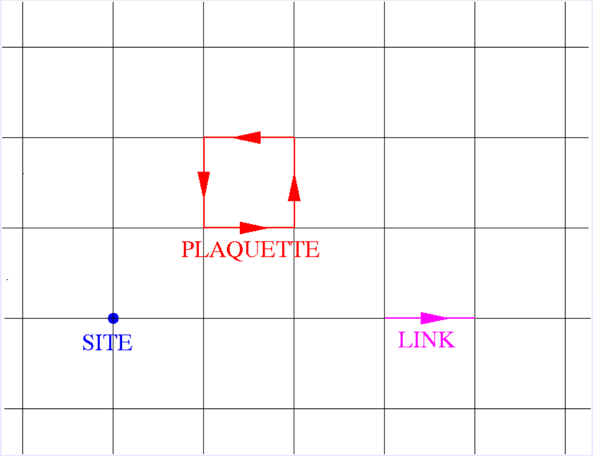
The Lattice



Duality between QFT and SM

The reformulation of Quantum Field Theory in Euclidean time allows to establish a correspondence between QFT quantities and Statistical Mechanics quantities

QFT		**SM**
Path integral	$\Leftrightarrow$	Partition Function
Vacuum	$\Leftrightarrow$	Equilibrium State
Mass gap	$\Leftrightarrow$	(Inverse) Correlation Length

Recovering the continuum theory

Lattice Gauge Theories aim at describing the continuum QFTs

- The lattice action of a theory must reproduce the continuum action (*naive continuum limit*)
- All lattice $n-$point functions must go to their continuum counterpart in the limit for the lattice spacing $a \to 0$
- if L is the lattice size, m a physical mass and a is the lattice spacing, the correct continuum physics is explored if

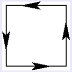
$$a \ll m^{-1} \ll L$$

- A continuum quantum field theory is realised near the second order phase transition points of the lattice theory

In QCD-like theories, the correctness of the continuum limit is a non-trivial consequence of asymptotic freedom

SU(N) Lattice Gauge Theories

- Link variables $U_\mu(i) = e^{ig_0 a A_\mu(i)}$
  Plaquettes $U_{\mu\nu}(i) = \prod_{U_\mu \in P_{\mu\nu}} U_\mu$



- Wilson action $\beta = (2N)/g_0^2$

$$S = \beta \sum_{i,\mu,\nu} \left(1 - \frac{1}{2N} \mathrm{Tr} \left(U_{\mu\nu}(i) + U^\dagger_{\mu\nu}(i) \right) \right)$$

- Path integral $Z = \int (\mathcal{D}U)\, \mathrm{e}^{-S}$
- Invariance under SU(N) gauge transformations

$$\tilde{U}_\mu(i) = G^\dagger(i) U_\mu(i) G(i + \hat{\mu})$$

Lattice action for QCD-like theories

Path integral

$$Z = \int (\mathcal{D}U_\mu(i)) \left(\det M(U_\mu)\right)^{N_f} e^{-S(U_{\mu\nu}(i))}$$

with

$$U_\mu(i) = Pexp\left(ig_0 \int_i^{i+a\hat{\mu}} A_\mu(x)\mathrm{d}x\right)$$

and

$$U_{\mu\nu}(i) = U_\mu(i)U_\nu(i+\hat{\mu})U^\dagger_\mu(i+\hat{\nu})U^\dagger_\nu(i)$$

Gauge part

$$S = \beta \sum_{i,\mu} \left(1 - \frac{1}{N}\mathcal{R}\mathrm{e}\ \mathrm{Tr}(\mathrm{U}_{\mu\nu}(\mathrm{i}))\right) \qquad , \qquad with\ \beta = 2N/g_0^2$$

Fermions on the lattice

Naive discretisation of the fermionic field gives rise to fermion doubling (16 species of fermions in $(3+1)$ dimensions)

No-go theorem (Nielsen-Ninomiya): no lattice formulation of fermions can be at the same time ultra-local, chirally symmetric and avoid fermion doubling

Solutions:

1. Wilson fermions $\leftarrow$ give up chirality
2. Staggered fermions $\leftarrow$ put up with doubling
3. Domain wall and overlap fermions $\leftarrow$ couple all sites

For SU(N) gauge theories at large N Wilson fermions have been used

Why Wilson fermions?

Different fermion formulations must give the same results, but a discretisation can provide some advantages

In our case, the Wilson formulation has been chosen because

1. Unlike non-local chiral fermions, Wilson fermions are cheap to simulate
2. Unlike the staggered fermions, a generic number of flavours can be simulated
3. Chiral symmetry can be recovered by tuning the hopping parameter
4. New technical breakthroughs allow to go close enough to the chiral limit (onset of χPT just around the corner?)

Wilson fermions

Take the naive Dirac fermions and add an irrelevant term that goes like the Laplacian

$$M_{\alpha\beta}(ij) = (M + 4r)\delta_{ij}\delta_{\alpha\beta} - \frac{1}{2}\left[(r-\gamma_\mu)_{\alpha\beta}U_\mu(i)\delta_{i,j+\mu} + (r+\gamma_\mu)_{\alpha\beta}U^\dagger_\mu(j)\delta_{i,i-\mu}\right]$$

This formulation breaks explicitly chiral symmetry

Define the hopping parameter

$$\kappa = \frac{1}{2(m+4r)}$$

Chiral symmetry recovered in the limit $\kappa \to \kappa_c$ (κ_c to be determined numerically)

Quenched approximation

For an observable $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \frac{\int (\mathcal{D}U_\mu(i)) \left(\det M(U_\mu)\right)^{N_f} f(M) e^{-S_g(U_{\mu\nu}(i))}}{\int (\mathcal{D}U_\mu(i)) \left(\det M(U_\mu)\right)^{N_f} e^{-S_g(U_{\mu\nu}(i))}}$$

Assume $\det M(U_\mu) \simeq 1$ i.e. fermions loops are removed from the action

The approximation is exact in the $m \to \infty$ and $N \to \infty$ limit ($g^2 N$ is fixed)
$\hookrightarrow$ the large N spectrum is quenched for $m \neq 0$

As N increases, unquenching effects are expected for smaller quark masses

Monte Carlo methods for Lattice Field Theories

- An evaluation of the path integral requires performing an integral in O($4N^2V$) dimensions
- Monte Carlo methods allow us to evaluate integrals in an high number of dimensions more efficiently than grid methods
- Underpinning concepts: Markov chains plus dynamics leading to the Boltzmann distribution
- In numerical simulations one obtains a sufficient number (generally of order 100-1000) of configurations distributed according to e^{-S} that allow to stochastically evaluate the path integral

Analysis of numerical data

Monte Carlo methods are importance sample methods $\Rightarrow$ expectation values of observables are given by simple averages of the numerical results

Due to the finiteness of the sampling, each observable carries a statistical error

Monte Carlo dynamics produces data that are correlated $\Rightarrow$ Gaussian statistics does not apply straightforwardly

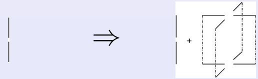
Correct data analysis keeps into account correlations in both averages (through binning) and fits (using correlated fits)

Bias in observables are dealt with methods such as bootstrap or jack-knife

Masses of states from correlators

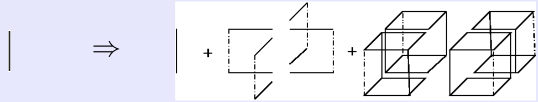
Masses of states extracted from two-point functions (*correlators*) of operators with the right quantum numbers
Starting from links, we can built those operators via

- Blocking



  Fast increase of the size of the operators

- Smearing



  Finer resolution

More modern approach: Wilson flow

Correlation matrix

Numerical signal improves considerably using the full correlation matrix

$$\begin{aligned} C_{ij}(t) &= \langle 0|\Phi_i(t)\left(\Phi_j(0)\right)^\dagger|0\rangle \\ &= \langle 0|e^{Ht}\Phi_i(0)e^{-Ht}\left(\Phi_j(0)\right)^\dagger|0\rangle \\ &= \sum_n \langle 0|e^{Ht}\Phi_i(0)e^{-Ht}|n\rangle\langle n|\left(\Phi_j(0)\right)^\dagger|0\rangle \\ &= \sum_n e^{-\Delta E_n t}\langle 0|\Phi_i(0)|n\rangle\langle n|\left(\Phi_j(0)\right)^\dagger|0\rangle \\ &= \sum_n c^*_{in}c_{jn}e^{-\Delta E_n t} \end{aligned}$$

After diagonalisation

$$C_{ij}(t) = \delta_{ij}\sum_n |c_{in}|^2 e^{-am_n t} \underset{t\to\infty}{\longrightarrow} \delta_{ij}|c_{i1}|^2 e^{-am_1 t}$$

Variational principle

1. Find the eigenvector v that minimises

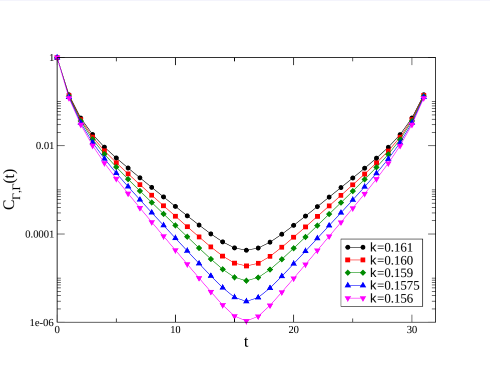
$$am_1(t_d) = -\frac{1}{t_d} \log \frac{v_i^* C_{ij}(t_d) v_j}{v_i^* C_{ij}(0) v_j}$$

   for some t_d
2. Fit $v(t)$ with the law $Ae^{-m_1 t}$ to extract m_1
3. Find the complement to the space generated by $v(t)$
4. Repeat 1-3 to extract $m_2, \ldots, m_n$

Need a good overlap with the state of interest

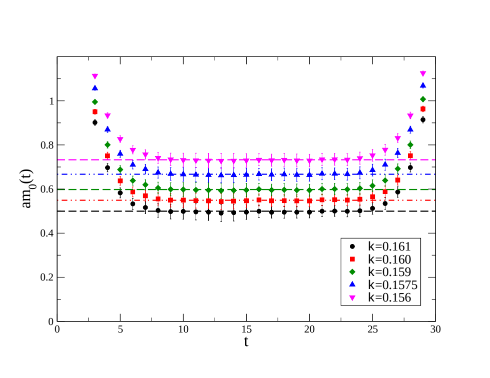
Correlator functions vs. Euclidean time

Example: fermionic bilinear with 0^{-+} quantum numbers



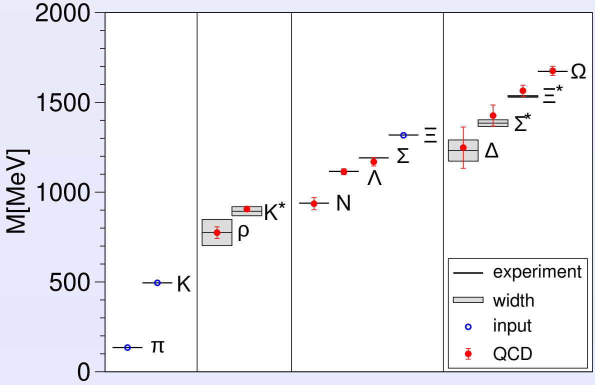
Expected behaviour: $C(T) = A\cosh(m(T - N_t/2))$

Effective mass



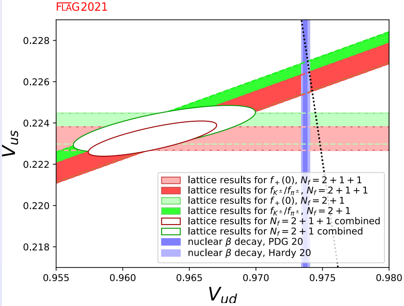
$$m_{eff}(T) = \text{acosh}\left(\frac{C(T-1)+C(T+1)}{2C(T+1)}\right)$$

The QCD Spectrum



(S. Durr *et al,* (BMW Collaboration), Science 347 (2015) 1452-1455 [arXiv:1406.4088])

Unitarity of the CKM matrix



Different **tensions** in the V_{us}-V_{ud} plane:

$$|V_u|^2 - 1 = 2.8\sigma$$

$$|V_u|^2 - 1 = 5.6\sigma \qquad |V_u|^2 - 1 = 3.3\sigma$$

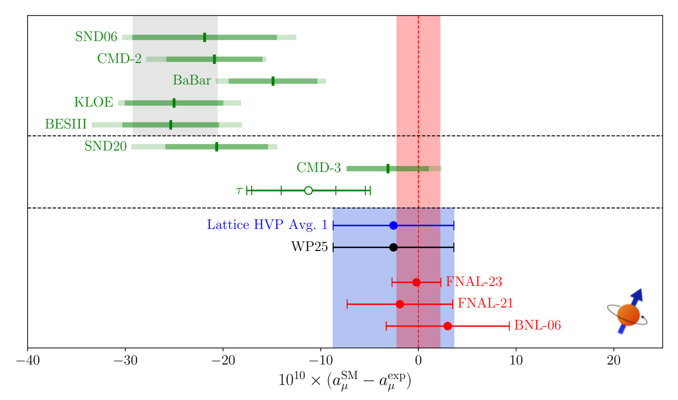
$$|V_u|^2 - 1 = 3.1\sigma \qquad |V_u|^2 - 1 = 1.7\sigma$$

Experimental and theoretical control of these quantities is of crucial importance to solve the issue

- **new measurements** (e.g. at NA62)
  (recent proposal in [V.Cirigliano et al., 2208.11707]: $K_{\mu 3}/K_{\mu 2}$)
- improve predictions of **radiative corrections and isospin-breaking effects**

(Credits: M. Di Carlo)

The muon $g - 2$



($g_\mu - 2$ white paper, arXiv:2505.21476)

The 't Hooft's large-N limit

The Conjecture

In the limit $N \to \infty$ and $g \to 0$ with $\lambda = g^2 N$ fixed, physical quantities in SU(N) gauge theories can be expressed as functions of $1/N$ (if N_f fermions in the fundamental representations are present) or $1/N^2$ (in the Yang-Mills case), with a finite large-N limit and a convergent series expansion about that limit down to some $N = N^\star$

Relevance

- Explanation of observed QCD features (OZI rules, stability of particles, . . .)
- Potential for analytic calculations
- Connection with gauge-string dualities

Support

Large-N extrapolation of lattice results

[For a review, see B. Lucini and M. Panero, Phys. Rept. 526 (2013) 93]

Large-N limit on the lattice

The lattice approach allows us to go beyond perturbative and diagrammatic arguments. In SU(N) YM, for a given observable

1. Continuum extrapolation
   - Determine its value at fixed a and N
   - Extrapolate to the continuum limit
   - Extrapolate to $N \to \infty$ using a power series in $1/N^2$
2. Fixed lattice spacing
   - Choose a in such a way that its value in physical units is common to the various N
   - Determine the value of the observable for that a at any N
   - Extrapolate to $N \to \infty$ using a power series in $1/N^2$

Study performed for various observables both at zero and finite temperature for $2 \leq N \leq 8$ (and $N = 17$!)

(k)-string tensions (closed string channel)

Confining potential: $V(R) = \sigma R$

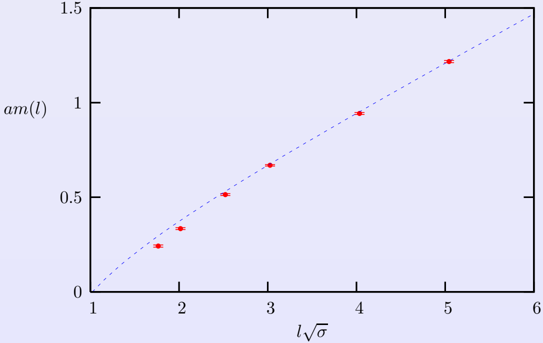
$V(R)$ can be extracted from Wilson loops (open string channel) or from Polyakov loops (closed string channel)

In the closed string channel

$$C^{\mathcal{R}\mathcal{R}}(t) = \left\langle \left(P^{\mathcal{R}}(0)\right)^{\dagger} P^{\mathcal{R}}(t) \right\rangle = \sum_j |c_j^{\mathcal{R}\mathcal{R}}|^2 e^{-am_j^{\mathcal{R}} t} \underset{t\to\infty}{\rightarrow} |c_l^{\mathcal{R}\mathcal{R}}|^2 e^{-am_l^{\mathcal{R}} t}$$

$$P^{\mathcal{R}}(t) = \frac{1}{d_{\mathcal{R}}} \sum_{\hat{k},\vec{n}\perp} \mathrm{Tr}_{\mathcal{R}} \prod_{j=0}^{l} U_k(\vec{n} + j\hat{k}, t) \; , \; am_l^{\mathcal{R}} \simeq a^2 \sigma_{\mathcal{R}} l - c_{\mathcal{R}} \frac{\pi(D-2)}{6l}$$

Confining flux tubes



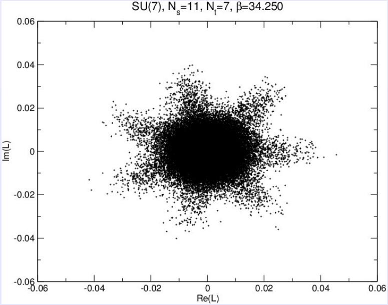
[H. Meyer and M. Teper, JHEP 0412 (2004) 031]

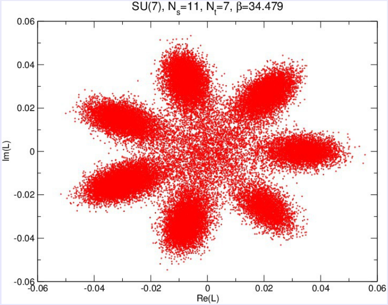
Deconfinement phase transition

- Temperature T introduced by taking a lattice $N_s \times N_t$, $N_s \gg N_t$ [$T = (N_t a(\beta))^{-1}$]
- A critical temperature T_c exists above which quarks and gluons are deconfined
- The phase transition can be seen as a change of symmetry in the ground state
- The relevant symmetry is $\mathbb{Z}(N)$, under which the ground state is not invariant above T_c
- The order parameter is the Polyakov loop

$$L = \frac{1}{N} \sum_{\vec{n}} \mathrm{Tr} \prod_{j=0}^{N_t - 1} U_4(\vec{n}, j)$$

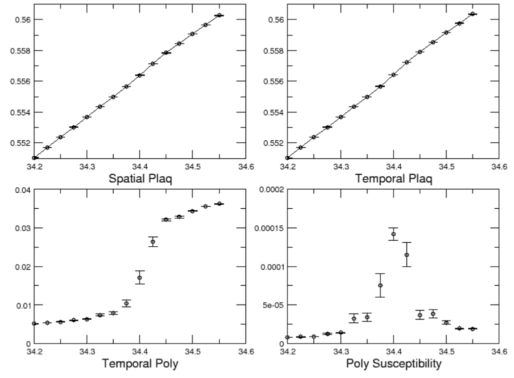
SU(7) near T_c





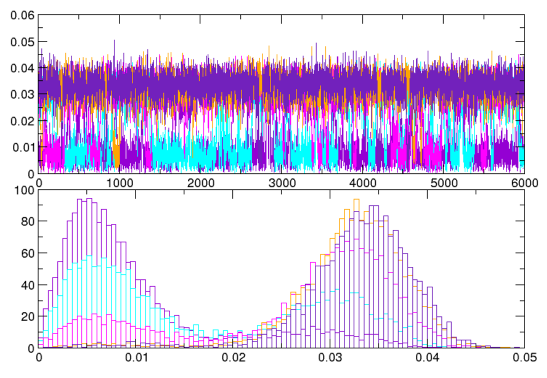
[BL, A. Rago, E. Rinaldi, Phys. Lett. B712 (2012) 279]

Order parameter near T_c



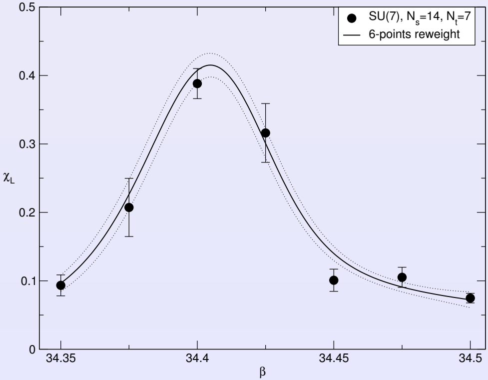
SU(7) theory, $N_t = 7$, $N_s = 14$

History and histogram of L near T_c



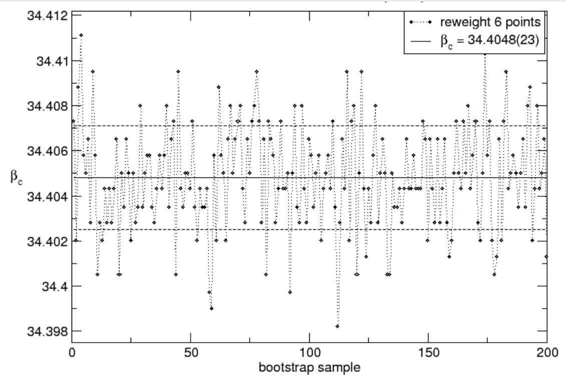
SU(7) theory, $N_t = 7$, $N_s = 14$

Susceptibility of L near T_c



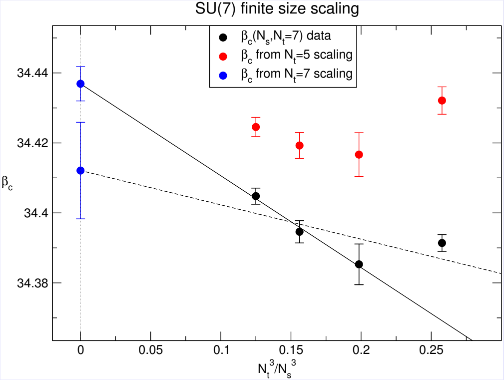
SU(7) theory, $N_t = 7$, $N_s = 14$

Error analysis of χ



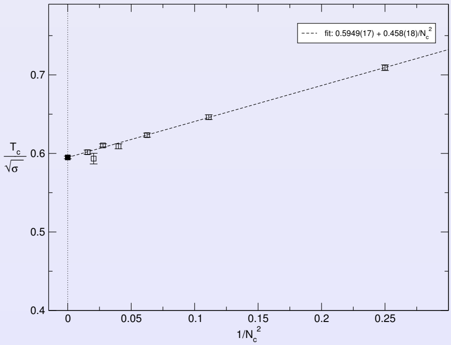
SU(7) theory, $N_t = 7$, $N_s = 14$

Infinite volume extrapolation of β_c



SU(7) theory, $N_t = 7$, various N_s

Deconfinement temperature



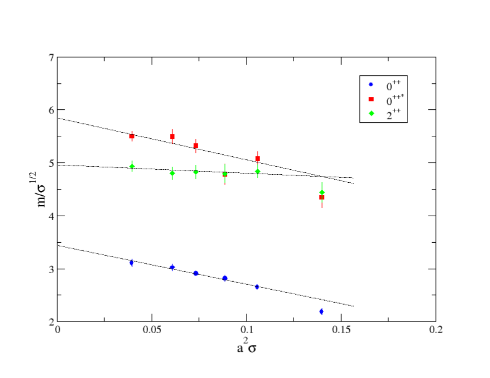
$$\frac{T_c}{\sqrt{\sigma}} = 0.5949(17) + \frac{0.458(18)}{N^2}$$

[See also B. Lucini, M. Teper and U. Wenger, JHEP 0401 (2004) 061]

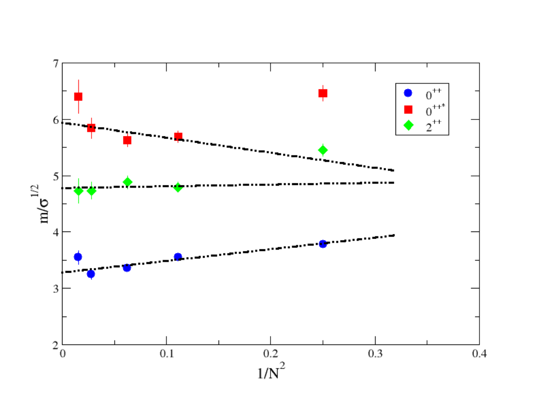
Extrapolation to the continuum limit

Example

Glueball masses in SU(4)



Glueball masses at large N



[B. Lucini, M. Teper and U. Wenger, JHEP 0406 (2004) 012]

Masses at $N = \infty$

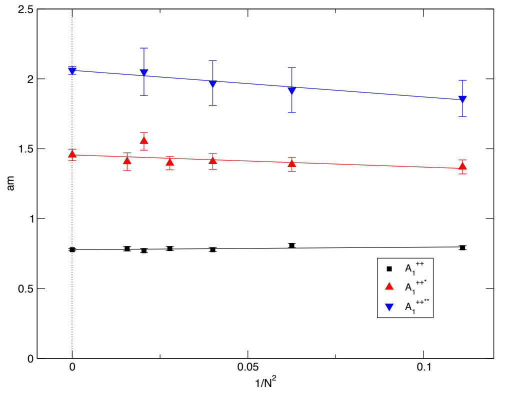
0^{++} $$\frac{m}{\sqrt{\sigma}} = 3.28(8) + \frac{2.1(1.1)}{N^2}$$

0^{++*} $$\frac{m}{\sqrt{\sigma}} = 5.93(17) - \frac{2.7(2.0)}{N^2}$$

2^{++} $$\frac{m}{\sqrt{\sigma}} = 4.78(14) + \frac{0.3(1.7)}{N^2}$$

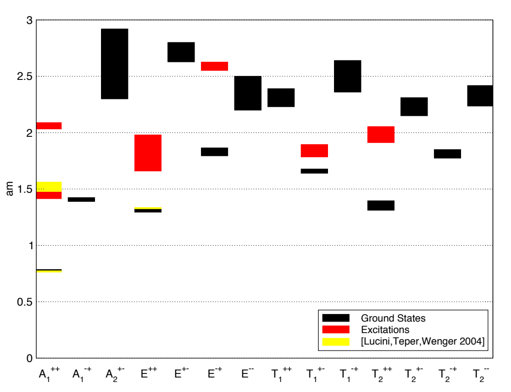
Accurate $N = \infty$ value, small $\mathcal{O}(1/N^2)$ correction

0^{++} excitations



Lattice spacing fixed by requiring $aT_c = 1/6$

Spectrum at $aT_c = 1/6$



[B. Lucini, A. Rago and E. Rinaldi, JHEP 1008 (2010) 119]

Fermionic operators (flavour triplet)

Meson masses extracted from large time behaviour of mesonic correlators

Particle	Bilinear	J^{PC}
a_0	$\mathbb{I}$, $\bar{\psi}_1\psi_2$	0^{++}
π	$\bar{\psi}_1\gamma_5\psi_2$, $\bar{\psi}_1\gamma_0\gamma_5\psi_2$	0^{-+}
ρ	$\bar{\psi}_1\gamma_i\psi_2$, $\bar{\psi}_1\gamma_0\gamma_i\psi_2$	1^{--}
a_1	$\bar{\psi}_1\gamma_5\gamma_i\psi_2$	1^{++}
b_1	$\bar{\psi}_1\gamma_i\gamma_j\psi_2$	1^{+-}

Fixing the bare parameter

β fixed by imposing that aT_c = 1/5

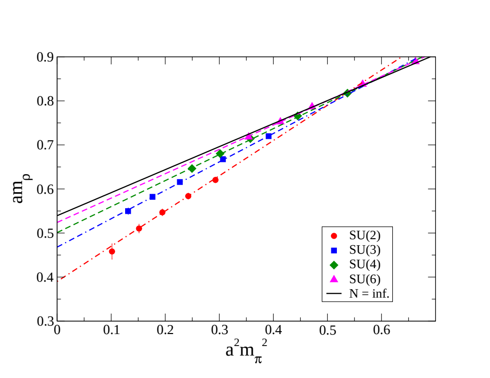
Another measured quantity (e.g. σ) could be used $\Rightarrow$ differences are $\mathcal{O}(1/N^2)$

Bare quark mass fixed (a posteriori!) by κ

Strategy

Study masses at fixed lattice spacing and various κ and fit to the expected behaviour to compare various N

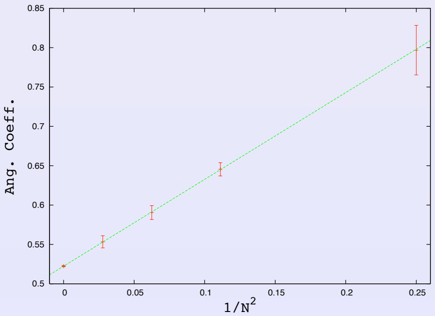
m_ρ vs. m_π^2



At small m_π

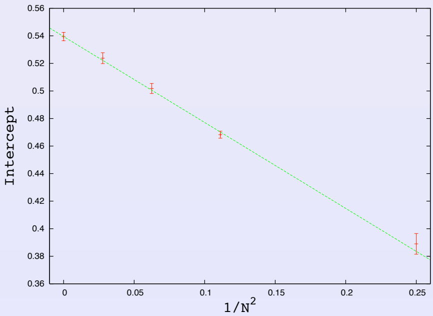
$$m_\rho = A m_\pi^2 + B$$

A vs. $1/N^2$



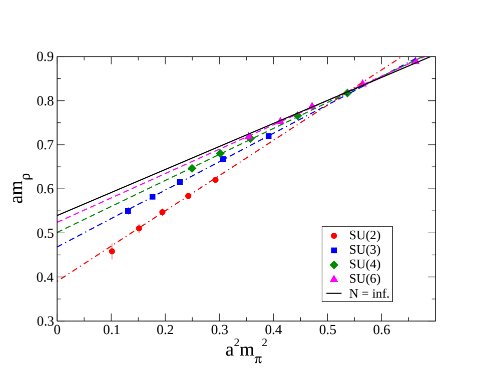
Fit with an $\mathcal{O}(1/N^2)$ correction only

B vs. $1/N^2$



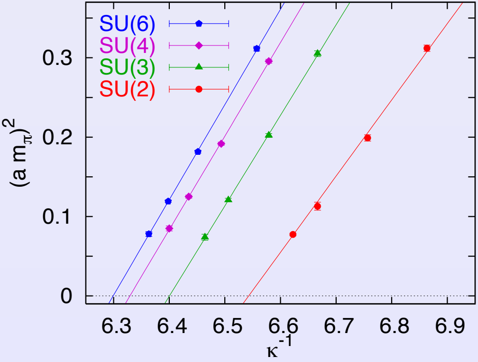
Fit with an $\mathcal{O}(1/N^2)$ correction only

m_ρ vs. m_π^2 at $N = \infty$



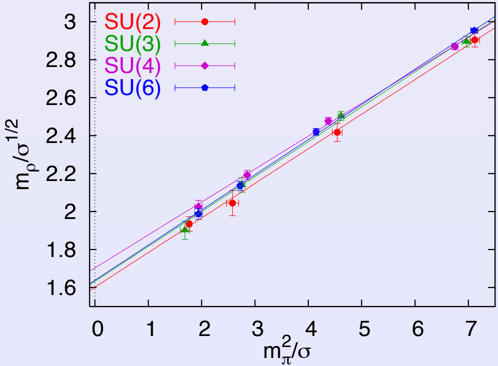
[L. Del Debbio, B. Lucini, A. Patella and C. Pica, JHEP 0803 (2008) 062]

Chiral extrapolation of m_π - fixing σ



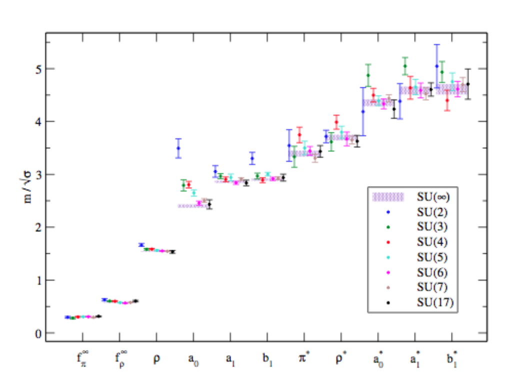
[G. Bali and F. Bursa, JHEP 0809 (2008) 110]

m_π vs. m_ρ - fixing σ



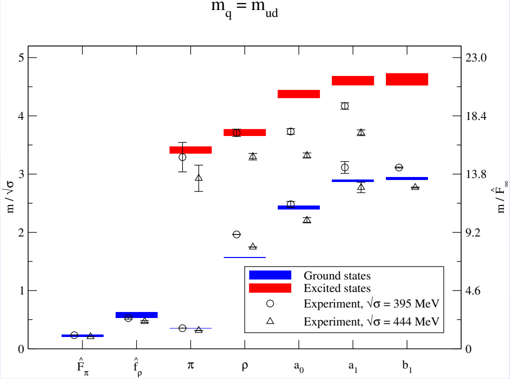
[G. Bali and F. Bursa, JHEP 0809 (2008) 110]

Approaching $N = \infty$ – The isotriplet spectrum



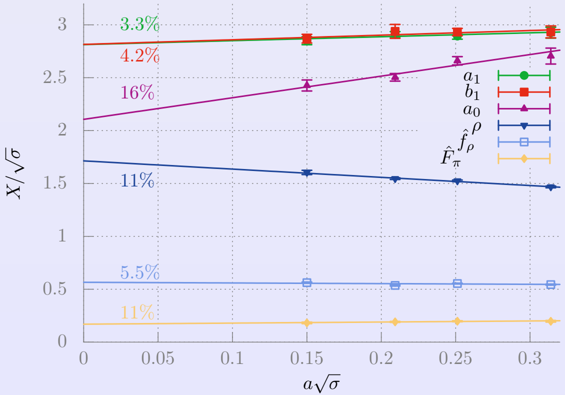
[G. Bali, F. Bursa, L. Castagnini, S. Collins, L. Del Debbio, B. Lucini and M. Panero, JHEP 1306 (2013) 071]

Comparison with QCD

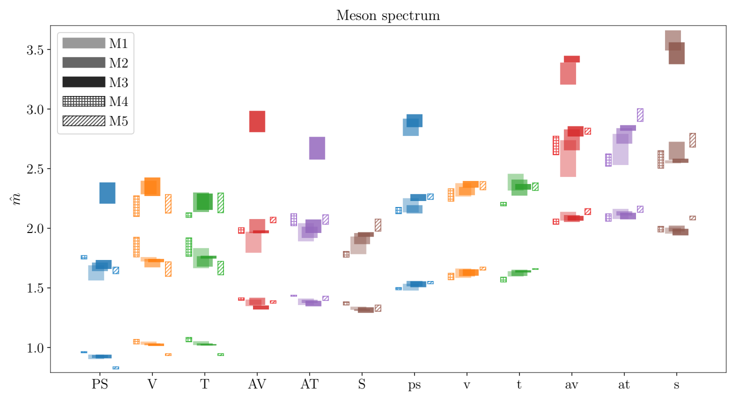


$\sqrt{\sigma}$ fixed from the condition $\hat{F}_\infty = 85.9$ MeV, m_{ud} from $m_\pi = 138$ MeV

Finite a corrections – SU(7)



Spectrum in a model for Higgs Compositeness



(E. Bennett, BL *et al.*, (the TELOS collaboration), Phys.Rev.D 110 (2024) 7, 074509, [arXiv:2405.01388])