SCHOOL ON EXPLICIT METHODS IN ARITHMETIC GEOMETRY: SUGGESTED READING AND TOPICS TO BE COVERED

ICTP TRIESTE, JUNE 30TH - JULY 4TH 2025

Contents

1. Javier Fresán: G-functions

The plan for the lectures are as follows:

(1) Definition and examples

(2) Period functions are G-functions

(3) The structure of the minimal differential equation of a G-function

(4) Special values of G-functions: irrationality and transcendence results

References

(1) B. Dwork, G. Gerotto, F. J. Sullivan, An introduction to G-functions

(2) Y. André, G-functions and Geometry

(3) T. Rivoal, Les E-fonctions et G-fonctions de Siegel

(4) G. Lepetit, Le théorème d'André–Chudnovsky–Katz

2. Jeanine van Order: Shimura Varieties

I will give a brief tour of the general theory of Shimura varieties, then focus on the special examples of Shimura curves and GSpin Shimura varieties, with links between special cycles and central derivative values of automorphic L-functions.

Date: Last Updated: April 22, 2025.

References

(1) F. Andreatta, E. Goren, B. Howard, and K. Madapusi Pera, Faltings heights of abelian varieties with complex multiplication, Ann. of Math., 187 (2018), 391–531

(2) F. Andreatta, E. Goren, B. Howard, and K. Madapusi Pera, Height pairings on orthogonal Shimura varieties, Compositio Math., vol. 153 (2017), 474-534

(3) J.H. Bruinier and T. Yang, Faltings heights of CM cycles and derivatives of L-functions, Invent. math. 177 (2009), 631-681.

(4) H. Carayol, Sur la mauvaise réduction des courbes de Shimura, Compos. Math. 59 (1986), no. 2, 151–230.

(5) P. Deligne, Travaux de Shimura, in "Séminaire Bourbaki, 23ème année" (1970/71), Exp. No. 389, Springer Lecture Notes in Math., 244, Berlin (1971), 123–165.

(6) S.S. Kudla, Analytic cycles on Shimura varieties of orthogonal type, Duke Math. J. 86 (1997), 39-78.

(7) J.S. Milne, Introduction to Shimura Varieties, in "Harmonic analysis, the trace formula, and Shimura varieties", Clay Math. Proc. 4, Amer. Math. Soc. Providence RI (2005).

(8) X. Yuan, S.-W. Zhang, and W. Zhang, The Gross–Zagier formula on Shimura curves, Ann. of Math. Stud. 184, Princeton University Press (2013).

3. Pietro Corvaja: Diophantine Geometry

The two main topics will be:

(1) Function fields analogues

(2) Improvements of the Mordell–Weil theorem and the Nagell–Lutz theorem

References

(1) J. Silverman, Arithmetic of elliptic curves

(2) D. Husemoller, Elliptic curves

Some prior knowledge of Galois Cohomology will be useful, but not indispensable.

4. Wadim Zudilin: Arithmetic of Differential Equations

Familiarity with the following topics will be useful.

References

(1) G. Almkvist, D. van Straten & W. Zudilin, Generalizations of Clausen's formula and algebraic transformations of Calabi–Yau differential equations
Proc. Edinburgh Math. Soc. 54:2 (2011), 273–295
(http://dx.doi.org/10.1017/S0013091509000959, http://www.mpim-bonn.mpg.de/preblob/4023) and references [2, 4, 8, 14, 15, 18, 20] there in.

(2) D. van Straten, Calabi-Yau operators.
https://arxiv.org/abs/1704.00164

5. Herberg Gangl: Algebraic K-Theory and Regulators

List of books/articles for the regulator lectures. For a first reading: A) in *Classical regulators* and one of B (i-iii) in *Classical regulators* for a somewhat different point of view. It would be beneficial to browse through one or two of the survey articles in *Survey/bigger picture/introduction*.

Classical regulators:

(A) Z. I. Borevich & I. R Shafarevich, Number Theory (particularly relevant is Ch.5 (needs Ch.2))

(B) Other nice articles on the relevant material, covering similar

  (i) Keith Conrad, Dirichlet's unit theorem
  https://kconrad.math.uconn.edu/blurbs/gradnumthy/unittheorem.pdf

  (ii) Drew Sutherland: Lecture 19-The analytic class number formula
  https://math.mit.edu/classes/18.785/2017fa/lectures.html

  (iii) Tom Weston: Lectures on the Dirichlet Class Number Formula for Imaginary Quadratic Fields
  http://www.pdmi.ras.ru/~lowdimma/BSD/weston_iqf.pdf

  (iv) Student report (Durham), John Rhodes: Polylogarithms https://www.maths.dur.ac.uk/users/herbert.gangl/Rhodes_Polylogarithms_report.pdf

Higher regulators: The abstract version
Seminal papers by Borel relating zeta values of number fields to algebraic K-theory; abstract K-groups, abstract regulator maps.

(i) Armand Borel: Stable real cohomology of arithmetic groups https://www.numdam.org/article/ASENS_1974_4_7_2_235_0.pdf

(ii) A. Borel, Cohomologie de SL_n et valeurs de fonctions zeta aux points entiers, Annales Sc. Norm. Sup. Pisa, C1 Sci. (4) 4, (1977), 613-636. Correction, ibid. 7, (1980), 373. http://www.numdam.org/item?id=ASNSP_1977_4_4_4_613_0

(iii) A. Borel, Values of zeta functions at integers, cohomology and polylogarithms, in Currents trends in mathematics and physics. A tribute to Harish-Chandra, S.D. Adhikari ed., 1-44, Narosa Publishing House, Bombay 1995.

The concrete version:

(i) Bloch's seminal notes introducing the first concrete candidate for $K_3(F)$, the dilogarithm as an explicit regulator map.
Spencer Bloch, Higher Regulators, Algebraic K-Theory, and Zeta Functions of Elliptic Curves. American Mathematical Society, 2000.

(ii) First concrete candidates for $K_{2n-1}(F), n > 2$, connecting polylogarithms to Dedekind zeta values, Zagier's Conjecture
Don Zagier: Polylogarithms, Dedekind zeta functions, and the algebraic K-theory of fields, (1990) https://people.mpim-bonn.mpg.de/zagier/ (Entry 55.)

(iii) Proof of Zagier's Conjecture for weight 3 Alexander Goncharov: Geometry of Configurations, Polylogarithms, and Motivic Cohomology (1995), https://www.sciencedirect.com/science/article/pii/S0001870885710456

Survey/bigger picture/introduction

(i) Introductory article on the first instances of higher regulator values
Don Zagier: The dilogarithm function
https://people.mpim-bonn.mpg.de/zagier/ (item 110.)

(ii) Requiring more substantial background: Dinakar Ramakrishnan: Regulators, algebraic cycles, and values of L-functions
Contemporary Mathematics, vol 83 (1989), pp.183-310.

(iii) Wilfred Hulsbergen, Conjectures in Arithmetic Algebraic Geometry https://link.springer.com/chapter/10.1007/978-3-663-09505-7_6

(iv) Joseph Oesterlé, Polylogarithmes (Bourbaki article)
https://www.numdam.org/item/SB_1992-1993__35__49_0.pdf (in French)

(v) Don Zagier & Herbert Gangl, Classical and elliptic polylogarithms and special values of L-series (2000)
https://people.mpim-bonn.mpg.de/zagier/ (item 94.)

(vi) Alexander Goncharov: Regulators (Handbook of K-theory)
https://arxiv.org/abs/math/0407308 (2005)

(vii) Clement Dupont: https://arxiv.org/pdf/2109.01702v2