Karlsruhe Institute of Technology

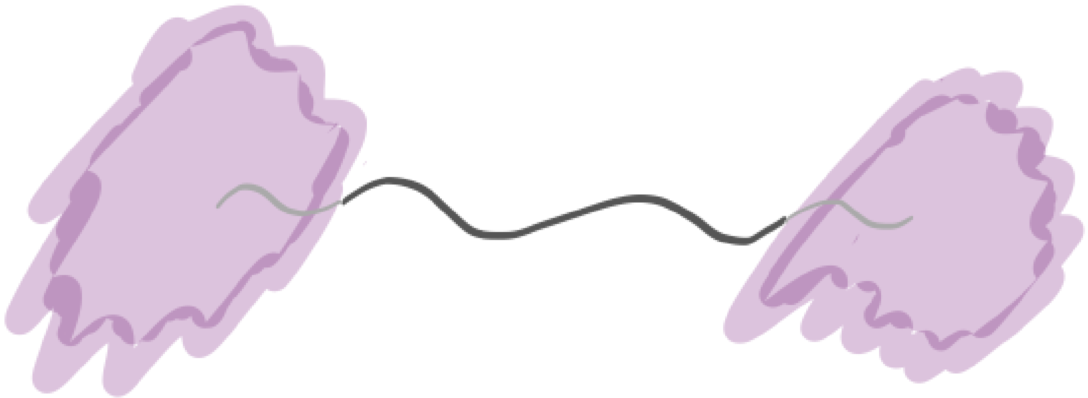
SYK Generalizations: A framework for strange transport and non-quasiparticle superconductivity

ICTP-WE Heraeus School and Workshop, August 4-15, Trieste

Jörg Schmalian
Karlsruhe Institute of Technology

last lectures

superconductivity without quasiparticles

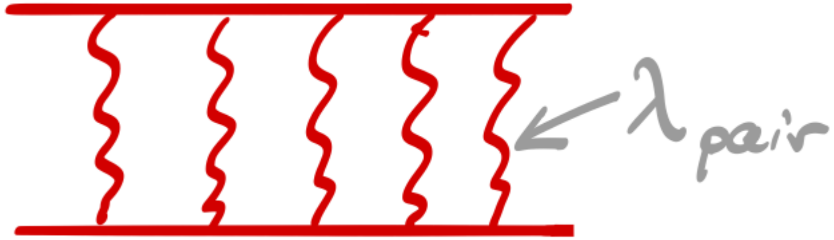


Can one make a pair of ill-defined entities?

superconductivity without quasiparticles

non-Fermi liquid $\Sigma(\omega) \propto -i\lambda\omega\,|\omega|^{-2\Delta}$

pairing susceptibility



$$\chi_{\text{pair}} = \frac{\chi^0_{\text{pair}}}{1-\lambda_{\text{pair}}\chi^0_{\text{pair}}}$$

$$\chi^0_{\text{pair}} = \int_{\boldsymbol{k}\omega} G(\boldsymbol{k},\omega)\, G(-\boldsymbol{k},-\omega)$$

$$\sim \rho_F Z \int_{T_c}^{\Lambda} \frac{d\omega}{\omega^{1-2\Delta}} \xrightarrow{T\to 0} \text{finite}$$

no weak-coupling instability:

A. Balatsky, Philos. Mag. Lett. 68, 251 (1993); A. Sudbo, Phys. Rev. Lett. 74, 2575 (1995);
L. Yin and S. Chakravarty, Int. J. Mod. Phys. B 10, 805 (1996).

zero space dimension, the Yukawa-SYK superconductor

S. Sachdev and J. Ye, PRL, 70 3339, (1993),, A. Georges, O. Parcollet, and S. Sachdev PRL 85, 840 (2000), A. Kitaev, KITP talk Feb., 2015.

$$S = \int d\tau \sum_{ijl} g_{ijl} c_i^\dagger(\tau) c_j(\tau) \phi_l(\tau)$$

I. Esterlis and J.S., Phys. Rev. B **100**, 115132 (2019) ,
Y. Wang, Phys. Rev. Lett. **124**, 017002 (2020).
L. Classen and A. Chubukov, Phys. Rev. B **104**, 125120 (2021).

critical normal state

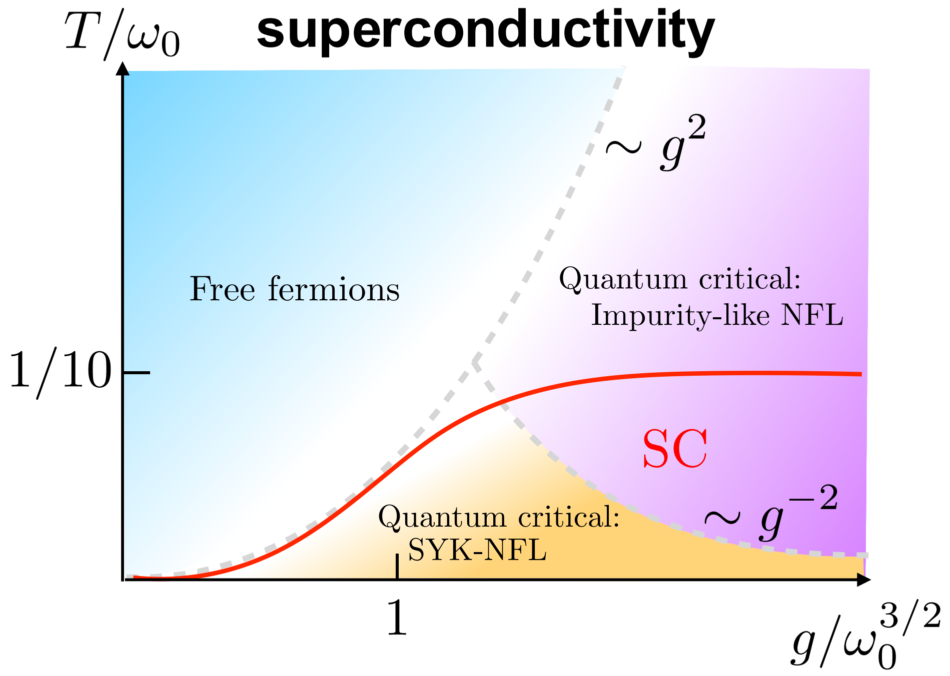
$$\hbar\tau^{-1} \sim |\omega|^{1-\gamma}$$

$$m^*/m \sim |\omega|^{-\gamma}$$

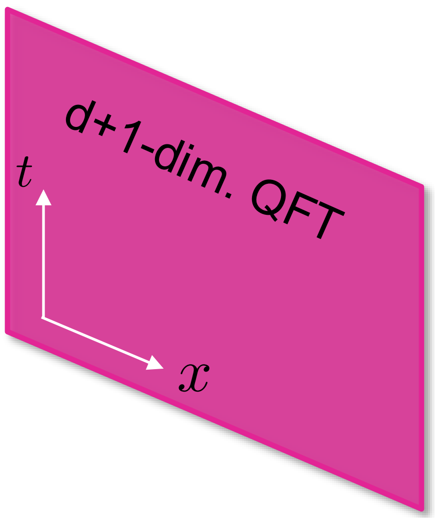
$$\lambda_{\text{pair}}(\omega) \sim |\omega|^{-\gamma}$$

$0 < \gamma < 1 \quad M = N: \ \gamma \approx 0.68$

superconductivity



holographic principle

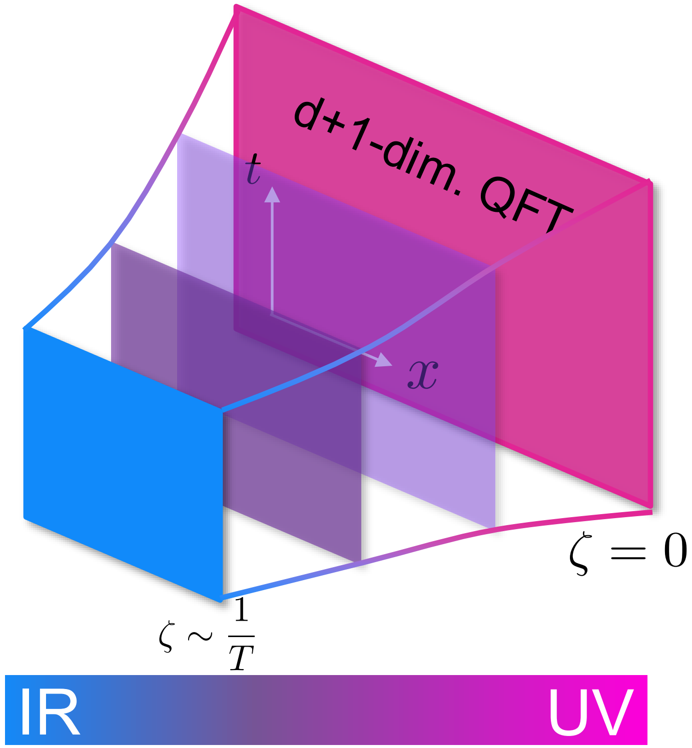


$$Z_{\mathrm{QFT}} \sim \int DgD\psi e^{-N^2 S_{\mathrm{AdS}_{d+2}}[g,\psi]}$$

J. M. Maldacena, *Adv. Theor. Math. Phys*. **2**, 231 (1998)
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Phys. Lett*. B **428**, 105 (1998)
E. Witten, *Adv. Theor. Math. Phys*. **2**, 253 (1998)

Condensed Matter:
C. P. Herzog, P. Kovtun, S. Sachdev, and D. T. Son, Phys. Rev. D **75**, 085020 (2007)

holographic principle



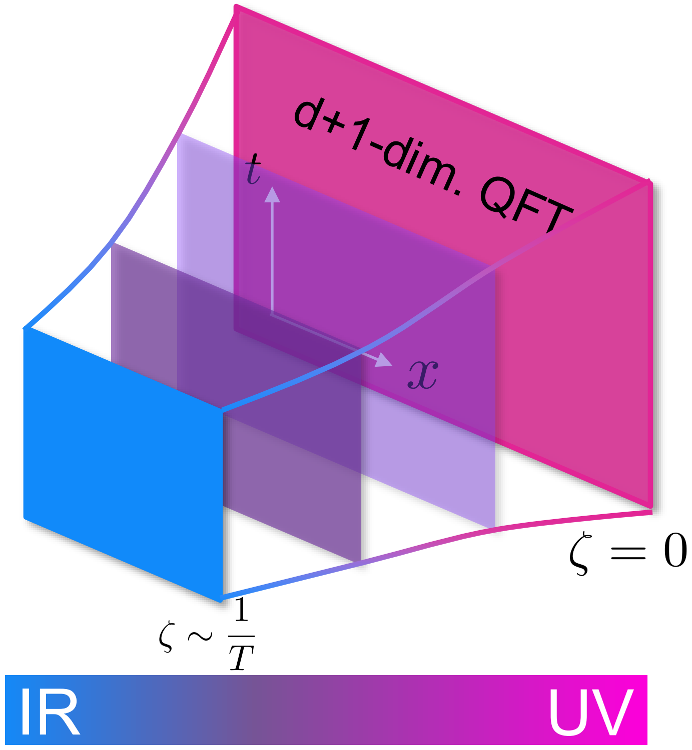
$$Z_{\mathrm{QFT}} \sim \int DgD\psi e^{-N^2 S_{\mathrm{AdS}_{d+2}}[g,\psi]}$$

J. M. Maldacena, *Adv. Theor. Math. Phys*. **2**, 231 (1998)
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Phys. Lett*. B **428**, 105 (1998)
E. Witten, *Adv. Theor. Math. Phys*. **2**, 253 (1998)

Condensed Matter:
C. P. Herzog, P. Kovtun, S. Sachdev, and D. T. Son, Phys. Rev. D **75**, 085020 (2007)

AdS = anti de Sitter space time (homogeneous space time of negative curvature)

holographic principle



selected applications:

- transport without quasiparticles

  $\eta/s \geq \frac{\hbar}{4\pi k_B}$ P. K. Kovtun, D. T. Son, and A. O. Starinets, PRL **94**, 111601 (2005)

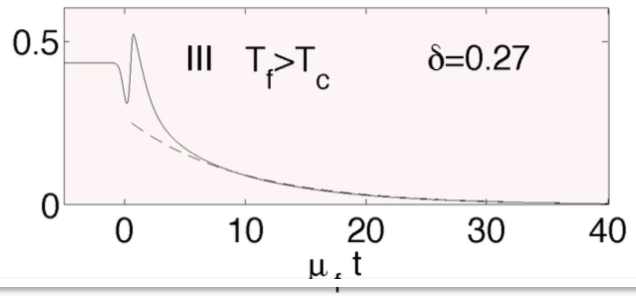
- holographic superconductivity (pairing without quasiparticles)

  S. S. Gubser, *Phys. Rev.* D **78**, 065034 (2008)
  S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, PRL **101**, 031601 (2008)

- charge density waves without quasipartices

  A. Donos and J.P. Gauntlett, JHEP **08**, 140 (2011).
  L. V. Delacrétaz, B. Goutéraux, S. A. Hartnoll, and A. Karlsso Phys. Rev. B **96**, 195128 (2017).

- non-equilibrium dynamics

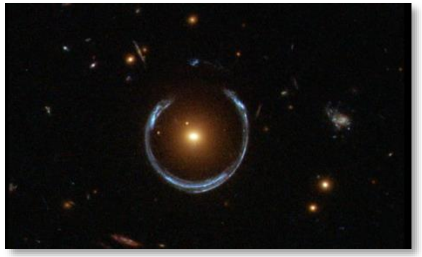


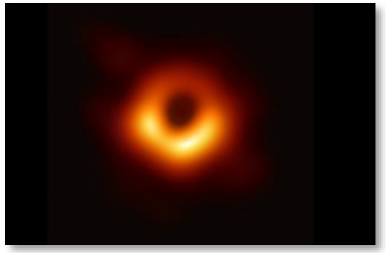
order parameter quench

M. Bhaseen, et al. PRL 110 (2013)

gravity and matter fields

field theory of gravity and matter





www.wikipedia.org

gravity

$$S_{\text{gravity}} = \frac{1}{2\kappa} \int \sqrt{-g}\,(R - 2\Lambda)$$

Einstein-Hilbert action

$$\frac{\delta\left(S_{\text{gravity}} + S_{\text{matter}}\right)}{\delta g^{\mu\nu}} = 0$$

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

Einstein equations

matter

$$S_{\text{matter}} = \frac{1}{2} \int \sqrt{-g}\mathcal{L}_{\text{matter}}$$

gauge, particles …

$$\frac{\delta S_{\text{matter}}}{\delta \psi} = 0$$

dynamics in the gravitational field

$$T_{\mu\nu} = \mathcal{L}_{\text{matter}} g_{\mu\nu} - 2\frac{\delta \mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}}$$

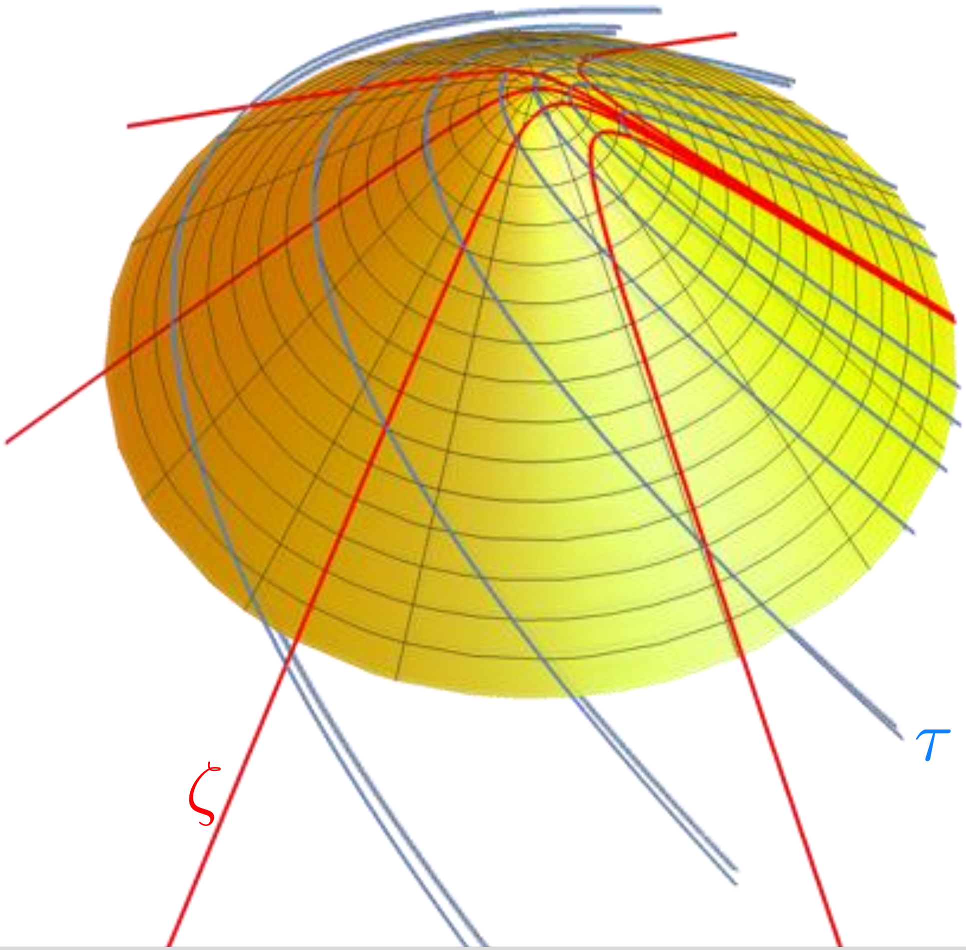
gravitational back reaction

cosmological constant

$\Lambda > 0$	de Sitter (if homogeneous)
$\Lambda = 0$	asymptotically flat
$\Lambda < 0$	Anti de Sitter (if homogeneous)

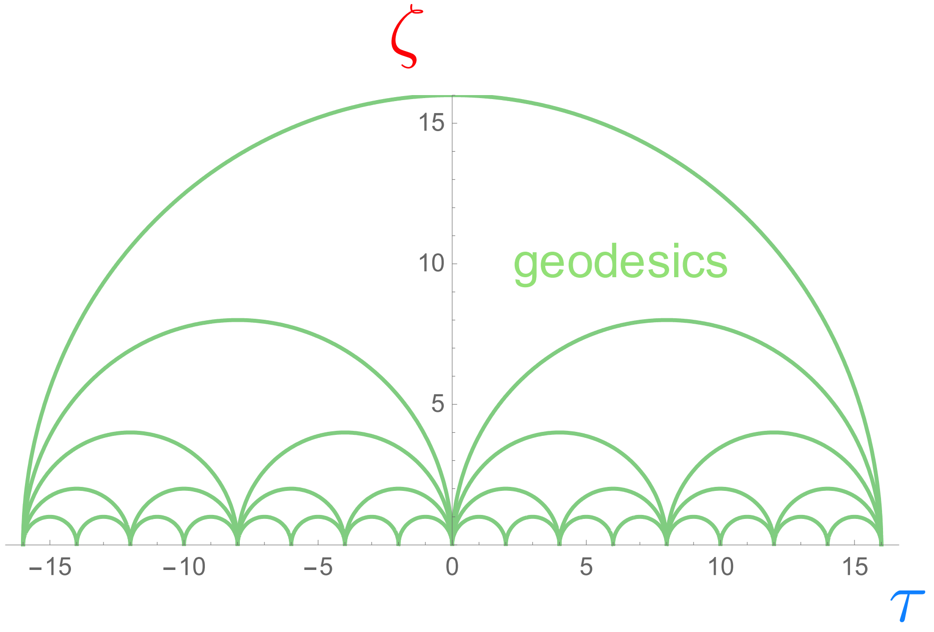
AdS_2: the simplest holographic space

Eucledian AdS_2=H_2 $X_1^2 + X_2^2 - X_3^2 = -1$



Poincaré half plane

$X_1 = \tau/\zeta \quad X_2 = \frac{\zeta}{2}\left(1+\zeta^{-2}(\tau^2-1)\right) \quad X_3 = \frac{\zeta}{2}\left(1+\zeta^{-2}(\tau^2+1)\right)$



Eucledian metric

$$ds^2 = g_{ab}dx^a dx^b = \frac{1}{\zeta^2}\left(d\tau^2 + d\zeta^2\right)$$

Eliashberg equation

Karlsruhe Institute of Technology

$$\Phi(\epsilon) = g^2 \int_T^{\Lambda} \frac{d\epsilon'}{2\pi} \frac{\Phi(\epsilon')}{|\epsilon - \epsilon'|^{\gamma} |\epsilon'|^{1-\gamma}} \quad \overset{\gamma \ll 1}{\Longrightarrow} \quad \frac{d}{d\epsilon}\epsilon^{1-\gamma}\frac{d}{d\epsilon}\epsilon^{\gamma}\Phi(\epsilon) = -\frac{\gamma g^2}{\pi}\frac{\Phi(\epsilon)}{\epsilon}$$

scalar field $\psi(\zeta) = \zeta^{\frac{1-\gamma}{2}} \Phi(1/\zeta)$ with coordinate $\zeta = 1/\epsilon$

$$-\partial_\zeta^2 \psi - \left(\frac{1}{4} - m^2\right)\frac{1}{\zeta^2}\psi = 0 \qquad m^2 = \frac{\gamma^2}{4} - \frac{g^2\gamma}{\pi}$$

static Klein-Gordon equation of pairs with mass m in AdS$_2$

D. T. Son, Phys. Rev. D **59**, 094019 (1999), A. V. Chubukov and J. S., PRB **72**, 174520 (2005), D. Hauck, M. J. Klug, I. Esterlis, J. S., Ann. of Phys. 417, 168120 (2020), A. Abanov, and A. V. Chubukov, Phys. Rev. B **102**, 024524 (2020), Y.-M. Wu, A. Abanov, and A. V. Chubukov, Phys. Rev. B 102, 094516 (2020)

remember: gravity + matter fields

$$Z = Z_0 \int Df e^{\alpha \frac{N}{U\beta} \int d\tau \{f,\tau\}} \times \int DF(\tau,\tau') e^{-S^{\mathrm{sc}}_{\mathrm{SYK}}}$$

fluctuating field: anomalous Gor'kov Green's function

$$F(\tau,\tau') \to F(\varepsilon,\omega)$$

FT of the relative time

FT of the absolute time

$$S^{(\mathrm{sc})}/N = \int_{\omega,\epsilon} \frac{F^\dagger(\omega,\epsilon) F(\omega,\epsilon)}{\Pi_{\mathrm{n.s.}}(\omega,\epsilon)} - \frac{\bar{g}^2\lambda_p}{2} \int_{\omega,\epsilon,\epsilon'} F^\dagger(\omega,\epsilon) D_{\mathrm{n.s.}}(\epsilon-\epsilon') F(\omega,\epsilon')$$

particle-particle propagator

$$\Pi_{\mathrm{n.s.}}(\omega,\epsilon) = G_{\mathrm{n.s.}}\left(\frac{\omega}{2}-\epsilon\right) G_{\mathrm{n.s.}}\left(\epsilon+\frac{\omega}{2}\right)$$

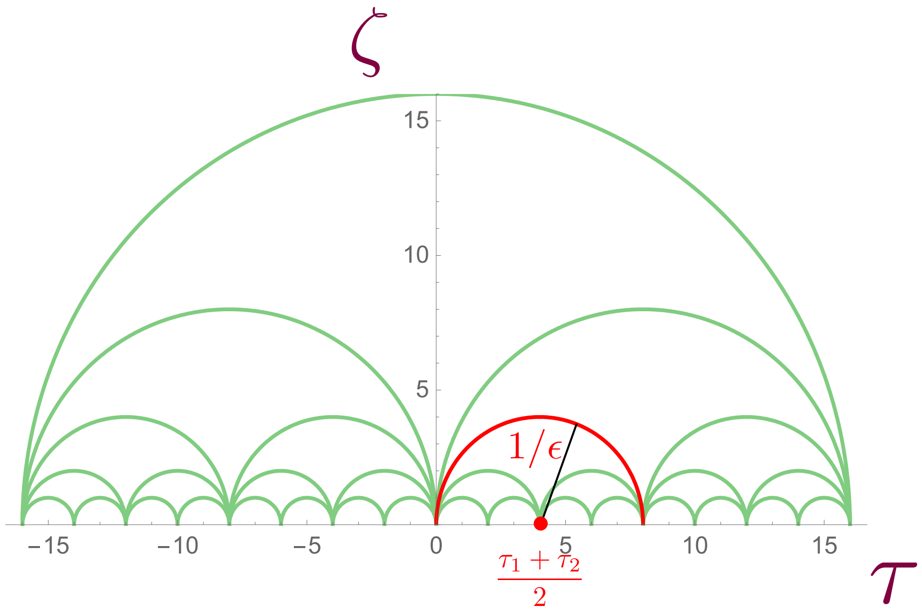
boson propagator
= singular paring interaction

$$D_{\mathrm{n.s.}}(\epsilon) \propto |\epsilon|^{-\gamma}$$

holographic map

$$F(\omega,\epsilon) \to F\left(\frac{\tau_1+\tau_2}{2},\epsilon\right)$$

$$F\left(\frac{\tau_1+\tau_2}{2},\epsilon\right) = |\epsilon|^{\frac{\gamma-1}{2}} \int_\Gamma \psi(\tau,\zeta)\, dl$$



Radon transformation

$$\Gamma: \quad |\epsilon|^{-1} = \sqrt{\left(\tau - \frac{\tau_1+\tau_2}{2}\right)^2 + \zeta^2}$$

geodesics of AdS_2

SYK superconductor = holographic superconductor

$$F\left(\frac{\tau_1+\tau_2}{2},\epsilon\right)=|\epsilon|^{\frac{\gamma-1}{2}}\int_\Gamma \psi\left(\tau,\zeta\right)dl$$

holographic superconductor in AdS_2 with Euclidean signature

$$S^{(\mathrm{sc})}=N\int d\tau d\zeta\left(\frac{m^2}{\zeta^2}\left|\psi\right|^2+\left|\partial_\tau\psi\right|^2+\left|\partial_\zeta\psi\right|^2\right)$$

dynamics

RG scale

$$S^{(\mathrm{sc})}/N=\int_{\omega,\epsilon}\frac{F^\dagger\left(\omega,\epsilon\right)F\left(\omega,\epsilon\right)}{\Pi_{\mathrm{n.s.}}\left(\omega,\epsilon\right)}-\frac{\bar{g}^2\lambda_p}{2}\int_{\omega,\epsilon,\epsilon'}F^\dagger\left(\omega,\epsilon\right)D_{\mathrm{n.s.}}\left(\epsilon-\epsilon'\right)F\left(\omega,\epsilon'\right)$$

SYK superconductor = holographic superconductor

$$F\left(\frac{\tau_1+\tau_2}{2},\epsilon\right)=|\epsilon|^{\frac{\gamma-1}{2}}\int_\Gamma \psi\left(\tau,\zeta\right)dl$$

holographic superconductor in AdS_2 with Euclidean signature at T=0

$$S^{(\mathrm{sc})}=N\int d\tau d\zeta\left(\frac{m^2}{\zeta^2}|\psi|^2+|\partial_\tau\psi|^2+|\partial_\zeta\psi|^2\right)$$

positive contribution to the mass
(no Cooper instability in NFL
with instantaneous pairing)

negative contribution to the mass
(generalized Cooper instab.)

$$S^{(\mathrm{sc})}/N=\int_{\omega,\epsilon}\frac{F^\dagger\left(\omega,\epsilon\right)F\left(\omega,\epsilon\right)}{\Pi_{\mathrm{n.s.}}\left(\omega,\epsilon\right)}-\frac{\bar{g}^2\lambda_p}{2}\int_{\omega,\epsilon,\epsilon'}F^\dagger\left(\omega,\epsilon\right)D_{\mathrm{n.s.}}\left(\epsilon-\epsilon'\right)F\left(\omega,\epsilon'\right)$$

holographic instability $m^2=m^2_{\mathrm{BF}}=-1/4$ (Breitenlohner Freedman condition)

=

Eliashberg instability

P. Breitenlohner and D. Z. Freedman, Ann. Phys. **144**, 249 (1982)

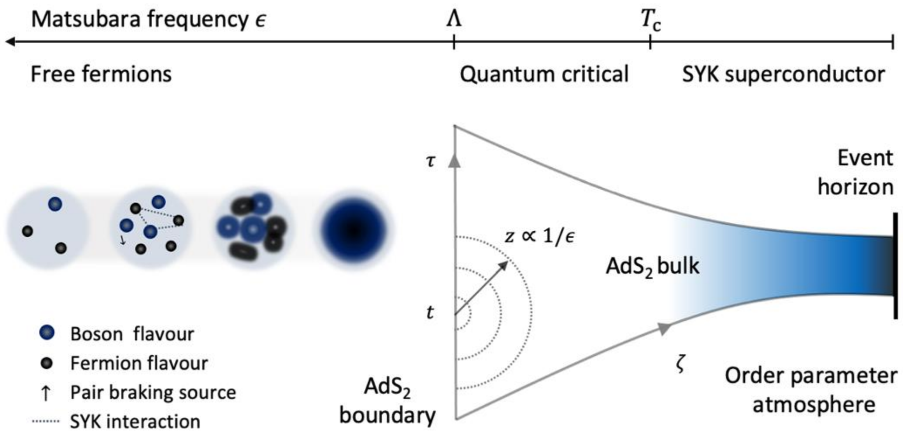
temperature, particle number

the holographic map can be extended

$$S^{(\mathrm{sc})} = S_{\mathrm{AdS}_2} = N \int d^2x \sqrt{g} \left(D_a \psi^* D^a \psi + m^2 |\psi|^2 \right)$$

- black hole horizon at finite T $\zeta_T^{-1} = 2\pi T$

$$ds^2 = g_{ab} dx^a dx^b = \frac{1}{\zeta^2} \left((1 - \zeta^2/\zeta_T^2) d\tau^2 + \frac{1}{(1 - \zeta^2/\zeta_T^2)} d\zeta^2 \right)$$



$$\frac{\delta S_{\mathrm{AdS}_2}}{\delta \psi} = 0$$

Eliashberg equations

temperature, particle number

the holographic map can be extended

$$S^{(\mathrm{sc})} = S_{\mathrm{AdS}_2} = N \int d^2x \sqrt{g} \left(D_a \psi^* D^a \psi + m^2 \left|\psi\right|^2 \right)$$

- black hole horizon at finite T $\zeta_T^{-1} = 2\pi T$

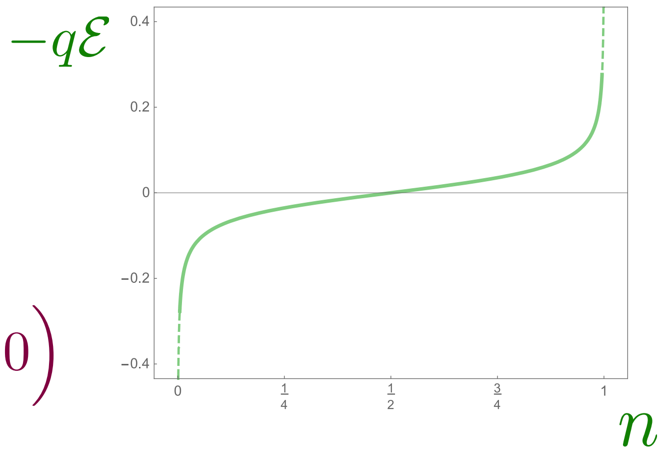
$$ds^2 = g_{ab} dx^a dx^b = \frac{1}{\zeta^2} \left((1 - \zeta^2/\zeta_T^2) d\tau^2 + \frac{1}{(1 - \zeta^2/\zeta_T^2)} d\zeta^2 \right)$$

- away from half filling

$$D_a = \partial_a - i q^\star A_a$$

Cooper pair charge: $q^\star = 2q$

boundary electric field: $A_a = \left(\frac{i\mathcal{E}}{\zeta} \left(1 - \zeta/\zeta_T\right), 0 \right)$



A. Georges, O. Parcollet, S. Sachdev,
PRB **63,** 134406 (2001)

Source fields

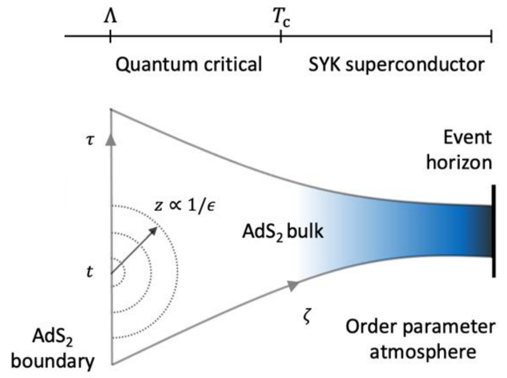
add an external pairing field
e.g. via Josephson coupling to another superconductor

$$S_J = -\int d\tau J_0(\tau) \frac{1}{N} \sum_i c_{i\uparrow}(\tau) c_{i\downarrow}(\tau) + h.c.$$

holographic map $$J(\zeta,\omega) = 2J_0(\omega)\,\zeta^{\frac{1-\gamma}{2}} \int_1^\infty \frac{dx}{x^{\frac{1+\gamma}{2}}} \frac{\cos\left(\omega\zeta\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}$$

$$S_{J,\mathrm{AdS}_2} = -\int d^2x \sqrt{g}\,\left(J^*(x)\,\psi(x) + h.c.\right)$$

source acts on the boundary



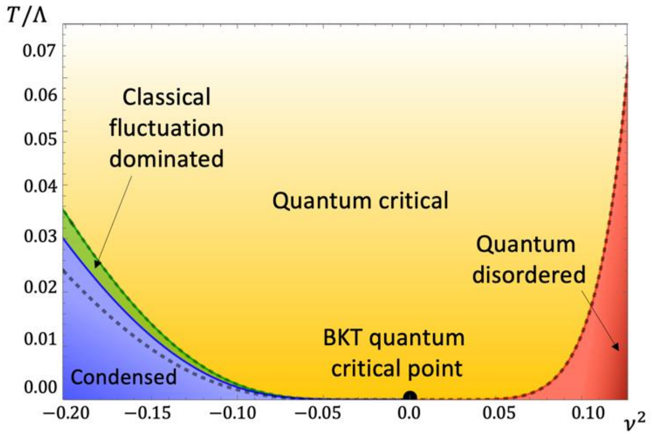
dynamic pairing susceptibility

not easy to calculate within Eliashberg approach, but easy in holography

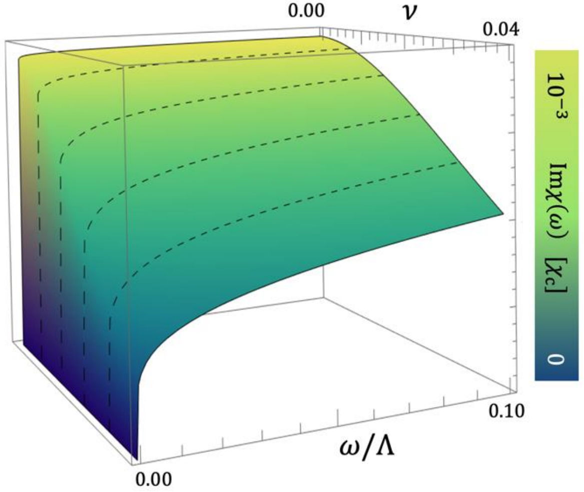
N. Iqbal, H. Liu, M. Mezei, Phys. Rev. D91, 025024 (2015)

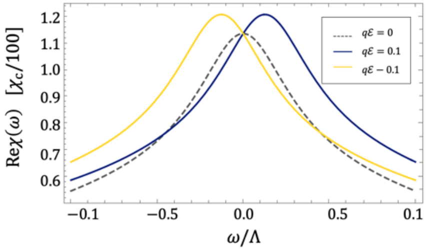
$$\chi_{\mathrm{AdS}_2} = \frac{1-g\mathcal{G}}{1-f\mathcal{G}}. \quad \mathcal{G}(T,\omega) = \frac{2\nu-\gamma}{2\nu+\gamma} T^{2\nu} \frac{\Gamma(u-\nu)\,\Gamma(v+\nu)}{\Gamma(u+\nu)\,\Gamma(v-\nu)} \quad u = \frac{1}{2} + i2q\mathcal{E} \quad v = \frac{1}{2} - i\frac{\omega-4\pi Tq\mathcal{E}}{2\pi T}$$

phase diagram



dynamic susceptibility





dynamic response, non-equilibrium behavior … fluctuations beyond Eliashberg, …

nonlinear effects and gravitational back reaction

mapping valid beyond the Gaussian regime

$$S^{(\mathrm{sc})} = S_{\mathrm{AdS}_2} = N \int d^2x\sqrt{g}\left(D_a\psi^* D^a\psi + m^2\left|\psi\right|^2 + u\left|\psi\right|^4 \cdots\right)$$

solving the coupled field theory beyond the linearized regimes

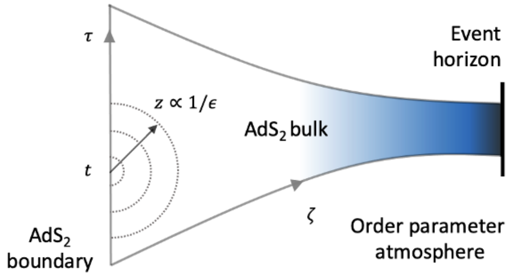
$$\delta\left\langle c_1^\dagger\left(\tau_1\right)c_2\left(\tau\right)\right\rangle \leftrightarrow \delta g_{\mu\nu}$$

as determined from corresponding Einstein eq.

$$R + 2\Lambda = \kappa T_{\mu\mu}$$

Conclusions

critical normal excitations provide the gravity; collective modes are matter fields



superconductivity via critical bosons ⟷ Yukawa-SYK superconductor ⟷ holographic superconductor

generalizations to finite dimensional problems

$$S = \int d^2r d\tau \left(\sum_{l\sigma} \psi^\dagger_{l\sigma} \left(\partial_\tau + \varepsilon\left(-i\nabla\right)\right) \psi_{l\sigma} + \frac{1}{2} \sum_s \phi_s \left(\omega_0^2 - \partial_\tau^2 - \nabla^2\right) \phi_s \right)$$

$$+ \frac{1}{\sqrt{N}} \int d^2r d\tau \sum_{ll'\sigma} v_{ll'}\left(\boldsymbol{r}\right) \psi^\dagger_{l\sigma} \psi_{l'\sigma} + \frac{1}{N} \int d^2r d\tau \left(\sum_{ll's\sigma} \left(g_{ijl} + g'_{ijl}\left(\boldsymbol{r}\right)\right) \psi^\dagger_{l\sigma} \psi_{l'\sigma} \phi_s. + h.c. \right)$$

potential scattering

position-independent
(does not break translation symmetry)

position-independent
(does not break translation symmetry)

I. Esterlis, H. Guo, A. A. Patel, and S. Sachdev, Phys. Rev. B **103**, 235129, **(**2021), A. A. Patel, H. Guo, I. Esterlis, S. Sachdev, Science 381, 790 (2023), C. Li, D. Valentinis, A. A. Patel, H. Guo, J. S., S. Sachdev, and I. Esterlis, PRL 133, 186502 (2024)

extreme limit: only spatially random interactions:

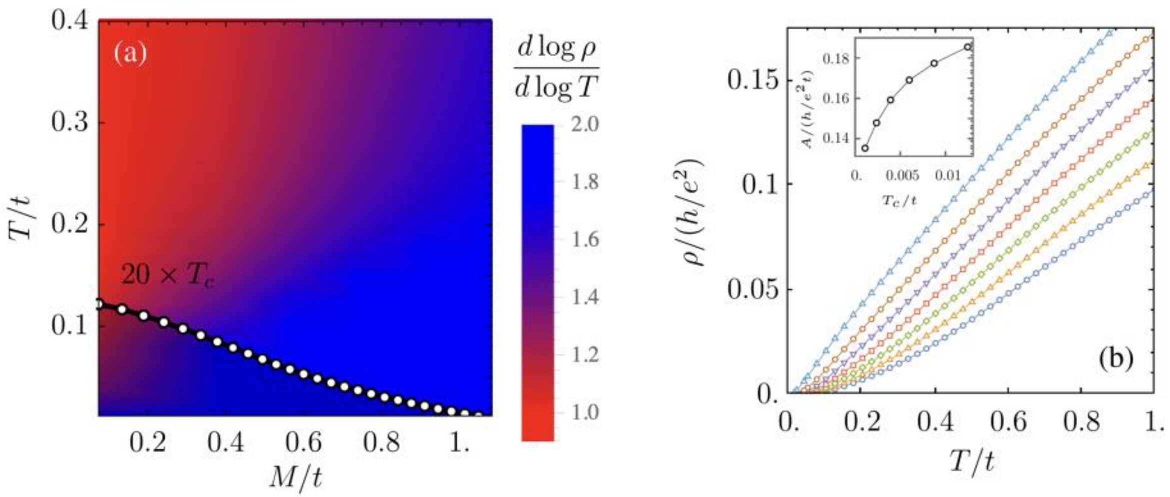
$$\Sigma(\omega) = g'^2 \int_{\omega'} G(\omega') D(\omega - \omega') + v^2 G(\omega)$$

$$\Pi(\omega) = -2g'^2 \int_{\omega'} G(\omega') G(\omega + \omega')$$

← momentum averaged propagators

$$G_{\boldsymbol{k}}(\omega) = \frac{1}{i\omega - \varepsilon_{\boldsymbol{k}} - \Sigma(\omega)}$$

$$D_{\boldsymbol{q}}(\omega) = \frac{1}{\omega^2 + \omega_{\boldsymbol{q}}^2 - \Pi(\omega)}$$



C. Li, D. Valentinis, A. A. Patel, H. Guo, J. S., S. Sachdev, and I. Esterlis, Phys. Rev. Lett. 133, 186502 (2024)

analytic analysis

$$\delta\Pi\left(\omega\right) = -8\pi^2\lambda\rho_F\left|\omega\right|$$

Landau damping

$$\Sigma\left(\omega\right) = -i\lambda\omega\log\frac{\Lambda^2}{m^2 + 8\pi^2\lambda\rho_F\left|\omega\right|}$$

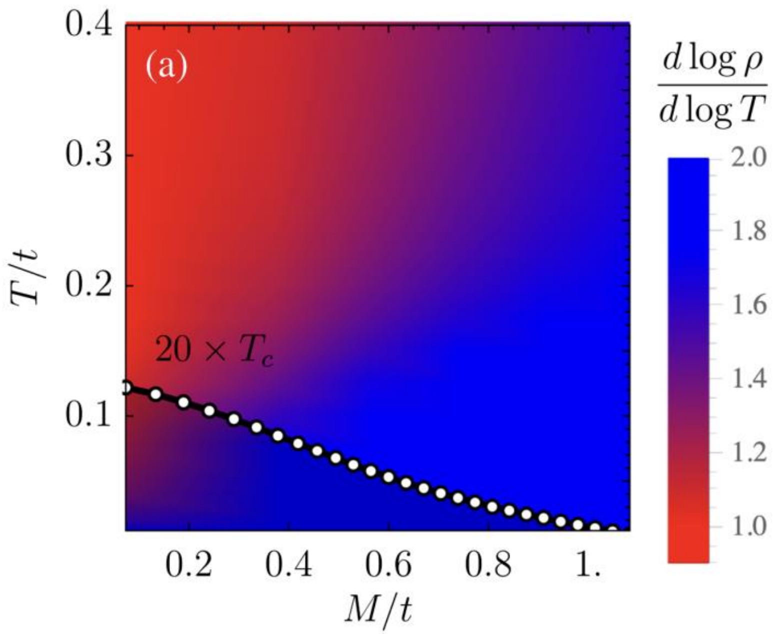
marginal Fermi liquid

C. M. Varma, et al.
PRL **63,** 1996 (1986)

$$\rho = \rho_0 + AT$$

strange metal behavior at a QCP

What is the gravity dual?



self energy is momentum dependent

$$ds^2 = \frac{1}{\zeta^2}\left(d\tau^2 + d\zeta^2\right) + d\boldsymbol{x}\cdot d\boldsymbol{x}$$

$$\mathrm{AdS}_2 \otimes \mathbb{R}_2$$

$$F\left(\frac{\tau_1+\tau_2}{2}, \frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}, \epsilon\right) = |\epsilon|^{\frac{\gamma-1}{2}} \int_\Gamma \psi\left(\tau, \frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}, \zeta\right) dl$$

holographic variable $\zeta \sim |\epsilon|^{-1}$

Generalized SYK models beyond the Gaussian saddle point

N. Bashan, E. Tulipman, J. S. and E. Berg, Phys. Rev. Lett. **132**, 236501 (2024)
E. Tulipman, N. Bashan, J. S. and E. Berg, Phys. Rev. B **110**, 155118 (2025)
N. Bashan, E. Tulipman, S. A. Kivelson, J. S., and E. Berg, arXiv:2502.08699

coupling of electrons electrons to pseudo spins

Karlsruhe Institute of Technology

$$H = \sum_{\boldsymbol{k},i<N} \varepsilon_{\boldsymbol{k}} c^{\dagger}_{i\boldsymbol{k}} c_{i\boldsymbol{k}} - \sum_{\boldsymbol{r},l<M} \boldsymbol{h}_{l,\boldsymbol{r}} \cdot \boldsymbol{\sigma}_{l,\boldsymbol{r}} + H_{\text{int}}$$

$$H_{\text{int}} = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{r},ij<N} \left[V_{ij,\boldsymbol{r}} c^{\dagger}_{i\boldsymbol{r}} c_{j\boldsymbol{r}} + \frac{1}{\sqrt{N}} \sum_{l=1}^{M} \boldsymbol{g}_{ijl,\boldsymbol{r}} \cdot \boldsymbol{\sigma}_{l,\boldsymbol{r}} c^{\dagger}_{i\boldsymbol{r}} c_{j\boldsymbol{r}} \right]$$

random couplings $\boldsymbol{g} = (g^x, g^y, g^z)$

$$\overline{g^a_{ijl,\boldsymbol{r}}} = 0$$

$$\overline{g^a_{ijl,\boldsymbol{r}} g^b_{ijl,\boldsymbol{r}'}} = \delta_{a,b} \delta(\boldsymbol{r} - \boldsymbol{r}') g_a^2$$

random TLS splittings $\boldsymbol{h} = (h^x, h^y, h^z)$

$$\boldsymbol{h}_{l,\boldsymbol{r}} \in \mathcal{P}(\boldsymbol{h})$$

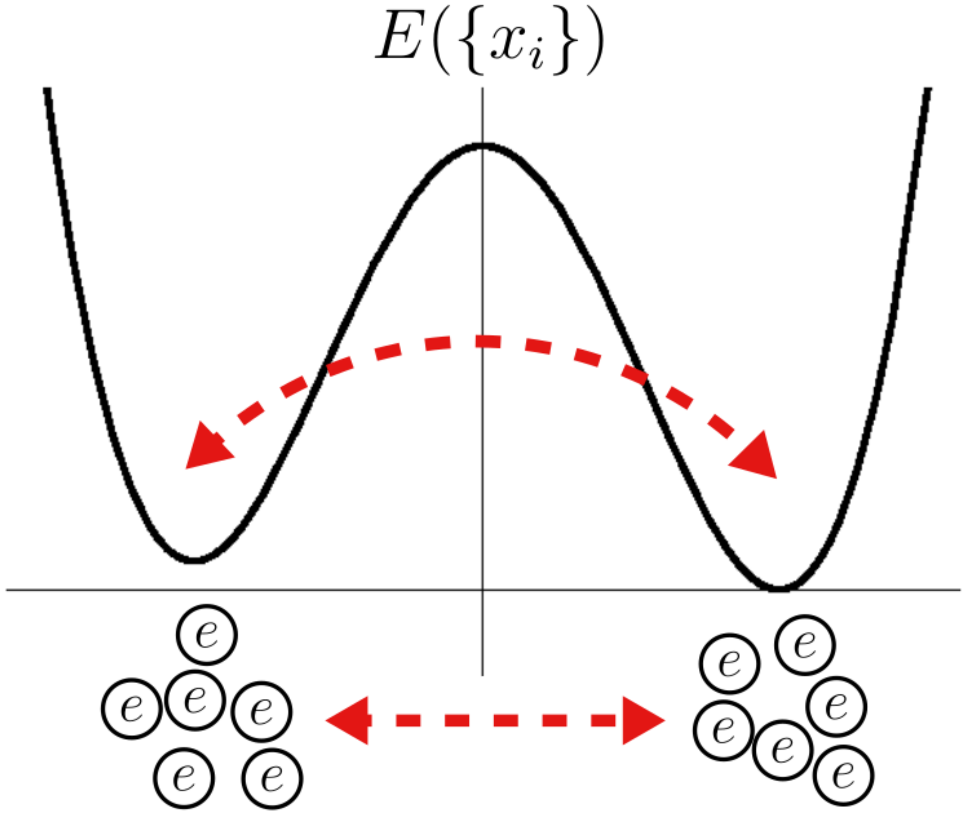
two level systems in glasses

low-energy behavior in a glass: tunneling between nearly degenerate configurations.

$$H_{\mathrm{TLS}} = \sum_{\boldsymbol{r}} h_{\boldsymbol{r}} \sigma_{z,\boldsymbol{r}} \qquad \mathcal{P}(h) \propto h^{\beta}$$

$$(\beta = 0)$$

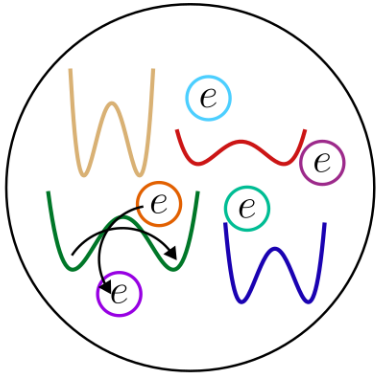
$$\Rightarrow C \approx \frac{\pi^2}{6} \mathcal{P}(0) T$$

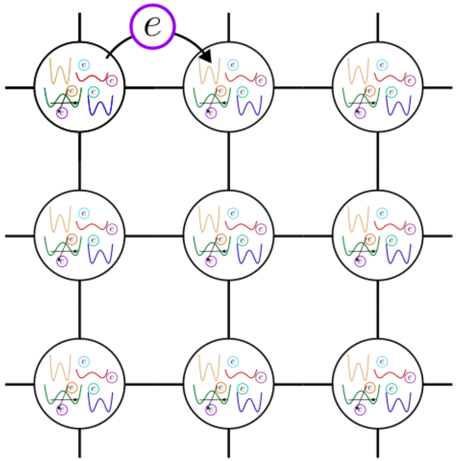


P. W. Anderson, B. I. Halperin, and C. M. Varma, Phil. Mag. **25**, 1 (1972),
W. A. Phillips, J. of Low Temp. Phys. **7**, 351 (1972)

KIT
Karlsruhe Institute of Technology

our model: strong coupling between TLS and electrons





$$H = \sum_{\boldsymbol{k}, i<N} \varepsilon_{\boldsymbol{k}} c^{\dagger}_{i\boldsymbol{k}} c_{i\boldsymbol{k}} - \sum_{\boldsymbol{r}, l<M} \boldsymbol{h}_{l,\boldsymbol{r}} \cdot \boldsymbol{\sigma}_{l,\boldsymbol{r}} + H_{\text{int}}$$

$$H_{\text{int}} = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{r}, ij<N} \left[V_{ij,\boldsymbol{r}} c^{\dagger}_{i\boldsymbol{r}} c_{j\boldsymbol{r}} + \frac{1}{\sqrt{N}} \sum_{l=1}^{M} \boldsymbol{g}_{ijl,\boldsymbol{r}} \cdot \boldsymbol{\sigma}_{l,\boldsymbol{r}} c^{\dagger}_{i\boldsymbol{r}} c_{j\boldsymbol{r}} \right]$$

random couplings $\boldsymbol{g} = (g, 0, 0)$

$$\overline{g_{ijl,\boldsymbol{r}}} = 0$$

$$\overline{g_{ijl,\boldsymbol{r}} g_{ijl,\boldsymbol{r}'}} = \delta(\boldsymbol{r} - \boldsymbol{r}') g^2$$

random TLS splittings $\boldsymbol{h} = (0, 0, h)$

$$h_{l,\boldsymbol{r}} \in \mathcal{P}(h) \sim h^{\beta}$$

$$\beta = 0 \text{ or } 1 \qquad h < h_c \ll E_F$$

similar in spirit to Yukawa-SYK models $\sigma^a_{l,\boldsymbol{r}} \to \phi_{l,\boldsymbol{r}}$ critical boson

0+1 dimensions

I. Esterlis and J.S., Phys. Rev. B **100**, 115132 (2019)
Y. Wang, PRL **124**, 017002 (2020)

2+1 dimensions

I. Esterlis, H. Guo, A. A. Patel, and S. Sachdev, PRB 103, 235129 (2021)
J. Kim, E. Altman, and X. Cao, Phys. Rev. B 103, L081113 (2021)
A. A. Patel, H. Guo, I. Esterlis, S. Sachdev, Science 381, 790 (2023)

important distinction: no Wick theorem for TLSs / spins
→ non-Gaussian saddle point
→ no limitation on space dimension, except d>1

clean system with related philosophy (non-Gaussian + effectively local)
Q Si, S Rabello, K Ingersent, JL Smith, Nature 413 (2001)

Large N, M limit

1. average over coupling constants and solve for fixed level splitting configuration

fermions: $$\Sigma_{\boldsymbol{r},\boldsymbol{r}'}\left(\tau\right)=\delta_{\boldsymbol{r},\boldsymbol{r}'}g^{2}G_{\boldsymbol{r},\boldsymbol{r}}\left(\tau\right)\chi_{\boldsymbol{r}}\left(\tau\right)$$ like in Migdal-Eliashberg

$$G_{\boldsymbol{r},\boldsymbol{r}'}\left(\omega\right)=\left(G_{0}^{-1}\left(\omega\right)-\Sigma\left(\omega\right)\right)^{-1}\Big|_{\boldsymbol{r},\boldsymbol{r}'}$$

TLS: $$\chi_{\boldsymbol{r}}\left(\tau\right)=\frac{1}{M}\sum_{l=1}^{M}\left\langle \sigma_{l,\boldsymbol{r}}^{x}\left(\tau\right)\sigma_{l,\boldsymbol{r}}^{x}\left(0\right)\right\rangle$$

spin-boson problem

$$S_{\mathrm{TLS}}\left[\boldsymbol{\sigma}_{l,\boldsymbol{r}}\right]=-\int d\tau h_{l,\boldsymbol{r}}\sigma_{l,\boldsymbol{r}}^{z}-\int d\tau d\tau'\Pi_{\boldsymbol{r}}\left(\tau'-\tau\right)\sigma_{l,\boldsymbol{r}}^{x}\left(\tau\right)\sigma_{l,\boldsymbol{r}}^{x}\left(\tau'\right)$$

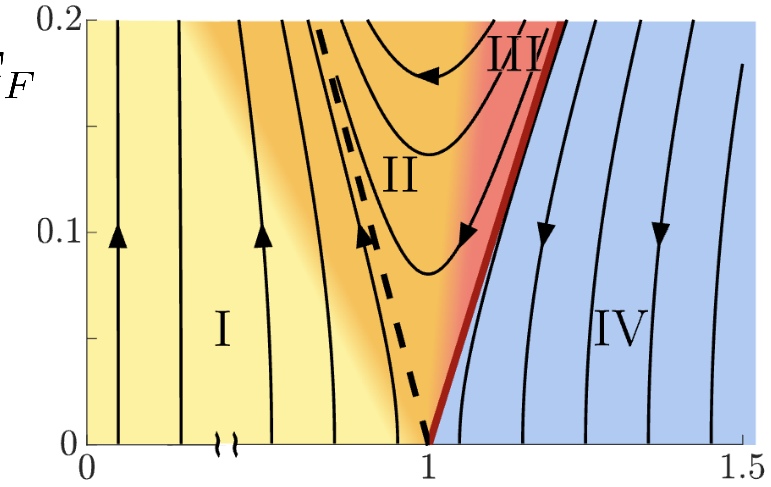
$$\Pi_{\boldsymbol{r}}\left(\tau\right)=-g^{2}G_{\boldsymbol{r},\boldsymbol{r}}\left(\tau\right)G_{\boldsymbol{r},\boldsymbol{r}}\left(-\tau\right)$$ like in Migdal-Eliashberg

solution of the spin-bose model

$$\frac{d\alpha}{d\ell} = -\alpha\tilde{h}^2 \qquad \frac{d\tilde{h}}{d\ell} = (1-\alpha)\tilde{h}$$

BKT-flow of the ohmic spin bose model

A. J. Leggett et al. Rev. Mod. Phys. **59**, 1 (1987)
U. Weiss, Quantum Dissipative Systems (1993)



$$h \to h_R \sim E_F \times \begin{cases} \tilde{h}^{\frac{1}{1-\alpha}} & \text{I} \\ e^{-b/\tilde{h}} & \text{II} \\ e^{-\pi/\sqrt{|\tilde{h}^2-(1-\alpha)^2|}} & \text{III} \\ 0 & \text{IV} \end{cases}$$

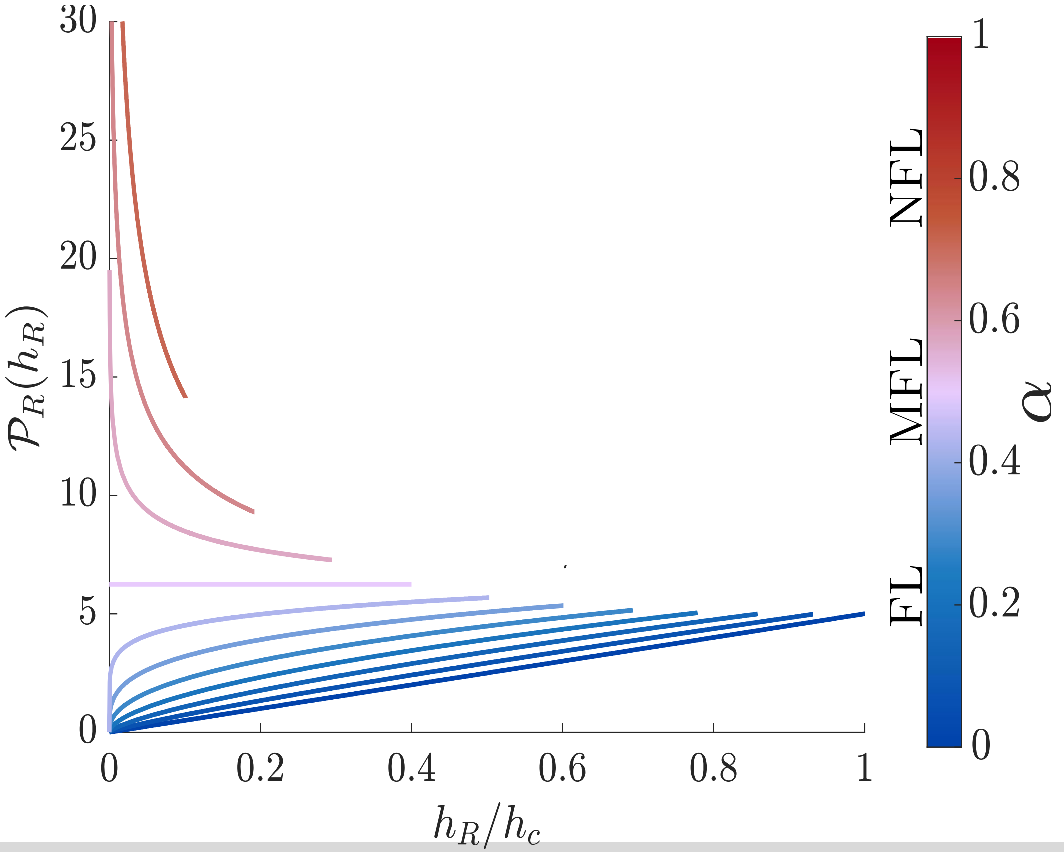
renormalized field

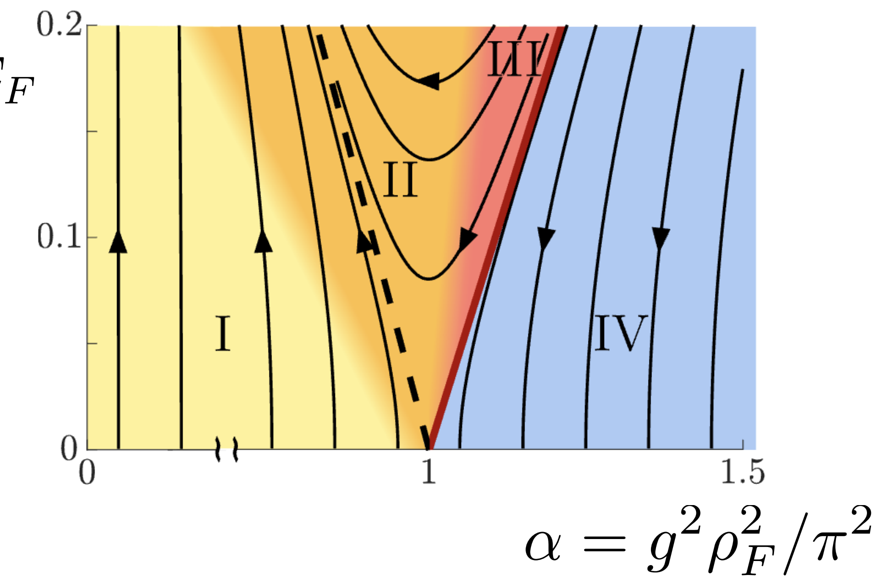
$$\chi''(\omega) = \frac{1}{\omega} f\left(\frac{\omega}{h_R}\right)$$

scaling form of the TLS propagator

solution of the spin-bose model

$$\frac{d\alpha}{d\ell} = -\alpha \tilde{h}^2 \qquad \frac{d\tilde{h}}{d\ell} = (1-\alpha)\tilde{h}$$





$$\mathcal{P}(h) \rightarrow \mathcal{P}_R(h_R) = \frac{dh_R}{dh}\mathcal{P}(h)$$

renormalized TLS distribution function

large N,M solution

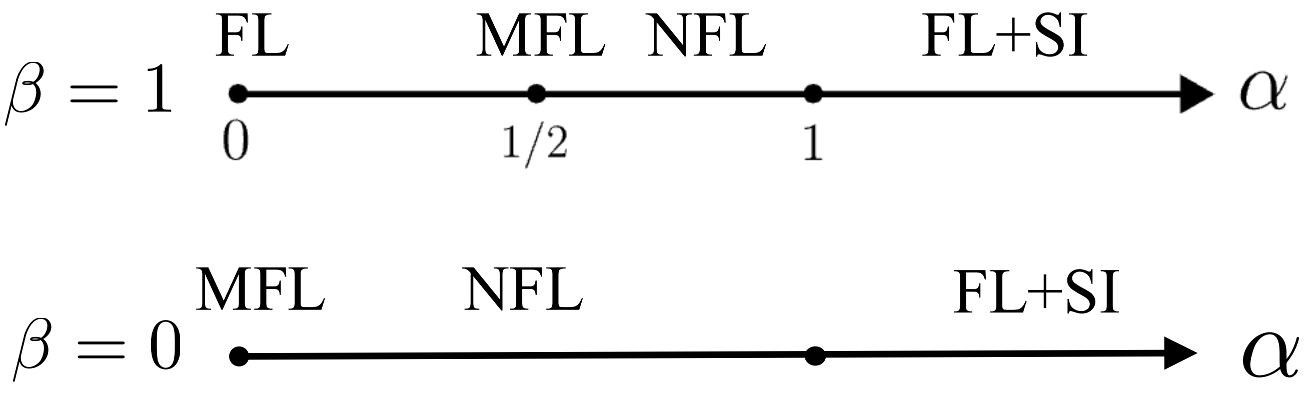
TLS-propagator

$$\mathrm{Im}\chi(\omega) = \int dh_r \mathcal{P}_r(h_r) \frac{1}{\omega} f(\omega/h_r) \sim \mathrm{sign}(\omega) \left| \frac{\omega}{h_{c,r}} \right|^{\gamma-1}$$

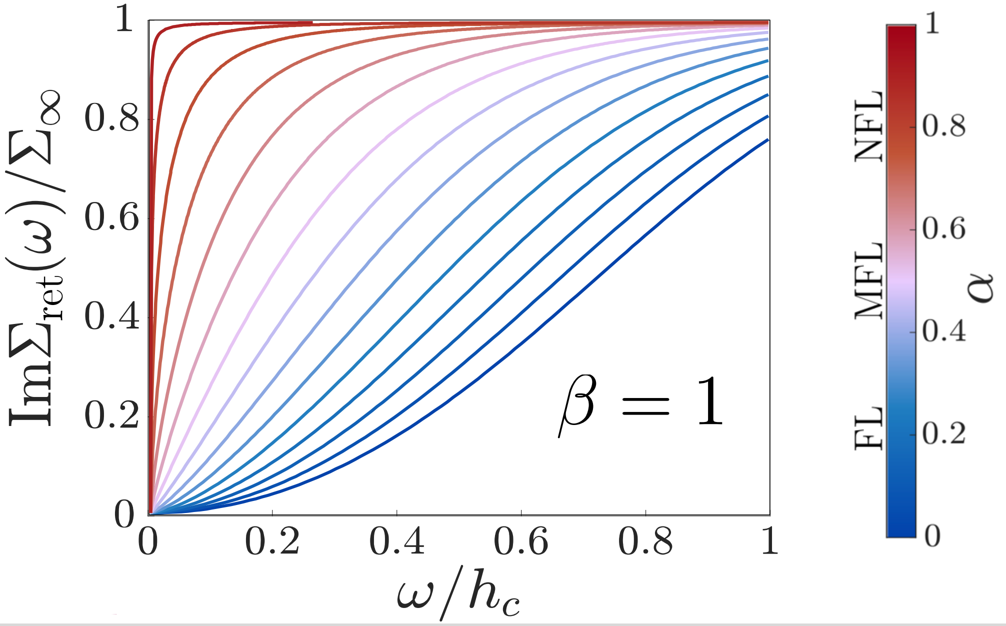
$$\gamma = (1+\beta)(1-\alpha)$$ coupling constant-dependent exponent

fermion self energy

$$\mathrm{Im}\Sigma(\omega) = -\rho_F g^2 \int_0^{|\omega|} d\omega' \mathrm{Im}\chi(\omega') \sim -\alpha E_F \left| \frac{\omega}{h_{c,r}} \right|^{\gamma}$$



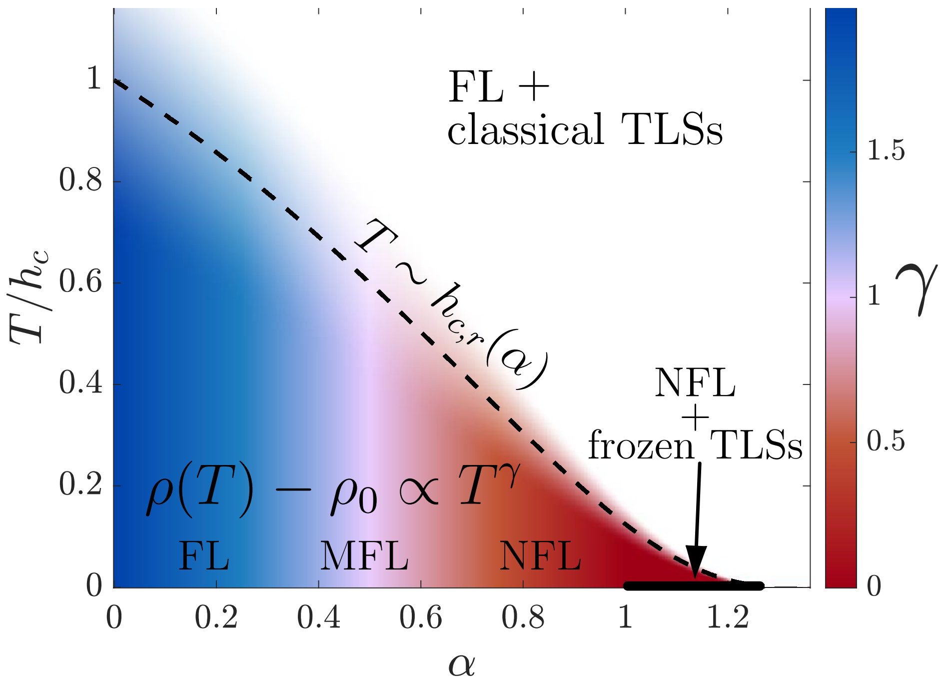
quantum critical phase!



implications

Phase diagram with critical phase and varying exponents

$\beta = 1$



transport $\rho(T \ll h_c) = \rho_0 + AT^\gamma,$

$$\gamma = (1+\beta)(1-\alpha)$$

consistent with weak coupling: $\rho(T) = \rho_0 + AT$

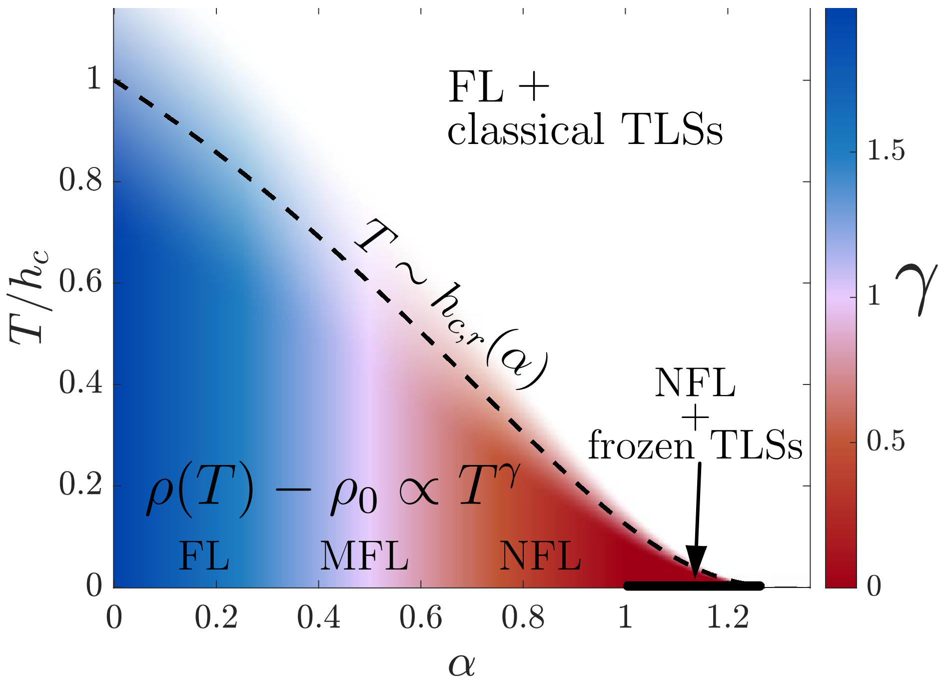
J. L. Black, B. L. Gyorffy, and J. Jäckle, Phil. Mag. B 40, 331 (1979)

$$\gamma(\beta = 0, \alpha \ll 1) \to 1$$

implications

Phase diagram with critical phase and varying exponents

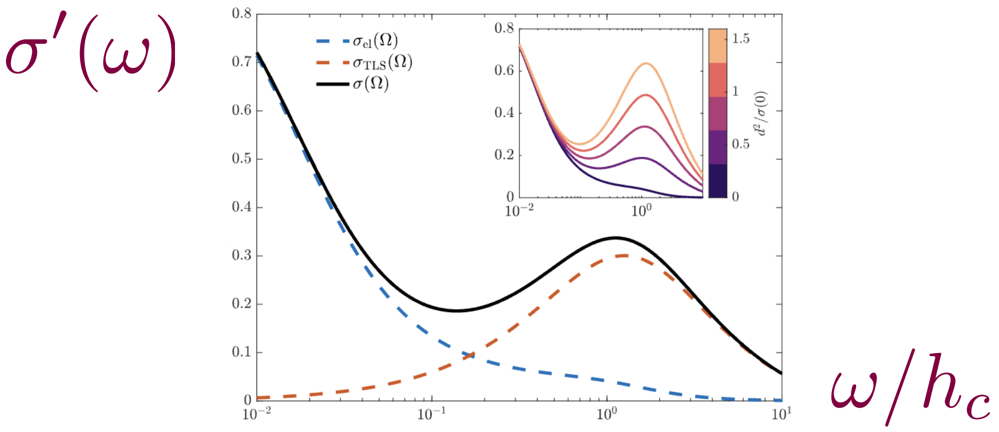
$\beta = 1$



transport $\rho(T \ll h_c) = \rho_0 + AT^\gamma,$

optics $\sigma'(\omega) = \frac{C}{\omega^\gamma} + B\omega Im\chi(\omega)$

$\sigma_{\text{el}}(\omega, T) = \omega^{-\gamma}\Phi\left(\frac{\omega}{T}\right)$

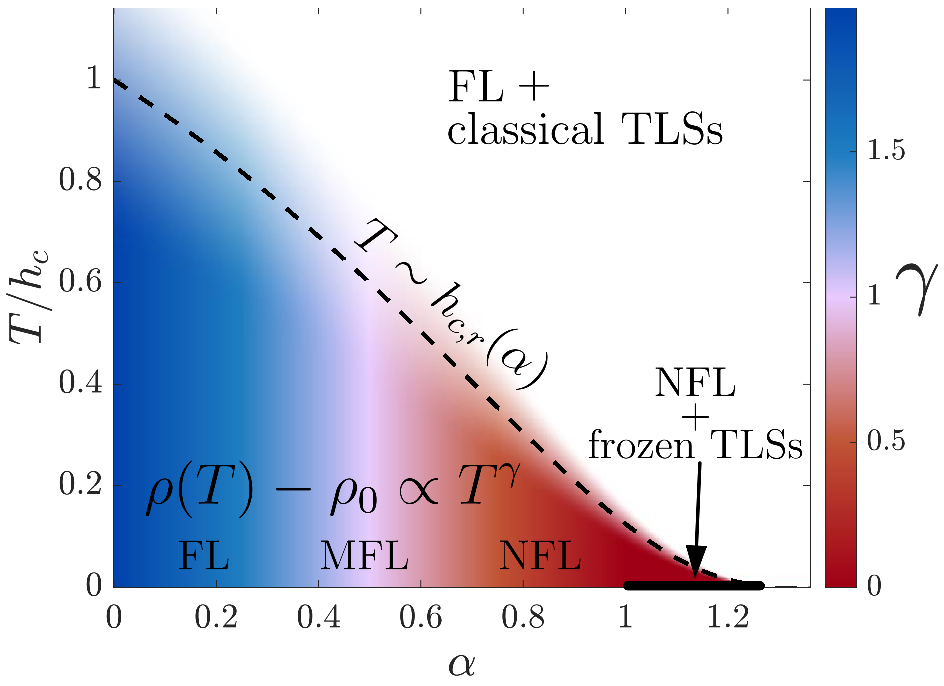


non-monotonic due to TLS dipole excitations

implications

Phase diagram with critical phase and varying exponents

$\beta = 1$



thermodynamics

$$C(T) \sim \begin{cases} T^\gamma & \text{if} \quad \gamma < 1 \\ T \log \frac{h_c}{T} & \text{if} \quad \gamma = 1 \\ T & \text{if} \quad \gamma > 1 \end{cases}$$