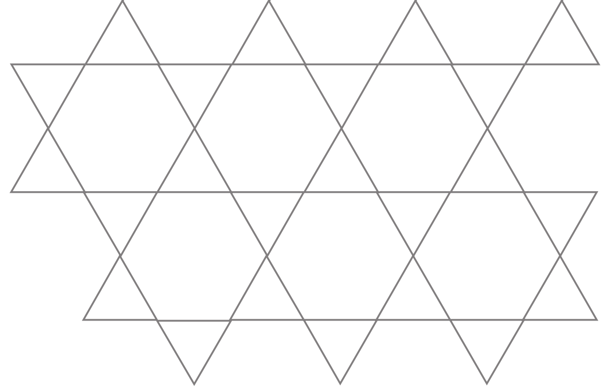

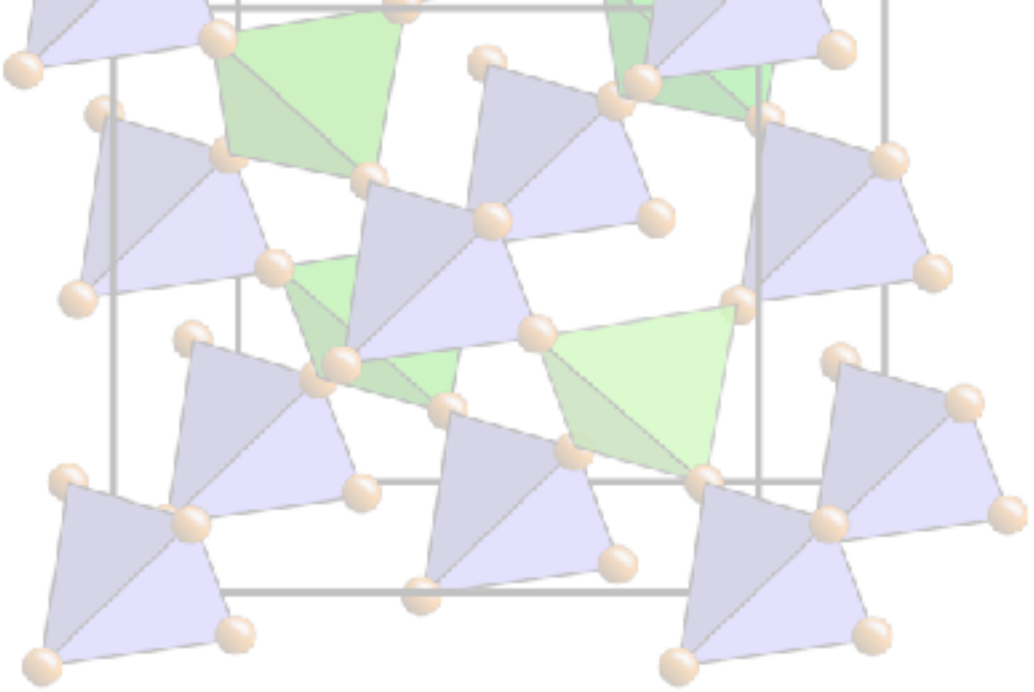

Quantum Spin Liquid and Quantum Spin Ice: Introduction II

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ICTP
August 8, 2025

NSERC
CRSNG

Center for Quantum Materials
UNIVERSITY OF TORONTO

Thermodynamic signatures in quantum spin liquids

Does spinon pairing exist ?

No → **U(1) Spin Liquid**

Emergent U(1) gauge field

- gapped spinon

  $C^{3D}_{\text{gauge}}(T) \sim T^3$

  (unstable in 2D)

- gapless spinon

  $C_{\text{dirac}}(T) \sim T^2$

  $C_{\text{FS}}(T) \sim T$ mean field

$C^{2D}_{FS} \sim T^{2/3}$ $C^{3D}_{FS} \propto T \ln T$	including fluctuations

Yes → **Z_2 Spin Liquid (BCS of spinons)**

U(1) gauge field gapped

(Meissner effect)

- gapped spinon

  $C(T) \sim e^{-\Delta/T}$

- gapless spinon

  $C_{\text{d-wave}}(T) \sim T^2$

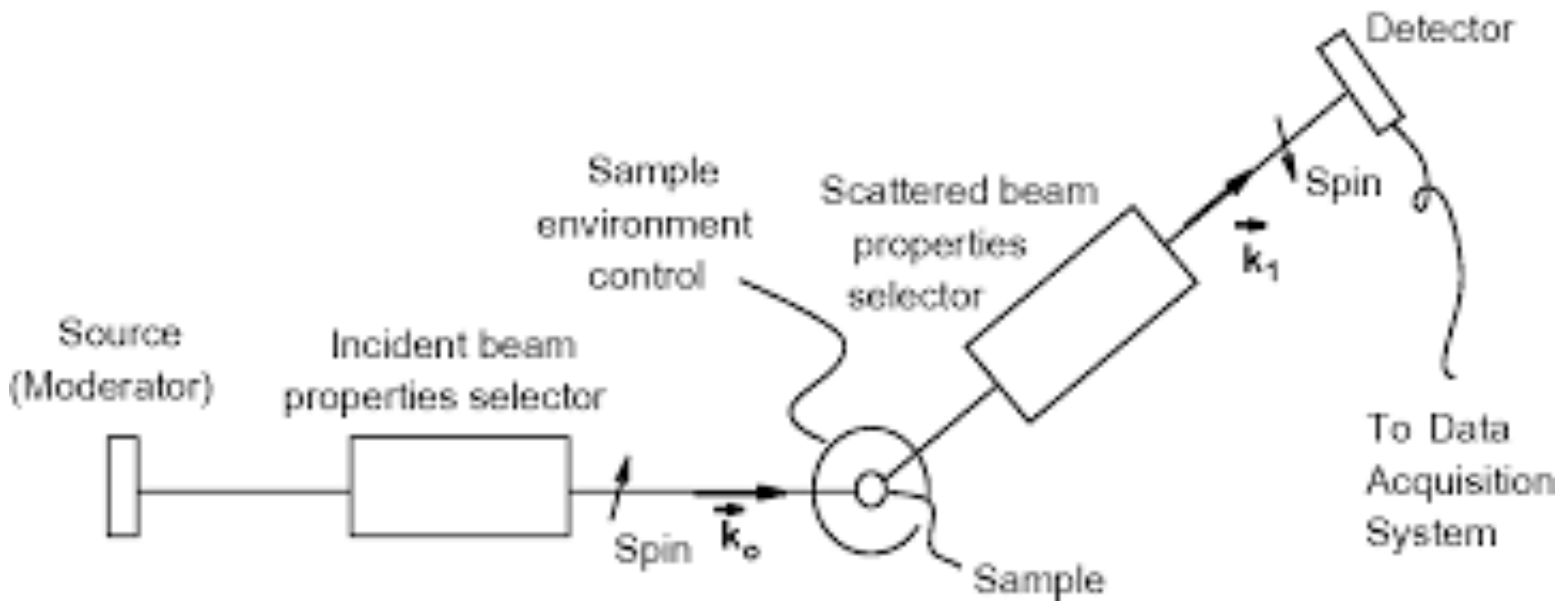
Neutron Scattering

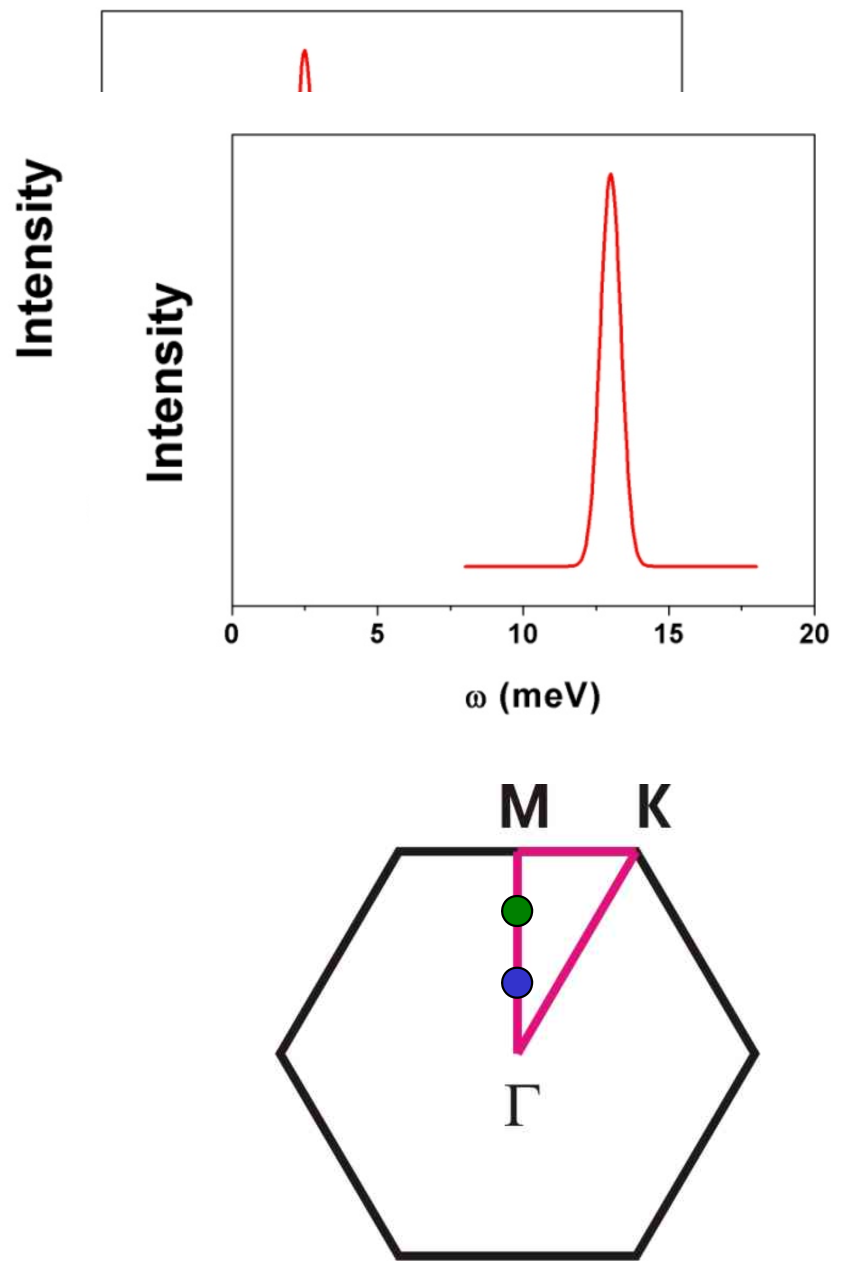
Time-dependent correlator

$$\sum_{\alpha=x,y,z} \langle S^\alpha(\mathbf{x},t) S^\alpha(\mathbf{x}=0,t=0)\rangle$$

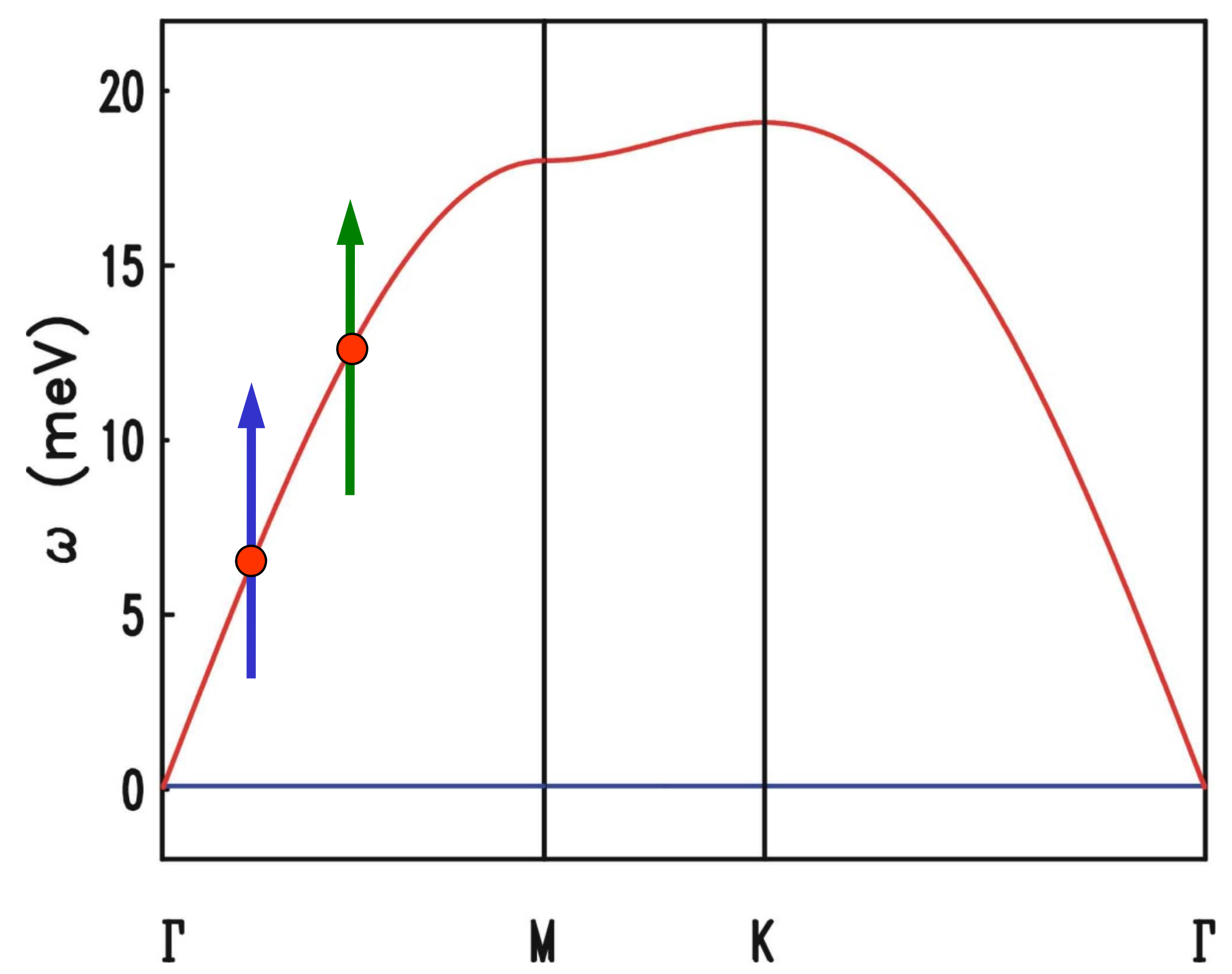
Dynamical spin structure factor (DSSF)

$$\longrightarrow \quad \mathcal{S}(\mathbf{q},\omega) \sim \int dt \int d^3\mathbf{x}\; e^{i\omega t}\; e^{i\mathbf{q}\cdot\mathbf{x}} \sum_{\alpha=x,y,z} \langle S^\alpha(\mathbf{x},t) S^\alpha(\mathbf{x}=0,t=0)\rangle$$





Magnons



How to detect excitations in quantum spin liquids ?

Neutron Scattering: Spin-1 excitations

Spinon-Antispinon pair excitations

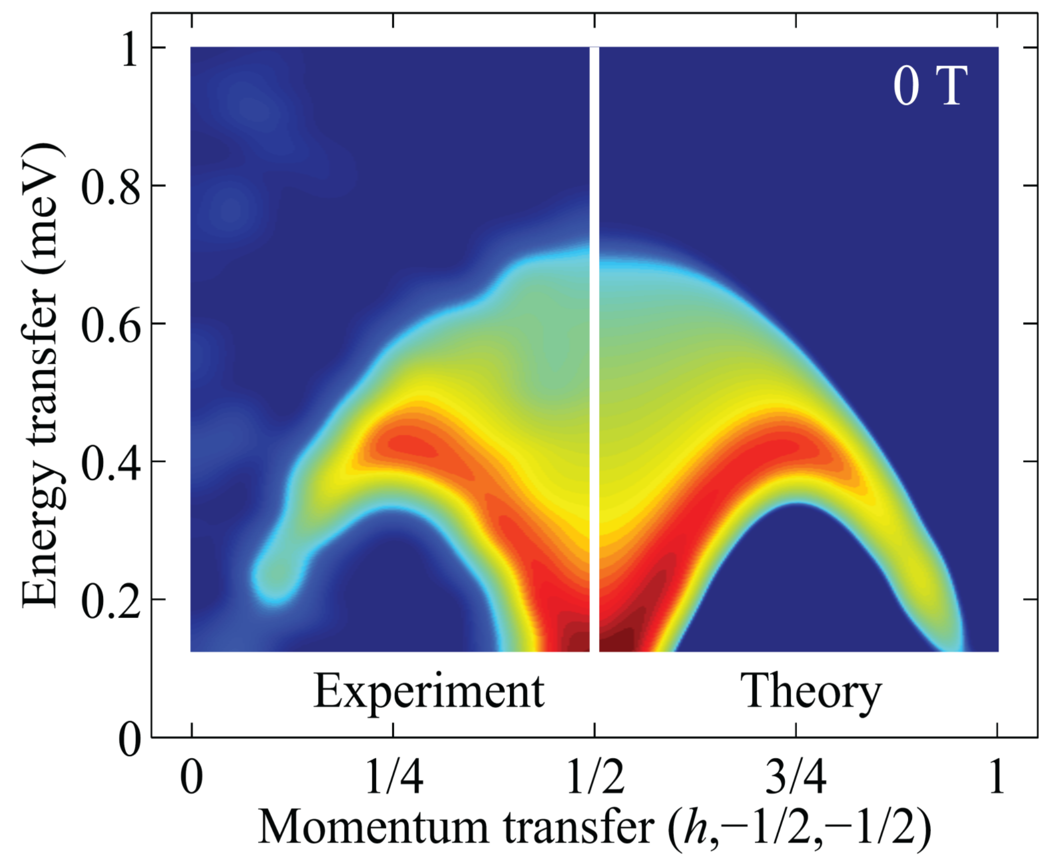
Scattering continuum

$$\omega_{\mathbf{q}} \sim \min \left[\varepsilon_{\frac{\mathbf{q}}{2}+\frac{\mathbf{p}}{2}} + \varepsilon_{\frac{\mathbf{q}}{2}-\frac{\mathbf{p}}{2}}\right]$$

for all possible $\mathbf{p}$

Well-defined dispersion -->
Threshold energy for pair excitations

$CuSO_4.5D_2O$

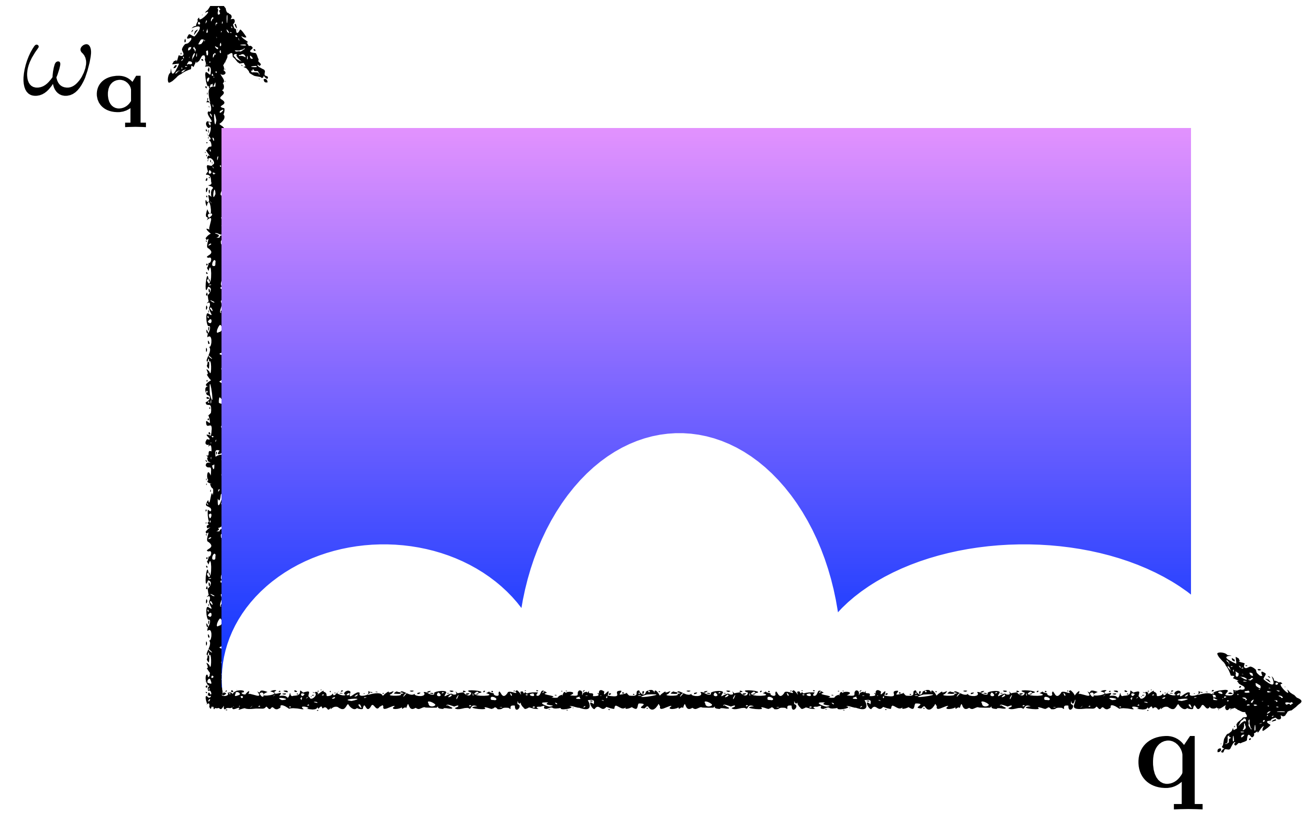


One dimensional spin liquid

Mourigal et al,
Nat. Phys. (2013)

Harder to confirm in 2D and 3D

Energy resolution may not be good enough



?

Spin flip excitations in Kitaev Model

Define bond fermions

$$\chi^{\alpha}_{\langle jk\rangle} = \frac{1}{2}\left(b^{\alpha}_j + i b^{\alpha}_k\right)$$

$$\chi^{\alpha\dagger}_{\langle jk\rangle} = \frac{1}{2}\left(b^{\alpha}_j - i b^{\alpha}_k\right)$$

Baskaran, Mandal, Shankar (2007)

$$\widetilde{H} = \sum_{\alpha\text{-bond}} i\hat{u}^{\alpha}_{jk} c_j c_k$$

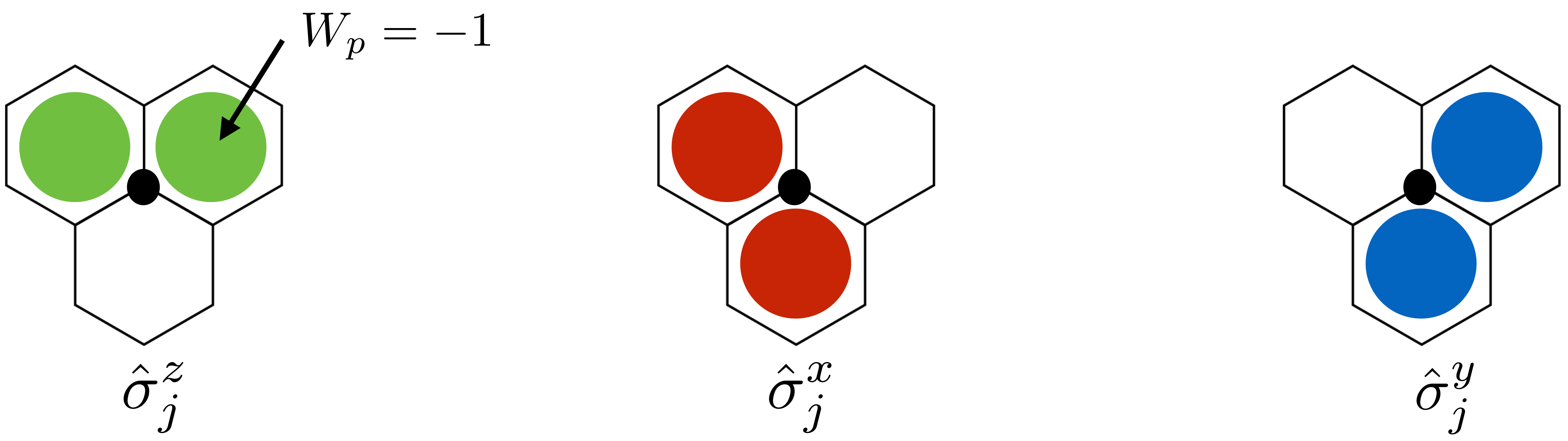
$$\hat{u}^{\alpha}_{jk} = 2\chi^{\dagger}_{\langle jk\rangle_\alpha}\chi_{\langle jk\rangle_\alpha} - 1$$

Ground state

$$\chi^{\dagger}_{\langle jk\rangle_\alpha}\chi_{\langle jk\rangle_\alpha} = 1 \qquad \hat{u}^{\alpha}_{jk} = +1$$

$$\chi^{\dagger}_{\langle jk\rangle_\alpha}\chi_{\langle jk\rangle_\alpha} = 1, 0 \longleftrightarrow \hat{u}^{\alpha}_{jk} = \pm 1$$

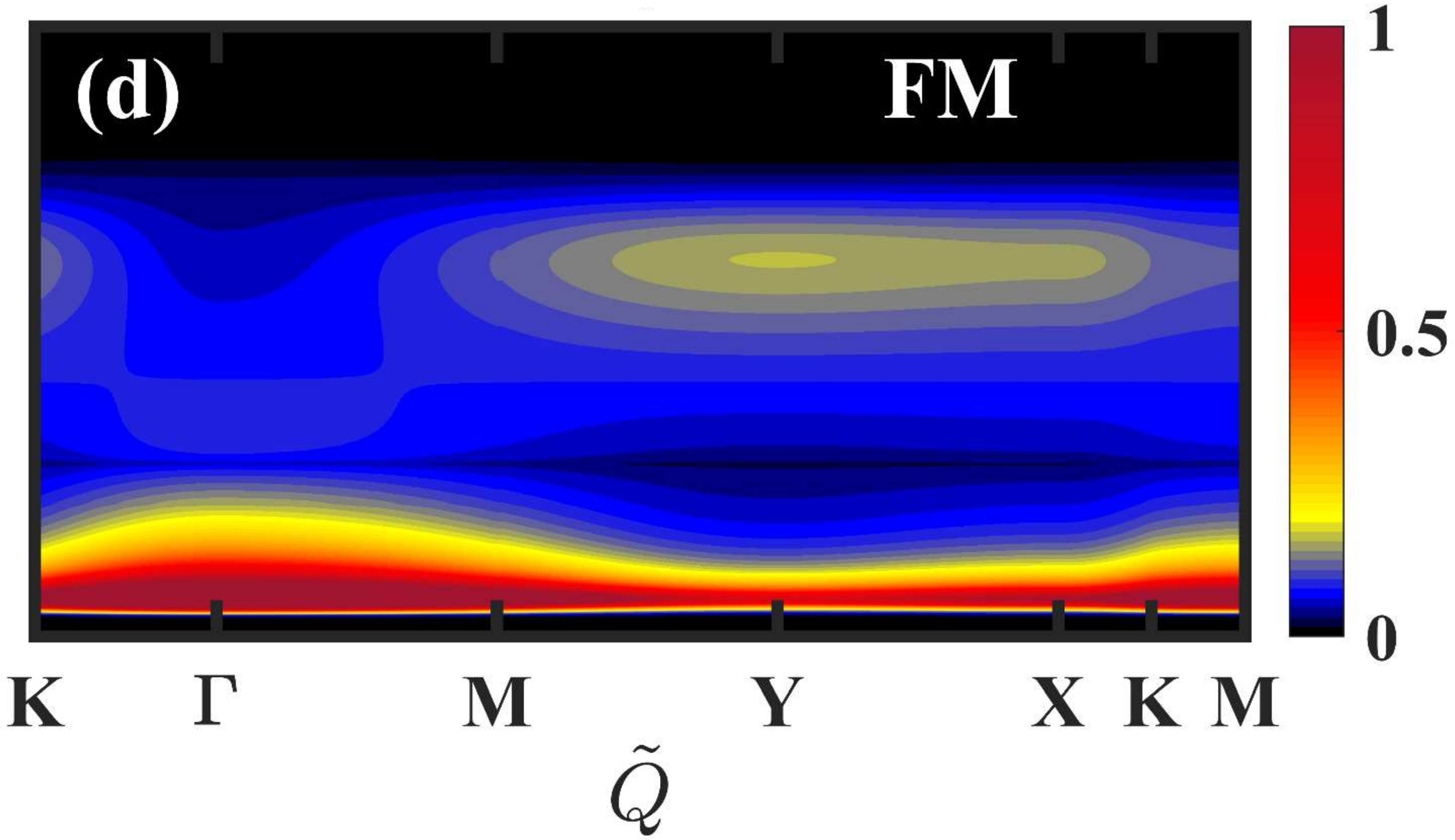
Single spin flip $\sigma^{\alpha}_j = i c_j \left(\chi^{\alpha}_{\langle jk\rangle} + \chi^{\alpha\dagger}_{\langle jk\rangle}\right)$ $\hat{u}^{\alpha}_{jk} \longrightarrow -\hat{u}^{\alpha}_{jk}$ ⟷ one fermion and two fluxes

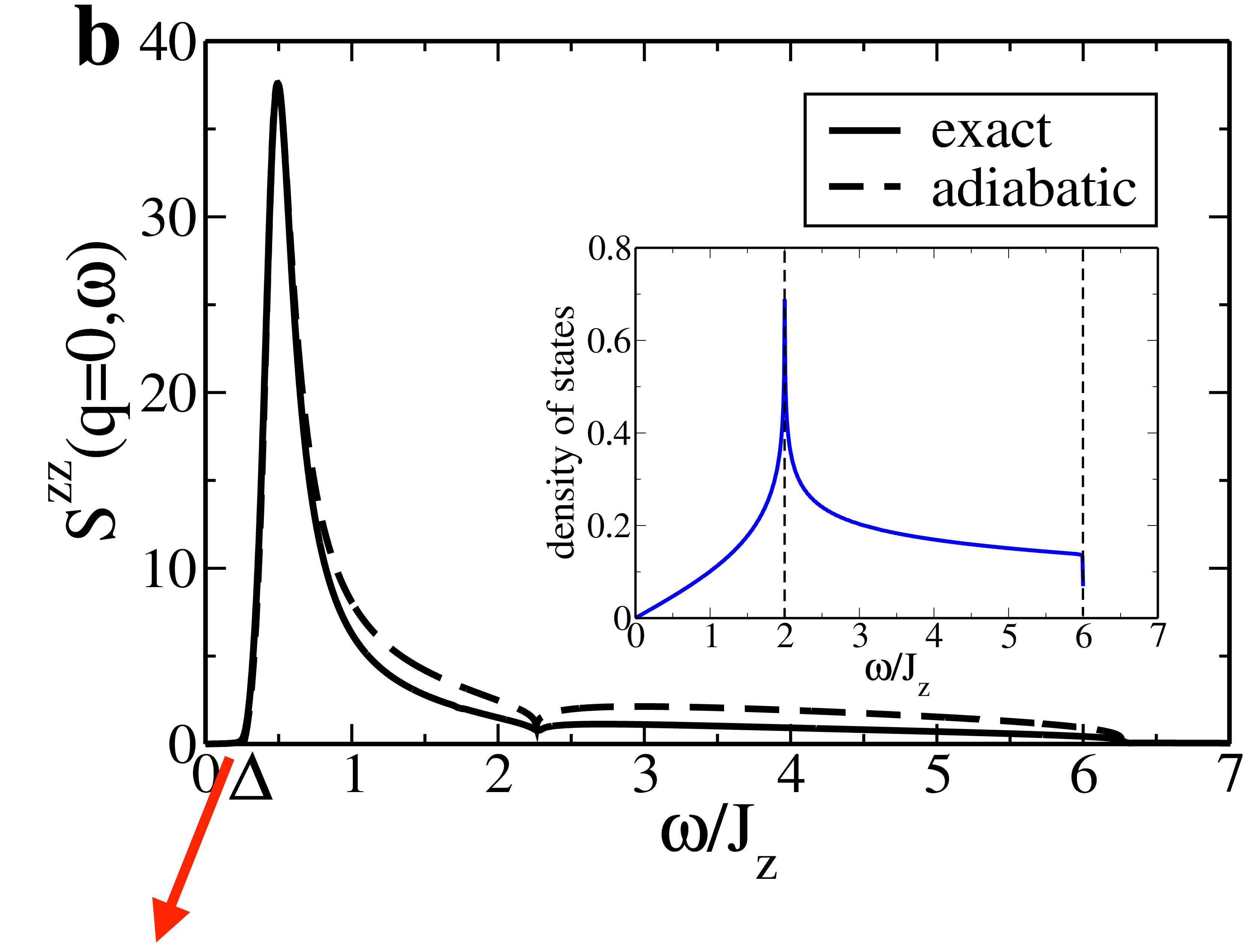


Dynamical Spin Structure Factor

$$S_{ij}^{ab}(t) = \langle 0|\hat{\sigma}_i^a(t)\hat{\sigma}_j^b(0)|0\rangle \xrightarrow{\text{F.T.}} S^{ab}(\mathbf{q},\omega)$$

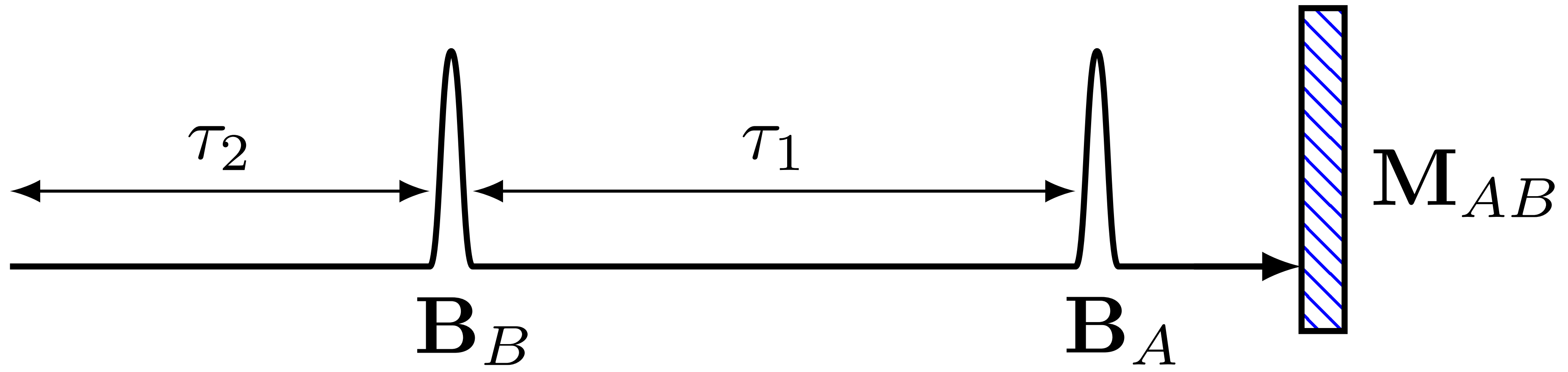
$$S(\mathbf{q},\omega) = \sum_a S^{aa}(\mathbf{q},\omega)$$





J. Knolle, D.L.Kovrizhin, J.T.Chalker, R. Moessner, PRL (2013)

2D nonlinear spectroscopy



$$\mathbf{B}(t) = \mathrm{B}_A^{\alpha} \hat{e}_{\alpha} \delta(t) + \mathrm{B}_B^{\beta} \hat{e}_{\beta} \delta(t - \tau_1)$$

Two pulses delayed by τ_1

Measure the nonlinear part of the transient magnetization

$$\mathbf{M}_{NL}(t) = \mathbf{M}_{AB}(t) - \mathbf{M}_A(t) - \mathbf{M}_B(t) \qquad t = \tau_1 + \tau_2$$

Nonlinear transient magnetization - nonlinear susceptibilities

$$H_{\text{tot}} = H - \mathbf{B}(t) \cdot \mathbf{M}(t)$$

$$M_{NL}^{\gamma}(\tau_1 + \tau_2)/N = \chi_{\alpha\beta}^{(2),\gamma}(\tau_2, \tau_1) B_A^{\alpha} B_B^{\beta} \quad \text{AB}$$

$$+ \chi_{\alpha\alpha\beta}^{(3),\gamma}(\tau_2, \tau_1, 0) B_A^{\alpha} B_A^{\alpha} B_B^{\beta} + \chi_{\alpha\beta\beta}^{(3),\gamma}(\tau_2, 0, \tau_1) B_A^{\alpha} B_B^{\beta} B_B^{\beta} + \mathcal{O}(B^4)$$

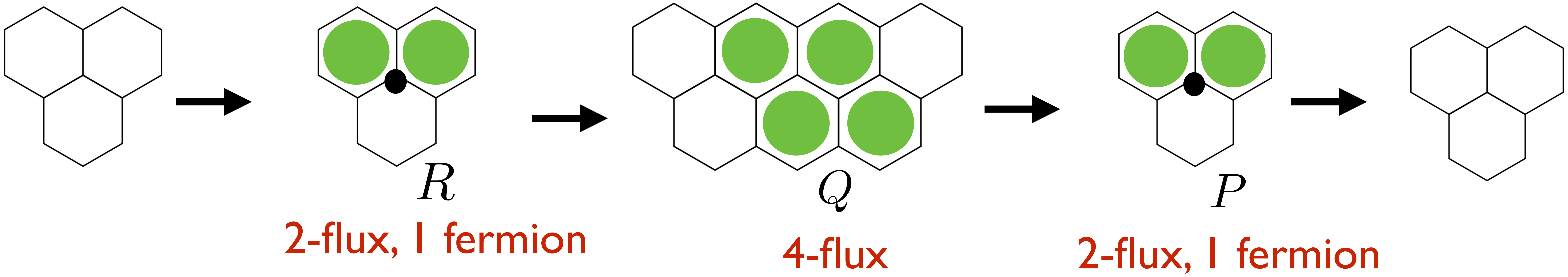
AAB ABB

$\alpha, \beta, \gamma = z, z, z$ case

3rd order susceptibility

$$R^{(1),z}_{zzz}(\tau_2, 0, \tau_1) = \sum_{jklm} \sum_{PQR} \langle 0|\hat{\sigma}^z_j|P\rangle\langle P|\hat{\sigma}^z_k|Q\rangle\langle Q|\hat{\sigma}^z_l|R\rangle\langle R|\hat{\sigma}^z_m|0\rangle e^{-i(E_R-E_0)\tau_2 - i(E_p-E_0)\tau_1}$$

ABB



$$(\omega_2, \omega_1) = (E_R - E_0, E_P - E_0) = ((E_2 - E_0) + \varepsilon_R, (E_2 - E_0) + \varepsilon_P)$$

$$E_R = E_2 + \varepsilon_R$$

$$E_P = E_2 + \varepsilon_P$$

E_2 Two flux energy

ε_R Fermion energy

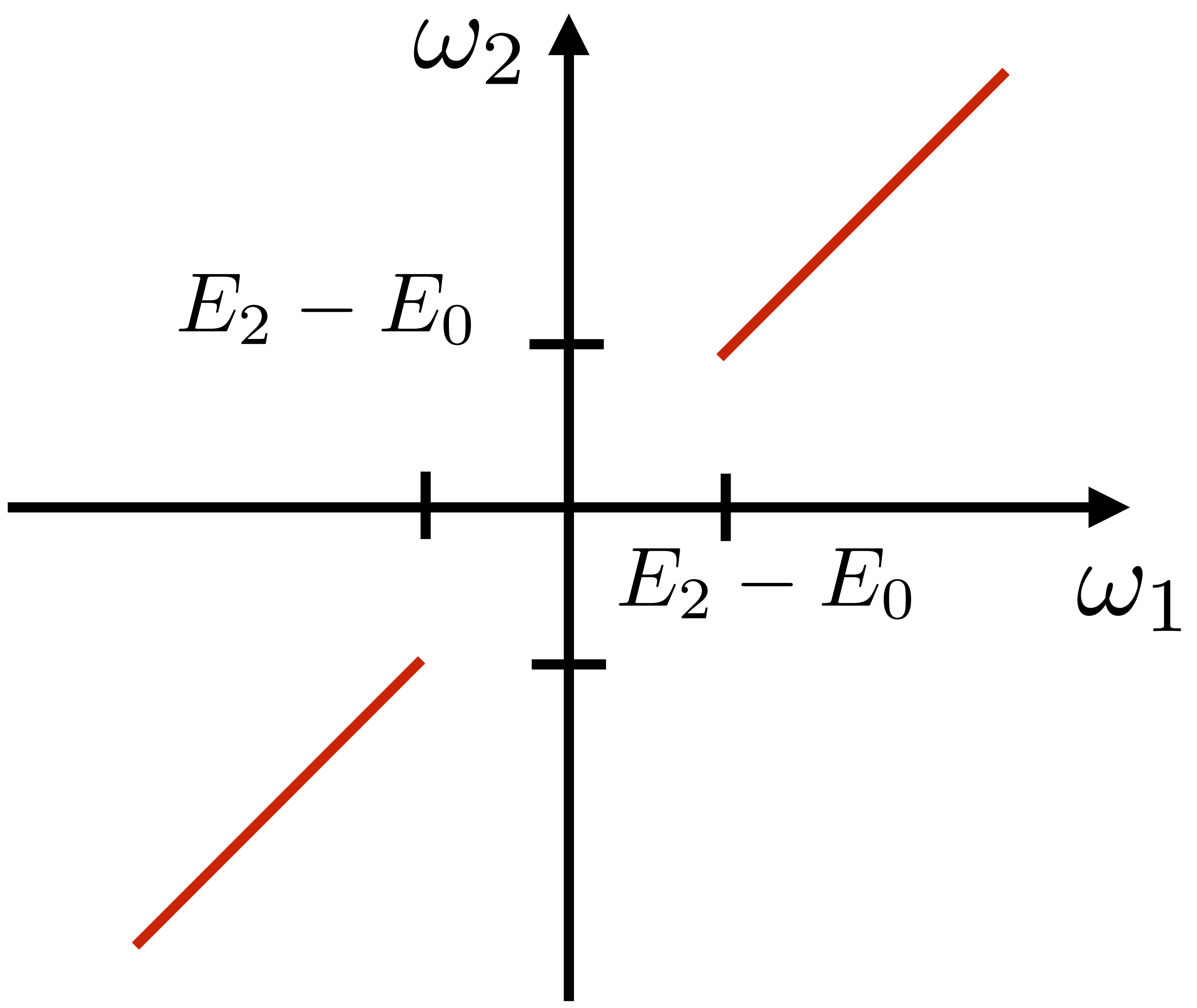
$\alpha, \beta, \gamma = z, z, z$ case

3rd order susceptibility

$$R^{(1),z}_{zzz}(\tau_2, 0, \tau_1) = \sum_{jklm} \sum_{PQR} \langle 0|\hat{\sigma}^z_j|P\rangle\langle P|\hat{\sigma}^z_k|Q\rangle\langle Q|\hat{\sigma}^z_l|R\rangle\langle R|\hat{\sigma}^z_m|0\rangle e^{-i(E_R-E_0)\tau_2-i(E_p-E_0)\tau_1}$$

ABB

$$(\omega_2, \omega_1) = (E_R - E_0, E_P - E_0) = ((E_2 - E_0) + \varepsilon_R, (E_2 - E_0) + \varepsilon_P)$$

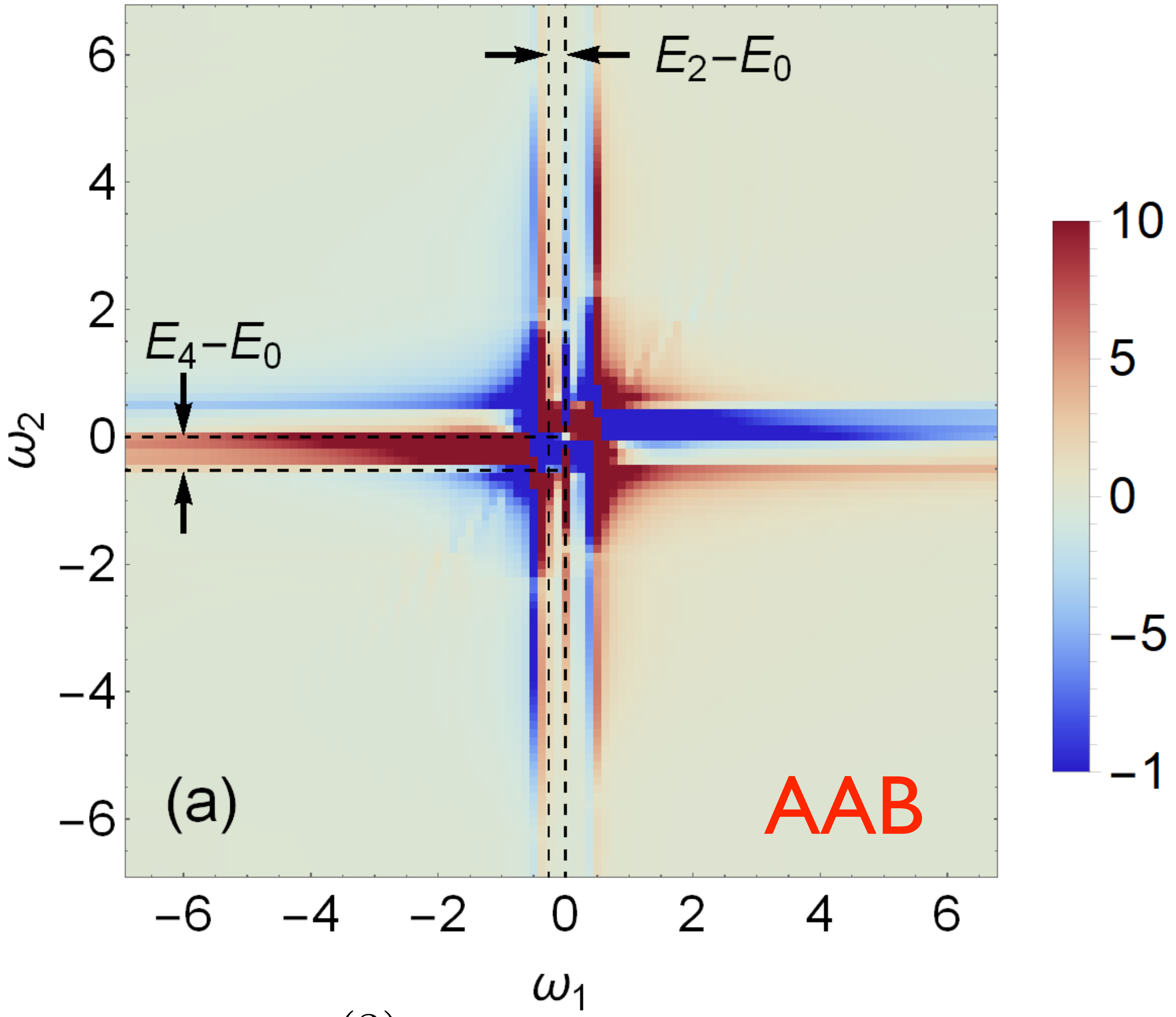


For $\varepsilon_R = \varepsilon_P$

$\omega_2 = \omega_1 \geq E_2 - E_0$

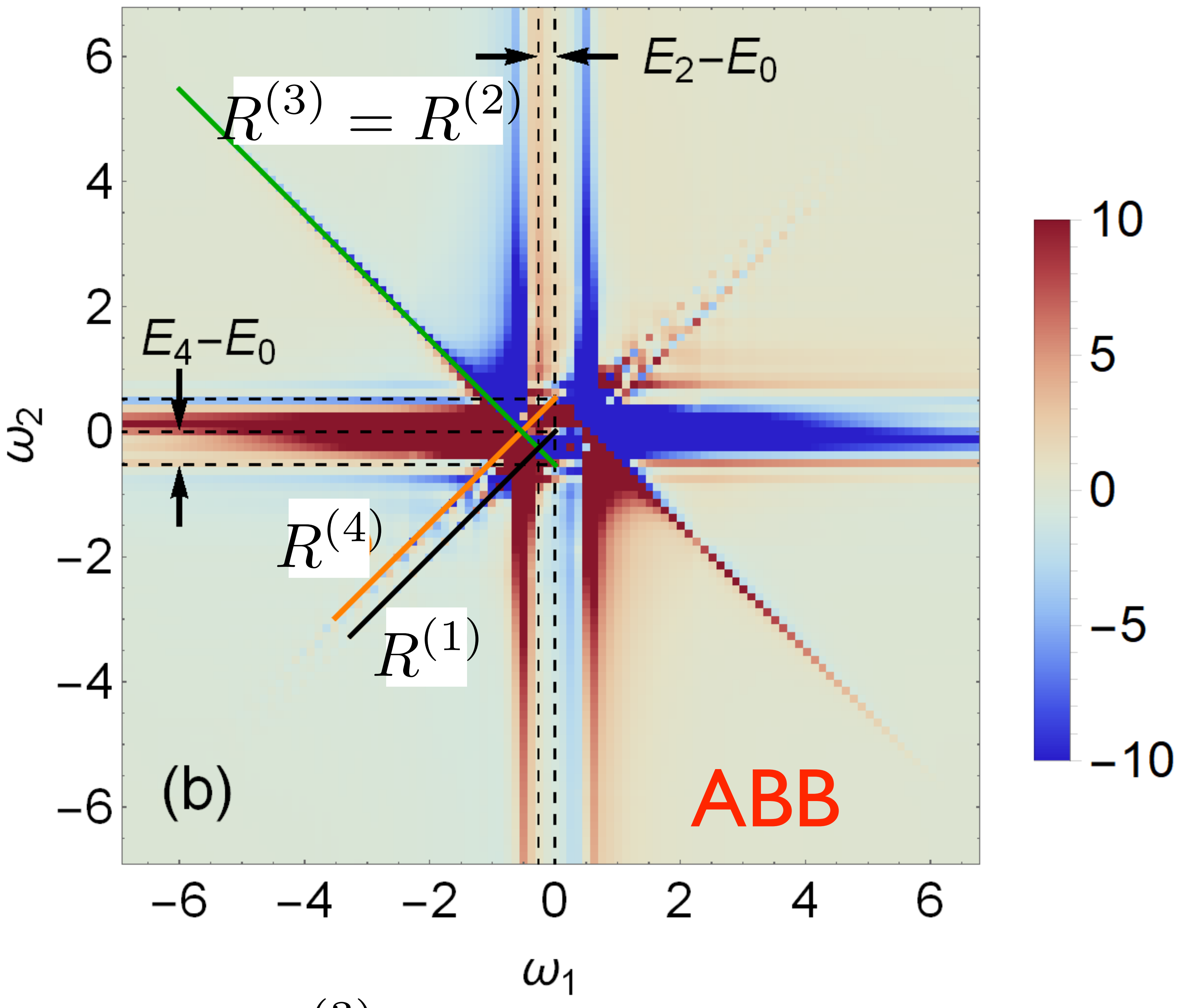
E_2 Two flux energy

2D Nonlinear susceptibility



$\chi^{(3),z}_{zzz}(\omega_2, \omega_1, 0)$

3rd order susceptibility



$\chi^{(3),z}_{zzz}(\omega_2, 0, \omega_1)$

$E_2 - E_0$ Two-flux gap

$E_4 - E_0$ Four-flux gap

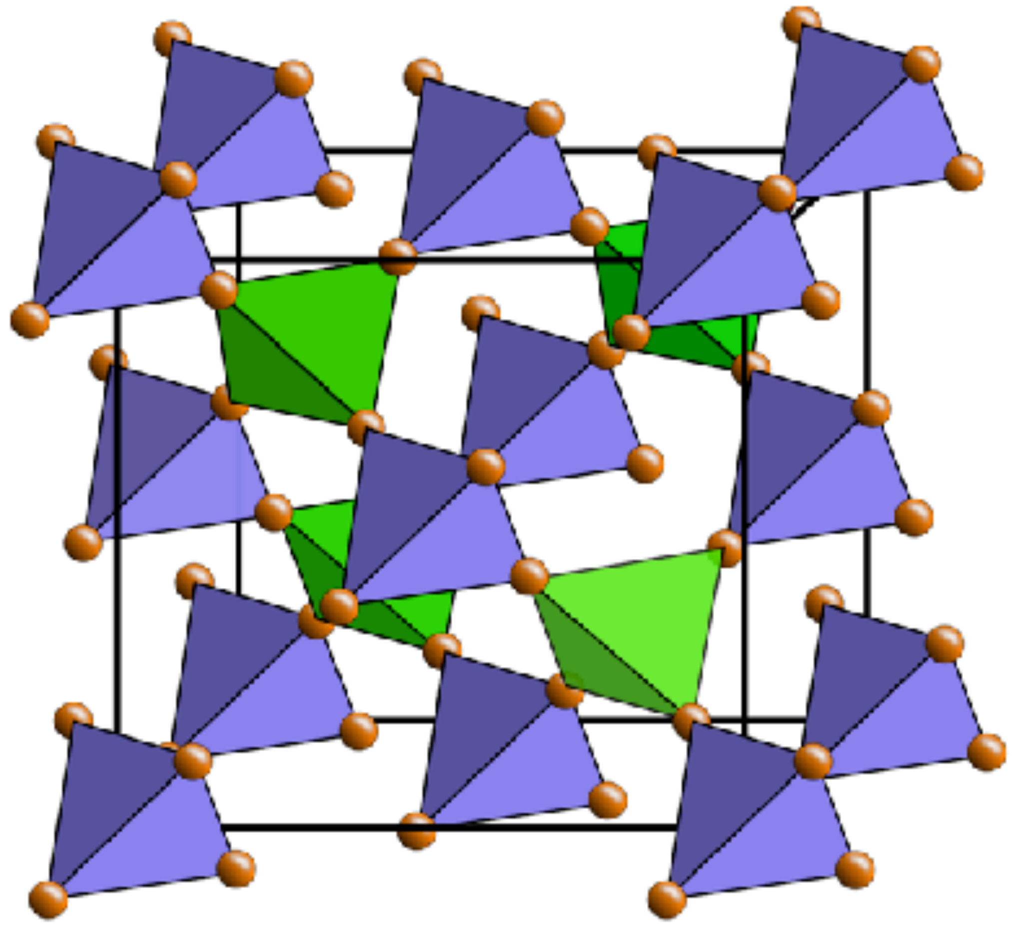
Wonjune Choi, Ki Hoon Lee, YBK, PRL 124, 117205, (2020)

Emily Z. Zhang, Ciaran Hickey, YBK, PRB 110, 104415, (2024)

Quantum Spin Ice:
a primer

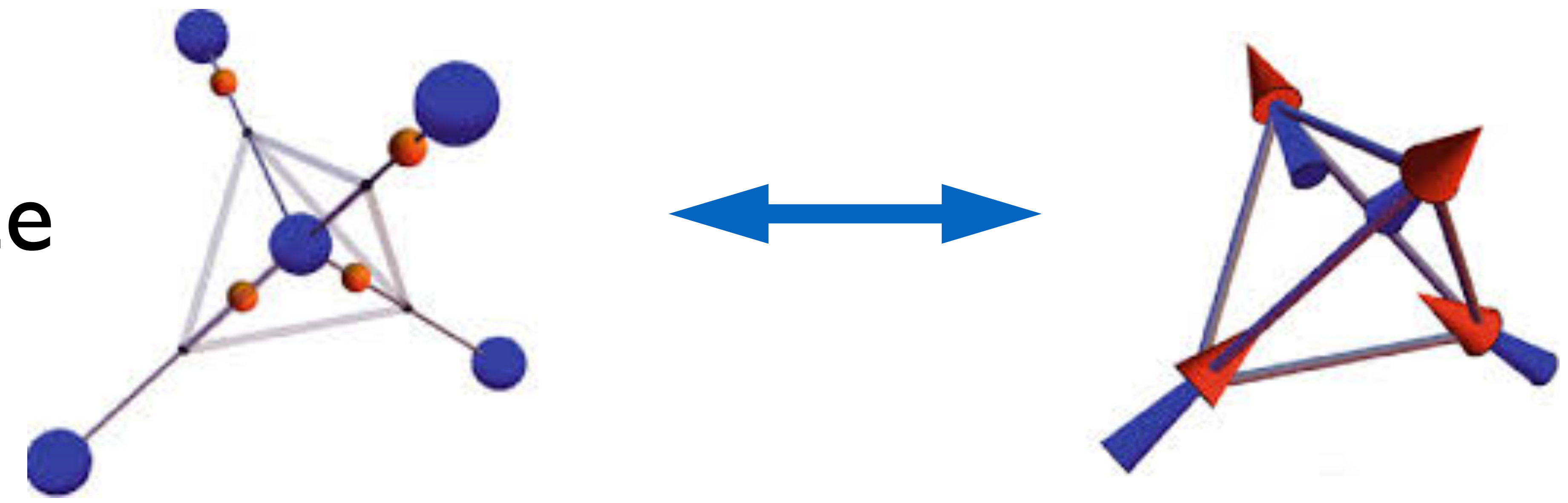
Classical Spin Ice

$$\mathcal{H}_I = J_z \sum_{\langle ij \rangle} S_i^z S_j^z = \frac{J_z}{2} \sum_{\triangleleft} (S_{\triangleleft}^z)^2$$ + constant

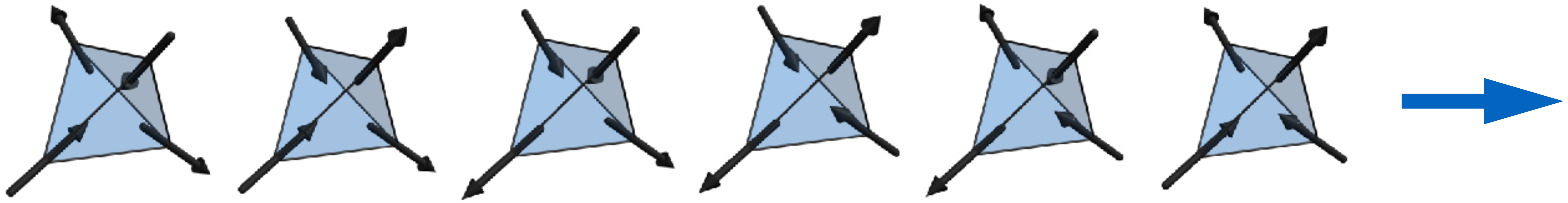


$$S_{\triangleleft}^z = \sum_{i \in \triangleleft} S_i^z = 0$$ 2-in/2-out

Water ice



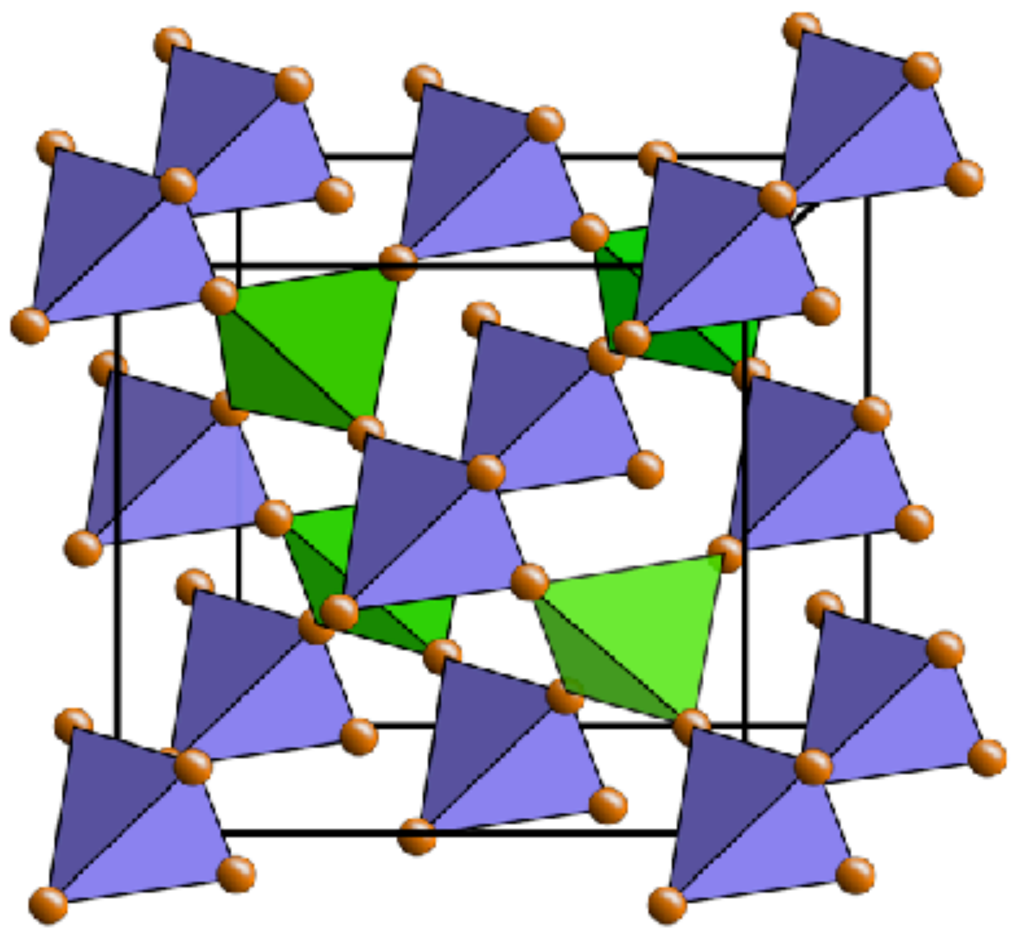
Spin ice



Massive degeneracy

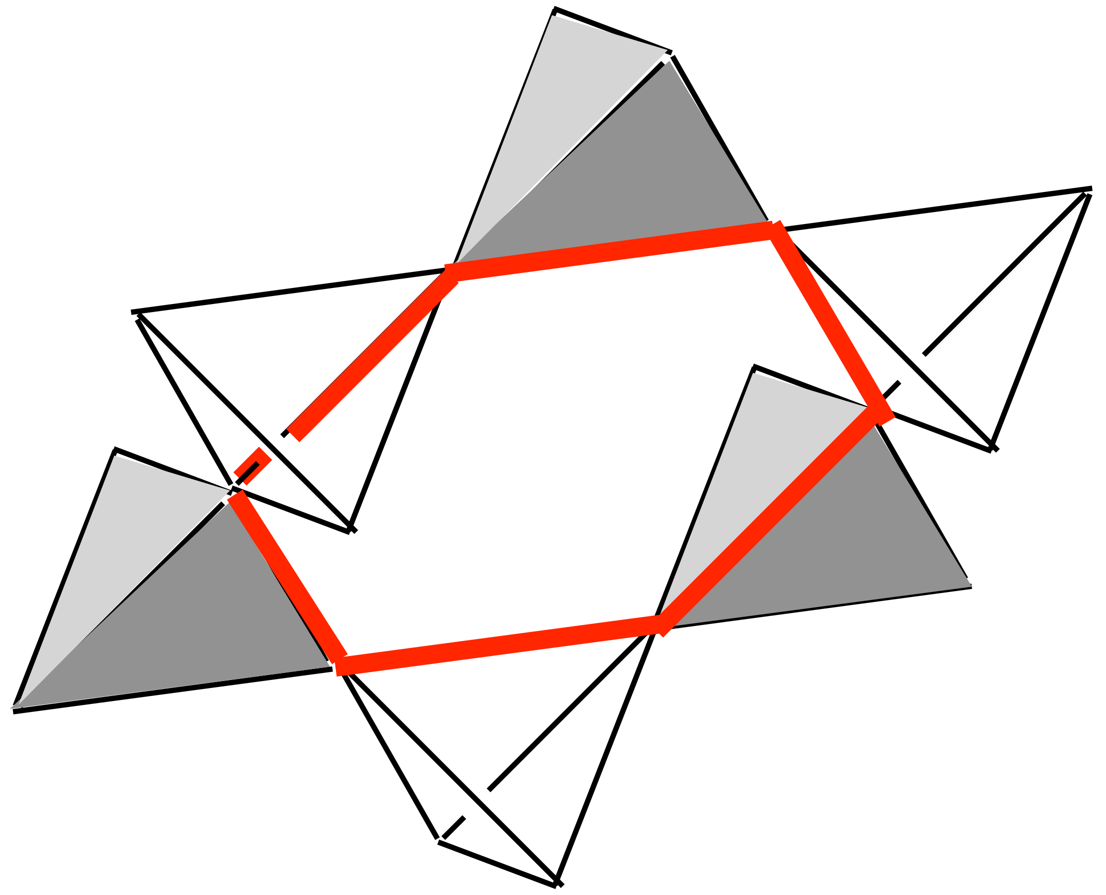
Quantum Spin Ice

$$\mathcal{H}_I = J_z \sum_{\langle ij \rangle} S_i^z S_j^z = \frac{J_z}{2} \sum_{\triangleleft} (S_{\triangleleft}^z)^2$$ + constant



$$S_{\triangleleft}^z = \sum_{i \in \triangleleft} S_i^z = 0$$ 2-in/2-out

$$\mathcal{H}' = -\frac{J_\perp}{2} \sum_{\langle ij \rangle} (S_i^+ S_j^- + h.c.)$$ Hermele, Balents, Fisher ’03



$J_z \gg J_\perp$ degenerate perturbation theory

$$\mathcal{H}_{eff} = -J_{ring} \sum_{\hexagon} (S_1^+ S_2^- S_3^+ S_4^- S_5^+ S_6^- + h.c.)$$

$$J_{ring} = 12 J_\perp^3 / J_z^2$$

Banerjee, Isakov, Damle, YBK ’08

Benton, Sikora, Shannon ’12

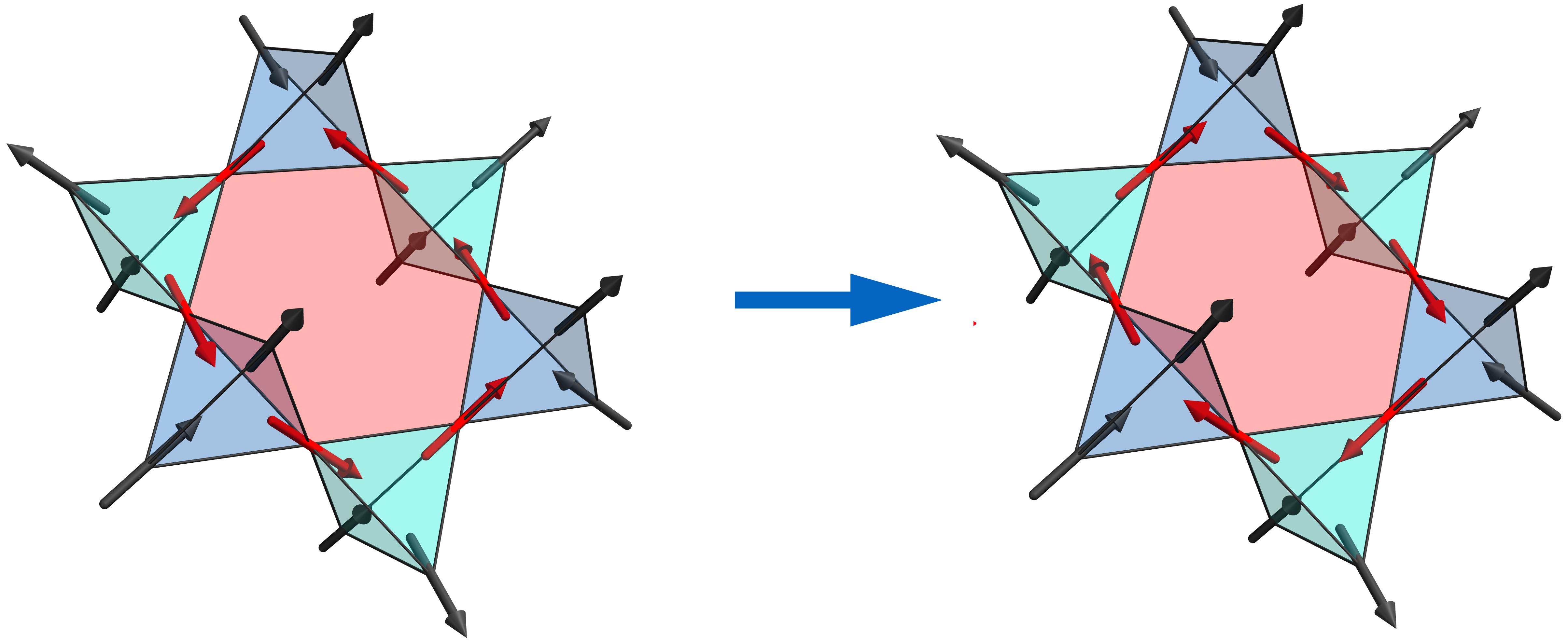
QMC

Connecting classically degenerate ground states

$$\mathcal{H}_{eff} = J_{ring} \sum_{\hexagon} (S_1^+ S_2^- S_3^+ S_4^- S_5^+ S_6^- + h.c.)$$

$$K \sim -J_{ring} \propto -J_\perp^3 / J_\parallel^2$$

Hermele, Balents, Fisher '03



Quantum ground state = massive superposition of classically degenerate states
=> Quantum Spin Liquid

Quantum Electrodynamics

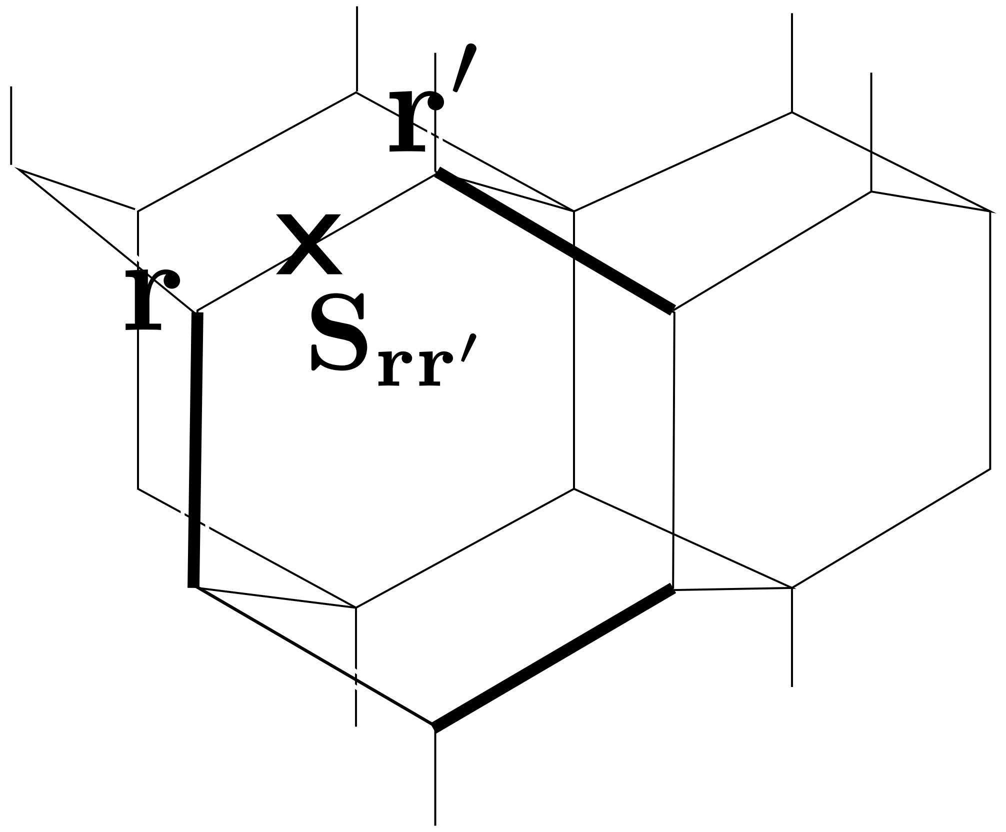
$$S_i^z = E_{rr'}$$

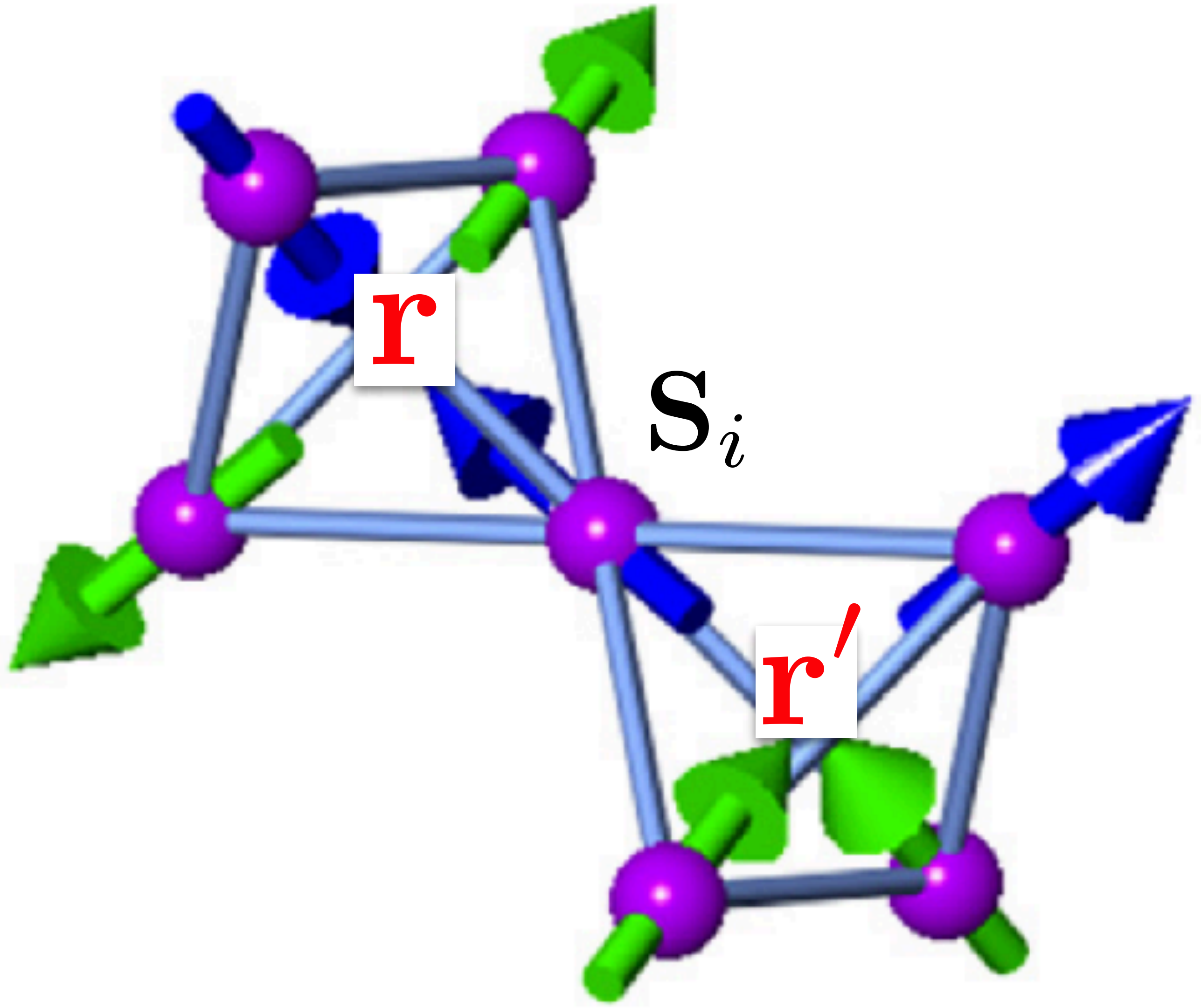
$$(\nabla \cdot E)_{\mathbf{r}} = \sum_{\mathbf{r}' \leftarrow \mathbf{r}} E_{\mathbf{rr}'} = \pm S^z_{\triangleleft}$$

$$(\nabla \cdot E)_{\mathbf{r}} = 0$$ Gauss's law

$$\mathbf{S}_i = \mathbf{S}_{\mathbf{rr}'}$$

link on the dual diamond lattice sites





$$\nabla \cdot E = 0$$

Quantum Electrodynamics

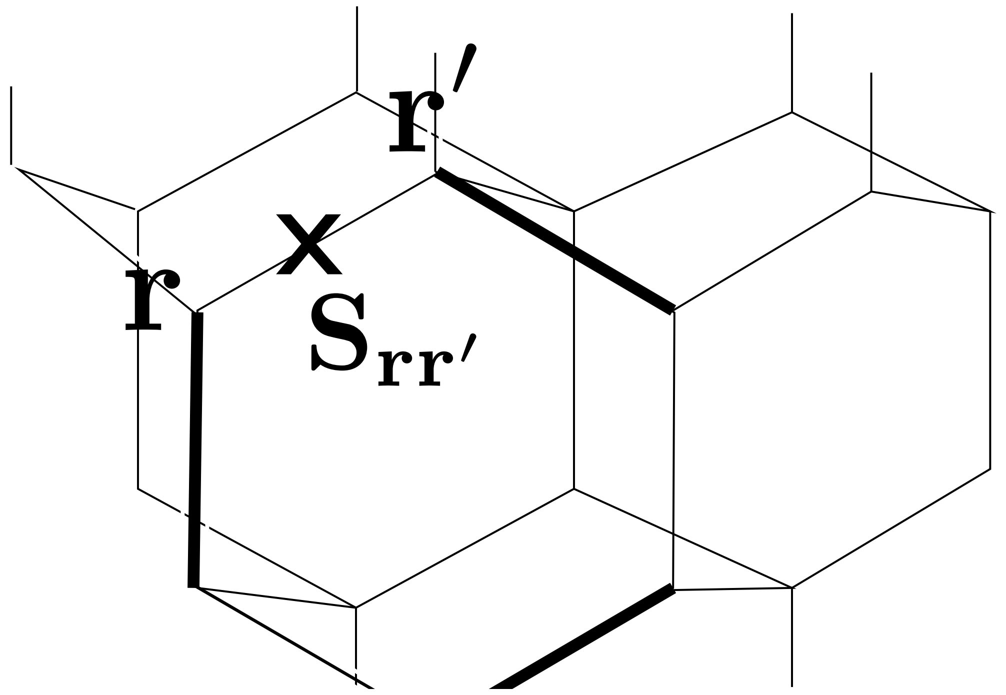
$S_i^z = E_{rr'}$ $\quad S_i^+ = \Phi_r^\dagger \, e^{iA_{rr'}} \Phi_{r'}$ $\quad \pm \, \mathbf{r} \in \mathrm{A/B}$ $\quad [S^+, S^z] = -S^+ \longrightarrow [A_{\mathbf{rr'}}, E_{\mathbf{rr'}}] = i$

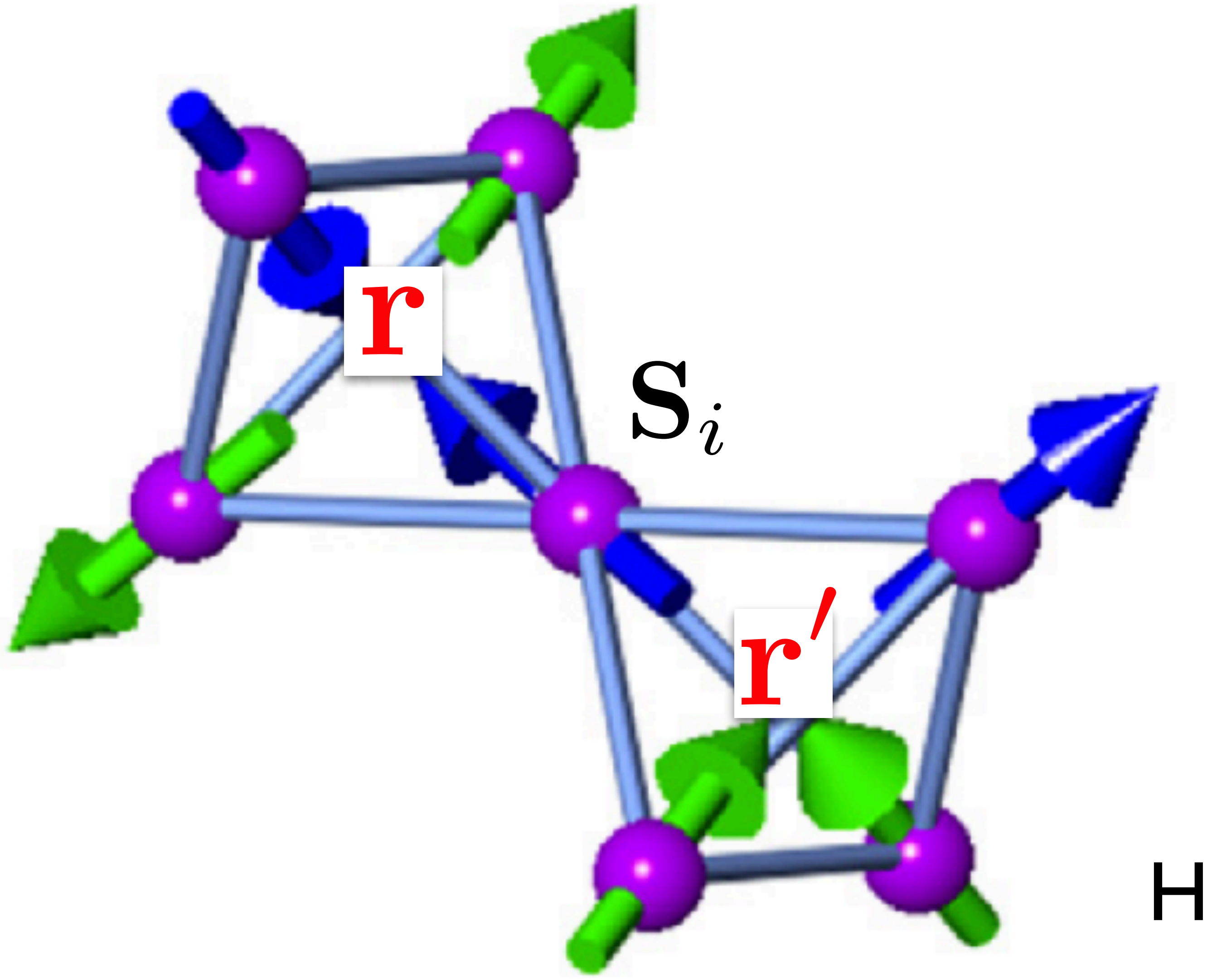
$$(\nabla \cdot E)_{\mathbf{r}} = \sum_{\mathbf{r}' \leftarrow \mathbf{r}} E_{\mathbf{rr'}} = \pm S^z_{\triangleleft}$$

$(\nabla \cdot E)_{\mathbf{r}} = 0$ Gauss's law

$\mathbf{S}_i = \mathbf{S}_{\mathbf{rr'}}$

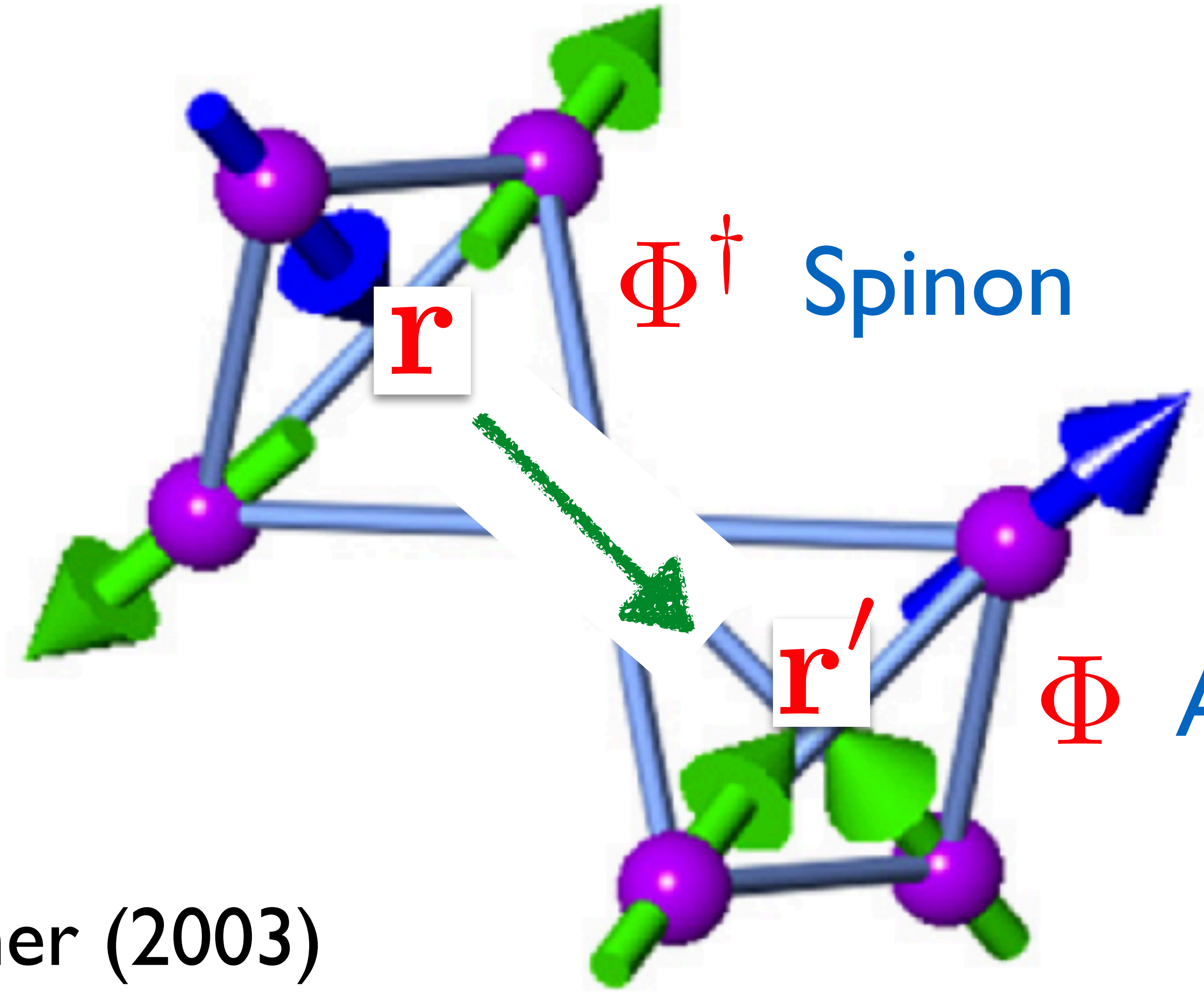
link on the dual diamond lattice sites





$\nabla \cdot E = 0$

$S_i^+ \longrightarrow$



Hermele, Balents, Fisher (2003)
Savary + Balents (2012)
S Lee, Onoda, Balents(2012)

Quantum Electrodynamics

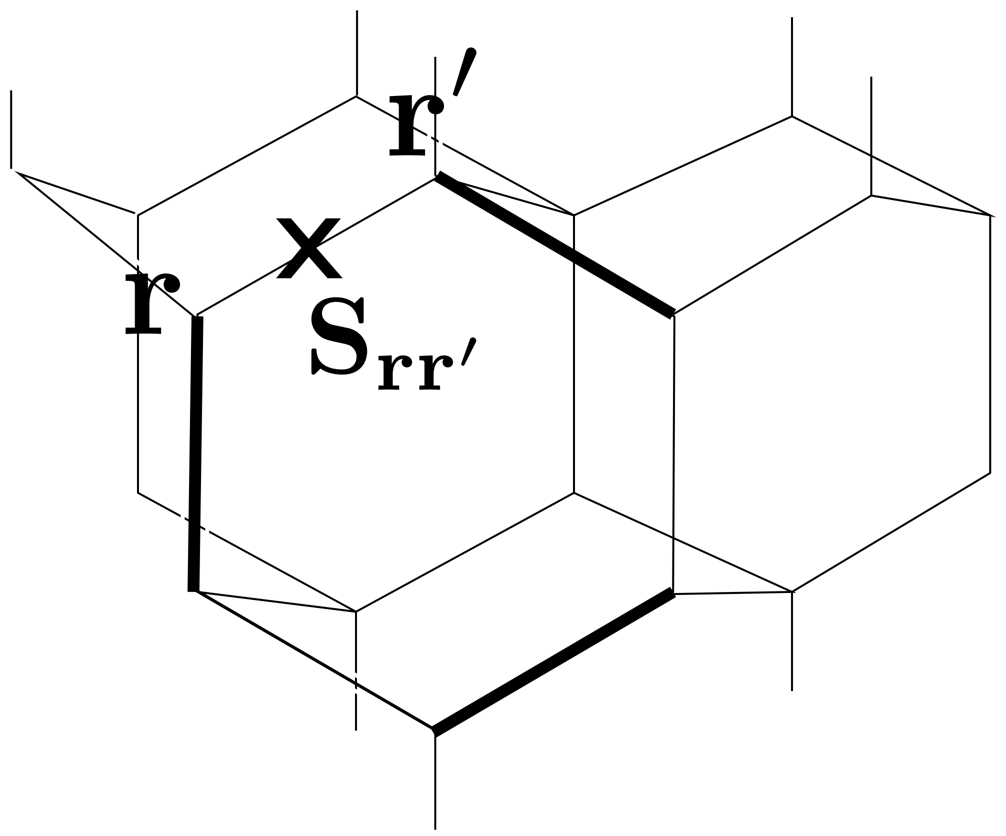
$S_i^z = E_{rr'} \qquad S_i^+ = \Phi_r^\dagger \, e^{iA_{rr'}} \Phi_{r'} \qquad \pm \, \mathbf{r} \in \mathrm{A/B} \qquad [S^+, S^z] = -S^+ \longrightarrow [A_{\mathbf{rr}'}, E_{\mathbf{rr}'}] = i$

$$(\nabla \cdot E)_{\mathbf{r}} = \sum_{\mathbf{r}' \leftarrow \mathbf{r}} E_{\mathbf{rr}'} = \pm S^z_{\triangleleft}$$

$$(\nabla \cdot E)_{\mathbf{r}} = 0$$ Gauss's law

$\mathbf{S}_i = \mathbf{S}_{\mathbf{rr}'}$

link on the dual diamond lattice sites



$$\mathcal{H}_{eff} = -J_{ring} \sum_{\hexagon} (S_1^+ S_2^- S_3^+ S_4^- S_5^+ S_6^- + h.c.) \longrightarrow \sum_{\hexagon} 2\cos(\nabla \times A)_{\hexagon}$$

$$\mathcal{H} = \frac{U}{2} \sum_{\langle \mathbf{rr}' \rangle} \left(E_{\mathbf{rr}'}^2 - \frac{1}{4} \right) - K \sum_{\hexagon} \cos(\nabla \times A)_{\hexagon}$$

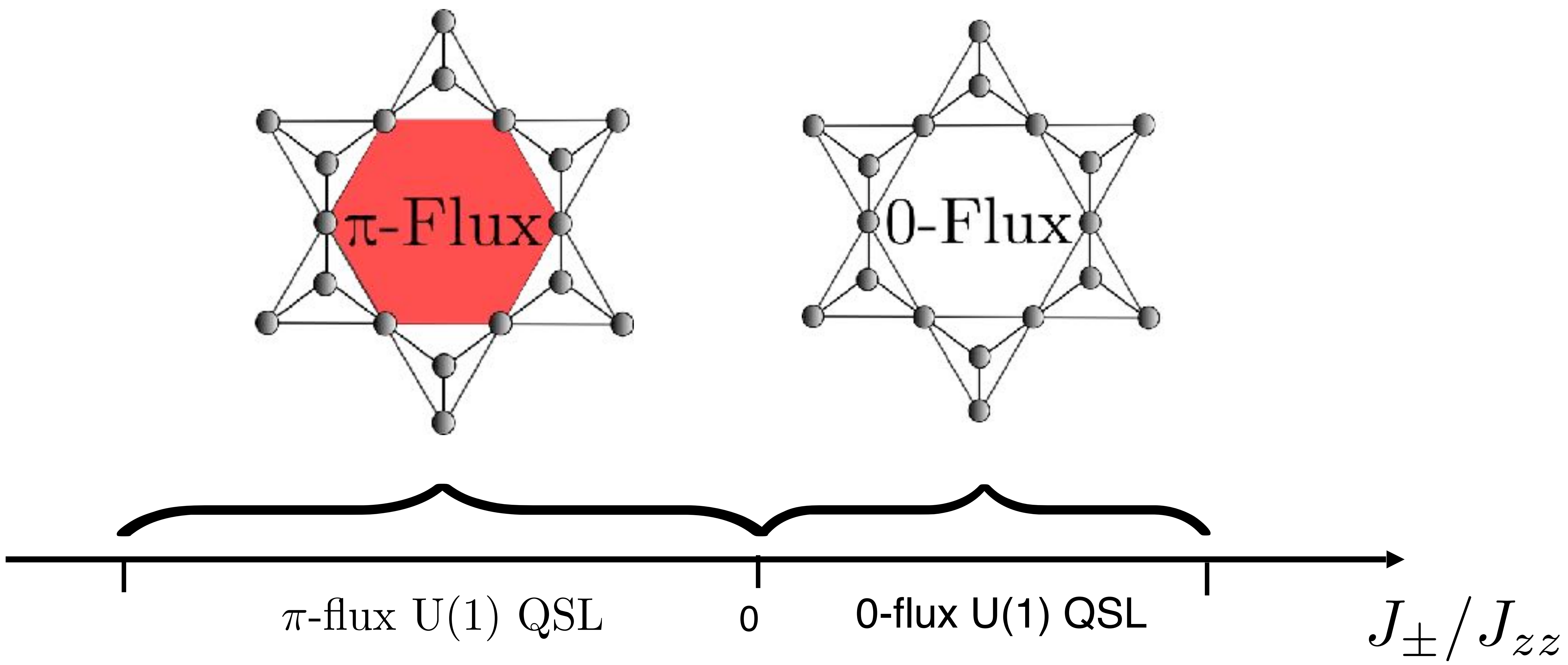
large U

$K \sim J_{ring}$

$$K \propto \frac{J_\pm^3}{J_{zz}^2}$$

0-flux and π-flux Quantum Spin Ice

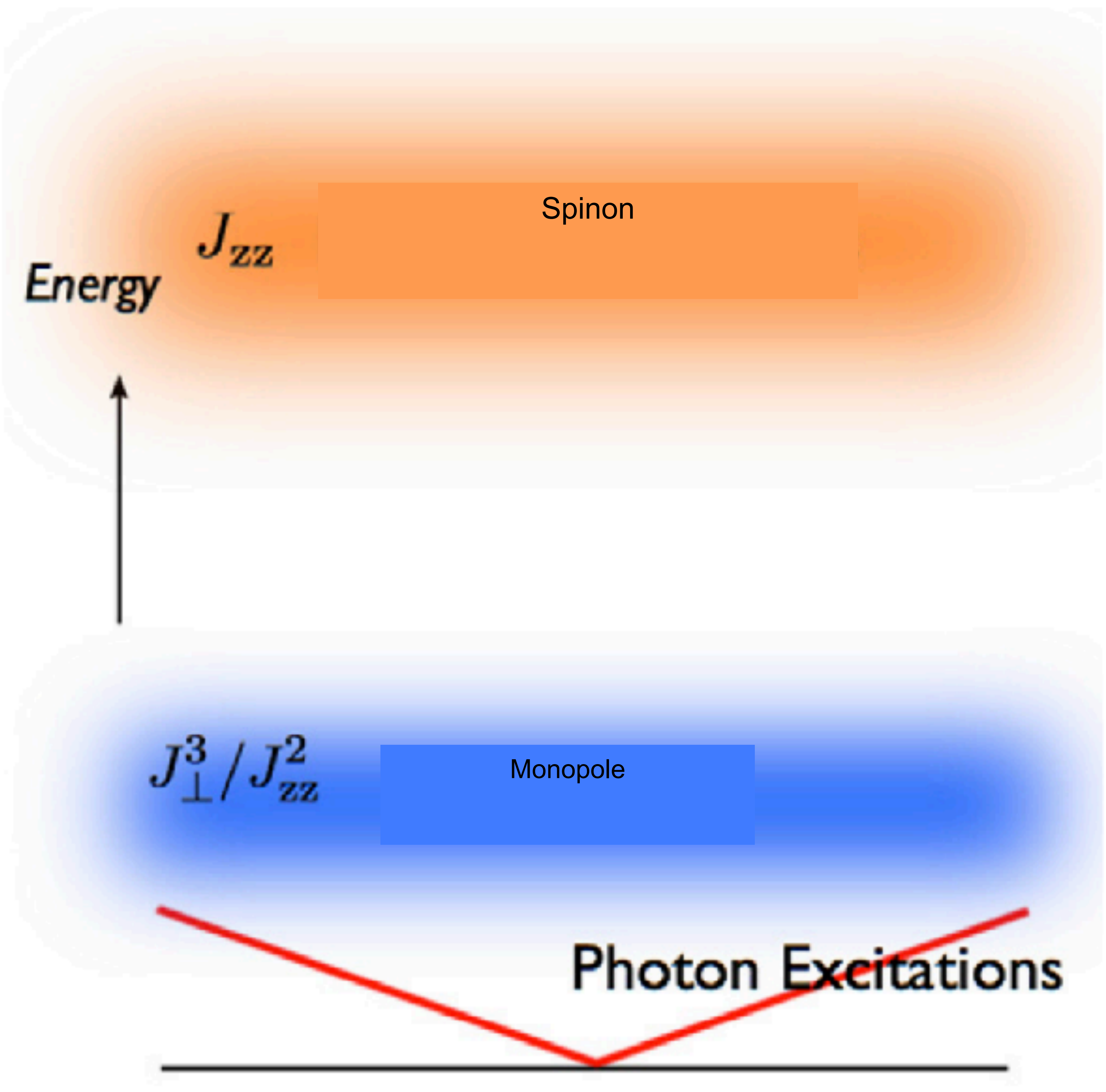
$$\mathcal{H} = \frac{U}{2} \sum_{\langle \mathbf{rr}' \rangle} \left(E^2_{\mathbf{rr}'} - \frac{1}{4} \right) - K \sum_{\hexagon} \cos(\nabla \times A)_{\hexagon} \qquad K \propto \frac{J^3_{\pm}}{J^2_{zz}}$$



M. Hermele, et al. PRB (2004)

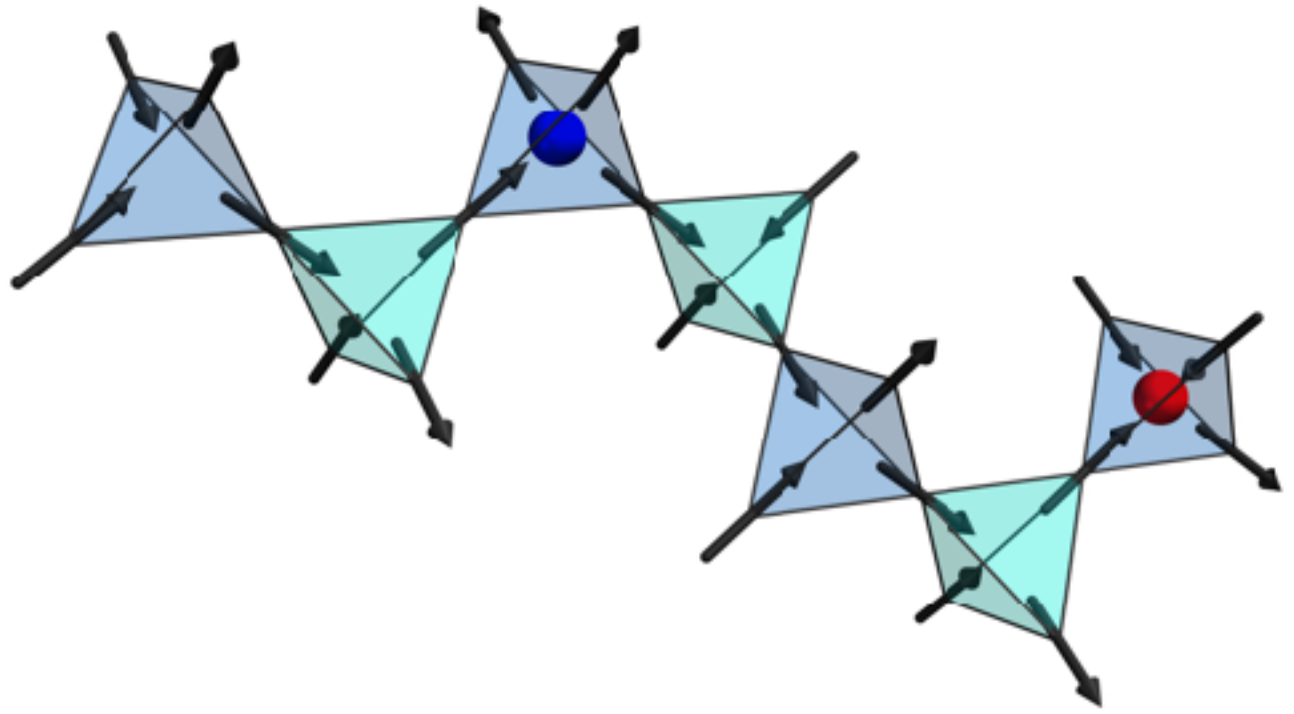
Excitations in the deconfined phase:
U(1) quantum spin liquid (Quantum Spin Ice)

$$\mathcal{H} = \frac{U}{2}\sum_{\langle \mathbf{r}\mathbf{r}'\rangle}\left(E^2_{\mathbf{r}\mathbf{r}'} - \frac{1}{4}\right) + \frac{K}{2}\sum_{\hexagon}[(\nabla \times A)_{\hexagon}]^2$$



Gingras & McClarty (2014)

electric monopoles (spinons)



$$2\Delta_{\text{spinon}} \sim J_z$$

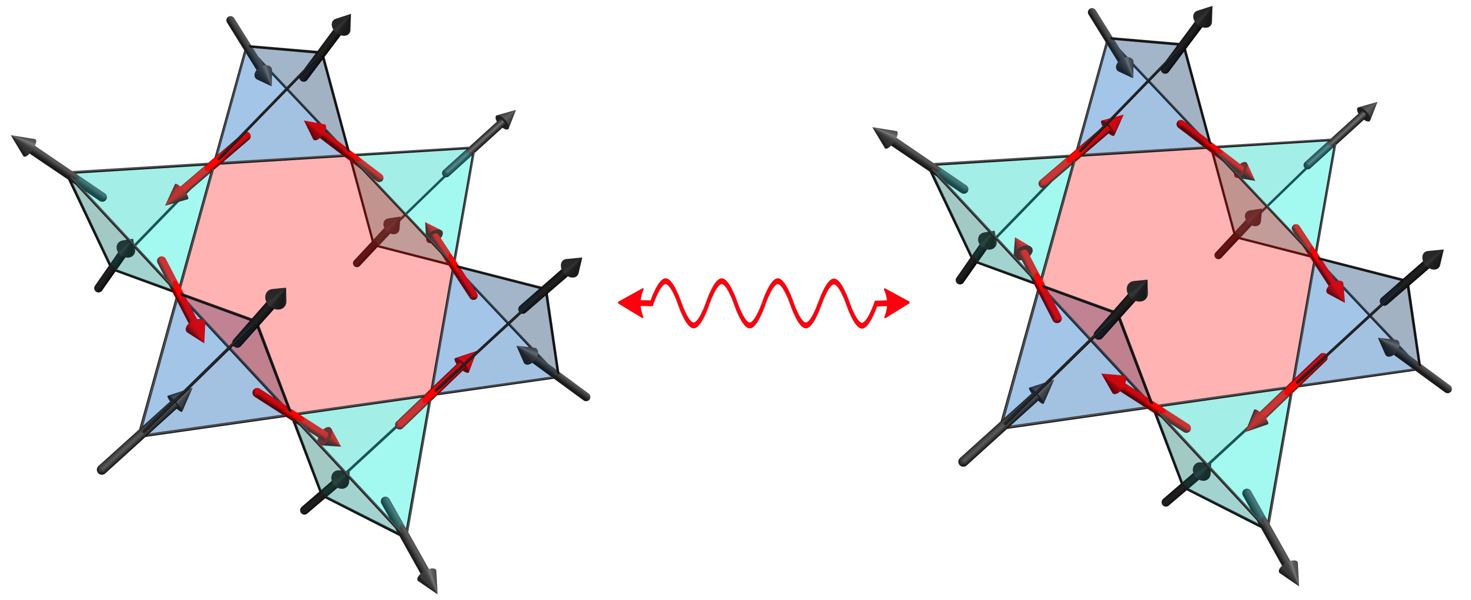
magnetic monopoles (visons)

$$\Delta_{\text{mon}} \sim J^3_{\pm}/J^2_z$$

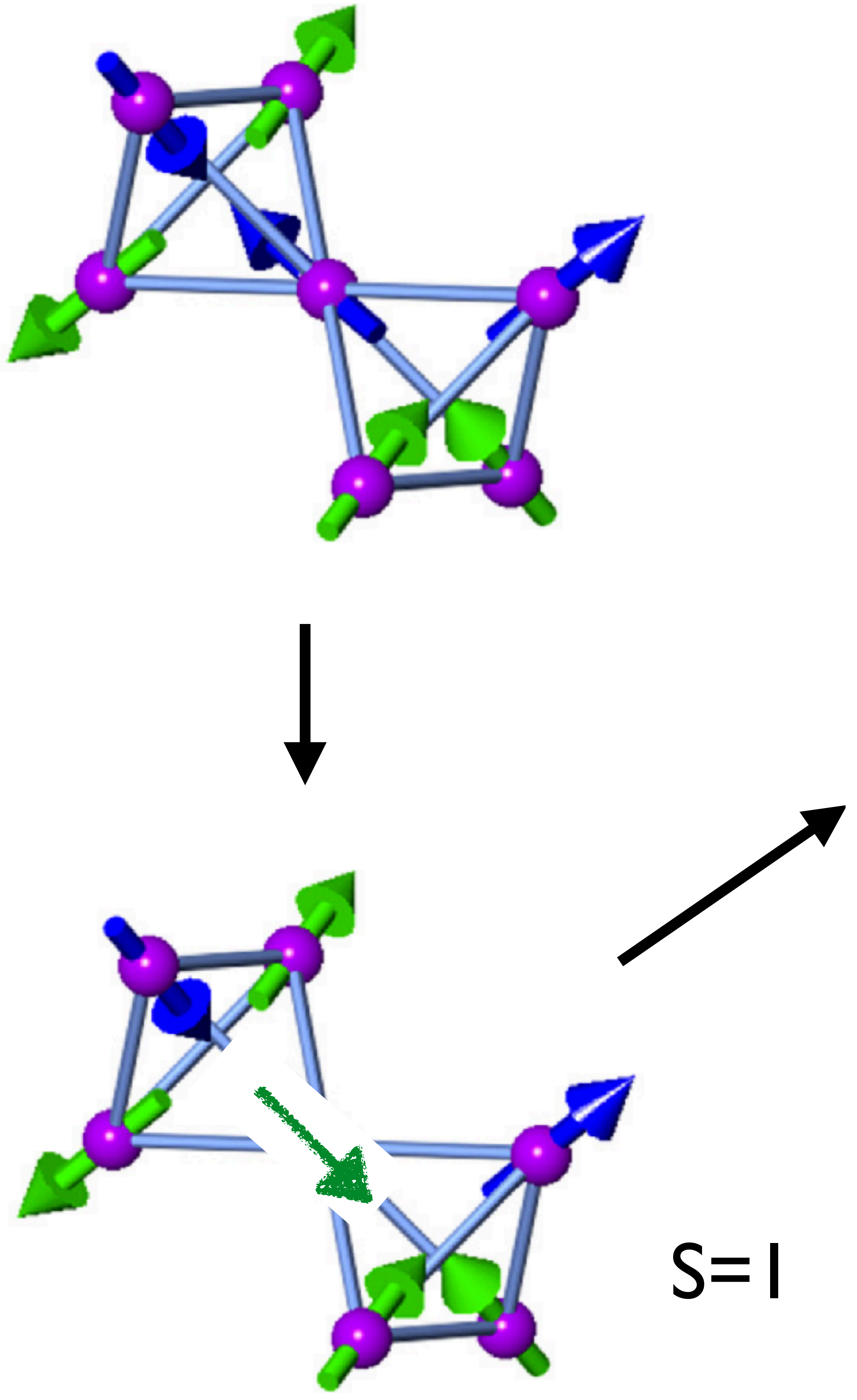
emergent photons

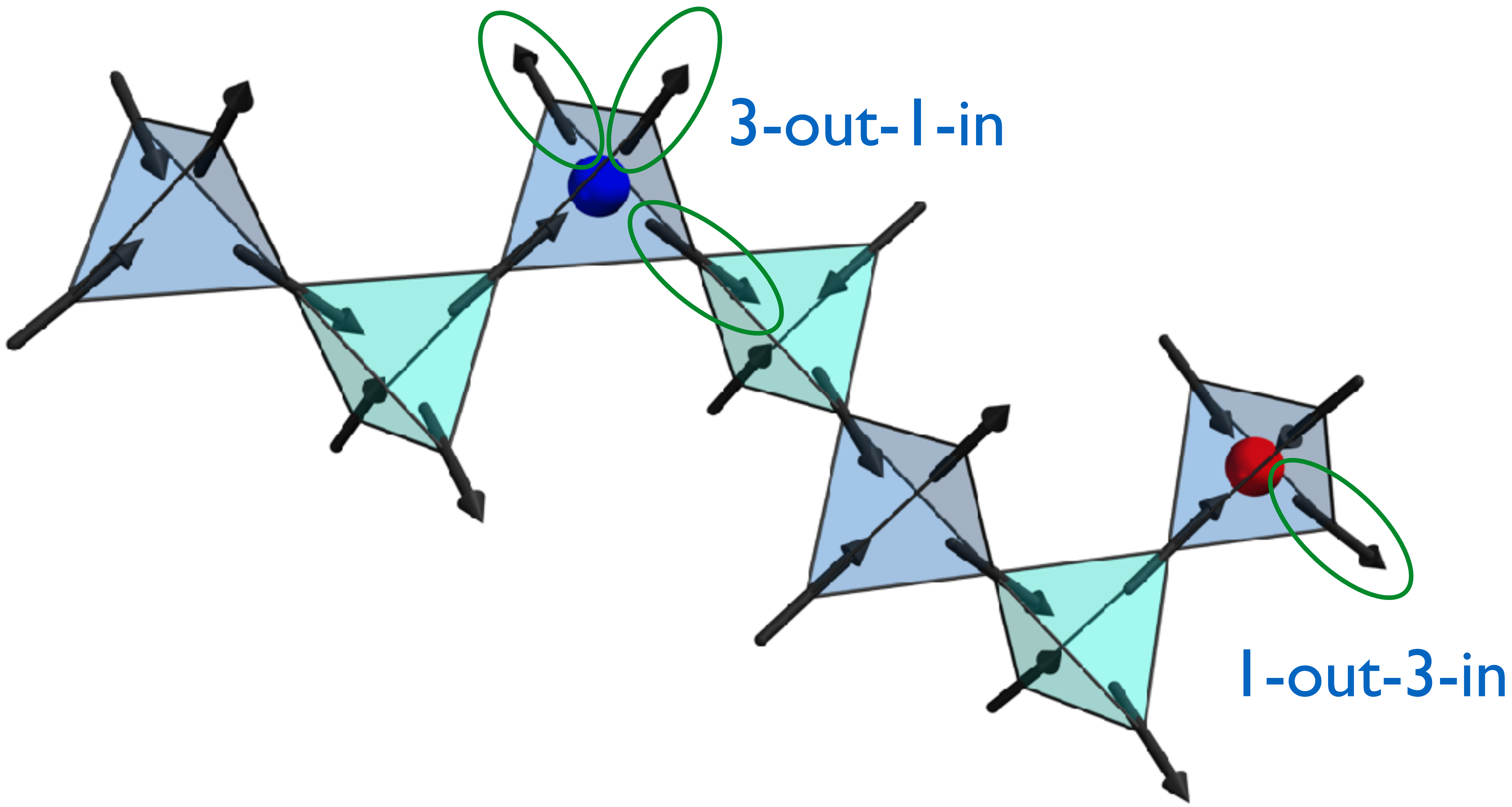
$$\omega(\mathbf{k}) \approx c|\mathbf{k}| \qquad c \propto \sqrt{UK}a_0/\hbar$$

$$C(T) \propto \frac{1}{c^3}T^3$$



Spinons





S=1

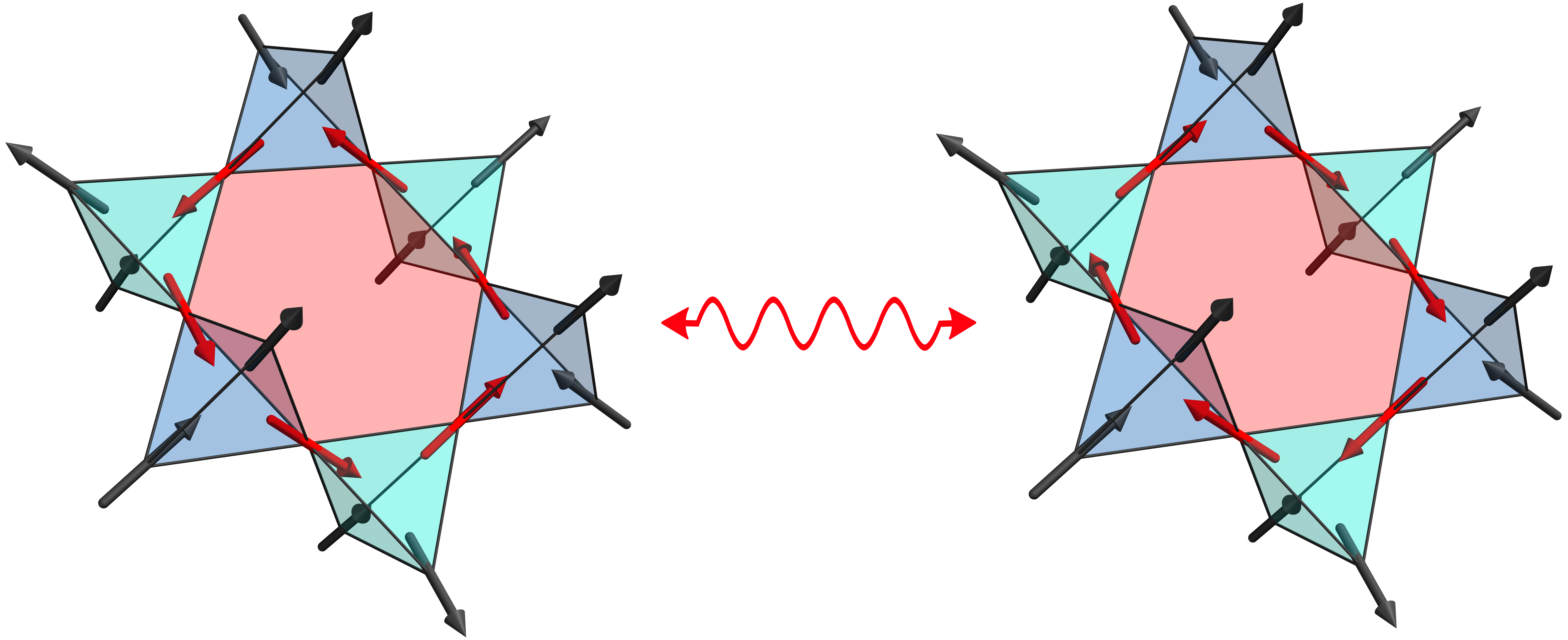
"Two" S=1/2 excitations

Spinon-Antispinon pair

Emergent Photons

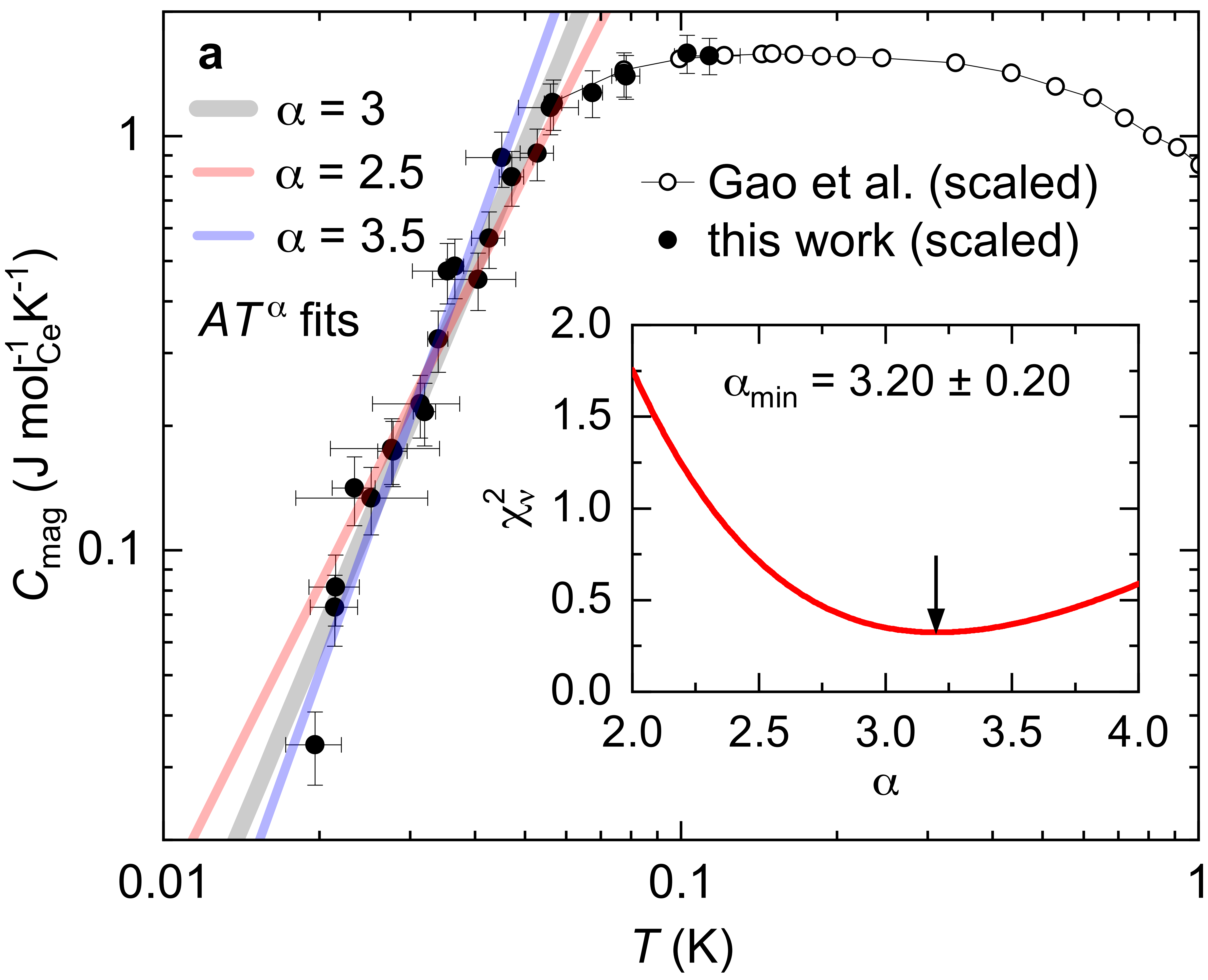
$$\mathcal{H}_{eff} = J_{ring} \sum_{\hexagon} (S_1^+ S_2^- S_3^+ S_4^- S_5^+ S_6^- + h.c.) \longrightarrow -K \cos(\nabla \times A)$$

$K \sim -J_{ring} \propto -J_\perp^3 / J_\parallel^2$ Hermele, Balents, Fisher '03



Specific heat data at low temperatures

$Ce_2Zr_2O_7$



T^3 from emergent photons ?

Photon velocity comparable to the estimation from neutron scattering

$$c_{QSI} = 7.9 \pm 0.4 \text{ m/s}$$

B.Gao, F.Desrochers, … S.Paschen, YBK, P.Dai.
arXiv:2404.04207, Nature Physics (2025)

Different routes to QSL: candidate materials

Candidate Materials (incomplete list)

Geometric

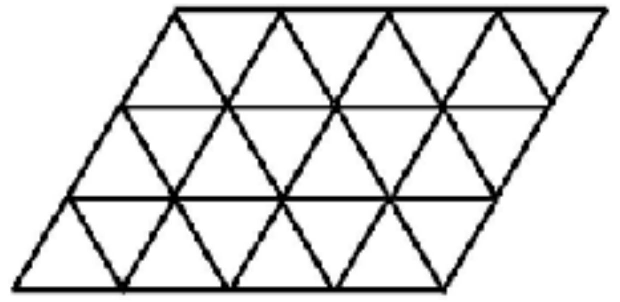
κ-$(BEDT\text{-}TTF)_2Cu_2(CN)_3$

$EtMe_3Sb[Pd(dmit)_2]_2$

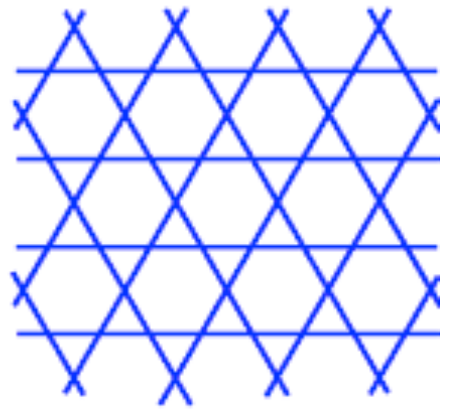
κ-$(BEDT\text{-}TTF)_2Ag_2(CN)_3$

κ-$H_3(Cat\text{-}EDT\text{-}TTF)_2$

Organic Materials
Triangular Lattice



Kagome

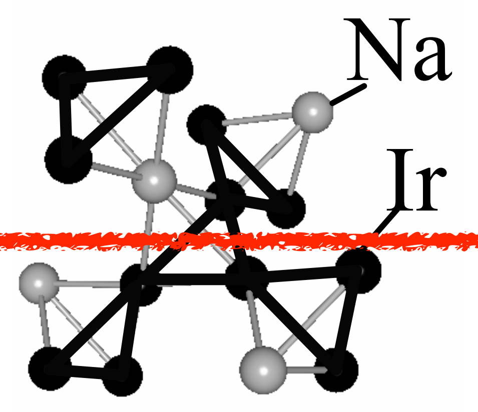


$ZnCu_3(OH)_6Cl_2$ herbertsmithtite

$Cu_3Zn(OH)_6FBr$ $Cu_4(OH)_6FBr$ barlowite

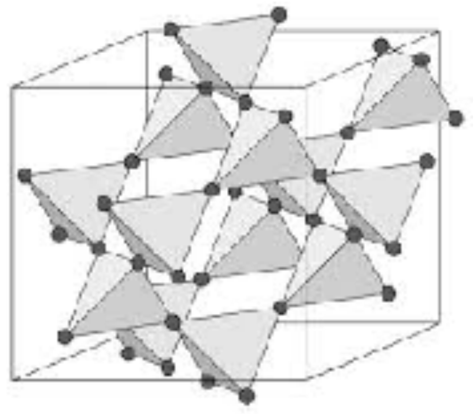
$Cu_3Zn(OH)_6FCl$ $Cu_4(OH)_6FCl$ claringbullite

$PbCuTe_2O_6$

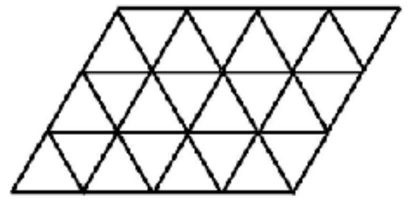


$Na_4Ir_3O_8$ hyper-kagome

$Ce_2Zr_2O_7$
$Pr_2Zr_2O_7$ pyrochlore



$YbMgGaO_4$ $NaYbSe_2$ triangular



Kitaev Materials honeycomb hyper-honeycomb

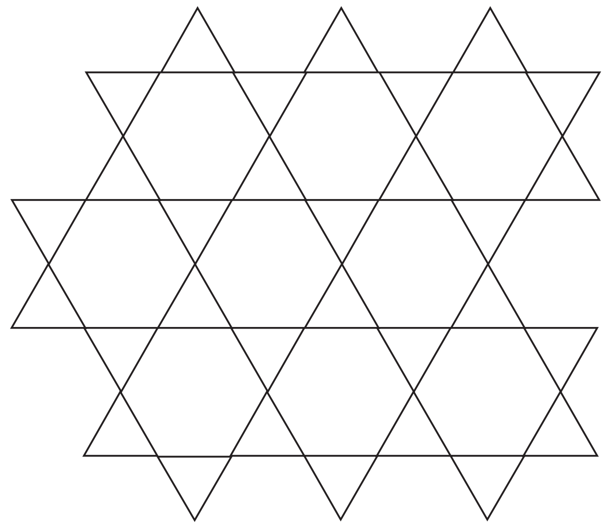
Spin-orbit

Different Routs to Quantum Spin Liquid

1. Geometric Frustration

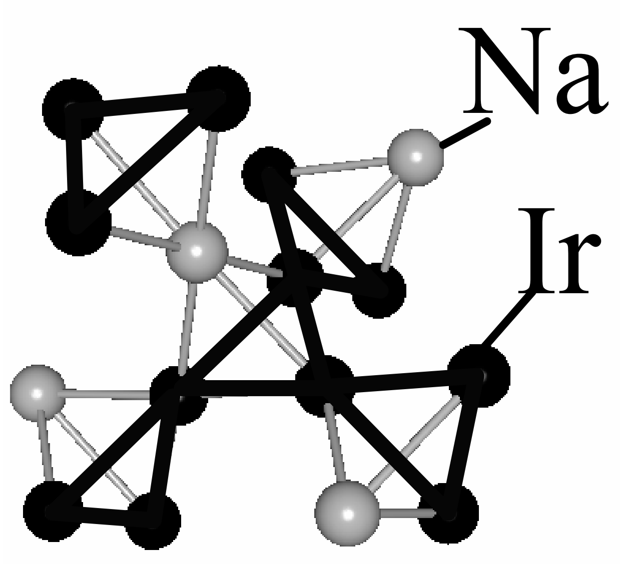
Herbertsmithite $ZnCu_3(OH)_6Cl_2$

"Ideal" Kagome lattice



D. G. Nocera, Y. S. Lee
P. Mendels

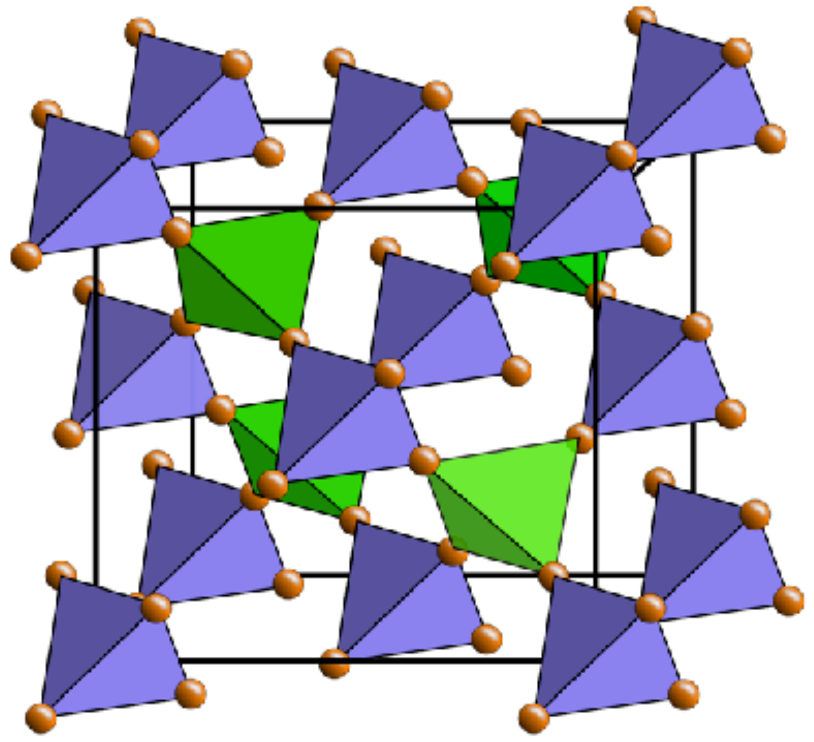
Hyperkagome $Na_4Ir_3O_8$



H. Takagi (2007)
Y. Singh, Y. Tokiwa, P. Gegenwart (2013)

Pyrochlore systems

Quantum spin ice $Ce_2Zr_2O_7$



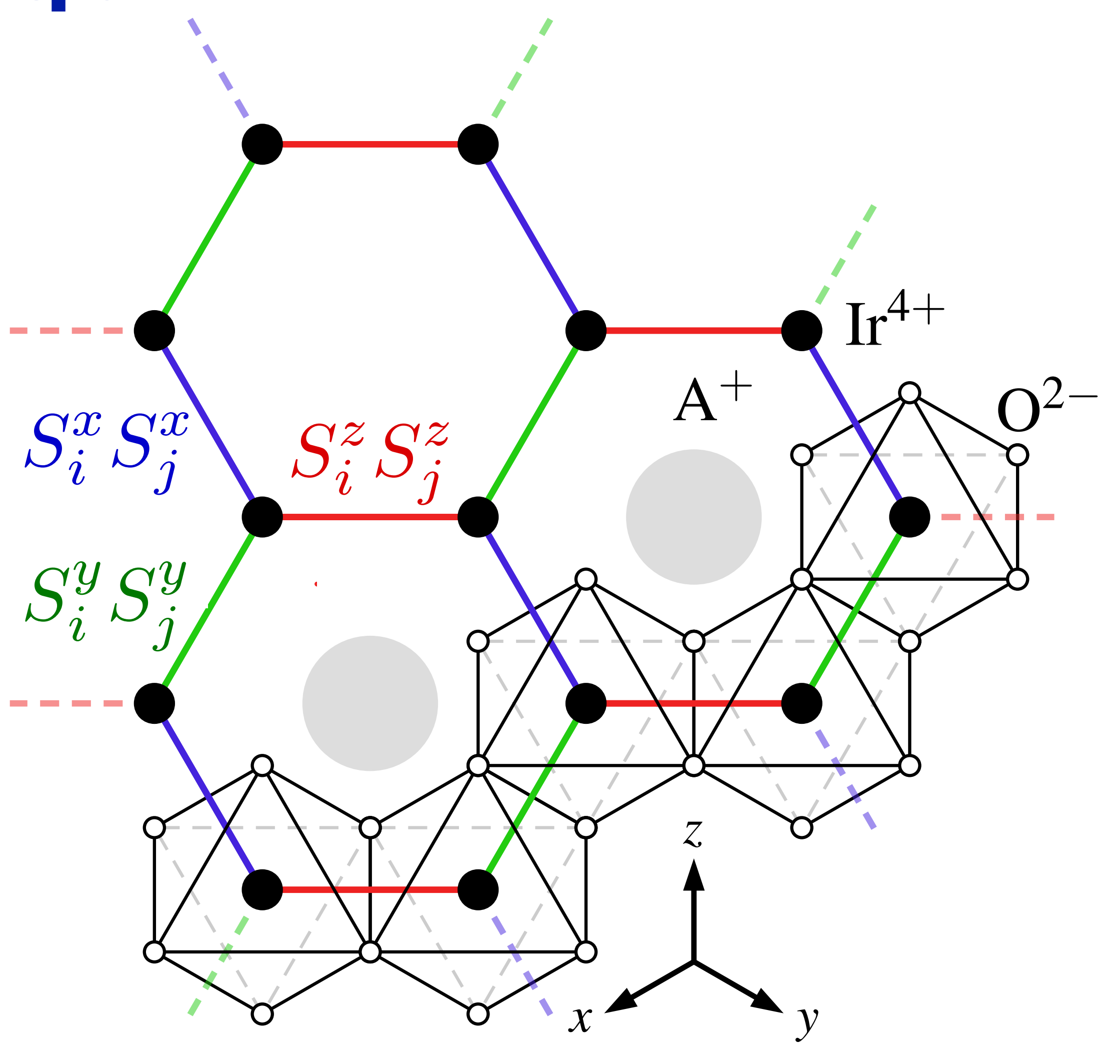
Pengcheng Dai, Bruce Gaulin, Romain Sibille, Elsa Lhotel, Sylvain Petit …

Different Routs to Quantum Spin Liquid

2. Frustrating Interactions

Kitaev Materials — α- Na_2IrO_3

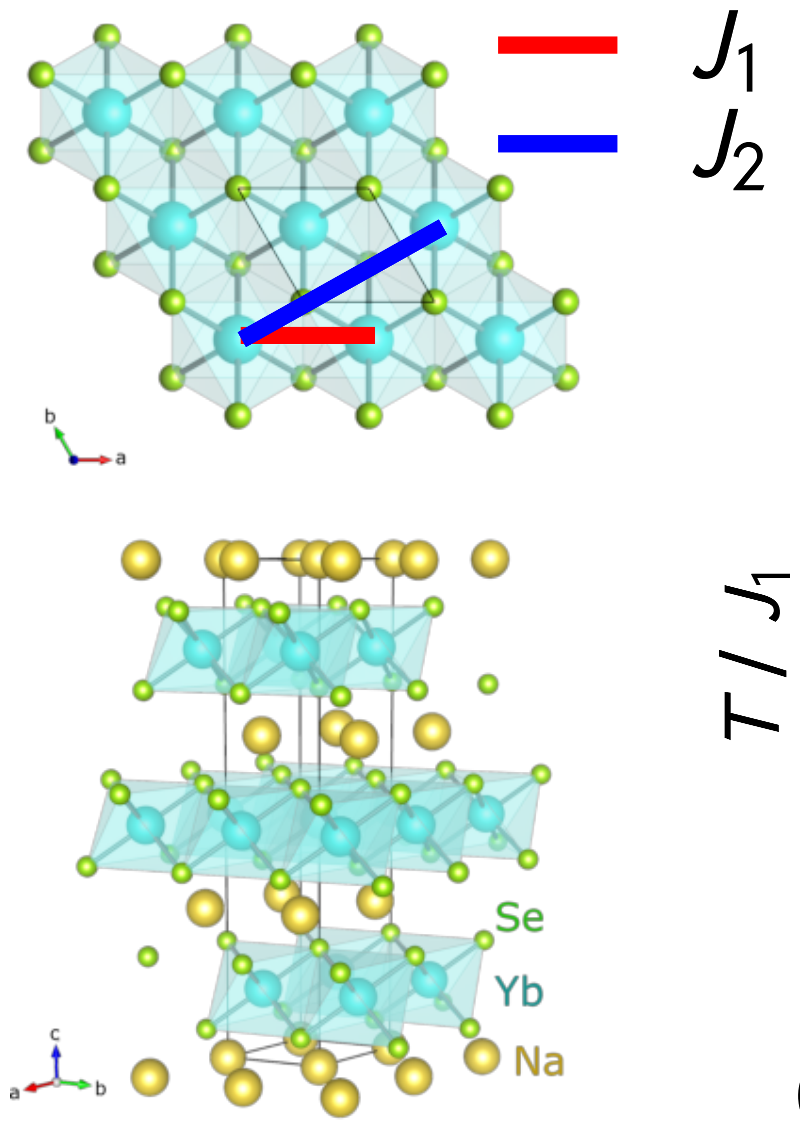
Bond-dependent interaction — α- $RuCl_3$



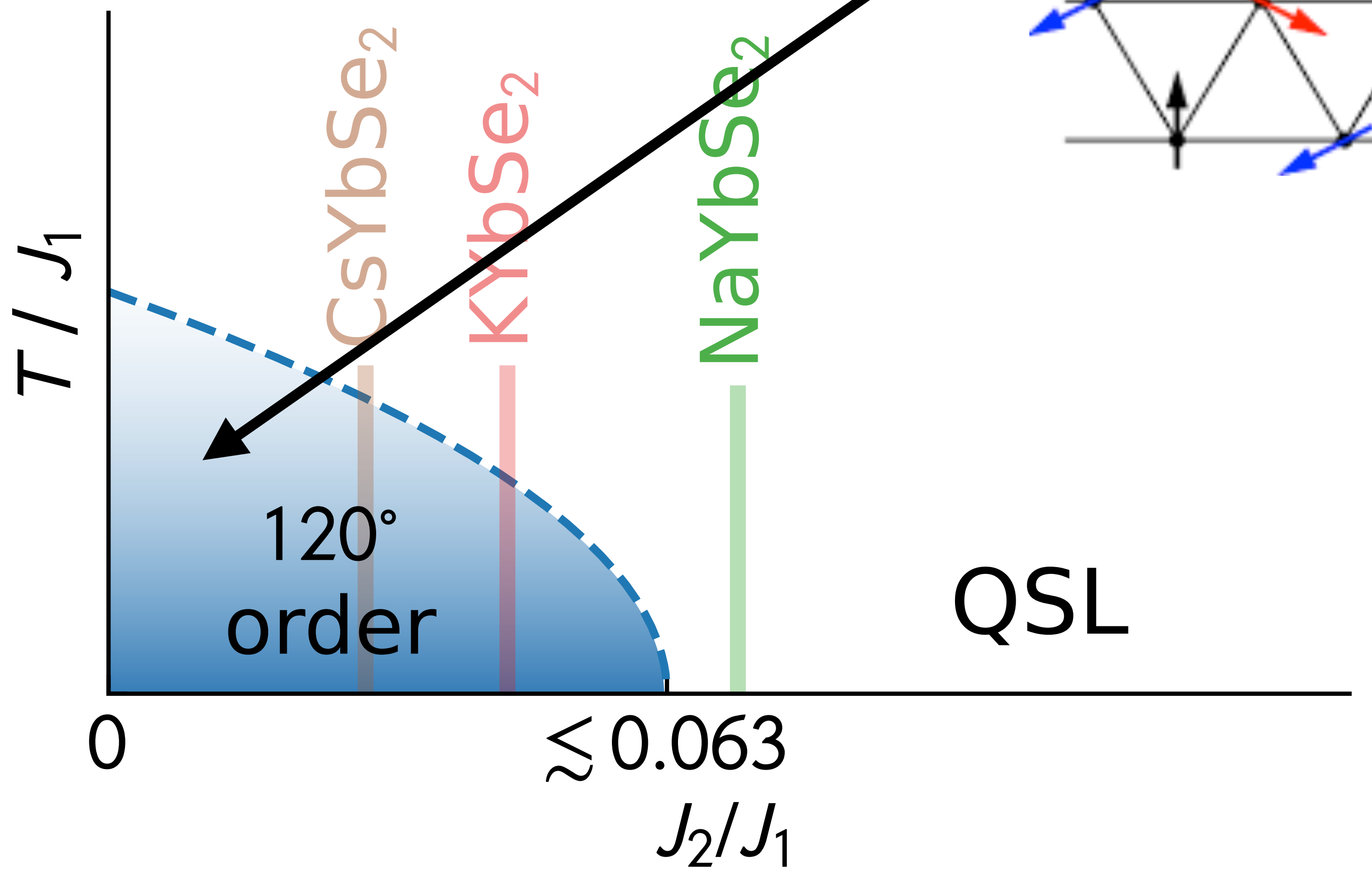
H.Takagi, R. Coldea, P. Geggenbart, YJ Kim, S. Nagler, …

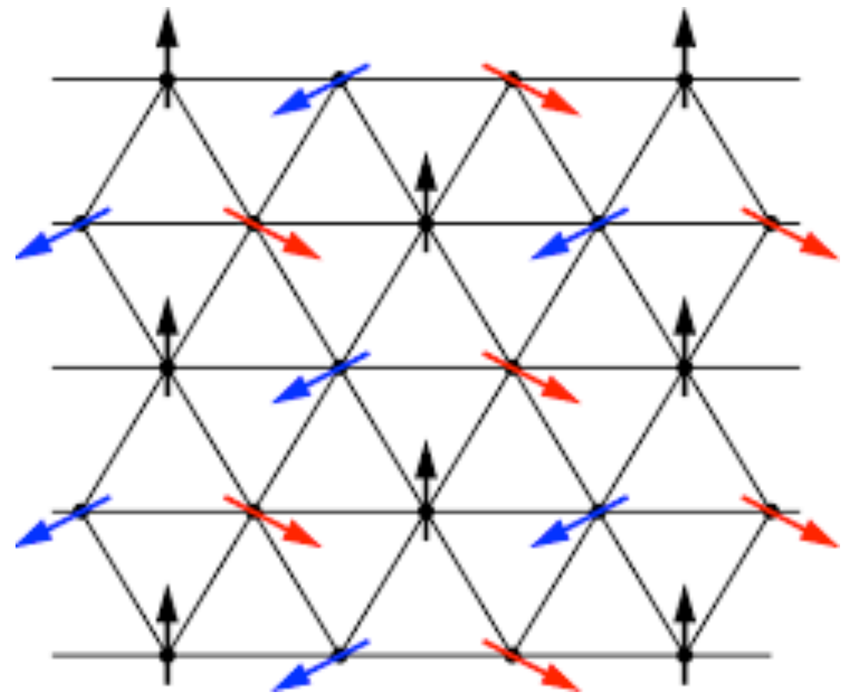
Longer range interactions

J_1-J_2 model — $NaYbSe_2$



$NaYbSe_2$





Different Routs to Quantum Spin Liquid

3. Weak Mott Insulator (charge fluctuations)

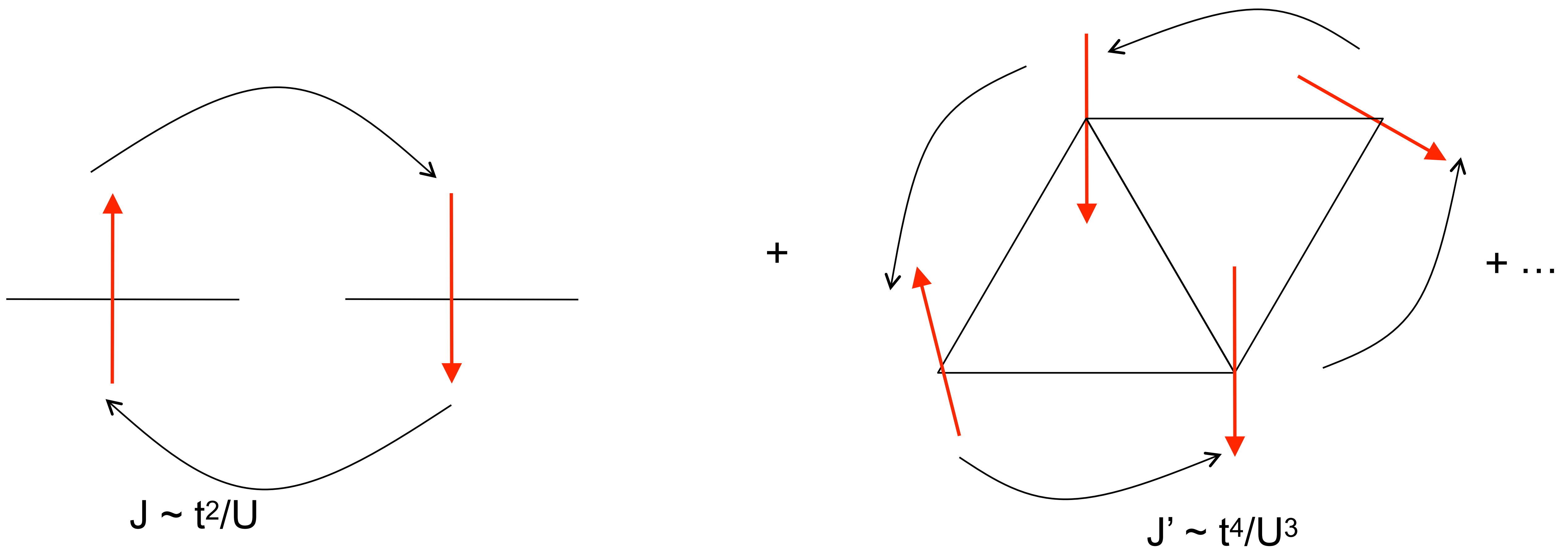
κ-$(BEDT\text{-}TTF)_2Cu_2(CN)_3$
$EtMe_3Sb[Pd(dmit)_2]_2$

K. Kanoda
R. Kato

M. Yamashita,
Y. Matsuda,

Organic Material
Triangular Lattice

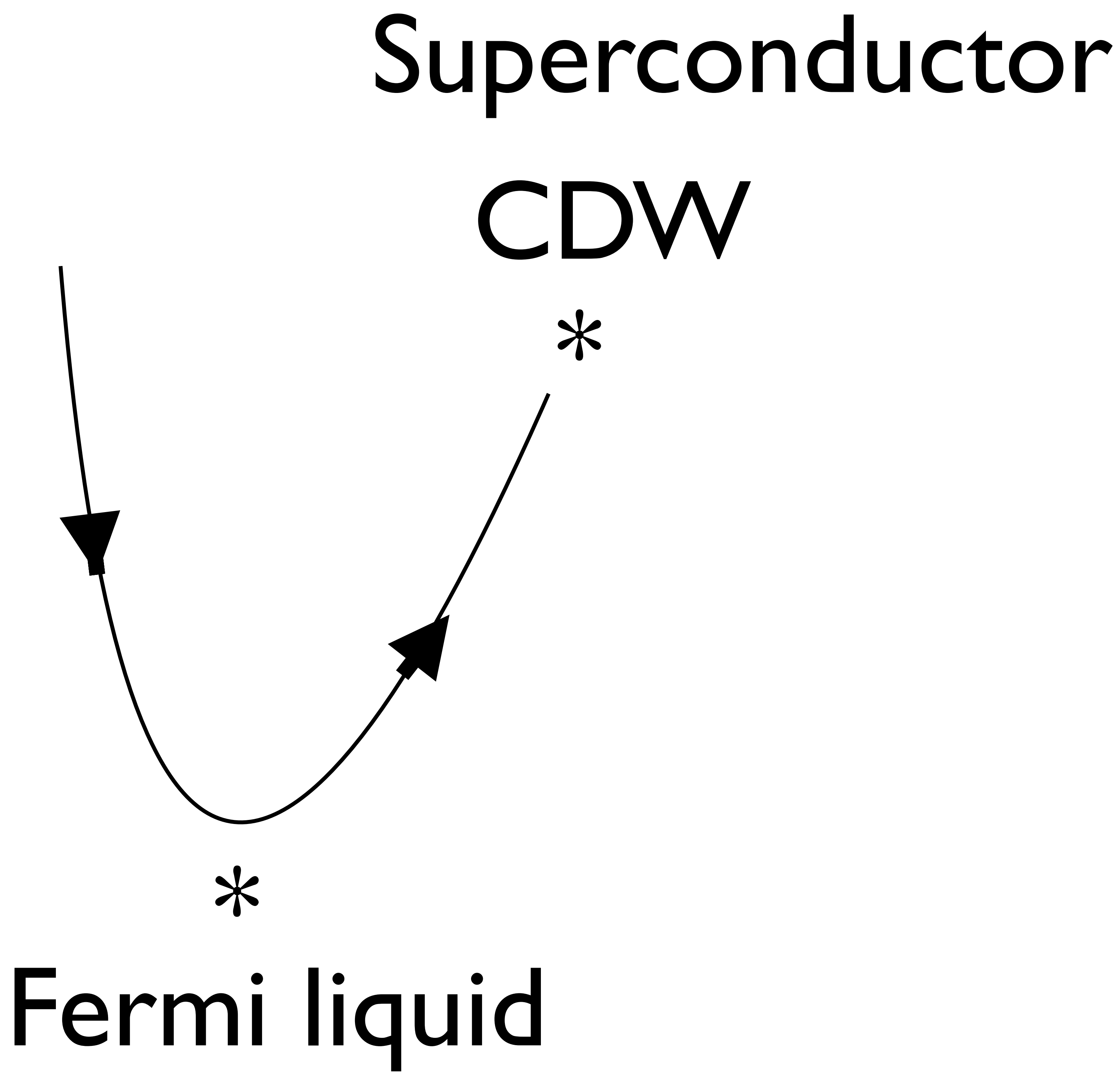
$$H \sim H_{\rm heisenberg} + H_{\rm ring} + \cdots$$



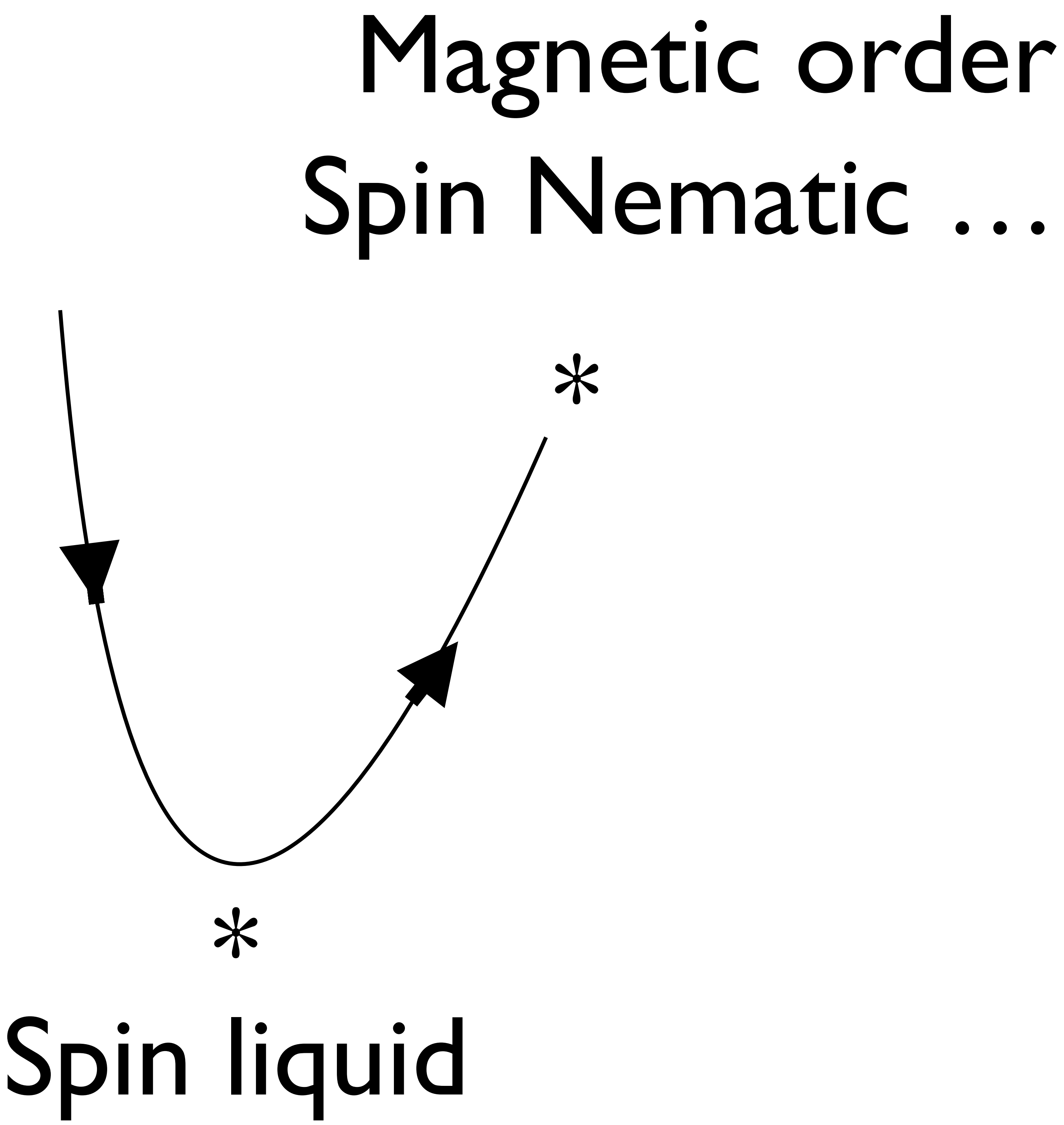
• • •

Charge fluctuations are important near the Mott transition even in the insulating phase

Fermi liquid behavior

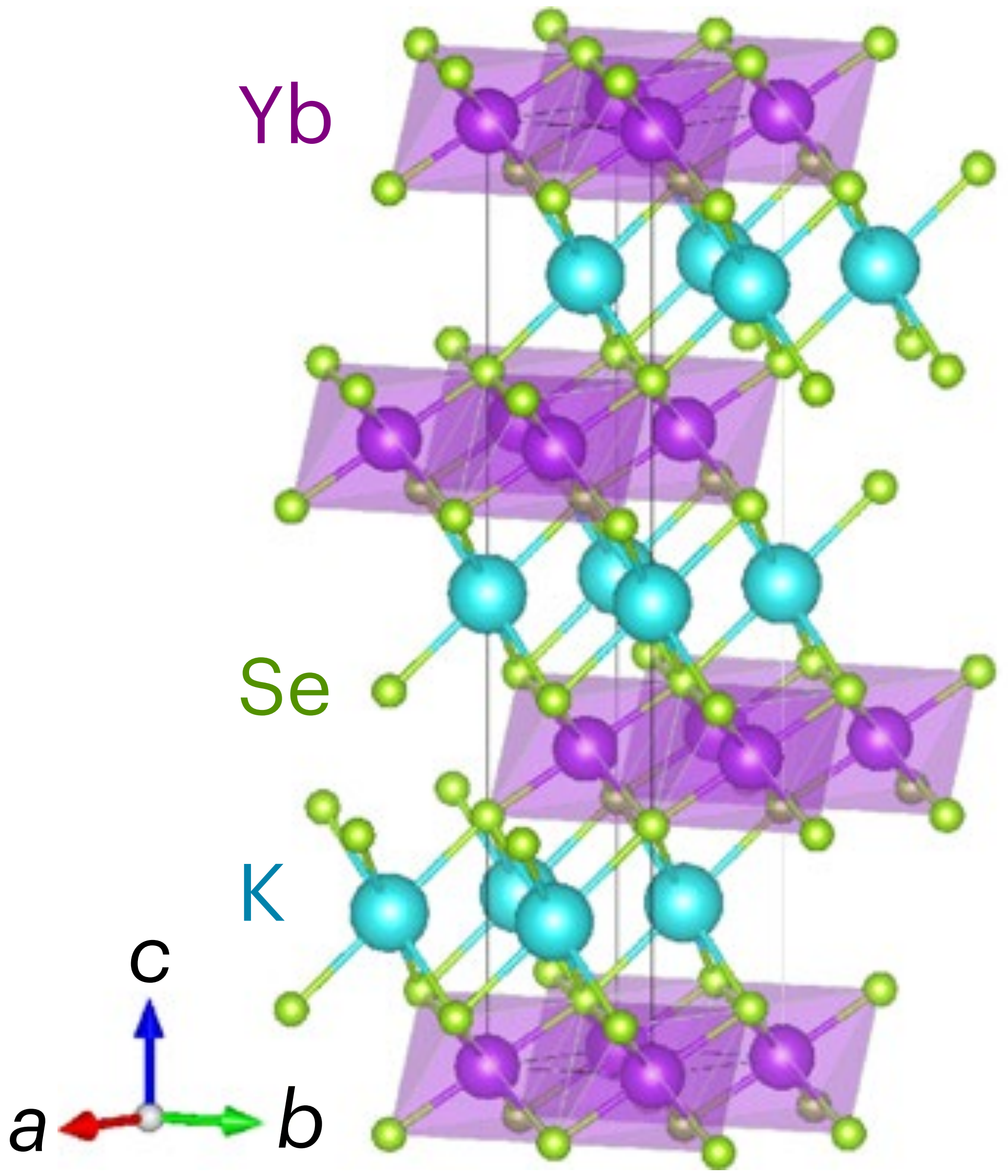


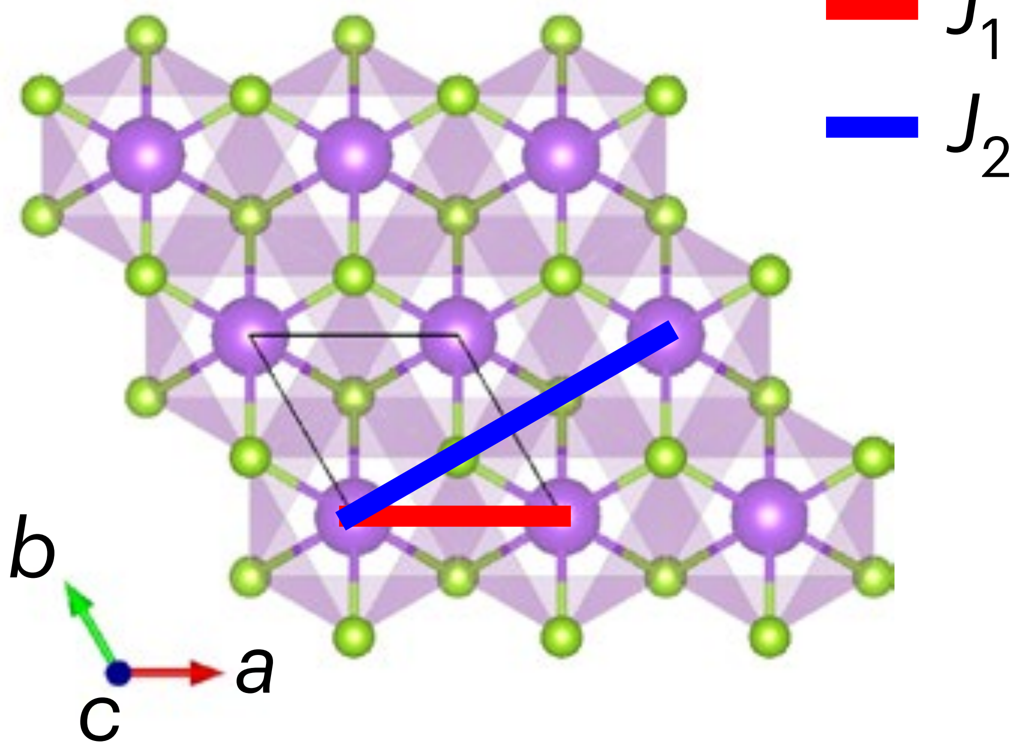
Spin liquid behavior ?



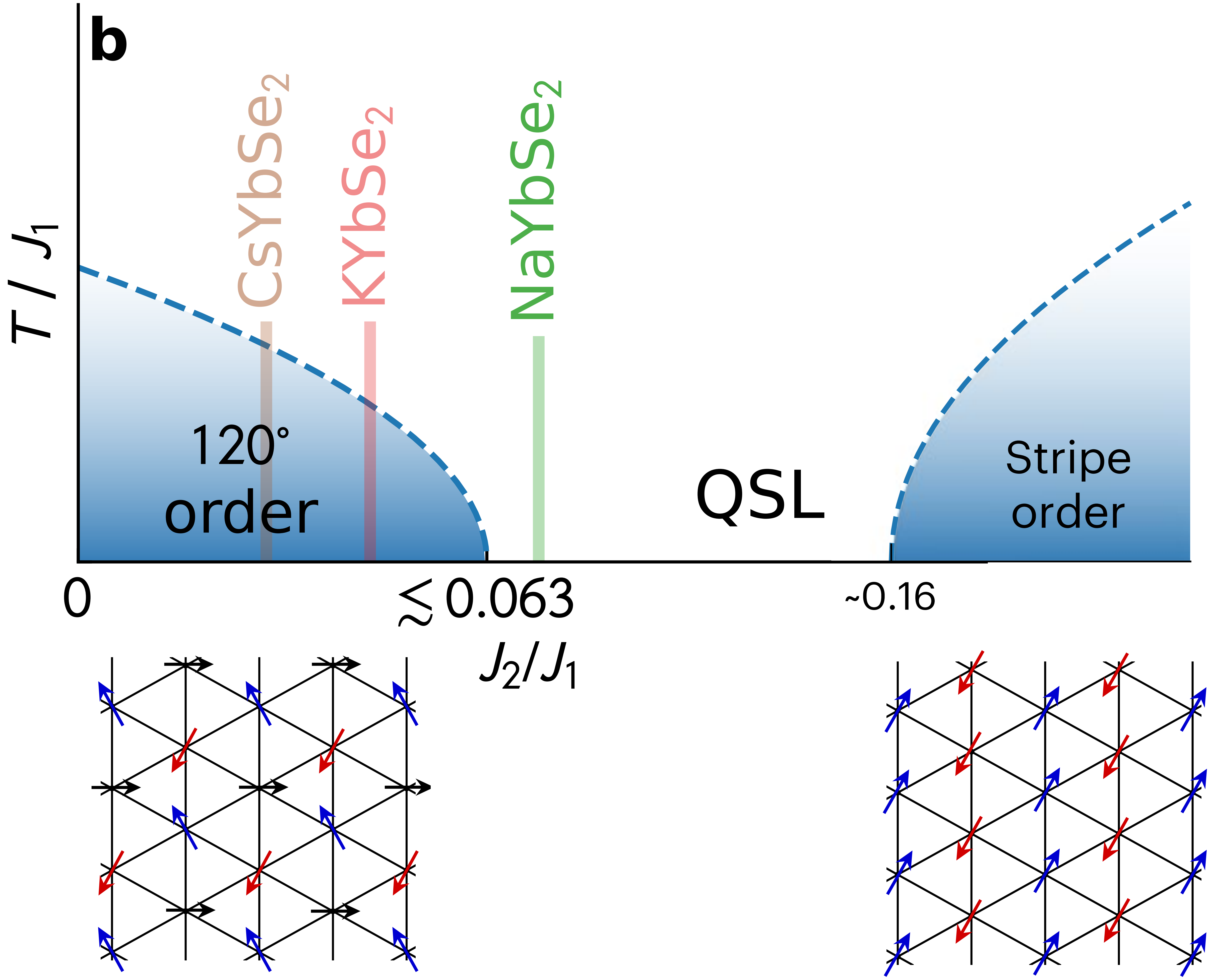
Triangular Lattice J_1-J_2

$AYbSe_2$ A=Cs, K, Na





A. O. Scheie et al, arXiv:2406.1777



$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

AYbSe$_2$ A=Cs, K, Na

Yb^{3+} $4f^{13}$ $S=1/2$ $L=3$ $J=7/2$

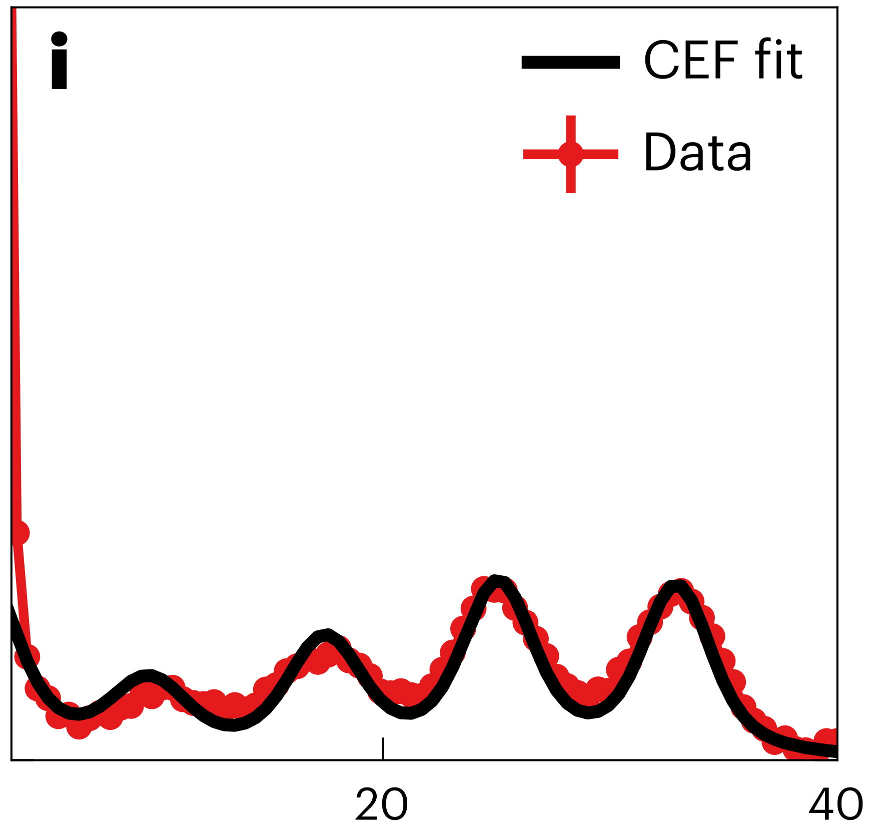
Lowest Kramers doublet in CEF

Pseudospin 1/2

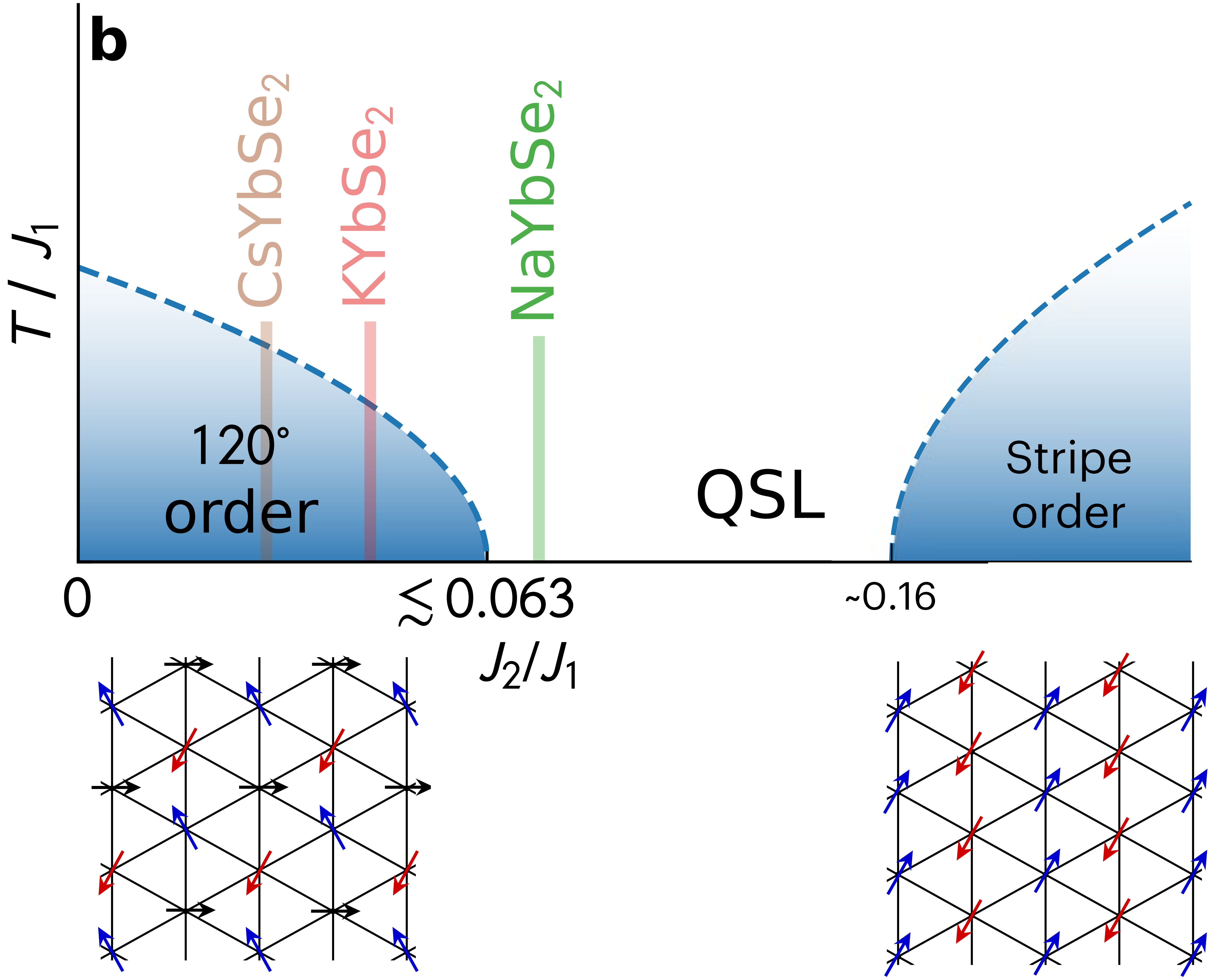
$$|\psi_\pm\rangle = 0.78(3)\left|\mp\frac{5}{2}\right\rangle \mp 0.44(4)\left|\pm\frac{1}{2}\right\rangle - 0.44(3)\left|\pm\frac{7}{2}\right\rangle$$

$g_{xx}=g_{yy}=3.0(2)$ $g_{zz}=1.8(6)$

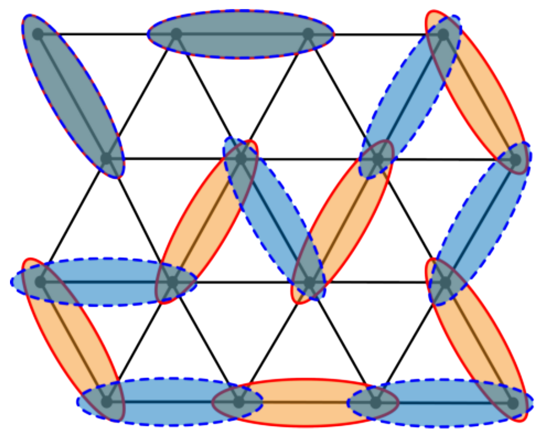
Easy plane anisotropy



A. O. Scheie et al, arXiv:2406.1777



$AYbSe_2$ A=Cs, K, Na


RVB

Gapped Z_2 SL

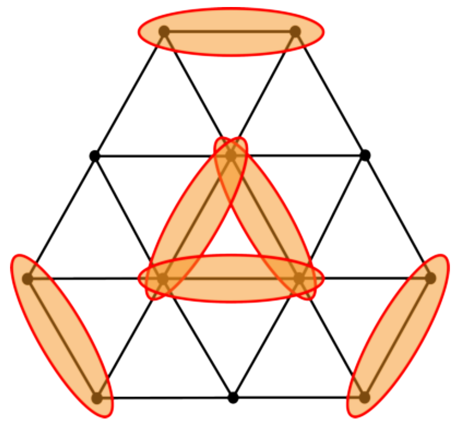
S. Sachdev, PRB 1992

Z. Zhu, S.R. White, PRB 2015 (DMRG)

U(1) Dirac SL

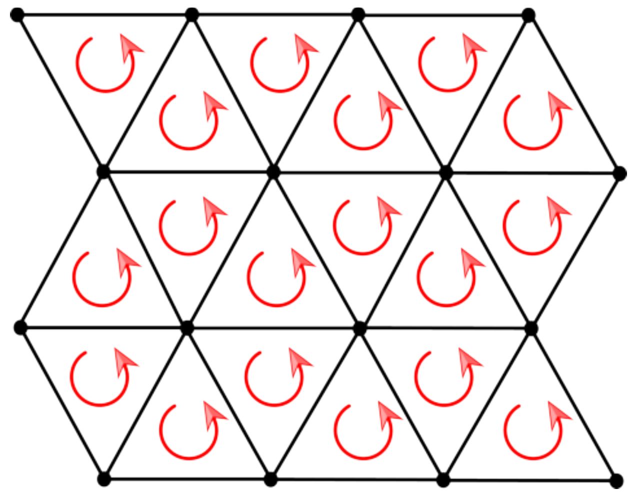
Y. Iqbal et al, PRB 2016 (VMC)

S. Hu et al, PRL 2019 (DMRG)


VBS

Valence Bond Solid

U. F. P. Seifert et al, arXiv:2307.12295

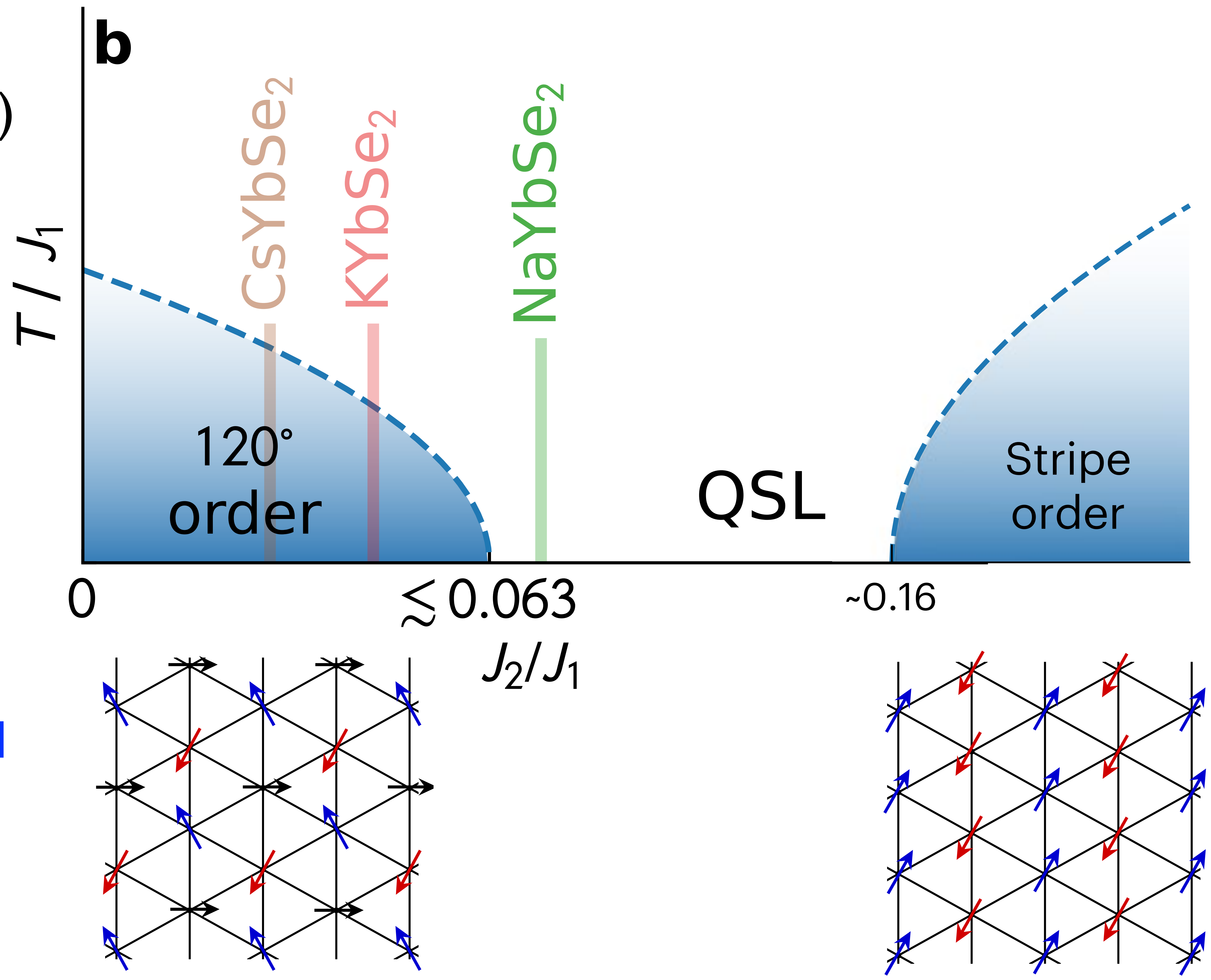

Chiral QSL

Hubbard Model Chiral Spin Liquid

A. Szasz et al, PRX (2020)

B.-B.Chen et al, PRB (2022)

A. O. Scheie et al, arXiv:2406.1777



Schwinger boson theory

$$H = J_1 \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle\langle ij \rangle\rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \cdots \qquad S_i^a = \frac{1}{2} \sum_{\alpha,\beta} b_{i\alpha}^\dagger \sigma_{\alpha\beta}^a b_{i\beta} \qquad \sum_\alpha b_{i\alpha}^\dagger b_{i\alpha} = 1$$

$$H_{MF} \sim \frac{1}{2} \sum_{ij} J_{ij} \left(-A_{ij}^* \sum_{\alpha,\beta} \epsilon_{\alpha\beta} b_{i\alpha} b_{j\beta} + B_{ij}^* \sum_\alpha b_{i\alpha}^\dagger b_{j\alpha} + h.c. \right)$$

$$A_{ij} = \frac{1}{2} \sum_{\alpha,\beta} \epsilon_{\alpha\beta} \langle b_{i\alpha} b_{j\beta} \rangle \qquad B_{ij} = \frac{1}{2} \sum_\alpha \langle b_{i\alpha}^\dagger b_{j\alpha} \rangle$$

$A_{ij} = 0$ U(1) spin liquid

$A_{ij} \neq 0$ Z_2 spin liquid

Bose condensation leads to magnetic ordering

Only gapped spin liquids are allowed

U(1) spin liquid possible only in 3D

Thermodynamic signatures in quantum spin liquids

Does spinon pairing exist ?

No → U(1) Spin Liquid

Emergent U(1) gauge field

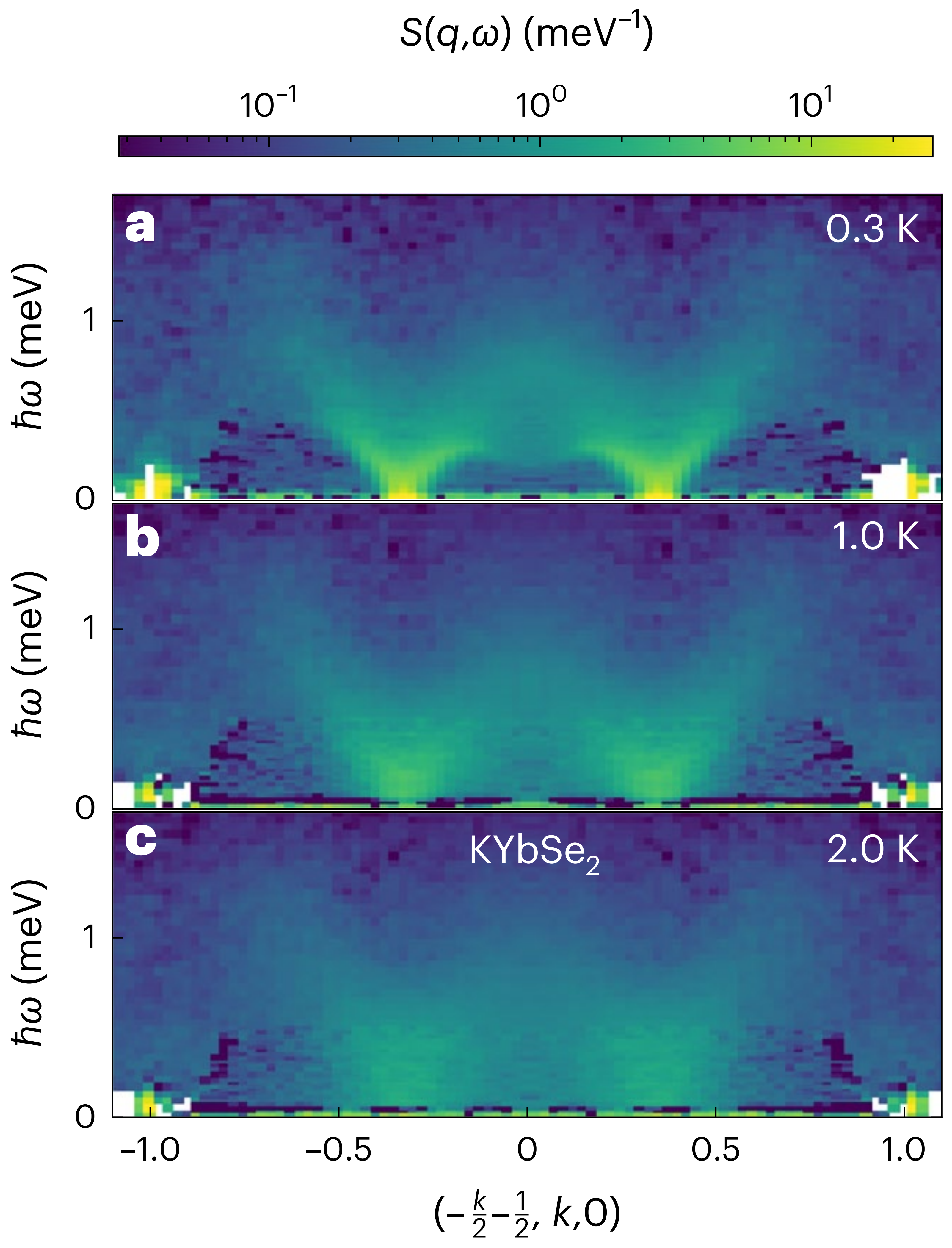
- gapped spinon
  - $C^{3D}_{\text{gauge}}(T) \sim T^3$
  - (unstable in 2D)
- ~~gapless spinon~~
  - $C_{\text{dirac}}(T) \sim T^2$
  - $C_{\text{FS}}(T) \sim T$ mean field
  - $C^{2D}_{FS} \sim T^{2/3}$, $C^{3D}_{FS} \propto T \ln T$ including fluctuations

Yes → Z_2 Spin Liquid

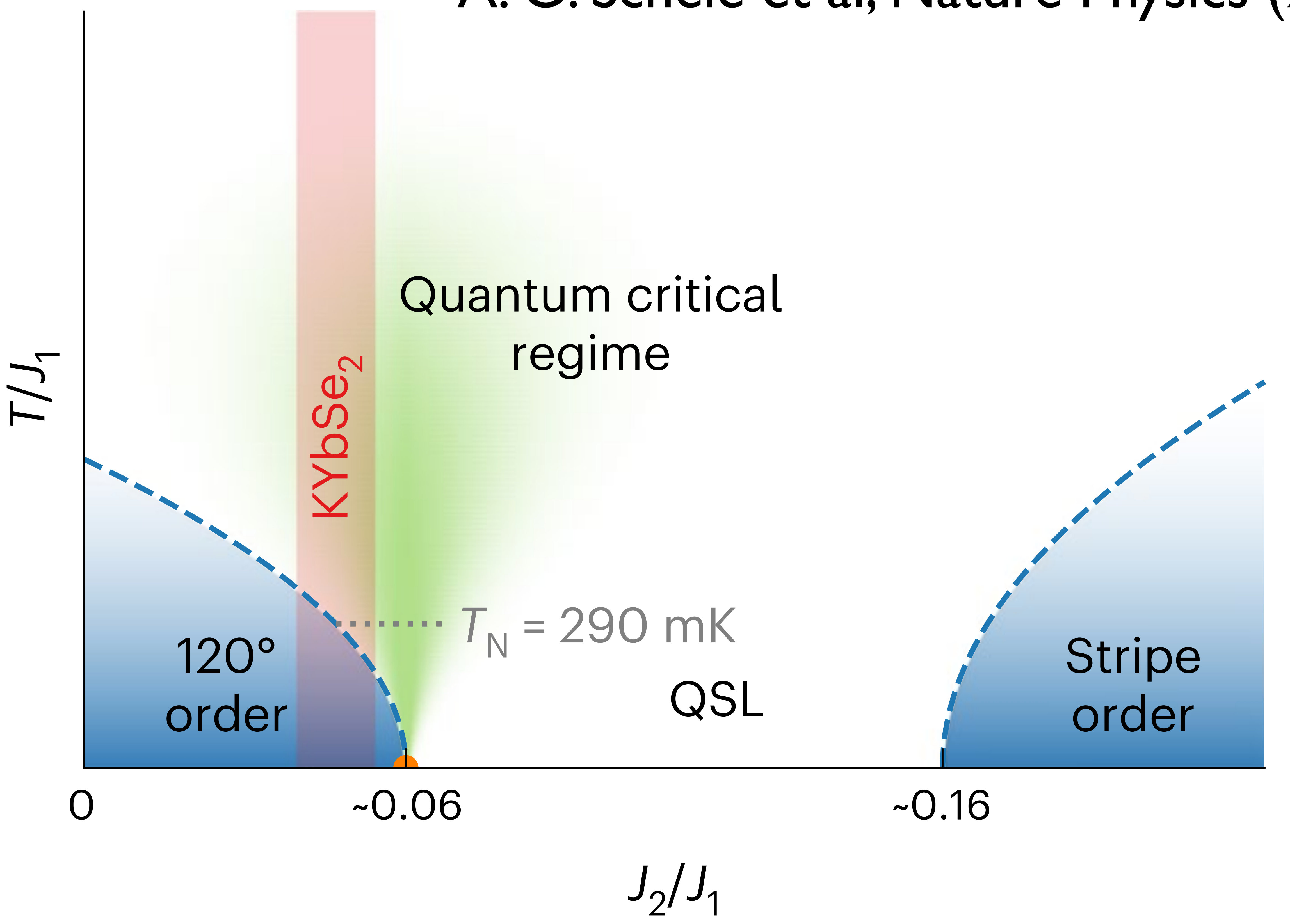
- gapped spinon
  - $C(T) \sim e^{-\Delta/T}$
- ~~gapless spinon~~
  - $C_{\text{d-wave}}(T) \sim T^2$

$KYbSe_2$

Theoretical technique	J_1 (meV)	J_2/J_1
Nonlinear spin waves	0.456 ± 0.013	0.043 ± 0.010
Heat capacity	0.429 ± 0.010	0.037 ± 0.013

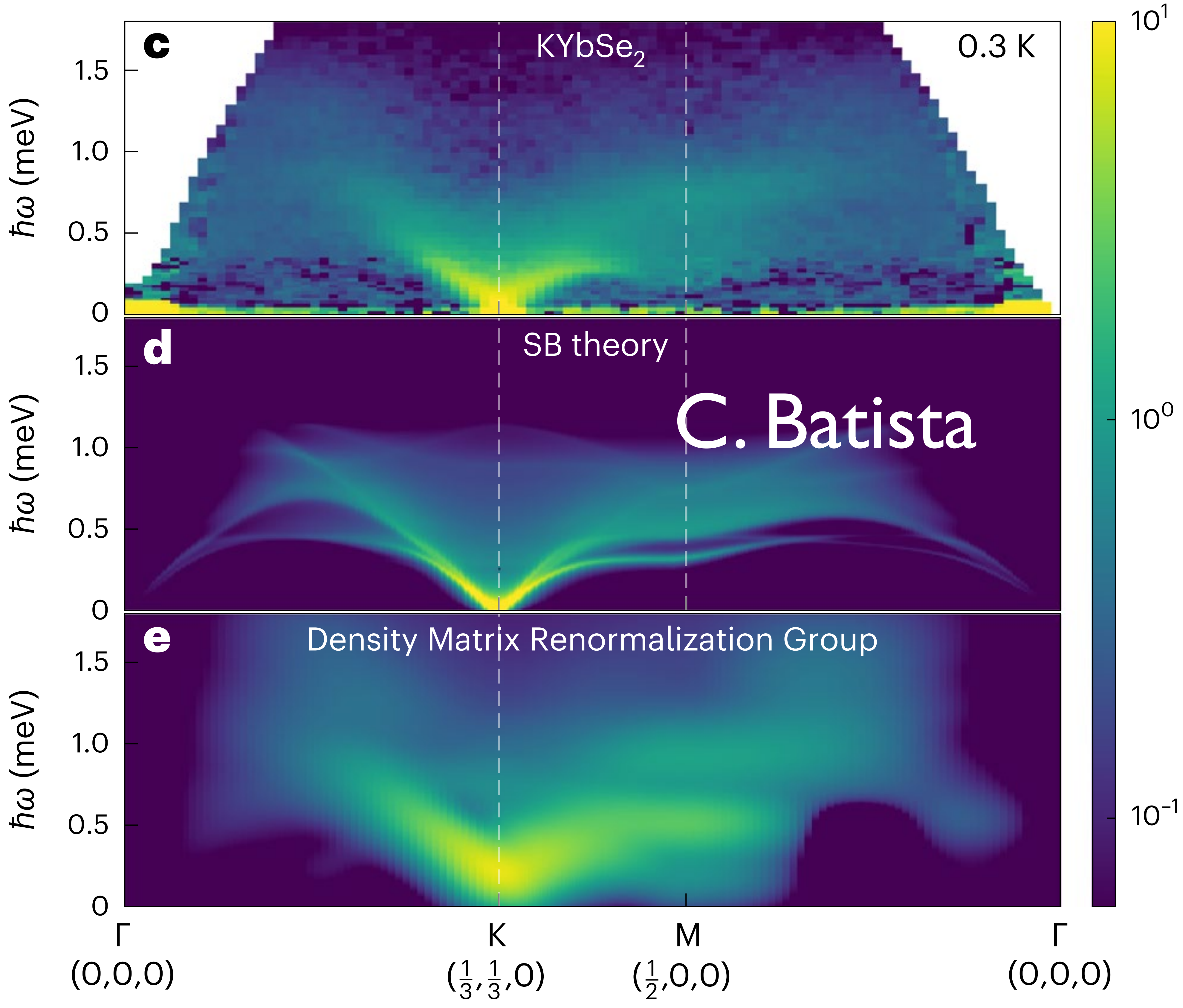


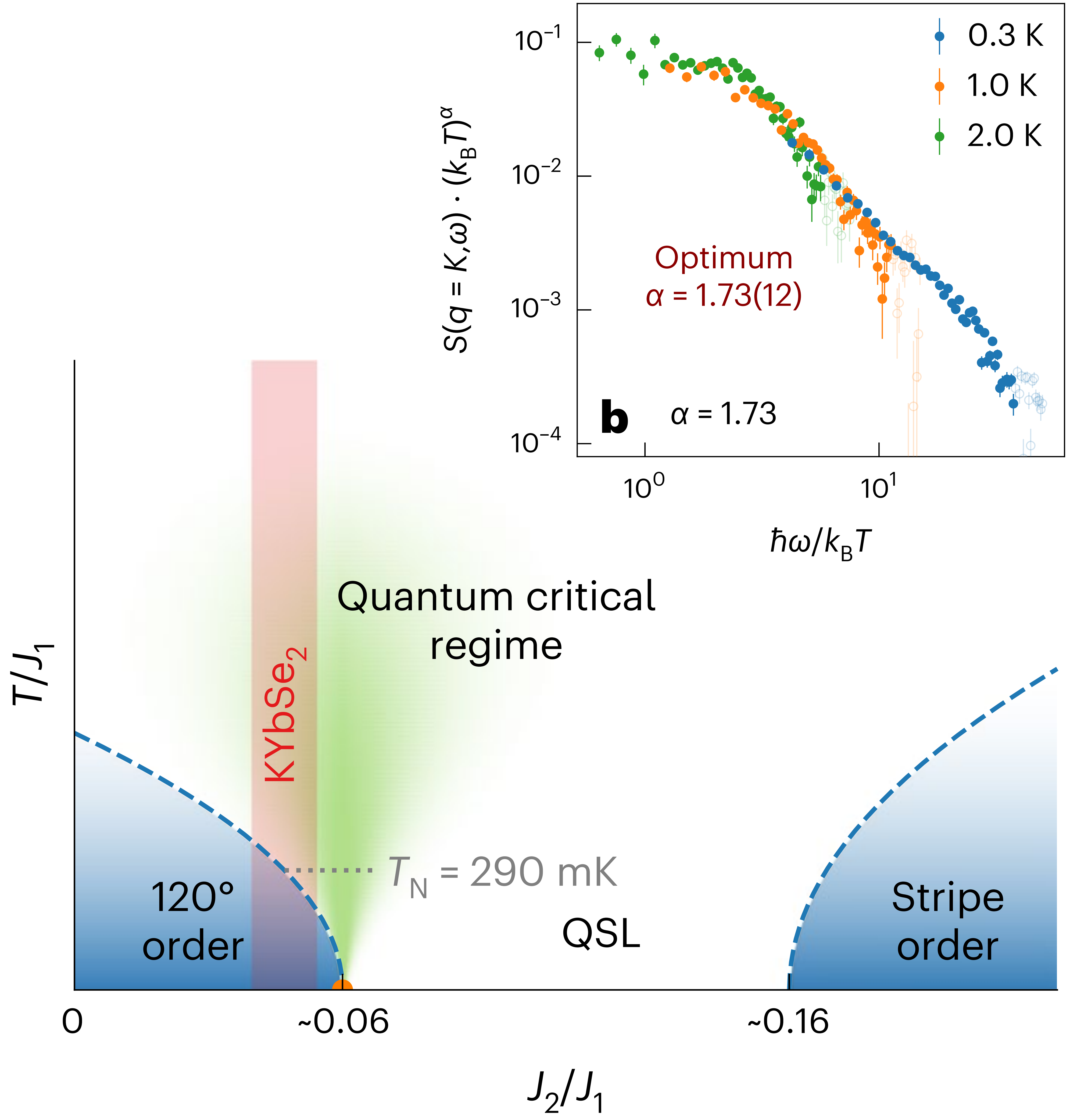
A. O. Scheie et al, Nature Physics (2023)



A. O. Scheie et al, Nature Physics (2023)

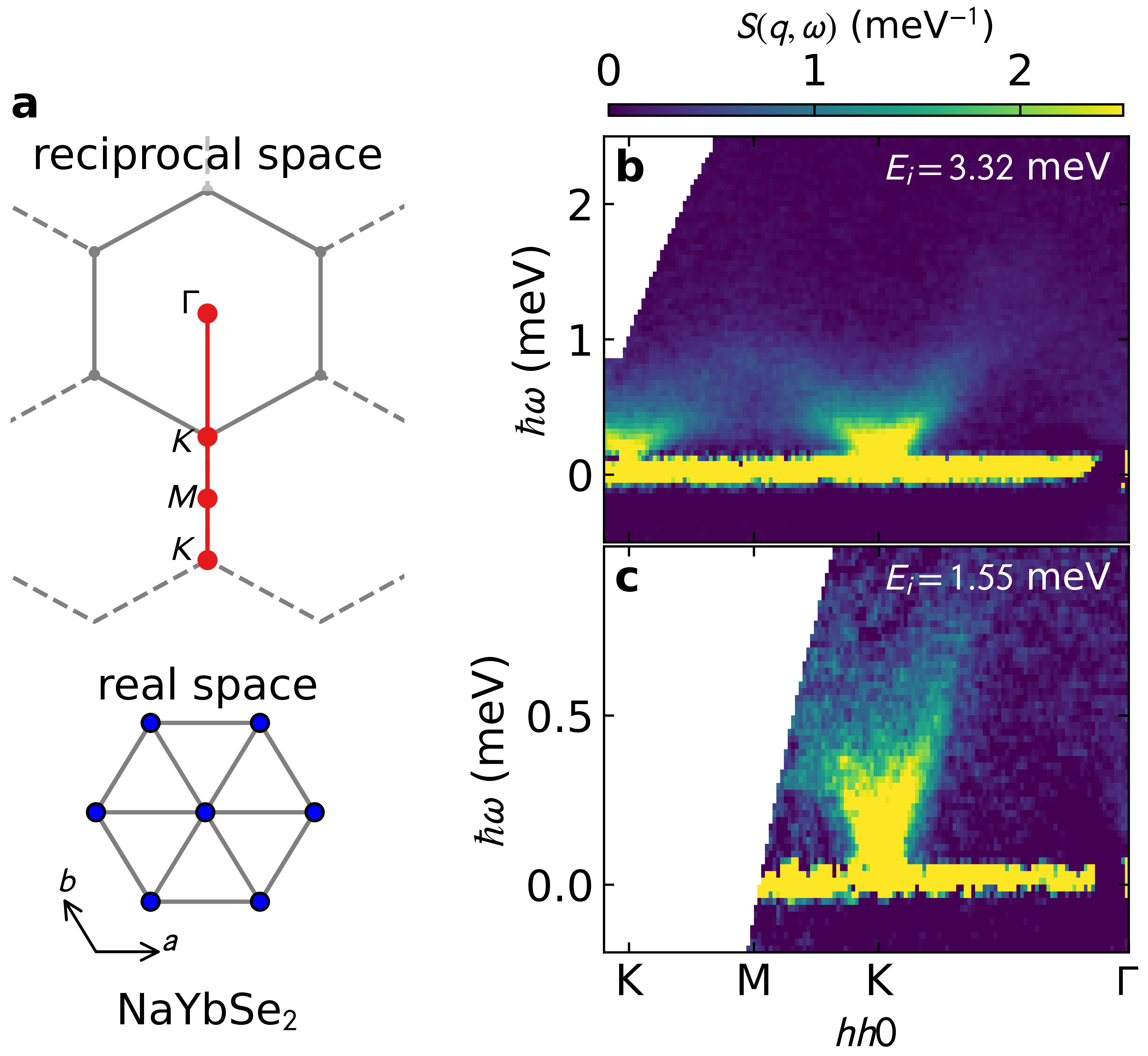
$KYbSe_2$



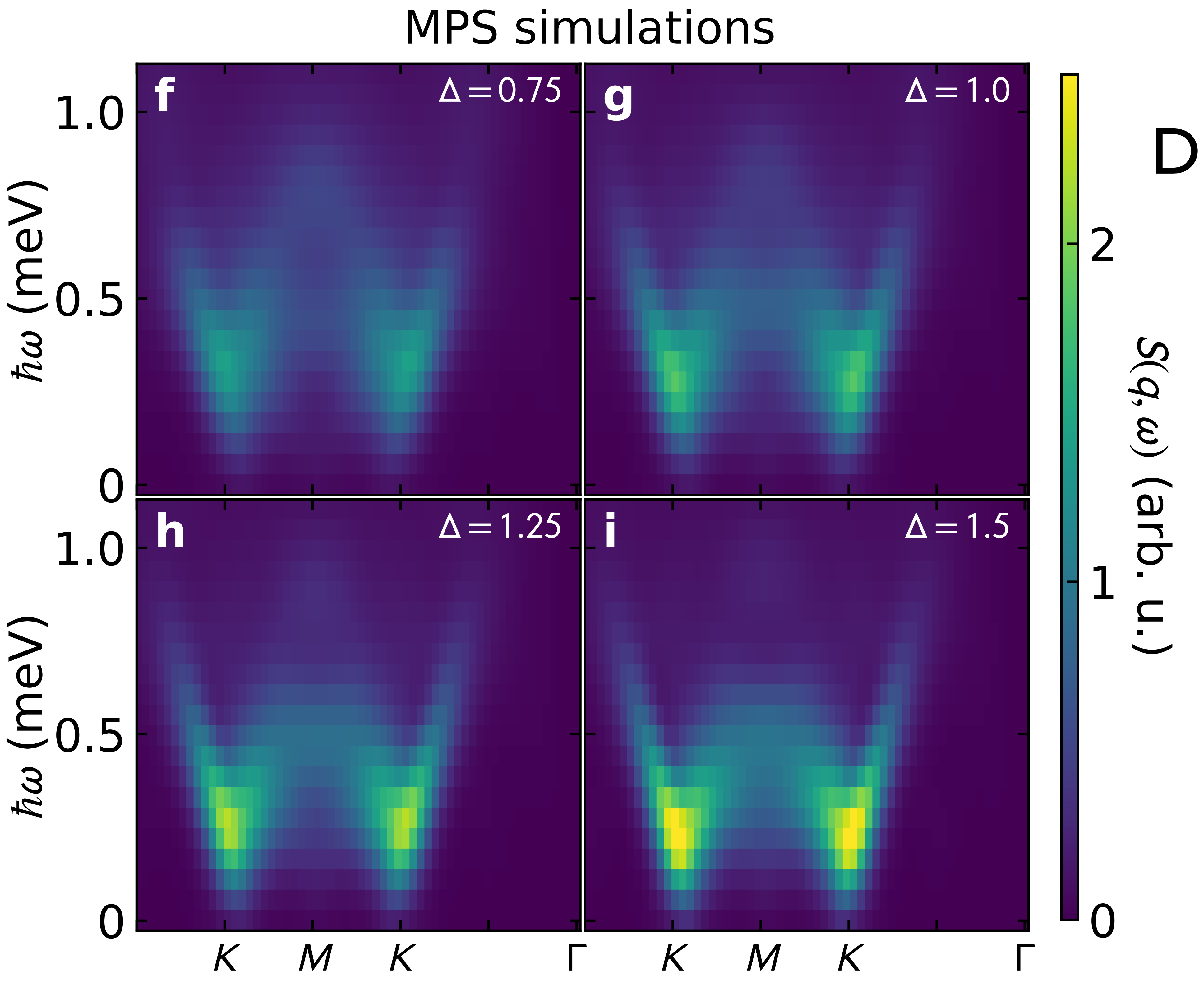


$NaYbSe_2$

No order down to $100\ mK$

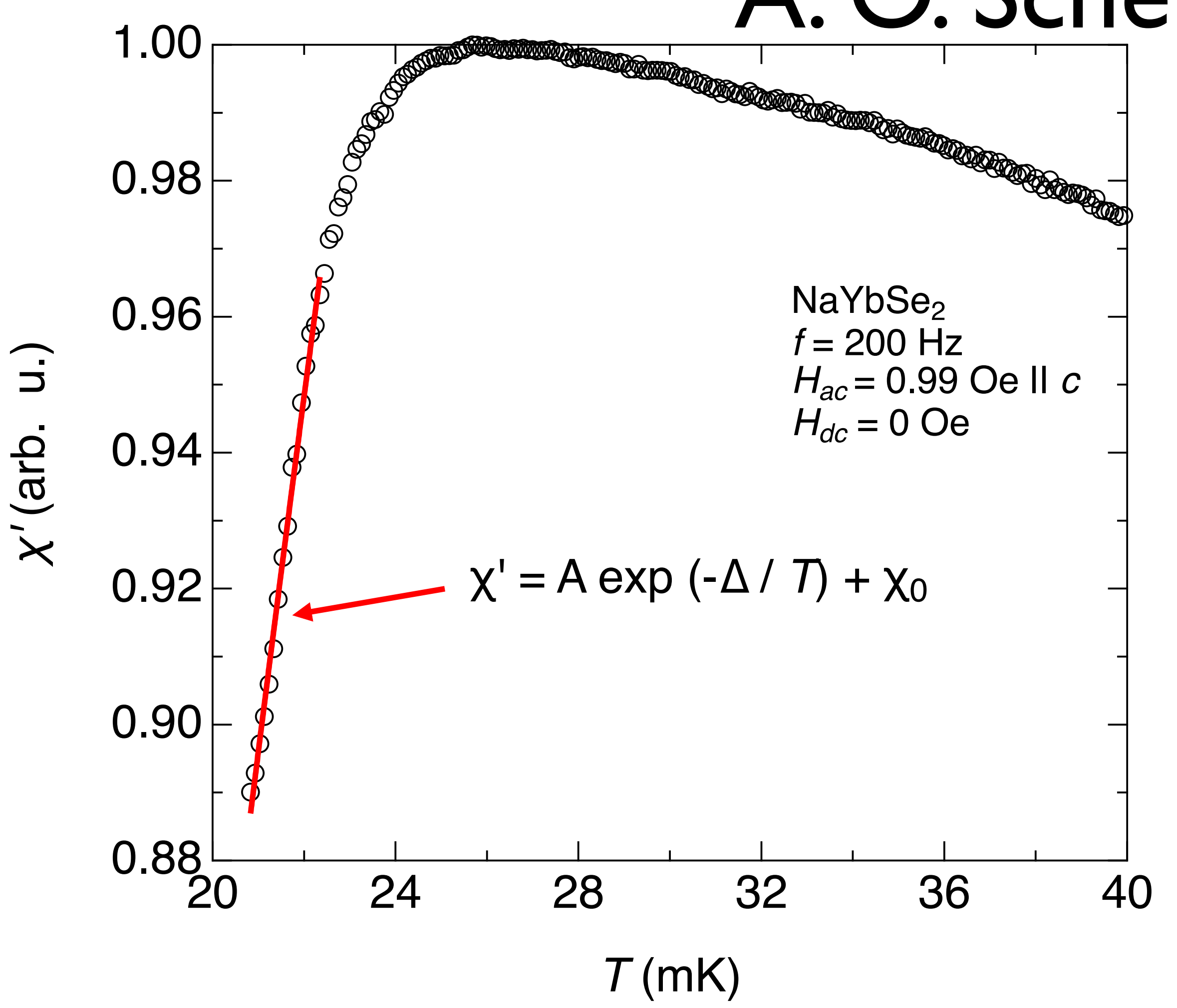


MPS simulations



DMRG

A. O. Scheie et al, arXiv:2406.1777



Excitation gap

$2.1\ \mu\text{eV}.$

$$H = J_1 \sum_{\langle i,j\rangle}(S_i^x S_j^x + S_i^y S_j^y + \Delta S_i^z S_j^z) + J_2 \sum_{\langle\langle i,j\rangle\rangle}(S_i^x S_j^x + S_i^y S_j^y + \Delta S_i^z S_j^z)$$

$$J_2/J_1 = 0.071$$