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M. K. Atatürk

SPATIO-TEMPORAL RECONSTRUCTION OF TEC MAPS USING SINGULAR VALUE DECOMPOSITION

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OUTLINE

- **INTRODUCTION**
- **SINGULAR VALUE DECOMPOSITION**
- **IONOLAB-SVD**
- **RESULTS**
- **CONCLUSIONS AND FUTURE DIRECTIONS**

INTRODUCTION (1)

- **Total Electron Content (TEC) is a key parameter for characterization of the ionospheric structure.**
- **TEC is defined as the line integral of electron density on a ray path**

$$\mathrm{TEC} = \int \mathrm{N_e dl} \ \ (\mathrm{TECU}) \qquad 1\ \mathrm{TECU} = 10^{16}\ \mathrm{el/m^2}$$

- **TEC can typically be estimated at only a limited number of spatial points.**
- **Accurate, reliable, robust and cost-efficient TEC mapping in near-real time is extremely important for nowcasting, forecasting and prediction purposes in modeling, and mitigation of errors in space-based positioning, navigation, guidance and remote sensing systems on LEO and MEO satellites.**
- **In this study, we introduce an interpolation/reconstruction method that works in the most cost-efficient closed form of Least Squares: IONOLAB-SVD.**
- **The bases are obtained using Singular Value Decomposition (SVD).**
- **We will present an example case over Europe where the model matrices are prepared within Solar Activity groups for every 24 h in a month.**
- **In order to demonstrate the accuracy, reliability and robustness of the proposed algorithm, we have used Global Ionospheric Maps (GIM) as ground truth.**

Singular Value Decomposition (SVD) (1)

- **In linear algebra, Singular Value Decomposition (SVD) is a factorization of a real or complex matrix into a rotation, followed by a rescaling, followed by another rotation.**
- **SVD generalizes the eigen-decomposition of a square normal matrix with an orthonormal eigenbasis to any size matrix.**
- **The SVD of an $N_l \times N_d$ matrix $\underline{\underline{G}}$ is a factorization of the form**

$$\underline{\underline{G}}_{N_l \times N_d} = \underline{\underline{U}}_{N_l \times N_d} \underline{\underline{\Sigma}}_{N_d \times N_d} \underline{\underline{V}}^T_{N_d \times N_d}$$

where $\underline{\underline{U}}$ is an $N_l \times N_d$ unitary matrix; $\underline{\underline{V}}$ is an $N_d \times N_d$ unitary matrix; and for real $\underline{\underline{G}}$, they are guaranteed to be real orthogonal matrices; $\underline{\underline{\Sigma}}$ is a diagonal matrix whose entries σ are uniquely determined by $\underline{\underline{G}}$.

The matrix $\underline{\underline{U}}$ is composed of left singular vectors $\underline{u}_{n_d}$ which are basis vectors for the column space of $\underline{\underline{G}}$. The matrix $\underline{\underline{V}}$ contains the right singular vectors $\underline{v}_{n_d}$, which make up the row space of $\underline{\underline{G}}$.

Singular Value Decomposition (SVD) (2)

- **SVD decomposes the model matrix into physical basis vectors:**

$$\underline{\underline{G}} = [\underline{u}_1 \dots \underline{u}_{n_d} \dots \underline{u}_{N_d}] \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & 0 & \\ & & \sigma_{N_\sigma} & & & \\ & & & \sigma_{N_\sigma+1} & & \\ & 0 & & & \ddots & \\ & & & & & \sigma_{N_d} \end{bmatrix} [\underline{v}_1 \dots \underline{v}_{n_d} \dots \underline{v}_{N_d}]^T$$

- **The singular values in matrix $\underline{\underline{\Sigma}}$ are arranged in descending order.**

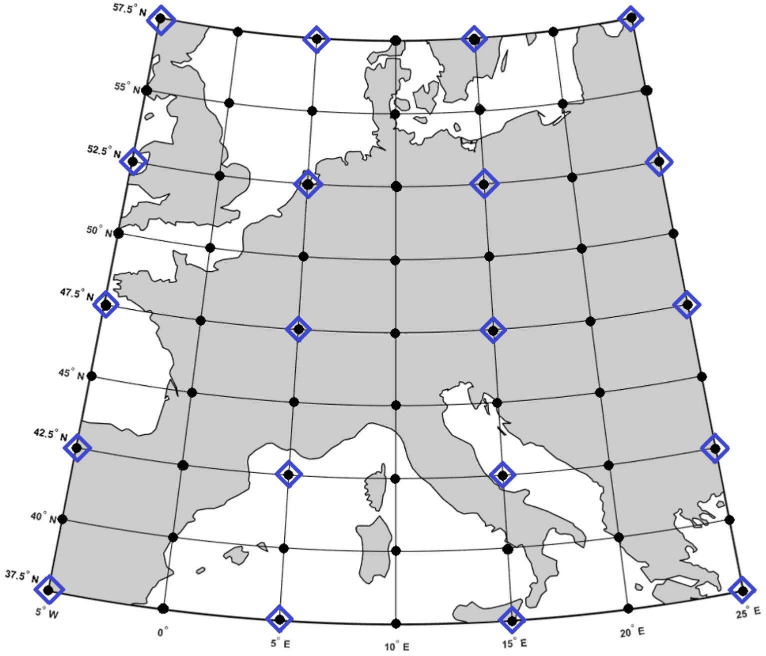
- **The decomposition produces an output that contains the maximum energy with the minimum basis vectors representing the underlying physics instead of a standard mathematical basis set.**

IONOLAB-SVD (1)

- The initial step of IONOLAB-SVD is the formation of model matrix for a given region.
- The entries of $\underline{\underline{G}}$ can be chosen to represent the underlying physical structure.
  - Years can be grouped in terms of Solar Activity (SA) level
  - The region to be discretized into a grid.
  - Values of TEC in a desired region can be arranged
    - monthly-daily, or
    - monthly-hourly
  - Each map in latitude and longitude can be vectorized using lexicographical indexing.
  - These vectors make up the columns of $\underline{\underline{G}}$

IONOLAB-SVD (2)

- TEC maps with desired resolution in space and time are lexicographically indexed as column vectors to generate Model Matrix ($\underline{\underline{G}}$).
- First N_σ column space basis vectors of the $\underline{\underline{G}}$ matrix that have the highest energies compose of the ‘Signal Subspace’, where the smaller rest represent the ‘Noise Subspace’.



$$\underline{\underline{G}} = \left[\underline{u}_1 \cdots \underline{u}_{n_d} \cdots \underline{u}_{N_d}\right] \left[\begin{array}{ccc|ccc} \sigma_1 & & & & & \\ & \ddots & & & 0 & \\ & & \sigma_{N_\sigma} & & & \\ \hline & & & \sigma_{N_\sigma+1} & & \\ & 0 & & & \ddots & \\ & & & & & \sigma_{N_d} \end{array}\right] \left[\underline{v}_1 \cdots \underline{v}_{n_d} \cdots \underline{v}_{N_d}\right]^T$$

$$\{\sigma_1, \ldots, \sigma_{N_\sigma}\} \subseteq \{\sigma_1, \ldots, \sigma_{N_d}\}$$

IONOLAB-SVD (3)

- The 'measurements' of TEC on a set of coordinates on the map can be represented by a sampling matrix from the vectorized TEC map, $\underline{g}_{N_l \times 1}$.

$$\underline{y}_{N_s \times 1} = \underline{\underline{S}}_{N_s \times N_l} \underline{g}_{N_l \times 1}$$

- In reconstruction step, the measurement vector is formulated to be obtained from left significant singular basis multiplied with coefficients to be determined in Least Square sense as

$$\underline{y} = \underline{\underline{S}}\, \underline{\bar{g}} = \sum_{n_d=1}^{N_\sigma} \underline{\underline{S}}\, \underline{u}_{n_d} \alpha_{n_d}$$

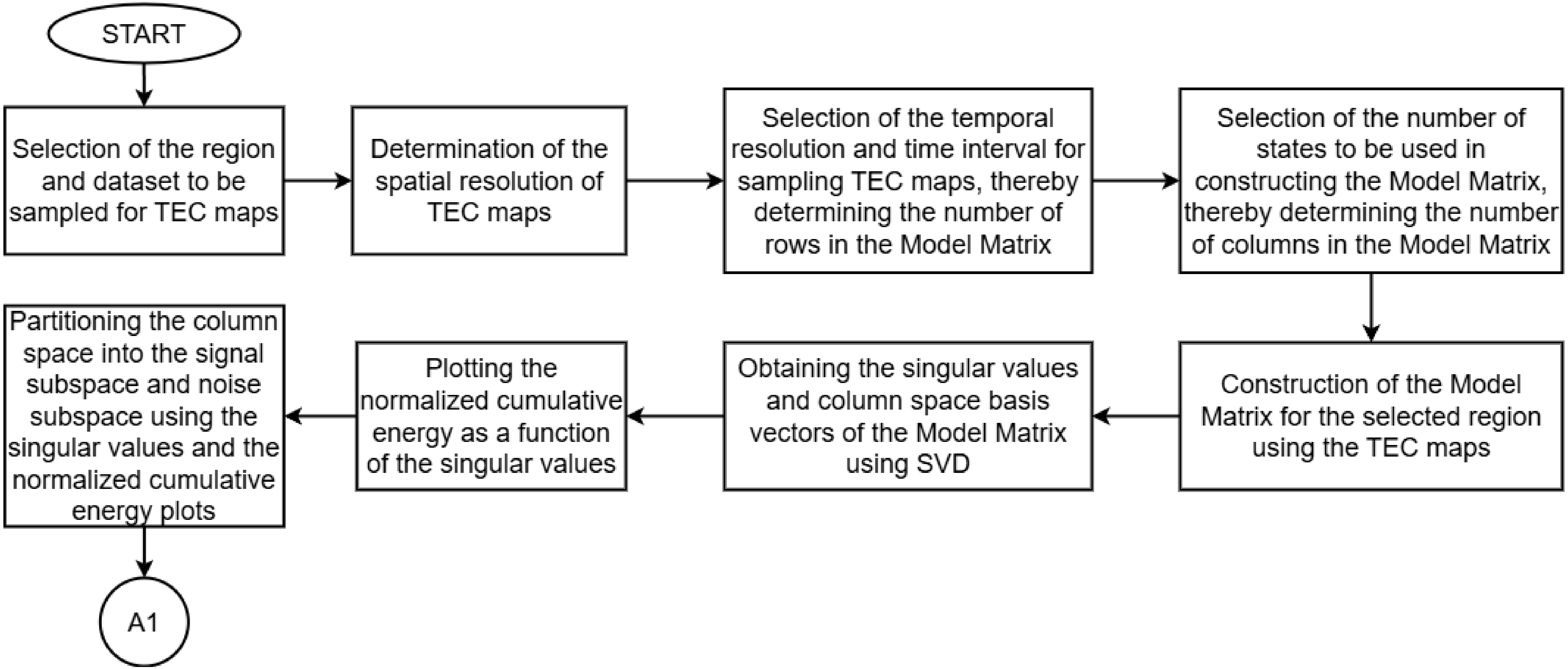
$$\underline{y} = \underbrace{\left[\underline{\underline{S}}\, \underline{u}_1 \cdots \underline{\underline{S}}\, \underline{u}_{n_d} \cdots \underline{\underline{S}}\, \underline{u}_{N_\sigma}\right]}_{\underline{\underline{B}}} \underline{\alpha}$$

$$\underline{\underline{B}} = \left[\underline{\underline{S}}\, \underline{u}_1 \cdots \underline{\underline{S}}\, \underline{u}_{n_d} \cdots \underline{\underline{S}}\, \underline{u}_{N_\sigma}\right]_{N_s \times N_\sigma}$$

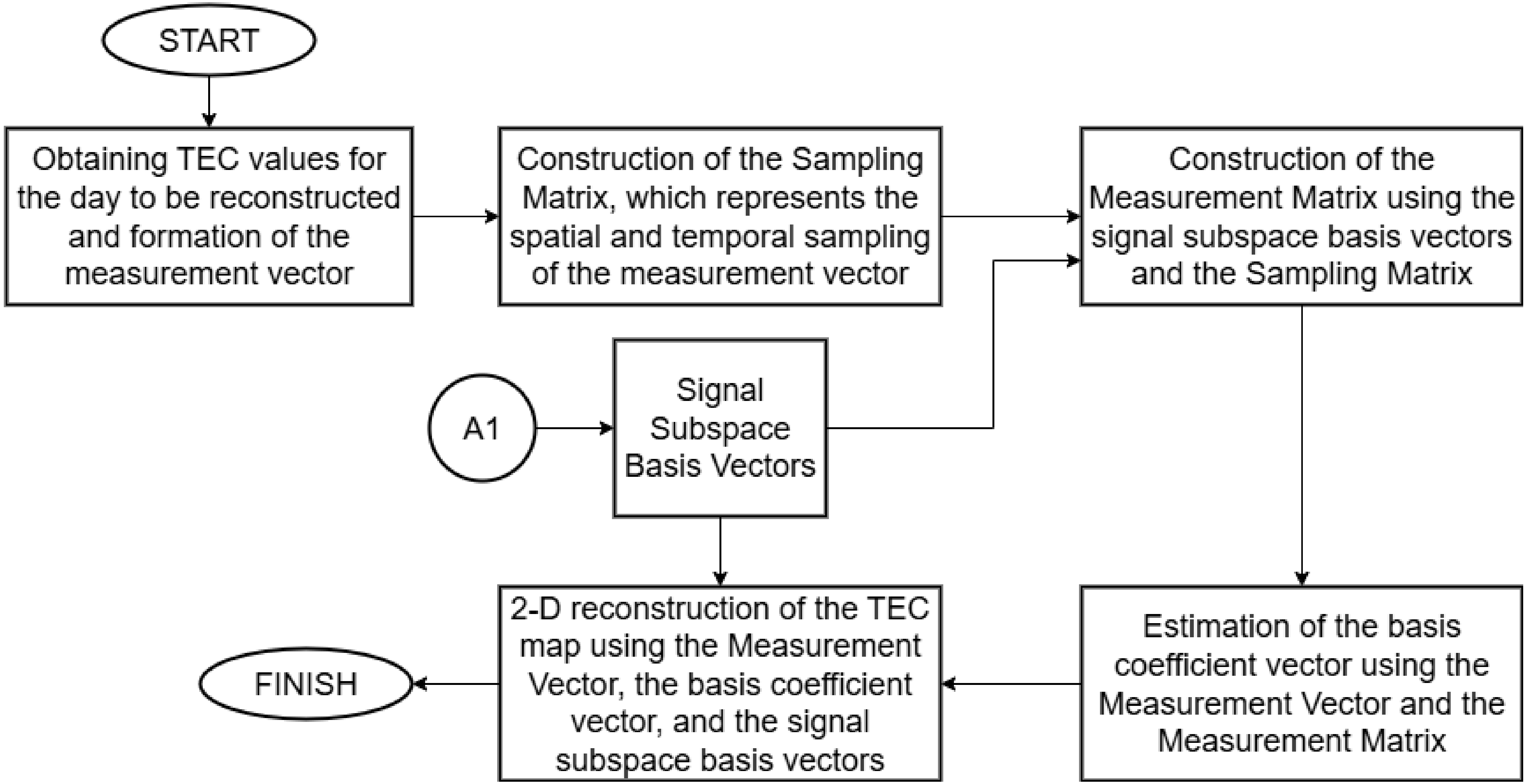
$$\underline{\hat{\alpha}} = \min_{\underline{\alpha}}((\underline{y} - \underline{\underline{B}}\ \underline{\alpha})^T (\underline{y} - \underline{\underline{B}}\ \underline{\alpha}))$$

$$\underline{\hat{\alpha}} = (\underline{\underline{B}}^T \underline{\underline{B}})^{-1} \underline{\underline{B}}^T \underline{y}$$

IONOLAB-SVD (4)
Model Matrix Generation and Basis Vector Extraction

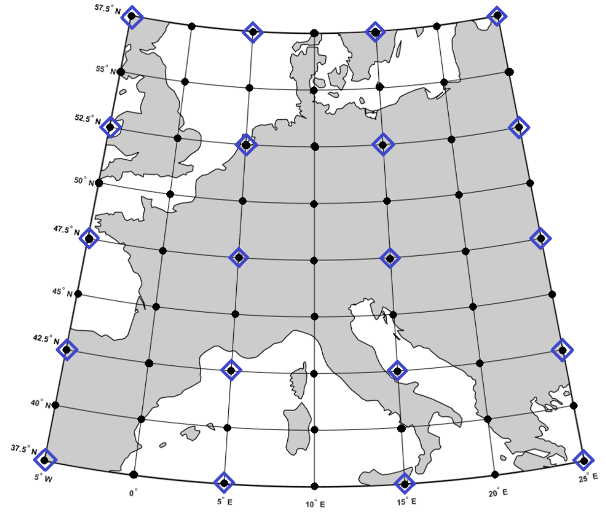


IONOLAB-SVD (5)
Reconstruction of TEC Maps

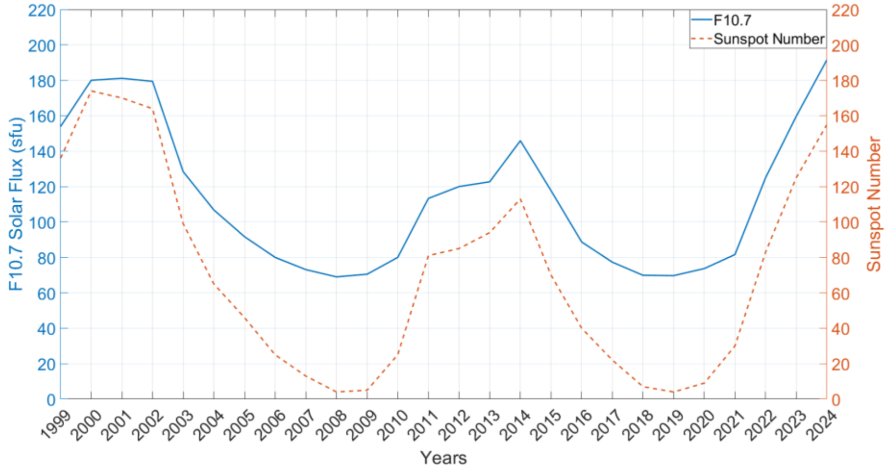


RESULTS (1)

- **European Region is selected for the application example.**
- **The years from 2000 to 2024 are grouped with respect to their solar activity level as**



Black dots indicate the coordinates whose TEC values are included in Model Matrix.
Blue diamonds denote the measurement coordinates.



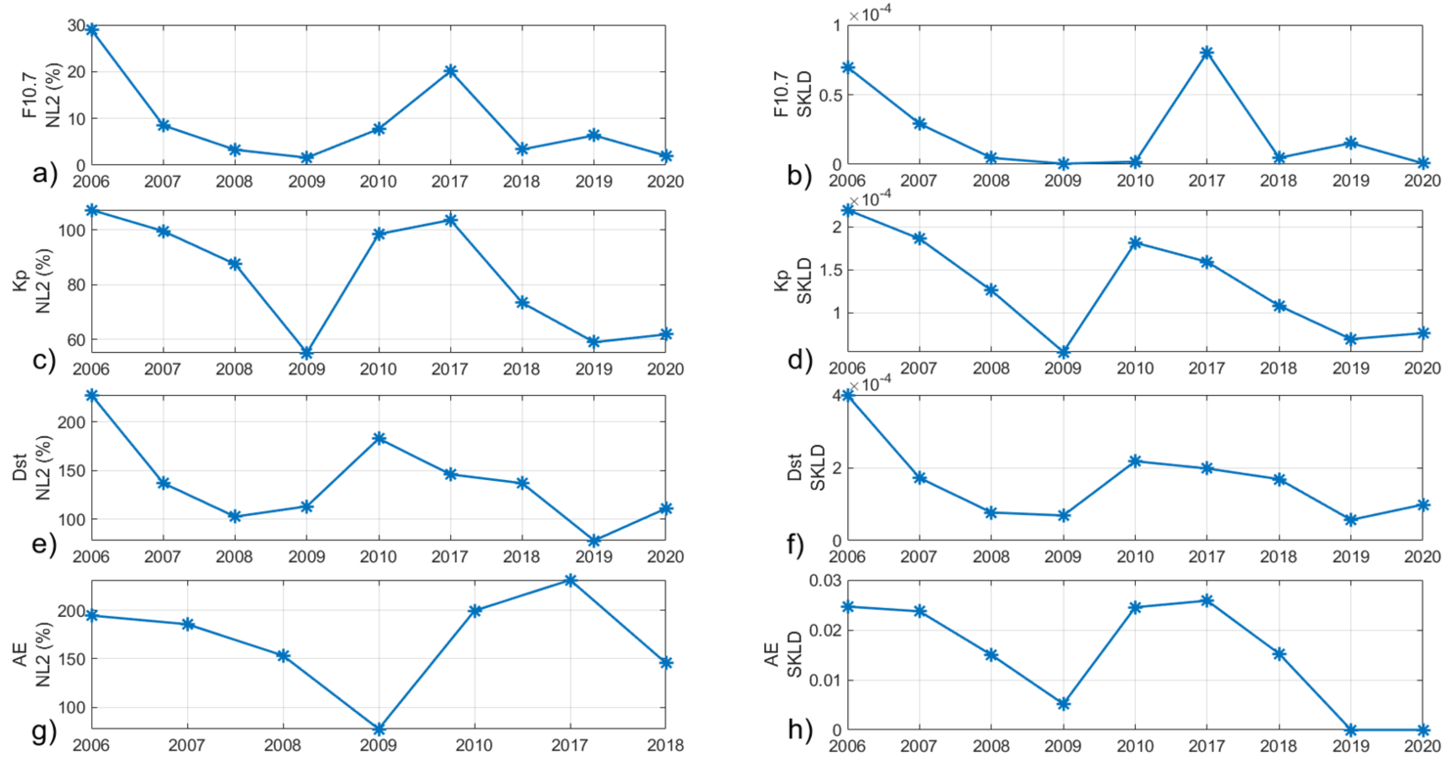
Annual average of F10.7 solar flux and Sunspot Number (SSN).

Solar Low (SL) ($F10.7 \leq 80$) ($SSN \leq 25$)	Solar Near Low (SNL) ($80 \leq F10.7 \leq 100$) ($25 \leq SSN \leq 50$)	Solar Moderate (SM) ($100 \leq F10.7 \leq 130$) ($50 \leq SSN \leq 100$)	Solar High (SH) ($140 \leq F10.7$) ($110 \leq SSN$)
2007	2005	2003	1999
2008	2016	2004	2000
2009	2006*	2011	2001
2017	2010*	2012	2002
2018		2013	2014
2019		2015	2023
2020		2022	2024
2006*			
2010*			

Years with an asterisk can be grouped in both Solar Low and Solar Near Low.

RESULTS (2)

- In each SA group, years are classified into model matrix, test and validation sets:



Example: Comparison of geomagnetic indices and F10.7 solar proxy with the respective median curve for April in SL group using Normalized L2 norm and Symmetric Kullback Leibler Distance (SKLD).

	SL	SNL	SM	SH
Model Matrix Years	2006 2009 2017	2005 2006 2010	2003 2004 2012	1999 2000 2001
Test Years	2018	2016	2013	2014
Validation Years	2020	2021	2022	2002

The model matrix is formed using data from the same month across three selected years.

RESULTS (3)

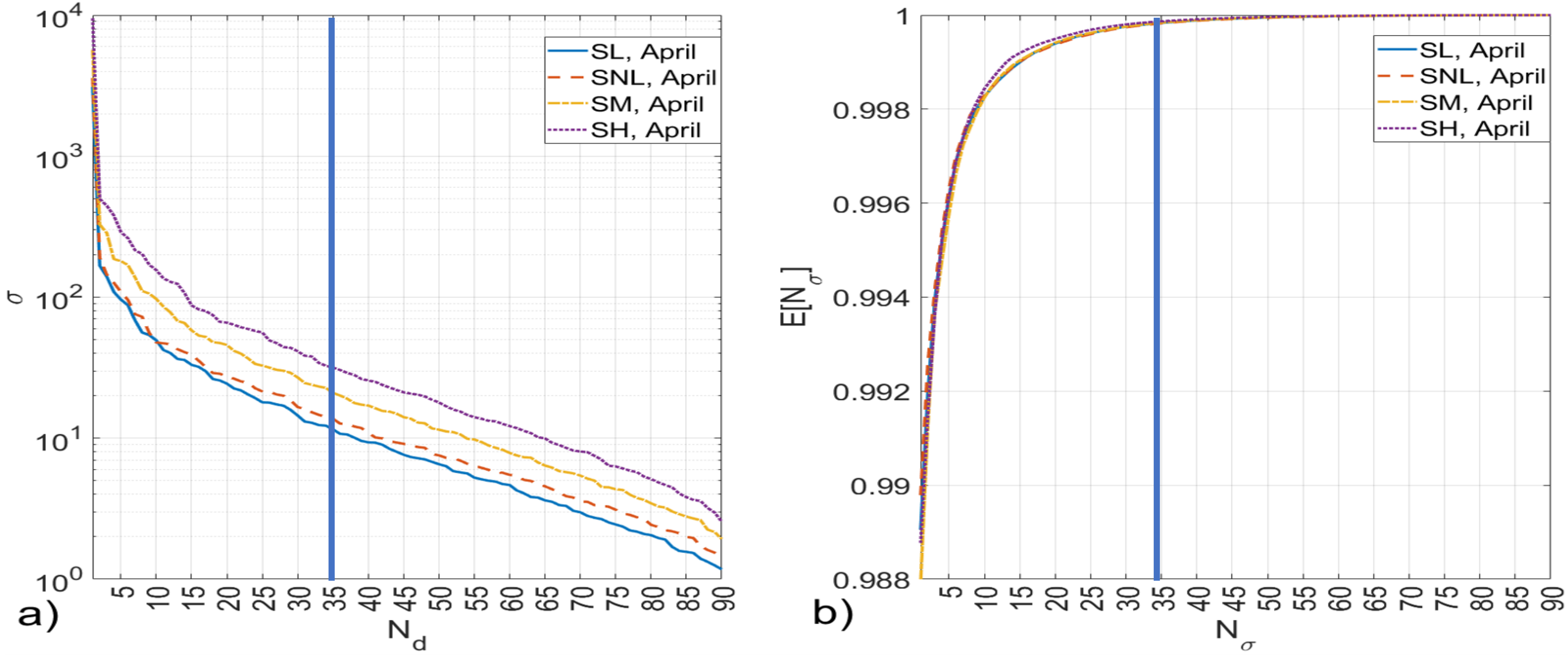
- Singular values of the Model Matrices

Cumulative Energy

$$E[N_\sigma] = \frac{\sum_{n_d=1}^{N_\sigma} \sigma_{n_d}^2}{\sum_{n_d=1}^{N_d} \sigma_{n_d}^2} \times 100 \ \ (\%)$$

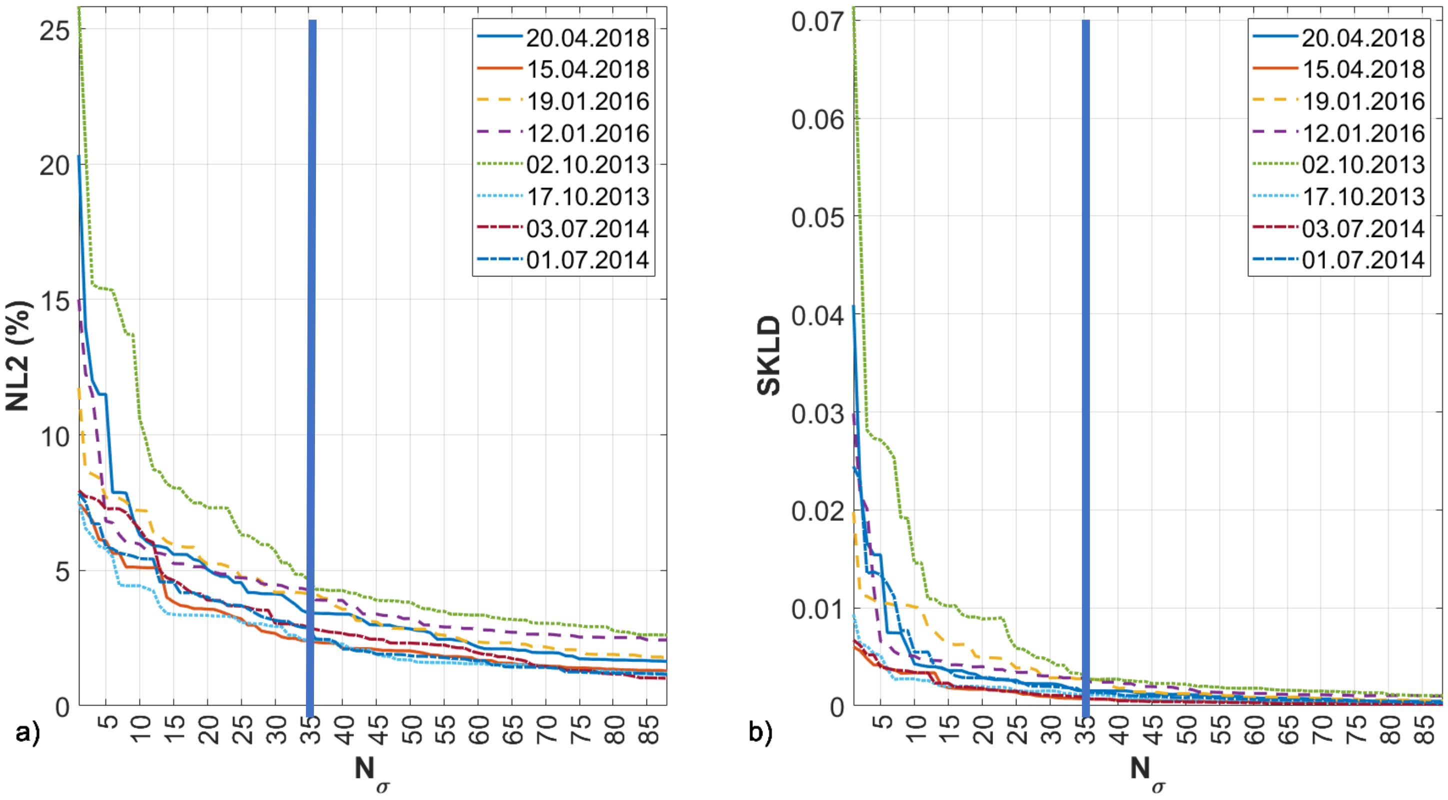
Example Singular Values for April in each SA group

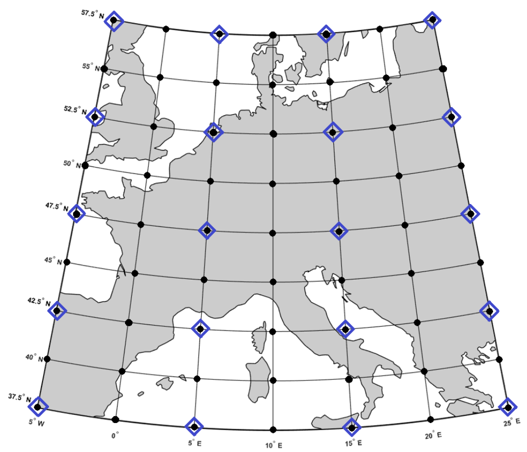
Nσ is chosen to be 35.



RESULTS (4)

- 4 calm and 4 disturbed days are selected as test days.
- The measurement vectors are formed by using the JPL-TEC values from coordinates denoted by blue diamonds.





**Differences between JPL-TEC and IONOLAB-SVD maps for various quiet and disturbed days with respect to the number of significant singular values in Normalized L2 in a)
SKLD in b)**

RESULTS (5)

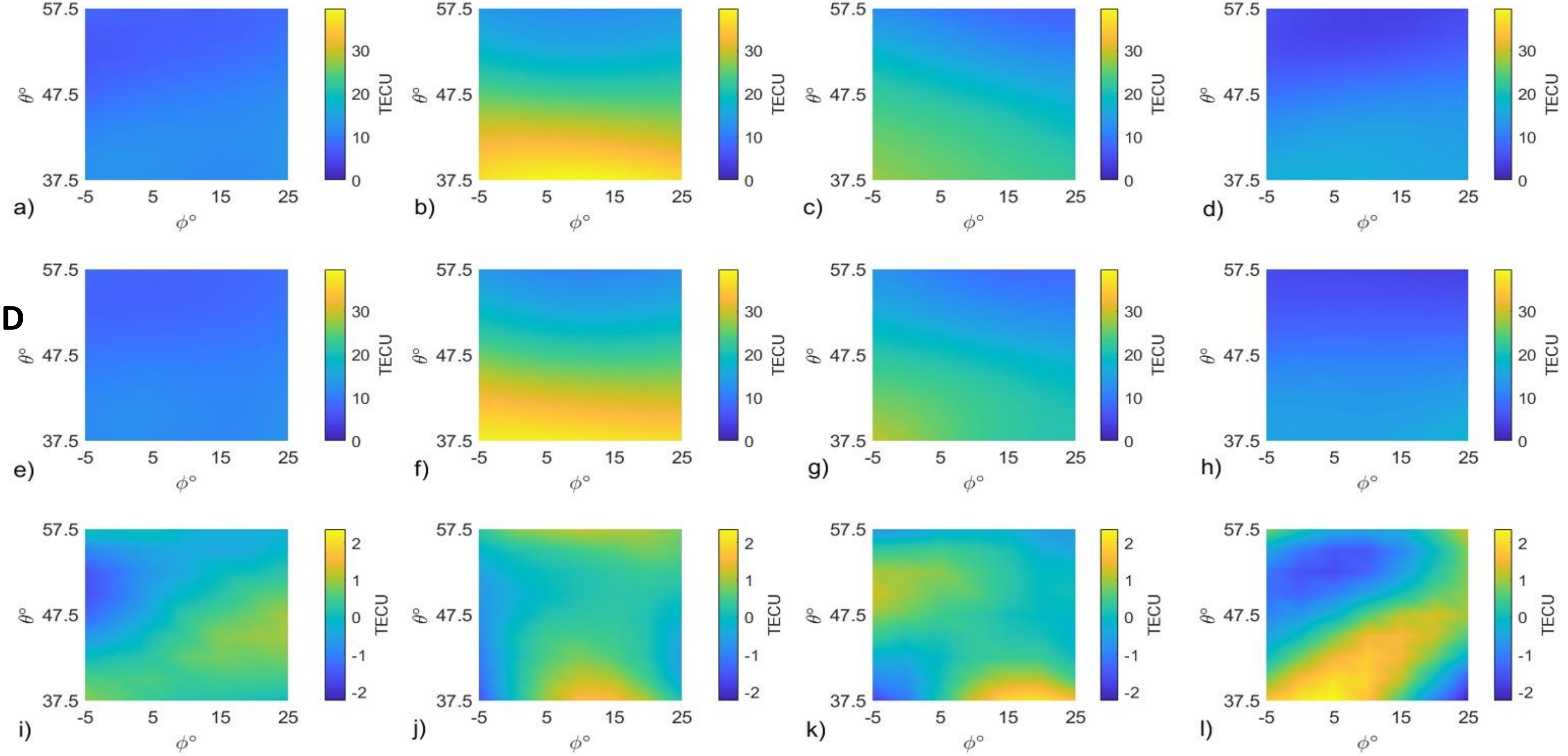
October 2, 2013 Disturbed Day
Kp = 7.7; Dst = -72 nT; AE = 108.3; F10.7= 129.9 sfu

04:00 UT | **12:00 UT** | **18:00 UT** | **22:00 UT**

JPL-TEC maps

IONOLAB-SVD maps

Difference maps



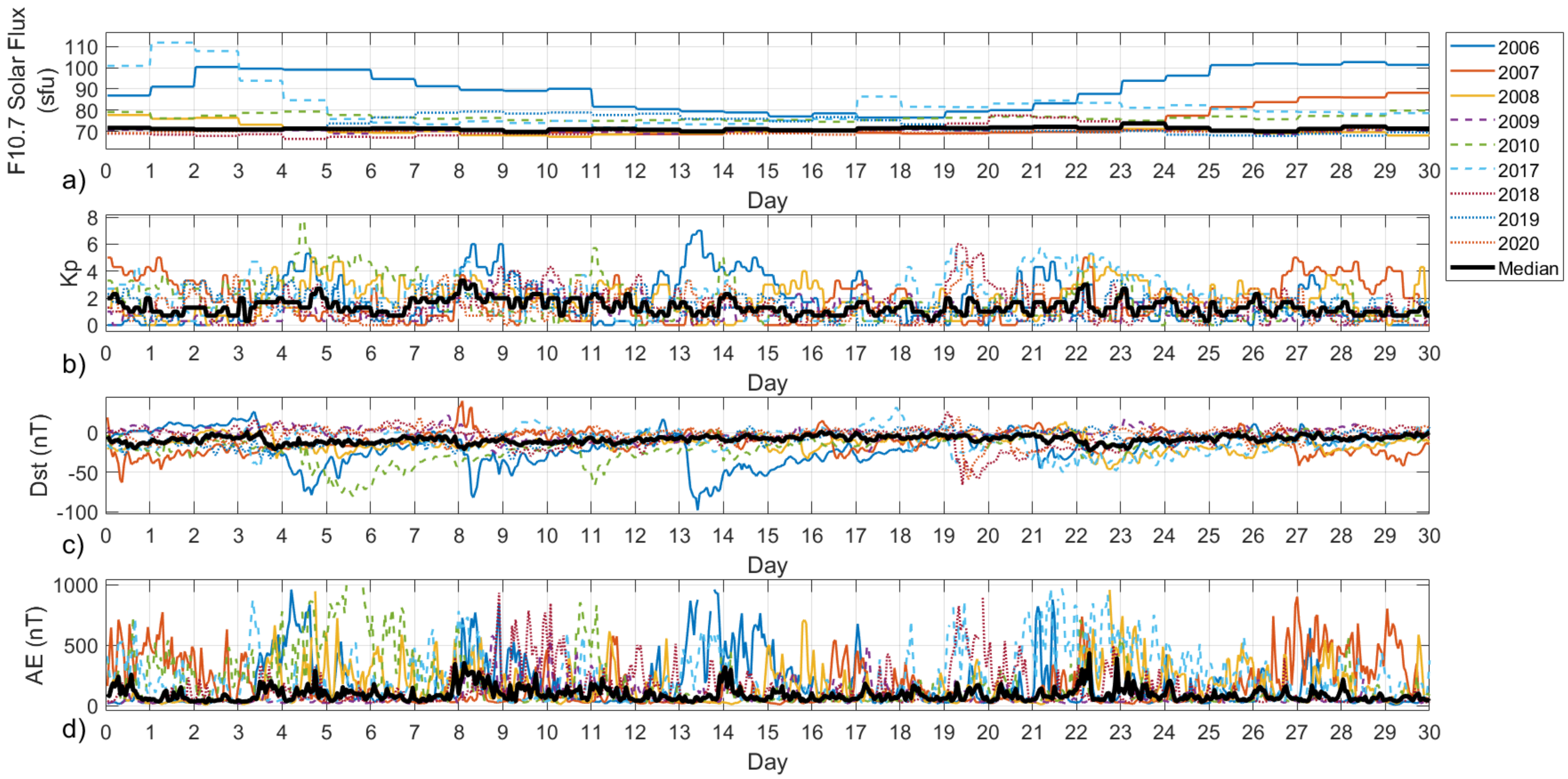
Conclusion

- **In this study, a closed-form reconstruction algorithm for 2-D reconstruction of TEC maps is proposed operating on user-defined regions, space and time resolutions.**
- **The developed algorithm is computationally efficient.**
- **Carefully constructed Model Matrices, encompassing diverse conditions, facilitate accurate reconstruction for both quite and disturbed days.**
- **The compherensive, multi-level classification of years for Model Matrix construction is a novel aspect of this study.**

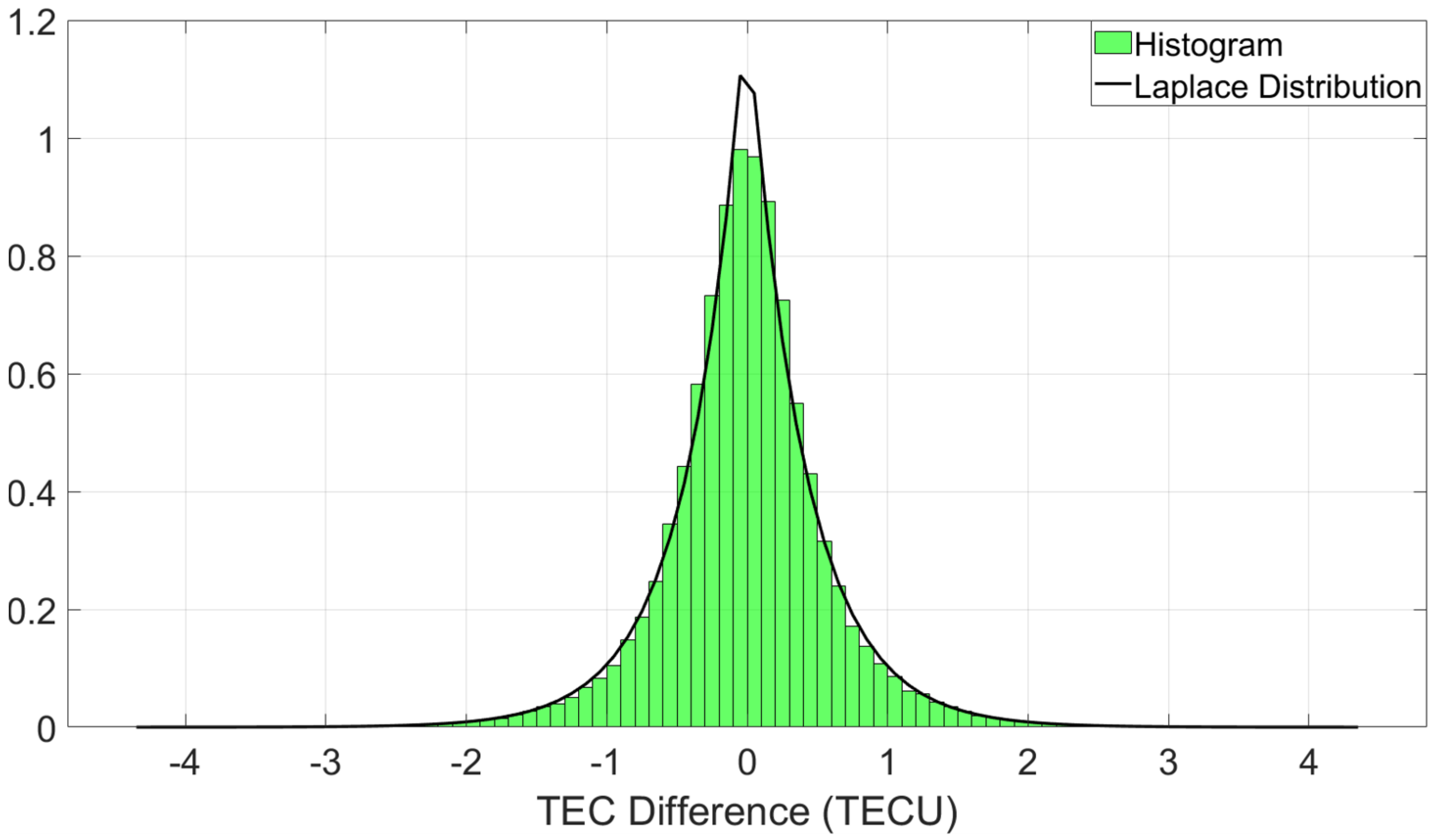
Future Work

- **Adoption of the algorithm for near-real time applications using Global Positioning System (GPS) and/or ionosonde measurements.**
- **Extending the algorithm for short-term TEC prediction using time-series TEC measurements as inputs.**
- **Analyzing singular values and temporal evolution of basis vectors from diffent Model Matrices to gain insight into ionospheric temporal variability.**
- **Potentially scaling the approach to generate Global Ionospheric Maps by adapting Model Matrix construction.**

Thank you

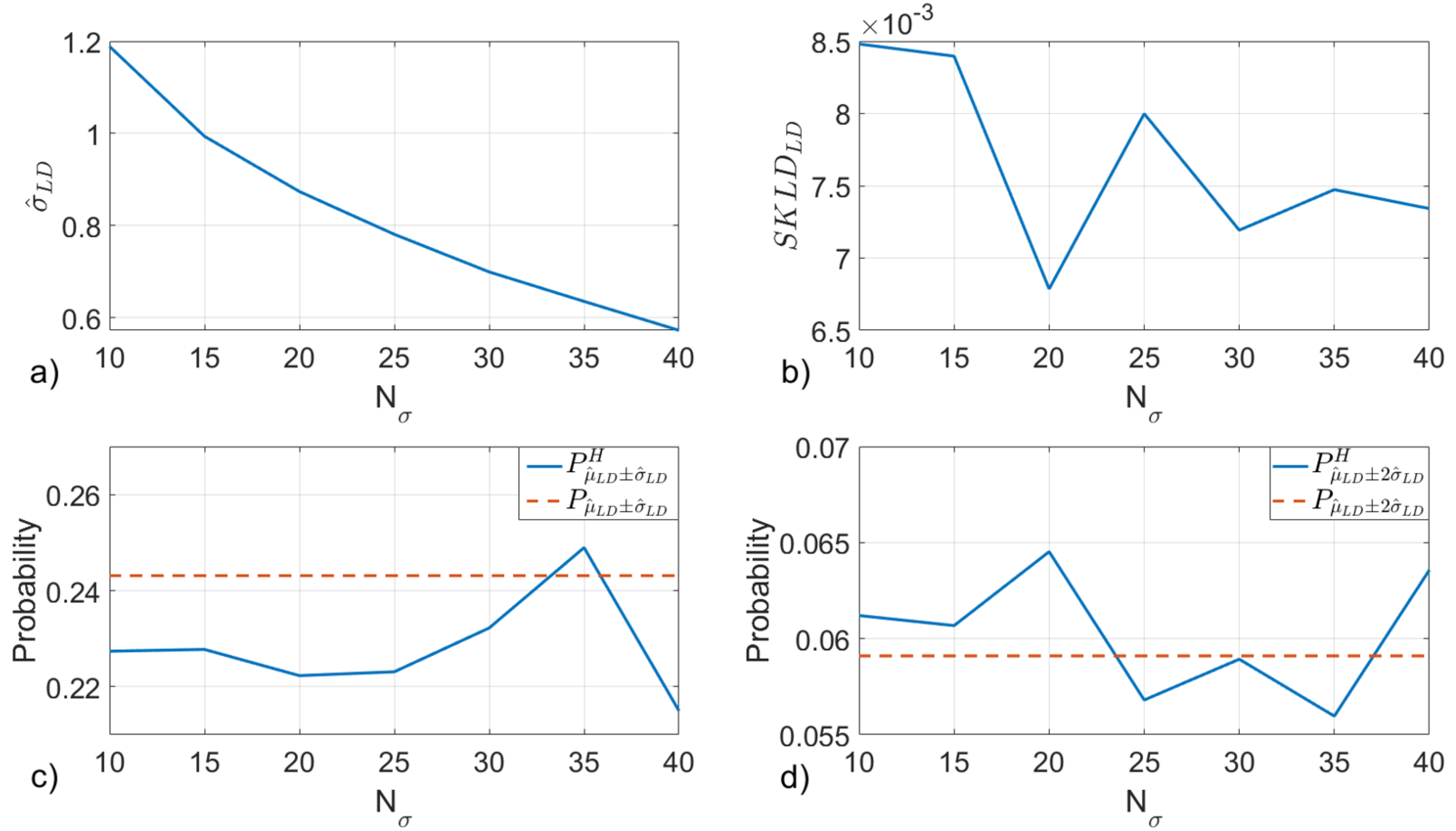


Hourly average of geomagnetic indices and Daily F10.7 Proxy for April in SL group. Black curve indicates the median curve.



Histogram for N_σ = 35 and Laplace distribution with estimated parameters for 96 test days.

Statistical Properties



Statistical properties for the fit of Laplace PDF for TEC differences, for N_σ = 10, 15, 20, 25, 30, 35, 40.
a) Standard deviation of the PDF, b) SKLD between fitted PDF and emprical distribution.
c) Probability for one sigma deviation, d) Probability for two sigma deviation