Lecture 1: parton construction for TOs

Monday, October 27, 2025 10:05 AM

Plan: how to construct different TOs using partons?

can parton w.f. be exact ground states of microscopic models?

what are the effective field theory for parton states? (Chern-Simons)

Two examples $\begin{cases} \text{Laughlin states} \longrightarrow \text{partons for superfluid, SPT, FQHs} \\ \text{toric code} \longrightarrow \text{partons for 16-fold way of } \mathbb{Z}_2 \text{ TOs} \end{cases}$ in $d=2$

1st example of TOs: FQHE at $\nu = 1/3$.

$$\hat{H} = \frac{(\vec{p} - e\vec{A})^2}{2m} + \sum_{i<j} V(\vec{r}_i - \vec{r}_j)$$

in symmetric gauge: $\vec{A}(x,y) = \frac{B}{2}(y, -x, 0)$

$$\text{LLL}: \quad \phi_m(x,y) \sim z^m e^{-|z|^2/4l_B^2} \qquad l_B = \sqrt{\frac{\hbar}{|e|\cdot B}}.$$

$m \geq 0$. $\quad z = x + iy$

below we will neglect the $e^{-|z|^2/4l_B^2}$ factor since it exists for all orbitals.

Laughlin state $\Psi_{\text{Laughlin}}(z_1, z_2, \cdots, z_N) \sim \prod_{i<j}(z_i - z_j)^3$

filling fraction $\nu = \lim_{N\to\infty} \frac{N}{\#\text{ of flux quanta}} = 1/3$

Recall that for a filled LLL.

$$\Psi_{LLL} = \det \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & & z_N \\ z_1^2 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ z_1^N & z_2^N & & z_N^N \end{vmatrix} = \prod_{i<j}(z_i - z_j)$$

$$= \langle 0 | \prod_j C(z_j) \, | \text{filled LLL} \rangle$$

so $\Psi_{\text{Laughlin}} \sim \Psi_{LLL}^3$

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$$= \langle 0 | \prod_j f_1(z_j) f_2(z_j) f_3(z_j) \left| \begin{matrix} \text{filled} \\ LLL \end{matrix} \right\rangle_1 \otimes \left| \begin{matrix} \text{filled} \\ LLL \end{matrix} \right\rangle_2 \otimes \left| \begin{matrix} \text{filled} \\ LLL \end{matrix} \right\rangle_3$$

$$= \langle 0 | \prod_j C(z_j) \, | \text{parton} \rangle$$

$C(z) \equiv \prod_{l=1}^{3} f_l(z)$. $\quad |\text{parton}\rangle = \left| \begin{matrix} \text{filled} \\ LLL \end{matrix} \right\rangle_1 \otimes \left| \begin{matrix} \text{filled} \\ LLL \end{matrix} \right\rangle_2 \otimes \left| \begin{matrix} \text{filled} \\ LLL \end{matrix} \right\rangle_3$,

Exact parent Hamiltonian : $V(\vec{r}) = \partial_z \partial_{\bar{z}} \delta^{(2)}(z)$

project into LLL $\longrightarrow$ "Haldane pseudopotential"

("The QHE" Prange, Girvin, 1990)

Parton construction :

microscopic d.o.f. is composed of parton operators,
(bosonic/fermionic) (bosonic/fermionic)

we consider non-interacting bands of partons to describe the strongly interacting states of microscopic d.o.f.'s

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below we consider $\nu = 1/2$ Laughlin state of hardcore bosons for the context of QSLs,

$$b(z) = f_1(z) f_2(z)$$

Constraint : $f_1^{+} f_1 - f_2^{+} f_2 = 0$

How to impose the constraint ?

Lagrangian multiplier : $\underset{\downarrow}{a_0}(f_1^{+} f_1 - f_2^{+} f_2)$

a U(1) gauge field

$$\mathcal{L} = \mathcal{L}_0 + a_\mu j^\mu.$$

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$$j^0 = f_1^\dagger f_1 - f_2^\dagger f_2 . \qquad \vec{j} \sim -i\left(f_1^\dagger \vec{\nabla} f_1 - f_2^\dagger \vec{\nabla} f_2\right)$$

3-currents in 2+1-D of the $U(1)$ gauge field.

$$a_\mu = (a_0, -a_x, -a_y)$$

① if $f_{1,2}$ both have Chern number 1 for a filled LLL.

$$\left(\sigma_{xy} = e^2/h\right)$$

$$\int dx\,dy\,dz\, \mathcal{L}_{eff}[a_\mu] = -\log\left(\int Df_1 Df_1^\dagger Df_2 Df_2^\dagger\, e^{-\int(\mathcal{L}_0 - a_\mu j^\mu)\,dx\,dy\,d\tau}\right)$$

$$\mathcal{L}_{eff}[a_\mu] = \frac{2\times 1}{4\pi} a_\mu \partial_\nu a_\rho \epsilon^{\mu\nu\rho} \equiv \frac{2}{4\pi} a\,d\,a$$

"Chern-Simons term"

Hall conductance?

charge density $\rho_0 = c^\dagger c = f_1^\dagger f_1 = f_2^\dagger f_2$

we can assign charge $1/2$ to both f_1 & f_2.

$$A_\mu J^\mu = \frac{1}{2} A_\mu \left(J_1^\mu + J_2^\mu\right)$$

$$\mathcal{L} = \mathcal{L}_0 + a_\mu j^\mu + A_\mu J^\mu$$

$$\mathcal{L}_{eff}[a_\mu, A_\mu] = \left(a+\frac{A}{2}\right)d\left(a+\frac{A}{2}\right) + \left(a-\frac{A}{2}\right)d\left(a-\frac{A}{2}\right)$$

$$= 2a\,d\,a + \frac{1}{2} A\,d\,A$$

$$\longrightarrow \sigma_{xy} = \frac{1}{2}\frac{e^2}{h}$$

* the constraint gauge field a_μ is not the dual gauge field.

in the dual description. $J_1 = \frac{da_1}{2\pi}$ (3-current of parton f_1)

$J_2 = \frac{da_2}{2\pi}$ (- - - - - - - - - - f_2)

$a\,da_1 + a_2\,da_2$ ... $a_1 da_1 + a_2 da_2$ $\quad a(da_1 - da_2)$

$$J_2 = \frac{da_2}{2\pi} \quad (\cdots\cdots J_2)$$

$$\mathcal{L} = \frac{a_1 da_1 + a_2 da_2}{4\pi} - a(J_1 - J_2) = \frac{a_1 da_1 + a_2 da_2}{4\pi} - \frac{a(da_1 - da_2)}{2\pi}$$

integrate out $a_\mu \longrightarrow \quad a_1 = a_2 \longrightarrow \mathcal{L}_{dual} = \frac{2}{4\pi} a_1 da_1 - A\frac{da_1}{2\pi}$

boson 3-current $J = J_1 = J_2 = \frac{da_1}{2\pi}$

* anyon statistics: $\mathcal{L}_{CS} = \frac{1}{4\pi}\sum_{IJ} a_I \, da_J \, K_{IJ} - \sum_I a_I \cdot j_I$, $\quad K = K^* = K^T$ (integer-valued)

$N = \dim K$

particles labeled by integer-valued vector $\vec{l} = (l_1, l_2, \cdots, l_N)$

half-braiding statistics (self) $\theta_{\vec{l}} = \pi \vec{l}^T K^{-1} \vec{l}$. $\quad \theta_1 = \frac{\pi}{2}$ for $K=2$ (semion)

full-braiding statistics: $\Theta_{\vec{l},\vec{l}'} = 2\pi l^T K^{-1} l'$

particles form a lattice $\sim \mathbb{Z}^N$

Wen, 9506066

electrons (no fractional statistics): columns of K

$\downarrow$

primitive vectors of the lattice.

$\#$ of different anyons $= |\det K|$ (size of unit cell)

② $SU(2)$ gauge field $\hat{a}_\mu = \sum_{\alpha=1}^{3} a_\mu^i \hat{\sigma}_i$ $\quad$ Pauli matrices

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \longrightarrow W \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad W \in SU(2).$$

$c = f_1 f_2 \longrightarrow c$ $\quad$ Wen & Zee 9711223

$SU(2)_q$ (level q) Chern-Simons terms:

$$\mathcal{L}_{SU(2)_q} = \frac{q}{4\pi} \epsilon^{\mu\nu\rho} \mathrm{Tr}\left(a_\mu \partial_\nu a_\rho + \frac{i}{3} a_\mu a_\nu a_\rho\right)$$

field strength $f_{\mu\nu} = [-i\partial_\mu - a_\mu, -i\partial_\nu - a_\nu]$

$$= i(\partial_\mu a_\nu - \partial_\nu a_\mu) + [a_\mu, a_\nu]$$

$q = 1, 2, 3, \ldots$ ("level") is the Chern # of each parton band.

Quantizing $SU(2)_q$ CS term on torus $\longrightarrow$ $(q+1)$-fold g.s.d.

$U(1)_2$ CS term on torus $\longrightarrow$ 2-fold g.s.d.

Consider Chern numbers C_1 for f_1 partons, C_2 for f_2 partons.

$$\mathcal{L}_{eff} = \frac{\mathrm{sgn}(C_1)}{4\pi} \sum_{i=1}^{|C_1|} a_i d a_i + \frac{\mathrm{sgn}(C_2)}{4\pi} \sum_{j=1}^{|C_2|} b_j d b_j + \frac{1}{2\pi} \lambda \left(\sum_i d a_i - \sum_j d b_j\right) + \frac{1}{2\pi} A \sum_i d a_i$$

Ugk

$$\det \begin{pmatrix} 1 & & & & 1 \\ & \ddots & & & \vdots \\ & & -1 & & \vdots \\ & & & \ddots & 1 \\ & & & -1 & 1 \\ 1 & \cdots & \cdots & 1 & 0 \end{pmatrix} = \mathrm{sgn}(C_1)^{|C_1|} \cdot \mathrm{sgn}(C_2)^{|C_2|} \cdot (-C_1 - C_2)$$

$\parallel$

K matrix

$$\Rightarrow \text{g.s.d.} = |C_1 + C_2| \quad \text{on torus}$$

$$\sigma_{xy} = -\frac{C_1 C_2}{C_1 + C_2}$$

$$-\frac{C_2}{4\pi} \lambda d\lambda - \frac{C_1}{4\pi} (\lambda + A) d(\lambda + A)$$

$$= -\frac{C_1 + C_2}{4\pi} \lambda d\lambda - \frac{C_1}{2\pi} \lambda dA - \frac{C_1}{4\pi} A dA$$

$$= -\frac{C_1 + C_2}{4\pi} \left(\lambda + \frac{C_1}{C_1 + C_2} A\right) d\left(\lambda + \frac{C_1}{C_1 + C_2} A\right) + \frac{AdA}{4\pi} \left(\frac{C_1^2}{C_1 + C_2} - C_1\right)$$

e.g. ① $C_2 = 1 - C_1 \longrightarrow$ bosonic SPT (IQHE for bosons)

$\sigma_{xy} = C_1(C_1 - 1) = 0 \mod 2$. YmL & Lee, 1210.0909

② $C_1 = 2$, $C_2 = 1 \Rightarrow K \simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $SU(3)_1$ state, g.s.d. $= 3$.

$\sigma_{xy} = -2/3$.

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③ $C_1 = C_2 = N$: $SU(2)_N$ state. g.s.d. $= N+1$.

non-Abelian if $N \geq 2$.

$\sigma_{xy} = -N$.

④ $C_1 = -C_2 = N$: confined superfluid phase.

$$\frac{N}{4\pi}(\lambda+A)d(\lambda+A) - \frac{N}{4\pi}\lambda d\lambda = \frac{N}{2\pi}\lambda dA + \frac{N}{4\pi}AdA$$

integrate out $\lambda \to$ $\mathcal{L}_{response} \sim A^2$ Higgs term.

Abrikosov-fermion rep. $\vec{S} = \frac{\hbar}{2} f_\alpha^\dagger \vec{\sigma}_{\alpha\beta} f_\beta$. $\alpha, \beta = \uparrow/\downarrow$.

$S^+ = f_\uparrow^\dagger f_\downarrow$ (compared to $b^+ = f_1^\dagger f_2^\dagger$ in previous case)

spin-1/2 can be mapped to hardcore bosons.

Previously. $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \to W \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $b^+ \to b^+$ for $\forall\, W \in SU(2)$.

this leads to $SU(2)$ gauge field.

in spin-1/2 case. $\hat{F} = \begin{pmatrix} f_\uparrow & f_\downarrow^\dagger \\ f_\downarrow & -f_\uparrow^\dagger \end{pmatrix}$

$$\vec{S}_i = \frac{1}{4}\mathrm{Tr}(\hat{F}_i^\dagger \vec{\sigma} \hat{F}_i), \Rightarrow \hat{F} \to \hat{F}W, \quad \vec{S} \to \vec{S}$$

for $\forall\, W \in SU(2)$.

$$\hat{H}_{MF} = \sum_{i,j} (\chi_{ij} f_{i\alpha}^\dagger f_{j\alpha} + h.c.) + (\Delta_{ij} f_{i\alpha}^\dagger \epsilon^{\alpha\beta} f_{j\beta}^\dagger + h.c.)$$

$$= \sum_{ij} \mathrm{Tr}(F_i^\dagger F_j T_{ij}) + h.c.$$

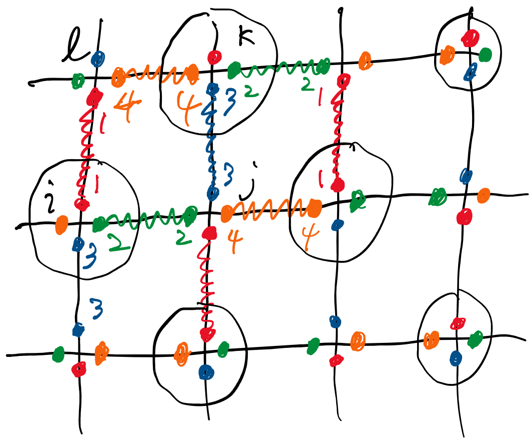
can break $SU(2)$ gauge group down to $U(1)$ if $\chi_{ij} \neq -\chi_{ij}^*$, $\Delta_{ij} = 0$

or Z_2 for generic $\Delta_{ij}, \chi_{ij} \neq 0$.

... provide exact g.s. of a parent Hamiltonian?

Can parton construction provide exact g.s. of a parent Hamiltonian?

Terhal, Hassler, DiVincenzo 1201.3757



$$\hat{H} = -\sum_{\langle i,j\rangle} \gamma_i^{\alpha_{ij}} \gamma_j^{\alpha_{ij}} (i t_{ij}) + U\sum_i (2n_\uparrow - 1)(1-2n_\downarrow)$$

$$\text{|||}$$

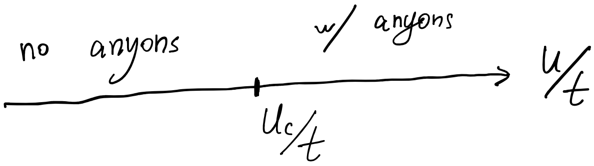
$$-U\prod_{\alpha=1}^{4}\gamma_{i,\alpha} = U(-1)^{\hat{n}_i}$$

$$f_\uparrow = \frac{\gamma_3 + i\gamma_4}{2}$$

$$f_\downarrow = \frac{\gamma_1 + i\gamma_2}{2}$$

$$\hat{F} = \begin{pmatrix} f_\uparrow & f_\downarrow^\dagger \\ f_\downarrow & -f_\uparrow^\dagger \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \gamma_3 + i\gamma_4 & \gamma_1 - i\gamma_2 \\ \gamma_1 + i\gamma_2 & i\gamma_4 - \gamma_3 \end{pmatrix} = \frac{\vec{\gamma}\cdot\vec{\sigma}}{2} + \frac{i}{2}\gamma_4$$

$$t_{ij} = t\cdot \mathrm{sgn}(t_{ij}), \quad t \equiv |t_{ij}|, \ \forall \langle i,j\rangle.$$



Consider $U/t \to \infty$ limit $\longrightarrow$ single occupancy 2-dimensional subspace. $\hat{n}_i = 1$.

$$\gamma_1\gamma_2\gamma_3\gamma_4 = 1 : \quad i\gamma_1\gamma_2 = -i\gamma_3\gamma_4 = \hat{Z}.$$

$$i\gamma_2\gamma_3 = \hat{X} = i\gamma_1\gamma_4$$

$$\hat{Y} = i\hat{X}\hat{Z} = i\gamma_3\gamma_1 = i\gamma_2\gamma_4$$

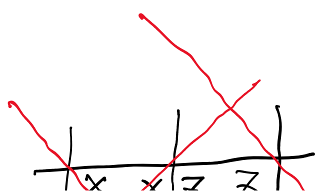
effective Hamiltonian in lowest order:

$$\hat{H}_{eff} = -\frac{t^4}{U^3}\eta_\square (i\gamma_1\gamma_2)_i \cdot (i\gamma_2\gamma_3)_j (i\gamma_3\gamma_4)_k (i\gamma_4\gamma_1)_l + \text{const.}$$

$$= -K\sum_{\substack{l\ \square\ k \\ i\ \ \ j}} \eta_{ijkl} Z_i X_j Z_k X_l \qquad \text{Wen's plaquette model.}$$

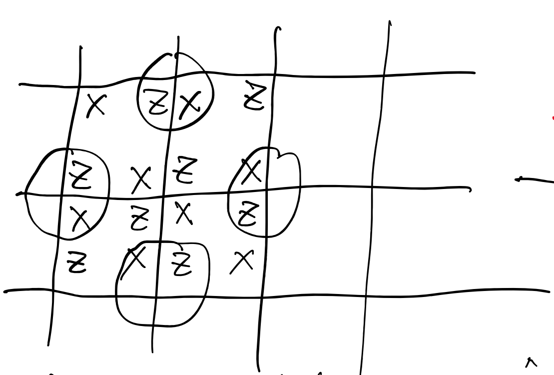
$$\eta_{\substack{l\ k\\ \square \\ i\ j}} = -\mathrm{sgn}(t_{ij})\,\mathrm{sgn}(t_{jk})\,\mathrm{sgn}(t_{kl})\,\mathrm{sgn}(t_{li})$$

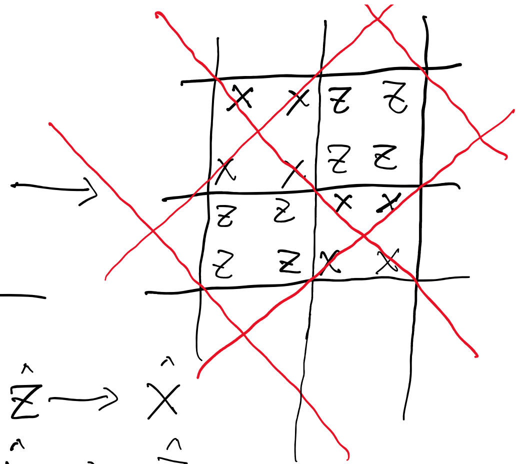
$K = t^4/U^3$



$\underset{i\ \ j}{\sqcup}$

$K = t^4/U^3$





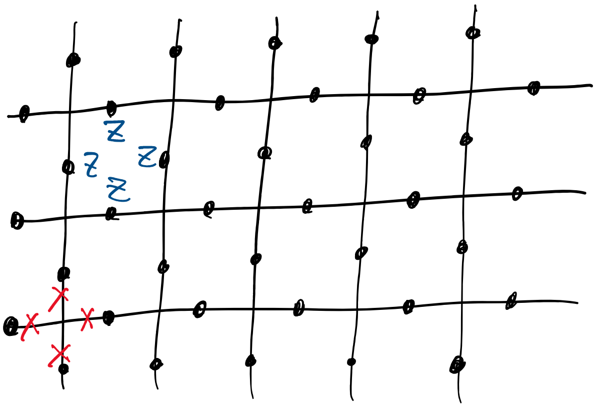
change of basis for the circled sites: $\hat{Z} \to \hat{X}$, $\hat{X} \to -\hat{Z}$, $\hat{Y} \to \hat{Y}$

(unitary transformation $\hat{U} = e^{i\frac{\pi}{4}\hat{Y}}$)

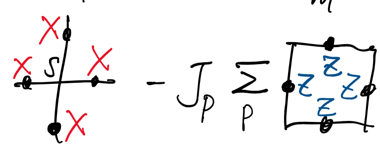
$$\hat{H}_{eff} \to \hat{H}_{T.C.} = -K\eta_v \sum_{+_v}\left(\prod_{l\in v}\hat{X}_l\right) - K\eta_p \sum_{\square_p}\left(\prod_{l\in\square}\hat{Z}_l\right)$$

Kitaev 9707021

Toric code (code distance $= L$)



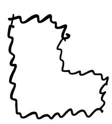
$$\hat{H} = -J_s\sum_s \hat{A}_s - J_p\sum_p \hat{B}_p$$



* ground state

$$|0\rangle = \prod_p \frac{1+\hat{B}_p}{2}\prod_s\frac{1+\hat{A}_s}{2}\bigotimes_l |Z_l = +1\rangle$$

$$\propto \sum_{\substack{\text{closed string}\\ \text{config.}}} |\ \ \rangle$$

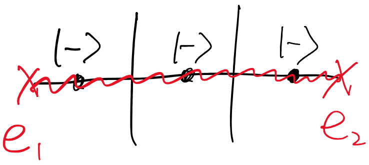


"loop soup"

$X_l = +1$: no string

$X_l = -1$: w/ string

* excitations: $\hat{A}_s$ violation $\longrightarrow$ open strings

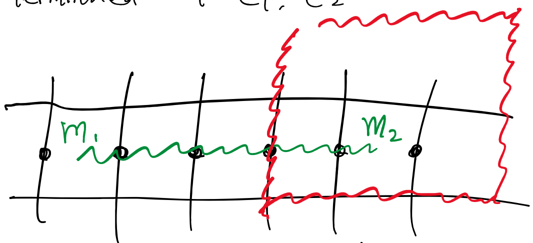


strings are deformable by acting $\hat{B}_p\,(=1)$ operators

$$|e_1\, e_2\rangle = \prod_{l\in\text{open}}\hat{Z}_l\,|0\rangle$$

$$|e_1 e_2\rangle = \prod_{\substack{l \in \text{open} \\ \text{string terminated at } e_1, e_2}} \hat{Z}_l |0\rangle$$

$\hat{B}_p$ violation $\longrightarrow$



a different type of open strings (on dual lattice)

strings are deformable by acting $\hat{A}_s (=1)$ operators.

$$|m_1 m_2\rangle = \prod_{\substack{\tilde{l} \in \text{open string} \\ \text{terminated at } m_1, m_2}} \hat{X}_{\tilde{l}} |0\rangle$$

* anyonic statistics:

full braiding e around m $\longrightarrow$ (-1) sign

"mutual semion statistics"

* fusion rules: $\hat{X}_{\tilde{l}}^2 = 1 = \hat{Z}_l^2 \Rightarrow e \times e = 1$ $\rightarrow$ Local excitations

$m \times m = 1$

$e \times m = \epsilon$ $\nearrow$ fermion

$\epsilon \times \epsilon = 1.$

Field theory description for paired superfluids of partons?

* Schwinger boson rep. $\vec{S} = \frac{1}{2} b_\alpha^\dagger \vec{\sigma}_{\alpha\beta} b_\beta$.

$$\hat{H}_{MF} = \sum_{\substack{ij \\ \sigma}} t_{ij} b_{i\sigma}^\dagger b_{j\sigma} + \Delta_{ij} b_{i\alpha}^\dagger b_{j\beta}^\dagger \epsilon_{\alpha\beta} + h.c.$$

conserved $U(1)$ current in a bosonic superfluid $\partial_\mu J^\mu = 0 \Rightarrow J^\mu = \epsilon^{\mu\nu\lambda} \frac{\partial_\nu a_\lambda}{2\pi}$

gapless Goldstone modes in 2+1-D $\longrightarrow$ $\mathcal{L}_{SF} = \rho_0 f_{\mu\nu} f^{\mu\nu} = \rho_0 (\partial_\mu a_\nu)^2$

if a bosonic superfluid couples to a dynamical gauge field b_μ.

$$\mathcal{L}_{eff} = \rho_0 (\partial_\mu a_\nu)^2 - b_\mu J^\mu = \rho_0 (\partial_\mu a_\nu)^2 - \frac{\epsilon^{\mu\nu\lambda}}{2\pi} b_\mu \partial_\nu a_\lambda$$

$$\mathcal{L}_{eff} = \rho_0 (\partial_\mu a_\nu) - \not{b}_\mu J - \rho_0 (\partial_\mu a_\nu) \quad 2\pi \quad)$$

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \text{ a Mott insulator in 2+1-D.}$$

for a non-chiral paired SF. $2J^\mu_{pair} = J^\mu_{boson} = 2\frac{1}{2\pi} da$

a paired SF coupled to a U(1) gauge field:

$$\mathcal{L}_{eff} = \rho_0 (\partial_\mu a_\nu)^2 - J^\mu_{boson} \not{b}_\mu = \rho_0 (\partial_\mu a_\nu)^2 - \frac{\epsilon^{\mu\nu\lambda}}{2\pi} 2 \not{b}_\mu \partial_\nu a_\lambda$$

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} : \quad e \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad m \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \epsilon \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hansson, Oganesyan, Sondhi, 0404327

* Next we switch to fermionic partons (e.g. Abrikosov fermions)

quadratic BdG Hamiltonian of superconductors in sym. class D

$\longrightarrow$ integer-valued topo. inv. $\nu \in \mathbb{Z}$.

Altland, Zirnbauer 9602137
Kitaev, 0901.2686
Schnyder et al, 0803.2786

$\nu = 0$: s-wave $\longrightarrow$ toric code

$\nu = 1$: spinless p-wave $\longrightarrow$ non-Abelian Ising anyons (Kitaev honeycomb model, gapped A phase)

chiral Majorana edge mode w/ $c_- = 1/2$

$\nu = 2$: spinful p-wave $\longrightarrow$ $U(1)_4$ CS theory

- - - - - - -

$\nu = 4$: $d + id$ $\longrightarrow$ $U(1)_2 \times U(1)_2$ CS theory. (semion)

we focus on the Abelian states w/ $\nu = 0 \mod 2$

strategy to obtain eff. field theory:

"gauging fermion parity" $\simeq$ coupling a $\mathbb{Z}_2$ gauge field to fermionic partons.

$\nu = 2n \Rightarrow$ Chern # $= n$ for fermions

$a\, da - a_2 da_2$

$\nu = 2n \Rightarrow$ Chern # $= n$ for fermions

$\nu = 0$: $J_{fermion} = \dfrac{da_1 + da_2}{2\pi}$, $\mathcal{L}_{parton} = \dfrac{a_1 da_1 - a_2 da_2}{4\pi}$

fermion : $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, π-flux : $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\theta_v = 0$.

$$K_{eff}^{-1} = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

$$\Rightarrow K_{eff} = \begin{pmatrix} 0 & 2 \\ 2 & -4 \end{pmatrix} \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$\nu = 2$: $J_{fermion} = \dfrac{da}{2\pi}$. $\mathcal{L}_{parton} = \dfrac{a da}{4\pi}$.

π-flux : $\frac{1}{2} = \vec{l}$ $\theta_v = \pi/4$.

$$K_{eff}^{-1} = \frac{1}{2} \cdot 1 \cdot \frac{1}{2} = 1/4 \Rightarrow K_{eff} = 4.$$

$K_{eff} = \dfrac{4}{4\pi} \tilde{a} d \tilde{a}$: $U(1)_4$ CS term

$\nu = 4$: $J_{fermion} = \dfrac{da_1 + da_2}{2\pi}$

$$\mathcal{L}_{parton} = \frac{1}{4\pi}(a_1 da_1 + a_2 da_2)$$

fermion : $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ π-flux : $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ $\theta_v = \pi/2$

$$K_{eff}^{-1} = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\Rightarrow K_{eff} = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix} \simeq \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

2-copies of semion theory

(Laughlin-Kalmeyer CSL)

$K^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \ldots \end{pmatrix} \begin{pmatrix} 1 & \ldots \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1/2 \\ \ldots \end{pmatrix}$

(Laughlin Kalmeyer ...)

$$\nu = 6 : \quad K_{eff}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$K_{eff} = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 4 \end{pmatrix} \simeq \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \theta_v = \frac{3\pi}{4}$$

generally $\theta_v = \frac{\nu}{8}\pi$, $\forall \nu \in \mathbb{Z}$.

$\nu \to \nu + 16$, vison statistics θ_v remains the same.

in fact all anyon statistics remain the same.

$\mathbb{Z}_2$ topological orders are classified by "16-fold way"

(fermion superconductors coupled to $\mathbb{Z}_2$ gauge field) Kitaev. 0506438