Lecture 2: Parton construction for symmetric spin liquids

Wednesday, December 3, 2025 11:24 AM

* symmetry fractionalization and its classification
  Essin, Hermele 1212.0593
  Tarantino, Lindner, Fidkowski 1506.06754
  Barkeshli et al 1410.4540
* projective symmetry group (PSG)
  Wen, 0107071
* triangular lattice as an example
  Schwinger boson : Z_2 SLs
  Abrikosov fermion: Z_2 & $U(1)$ SLs.
  Unification of the two different representations.

In $d=2$,

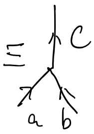
invertible (no fractional statistics)	SSB
non-invertible (intrinsic topo. order)	gapless

Topo. order is mathematically described by algebraic theory of anyons, known as Unitary Modular Tensor Category (UMTC)

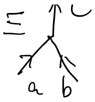
(Kitaev, 0506438, Appendix E ; Kong, Zhang, 2205.05565)

* anyon types $a, b, c, \cdots$
* fusion rule $a\times b = \sum_c N_{ab}^c\, c$

$$N_{ab}^c = \dim\left(V_{ab}^c \equiv \right) \in \mathbb{Z}_+ .$$

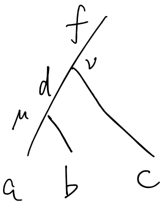


fusion space

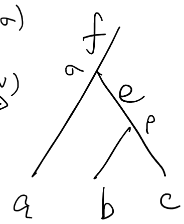
$N_{ab}^{c} = \dim\left(V_{ab}^{c} \equiv \right)$  $\in \mathbb{Z}_+$.

fusion space

* associator (F-symbol)

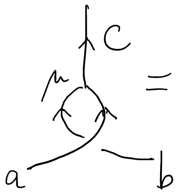


$\xrightarrow{\left(F^{f}_{abc}\right)^{(e\rho\sigma)}_{(d\mu\nu)}}$

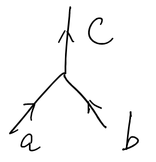


(ρ, σ, μ, ν) supressed if $N_{ab}^{c} \le 1$, $\forall\, a, b, c$

* braiding



$= \sum_{\nu} \left(R^{c}_{ab}\right)_{\mu,\nu}$



Generally anyons form a ring where multiplication is defined by fusion rules,
("fusion ring")

Anyon a is Abelian if there is a unique c s.t. $N_{a,b}^{c} = 1$, $\forall\, b$.
$(a \times b = c)$

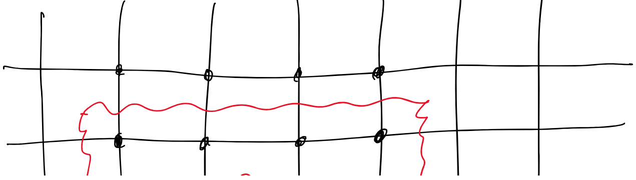
Abelian anyons form a group $\mathcal{A}$ under fusion,

e.g. $\mathcal{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$ for toric code,

Thm.: The unitary string operator of a non-Abelian anyon is a quantum circuit, whose depth is lower bounded by the length of the string. (Shi, 1810.01986)

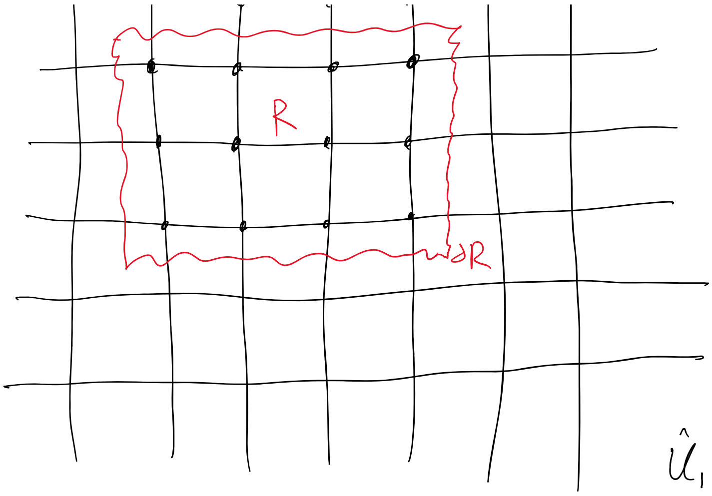
* How does sym. act on a topological order locally ?

e.g. TFIM in 2d , $\hat{H} = -\sum_{\langle ij\rangle} J_{ij}\, Z_i Z_j - h \sum_i X_i$



- one qubit on each site

$h \gtrsim |J_{ij}|$ in the sym. phase



$h \gtrsim |J_{ij}|$ in the sym. phase

continuous family of gapped Ham.

$J_{ij}(t=0) = J_{ij}$

$J_{ij}(t=1) = -J_{ij}$ for $\langle ij \rangle \in \partial R$

$\hat{U}_1 = \hat{T} e^{-i\int_0^1 \tilde{H}(t)\,dt} \simeq$ finite depth quantum circuit (FDQC)

Bravyi et al, 0603121, Chen et al 1004.3835

$\hat{U}_R(g) \equiv \left(\prod_{i\in R} X_i\right) \cdot \hat{U}_1$ is a local action of sym. g inside region R.

$\boxed{U_R(g) \text{ is a FDQC !}}$

consider $U_{\partial R}(g,h) = U_R(g)\, U_R(h)\, U_R^{-1}(gh)$, which is FDQC on ∂R.

$[U_{\partial R}(g,h), \hat{P}_{low}\hat{H}\hat{P}_{low}] = 0 \implies U_{\partial R}(g,h) \simeq L_{\partial R}(a)$.

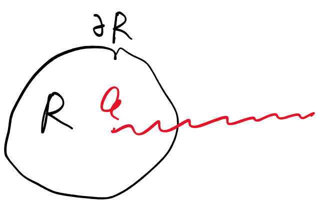
Abelian anyon string operator since it's FDQC.

$a \equiv \omega(g,h) \in \mathcal{A}$ (the set of Abelian anyons)

$[U_R(g)\cdot U_R(h)]\cdot U_R(k) = U_R(g)\,[U_R(h)\cdot U_R(k)]$

$\implies \omega(g,h)\times\omega(gh,k) = \omega(g)\times\omega(g,hk)$

$\implies \{\omega(g,h)\} \in \mathcal{H}^2(G,\mathcal{A})$ 2nd group cohomology



for anyon a inside region R.

$U_R^{(a)}(g)\cdot U_R^{(a)}(h) = \Theta_{\omega(g,h),a}\, U_R^{(a)}(gh)$

$\Theta_{a,\omega(g,h)} \in U(1)$ is the full braiding phase b/w anyon a and $\omega(g,h)$

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How to realize distinct sym. fractionalization classes in partons?

projective sym. group classification

we will consider SG = space group $\times Z_2^T$ for symmetric spin liquids.
(time reversal)

Schwinger boson: $\begin{pmatrix} b_{r\uparrow} \\ b_{r\downarrow} \end{pmatrix} = B_{\vec r} \xrightarrow{g \in SG} \underbrace{U_g(g\vec r)}_{\text{spin rotation}} \underbrace{e^{i\theta_g(g\vec r)}}_{U(1)\text{ gauge rotation}} B_{\vec r}$

$\hat F_r^+ = \begin{pmatrix} f_{r\uparrow} & f_{r\downarrow} \\ f^+_{r\downarrow} & -f^+_{r\uparrow} \end{pmatrix} \xrightarrow{g \in SG} G_g(g\vec r) \hat F^+_{g\vec r} U_g(g\vec r)$

$G_g(g\vec r)$: SU(2) gauge rotation, $e^{i\theta \vec n \cdot \vec \tau}$

$U_g(g\vec r)$: spin rotation

in our case of SG = space group $\times Z_2^T$.

$U_g(\vec r) \equiv 1$ if $g \in$ space group. $\quad U_T(\vec r) = i\sigma_y$.

$G_e(\vec r) \equiv G_e \in IGG \qquad$ (invariant gauge group)

$$G^{-1}_{gh}(gh\vec r)\, G_g(g\vec r) \cdot G_h(gh\vec r) = W_{g,h} \in IGG$$

$W_{g,h}$: independent of $\vec r$

for anyon a.

$IGG_a = \{ \Theta_{\omega(g,h),a} \mid g,h \in SG \}$.

for Z_2 spin liquids ($\simeq$ toric code), e is boson, ϵ is fermion.

Schwinger boson: $IGG_e = \{\pm 1\} = Z_2$.

Abrikosov fermion: $IGG_\epsilon = \{\pm 1\} = Z_2$

$\ldots IGG \ldots U(1) = \{ e^{i\theta \hat\tau_z} \mid 0 \le \theta < 2\pi \}$

Abrikosov fermion: large ...

$U(1)$ spin liquids: $\quad IGG = U(1) = \{ e^{i\theta \hat{\tau}_z} \mid 0 \le \theta < 2\pi \}$

$SU(2)$ spin liquid: $\quad IGG = SU(2) = \{ e^{i\theta \vec{n}\cdot\vec{\tau}} \}$

$$1 \to IGG \longrightarrow PSG \longrightarrow SG \longrightarrow 1, \quad \text{classified by } \mathcal{H}^2(SG, IGG)$$

"projective symmetry group"

$$PSG = \{ G_e \cdot G_g(g\vec{r}) \mid g \in SG,\ G_e \in IGG \}.$$

$$\hat{H}_{MF} = \sum_{i,j} tr(\hat{F}_i^{+} \hat{F}_j T_{ji}) + h.c. \qquad T_{ij} = \begin{pmatrix} \chi_{ij} & \Delta_{ij} \\ \Delta_{ij}^* & -\chi_{ij}^* \end{pmatrix}$$

$\hat{H}_{MF}$ preserves $SG \Rightarrow G_g^{+}(g\vec{r}') T_{r',r} G_g(g\vec{r}) = T_{g\vec{r}', g\vec{r}}$

so PSG determines all possible $\hat{H}_{MF}$.

(1) How to determine the distinct $\{W_{g,h}\}$ and PSGs?
(gauge-inequivalent)

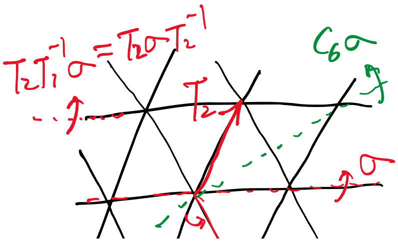
gauge transformation:

$$F(\vec{r}) \longrightarrow F(\vec{r}) W^{+}(\vec{r}) \Rightarrow G_g(g\vec{r}) \longrightarrow W(g\vec{r}) G_g(g\vec{r}) W^{+}(\vec{r})$$

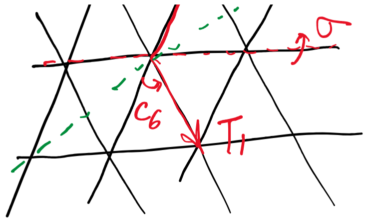
(2) How to constrain $\hat{H}_{MF}$ by the PSG?

$$G_g^{+}(g\vec{r}') T_{r',r} G_g(g\vec{r}) = T_{g\vec{r}', g\vec{r}}$$

$*$ a baby example: triangular lattice spin liquids enriched by space group symmetry $SG = p6m$



$$p6m = \{ \sigma^{n_\sigma} C_6^{n_6} T_1^{n_1} T_2^{n_2} \mid n_\sigma = 0, 1,\ n_6 = 0, 1, \cdots, 5,\ n_1, n_2 \in \mathbb{Z} \}$$



$n_1, n_2 \in \mathbb{Z}$

$$\begin{cases} C_6\sigma = \sigma C_6^{-1} \Rightarrow (\sigma C_6)^2 = e \\ \sigma^2 = e \\ C_6^6 = e \\ T_1\sigma = \sigma T_2 \Longleftrightarrow T_1^{-1}\sigma = \sigma T_2^{-1} \\ T_2\sigma = \sigma T_1 \Longleftrightarrow T_2^{-1}\sigma = \sigma T_1^{-1} \\ T_1 C_6 = C_6 T_2^{-1} \\ T_2 C_6 = C_6 T_1 T_2 \\ T_2 T_1 = T_1 T_2 \end{cases}$$

$$\begin{cases} G_\sigma(\sigma C_6\vec{r})\, G_{C_6}(C_6\vec{r})\, G_\sigma(\vec{r})\, G_{C_6}(\sigma^{-1}\vec{r}) = \eta_{C_6\sigma} \rightarrow \omega(C_6\sigma, C_6\sigma) \\ G_\sigma(\sigma\vec{r})\, G_\sigma(\vec{r}) = \eta_\sigma \rightarrow \omega(\sigma,\sigma) \\ G_{C_6}(C_6^3\vec{r})\, G_{C_6}(C_6^2\vec{r})\, G_{C_6}(C_6\vec{r})\, G_{C_6}(\vec{r})\, G_{C_6}(C_6^{-1}\vec{r})\, G_{C_6}(C_6^{-2}\vec{r}) = \eta_{C_6} \rightarrow \omega(C_6^3, C_6^3) \\ G_{T_2}^{-1}(\sigma\vec{r})\, G_\sigma^{-1}(\vec{r})\, G_{T_1}(\vec{r})\, G_\sigma(T_1^{-1}\vec{r}) = \eta_{T_2\sigma} = 1 \\ G_{T_1}^{-1}(\sigma\vec{r})\, G_\sigma^{-1}(\vec{r})\, G_{T_2}(\vec{r})\, G_\sigma(T_2^{-1}\vec{r}) = \eta_{T_1\sigma} = 1 \\ G_{T_2}(C_6^{-1}\vec{r})\, G_{C_6}^{-1}(\vec{r})\, G_{T_1}(\vec{r})\, G_{C_6}(T_1^{-1}\vec{r}) = \eta_{T_1C_6} = 1 \\ G_{T_2}^{-1}(T_1^{-1}C_6^{-1}\vec{r})\, G_{T_1}^{-1}(C_6^{-1}\vec{r})\, G_{C_6}^{-1}(\vec{r})\, G_{T_2}(\vec{r})\, G_{C_6}(T_2^{-1}\vec{r}) = \eta_{T_2C_6} = 1 \\ G_{T_2}^{-1}(T_1^{-1}\vec{r})\, G_{T_1}^{-1}(\vec{r})\, G_{T_2}(\vec{r})\, G_{T_1}(T_2^{-1}\vec{r}) = \eta_{12} \rightarrow \dfrac{\omega(T_1,T_2)}{\omega(T_2,T_1)} \end{cases}$$

$\vec{r} = (r_1, r_2) \xrightarrow{T_1} (r_1+1, r_2)$. $\qquad \vec{r} \xrightarrow{T_2} (r_1, r_2+1)$

$\vec{r} \xrightarrow{\sigma} (r_2, r_1)$ $\qquad \vec{r} \xrightarrow{C_6} (r_1 - r_2, r_1)$

$\vec{r} \xrightarrow{C_6^{-1}} (r_2, r_2 - r_1)$

choose Landau gauge: $\underset{\forall \vec{r}}{G_{T_1}(\vec{r}) \equiv 1}$, $\quad \underset{\forall r_2}{G_{T_2}(r_1 = 0, r_2) \equiv 1}$.

we can set $\eta_{T_1C_6}=\eta_{T_2C_6}=1$ by absorbing them into $G_{T_{1,2}}$.

$$G_{T_2}(r_1,r_2)=\eta_{12}^{r_1}\,,\qquad G_{T_1}(r_1,r_2)=1\,.$$

$$\begin{cases} G_{C_6}(\vec r)=G_{C_6}(T_1^{-1}\vec r)\,G_{T_2}(C_6^{-1}\vec r)=G_{C_6}(T_1^{-1}\vec r)\,\eta_{12}^{r_2} \\ G_{C_6}(\vec r)=G_{T_2}(\vec r)\,G_{C_6}(T_2^{-1}\vec r)\,G_{T_2}^{-1}(T_1^{-1}C_6^{-1}\vec r)=\eta_{12}^{r_1}\,G_{C_6}(T_2^{-1}\vec r)\,\eta_{12}^{1-r_2} \end{cases}$$

$$\Rightarrow G_{C_6}(r_1,r_2)=G_{C_6}(0,r_2)\,\eta_{12}^{r_1r_2}=G_{C_6}(0,0)\,\eta_{12}^{r_1r_2+\frac{r_2(1-r_2)}{2}}$$

$$\sum_{x=1}^{r_2}(1-x)=r_2-\frac{r_2(r_2+1)}{2}=\frac{(1-r_2)r_2}{2}$$

now consider $\eta_{T_{1,2}\sigma}$. $\Rightarrow$
$$\begin{cases} G_\sigma(\vec r)=G_\sigma(T_1^{-1}\vec r)\,\eta_{T_2\sigma}^{-1}\,G_{T_2}^{-1}(\sigma\vec r) \\ G_\sigma(\vec r)=G_{T_2}(\vec r)\,G_\sigma(T_2^{-1}\vec r)\,\eta_{T_1\sigma}^{-1} \end{cases}$$

$$\Rightarrow G_\sigma(\vec r)=G_\sigma(0,r_2)\,\eta_{T_2\sigma}^{-r_1}\,\eta_{12}^{-r_1r_2}=G_\sigma(0,0)\,\eta_{T_1\sigma}^{-r_2}\,\eta_{T_2\sigma}^{-r_1}\,\eta_{12}^{-r_1r_2}$$

$$=\eta_{12}^{r_1r_2}\,G_\sigma(r_1,0)\,\eta_{T_1\sigma}^{-r_2}=\eta_{12}^{r_1r_2}\,G_\sigma(0,0)\,\eta_{T_2\sigma}^{-r_1}\,\eta_{T_1\sigma}^{-r_2}$$

$$g_\sigma\equiv G_\sigma(0,0)\,,\qquad g_{C_6}\equiv G_{C_6}(0,0)$$

$$G_\sigma(r_1,r_2)\,G_\sigma(r_2,r_1)=\eta_\sigma\Rightarrow \eta_{12}^{r_1r_2}\,g_\sigma\,\eta_{T_2\sigma}^{-r_1}\,\eta_{T_1\sigma}^{-r_2}\,\eta_{12}^{r_1r_2}\,g_\sigma\,\eta_{T_2\sigma}^{-r_2}\,\eta_{T_1\sigma}^{-r_1}=\eta_\sigma\,.$$

* for $IGG=U(1)$, Zhou & Wen, 0210662

Li, YML, Chen, 1612.03447

8 solutions: $\eta_{12}=\pm1$. $g_\sigma=\tau_x^{n_\sigma}$. $g_{C_6}=\tau_x^{n_{C_6}}$. $n_\sigma, n_{C_6}=0,1$.

π-flux state: $\eta_{12}=-1$. $g_\sigma=g_{C_6}=1$.

* below we consider $Z_2=IGG$, all $\eta=\pm1$.

$\sigma^2=e \longrightarrow \eta_{T_1\sigma}=\eta_{T_2\sigma}\qquad g_\sigma^2=\eta_\sigma$

below

$$\sigma^2 = e \longrightarrow \eta_{T_1\sigma} = \eta_{T_2\sigma}, \quad g_\sigma^2 = \eta_\sigma$$

$$(C_6\sigma)^2 = e \longrightarrow \eta_{T_1\sigma} = 1, \quad (g_{C_6} g_\sigma)^2 = \eta_{\sigma C_6}$$

$$C_6^6 = e \longrightarrow (g_{C_6})^6 = \eta_{C_6}$$

① Schwinger boson. Sachdev, PRB 1992

Wang, Vishwanath, 0608129

$$g_\sigma = e^{i\theta_\sigma}, \quad g_{C_6} = e^{i\theta_{C_6}}$$

$$\Rightarrow \eta_{C_6} = \eta_{\sigma C_6}^3 \cdot \eta_\sigma^{-3} = \eta_{\sigma C_6} \cdot \eta_\sigma$$

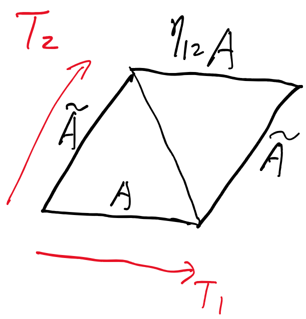
only 3 independent $\mathbb{Z}_2$-valued ~~topo.~~ invariants: $\eta_{12}, \eta_\sigma, \eta_{\sigma C_6} = \pm 1$.

$\mathbb{Z}_2^3$ classification : 8 distinct $\mathbb{Z}_2$ SLs.

$\eta_{12} = 1$: 0-flux state

flux per u.c. $= \eta_{12}$.

$\eta_{12} = -1$: π-flux state.



$$A_{i,j} = \begin{pmatrix} \chi_{i,j} \\ \Delta_{i,j} \end{pmatrix} \xrightarrow{G_{T_1}(\vec{r}) = 1} A_{T_1 i, T_1 j} = A_{i,j}$$

$$A_{i,j} \xrightarrow{G_{T_2}(\vec{r}) = \eta_{12}^{r_1}} A_{T_2 i, T_2 j} = \eta_{12}^{r_{1,i}} A_{i,j}\, \eta_{12}^{-r_{1,j}}$$

② Abrikosov-fermion : Zhou, Wen 0210662

YML 1505.06495

$$g_\sigma, g_{C_6} \in SU(2)$$

$2^4 = 16$ inequivalent solutions. (20 w/ time reversal)

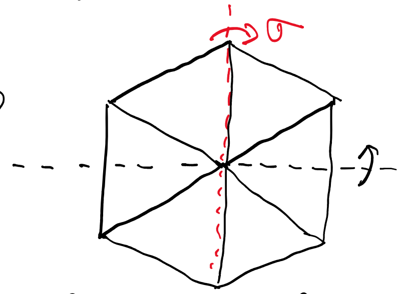
$$\eta_{12}, \eta_{C_6}, \eta_\sigma, \eta_{\sigma C_6} = \pm 1 \qquad H^2(P6m, \mathbb{Z}_2) = \mathbb{Z}_2^4$$

if $\eta_{C_6} = \eta_\sigma \cdot \eta_{\sigma C_6}$. gapped. Z_2 SLs.

if $\eta_{C_6} \neq \eta_\sigma \cdot \eta_{\sigma C_6}$ gapless Z_2 SLs. YML, 1606.05652

example of $\eta_{C_6} \neq \eta_\sigma \cdot \eta_{\sigma C_6}$: $\begin{cases} g_\sigma = i\tau_x \\ g_{C_6} = i\tau_z \end{cases}$.

Why gapless?



$R = C_6^3 \sigma = C_6 (C_6 \sigma) C_6^{-1} \simeq C_6 \sigma$.

$$(C_6)^6 = (C_6^3)^2 = (R \cdot \sigma)^2 = R^2 \cdot \sigma^2 \cdot (R^{-1} \sigma^{-1} R \sigma)$$

$$\eta_{C_6} = \eta_{\sigma C_6} \cdot \eta_\sigma \cdot (R^{-1} \sigma^{-1} R \sigma)$$

if $R^{-1}\sigma^{-1} R \sigma = -1$. $\longrightarrow$ degeneracy in parton spectrum hence gaplessness.

③ Unification b/w Schwinger boson & Abrikosov fermion

YML, Cho, Vishwanath. 1403.0575

Zaletel, YML, Vishwanath 1501.01395

Schwinger boson : $W^b_{g,h} = \Theta_{\omega(g,h), e}$ where e is bosonic spinon.

Abrikosov fermion $W^f_{g,h} = \Theta_{\omega(g,h), \epsilon}$ where ϵ is fermionic spinon.

vison $m = e \times \epsilon$: $W^v_{g,h} = \Theta_{\omega(g,h), v} = W^b_{g,h} \cdot W^f_{g,h}$

for vison : $\eta^v_{C_6} = -1 = -\eta^b_{C_6} \cdot \eta^f_{C_6}$

$\eta^v_{12} = -1 = \eta^b_{12} \cdot \eta^f_{12}$

Berry phase of braiding m around e or ϵ.

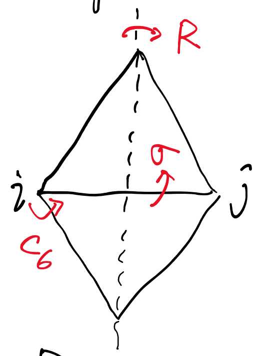
$\eta^v_{\sigma C_6} = 1 = -\eta^b_{\sigma C_6} \cdot \eta^f_{\sigma C_6}$

$\eta^v_{\sigma} = 1 = -\eta^b_{\sigma} \cdot \eta^f_{\sigma}$

otherwise protected edge states (1811.00434)

How to constrain mean-field amplitudes by symmetry?

e.g. NN amplitude $T_{i,j}$



Link sym. group

$\{ \sigma^n R^{n'} \mid n, n' = 0, 1 \} \simeq D_2$

$G_g^\dagger(g\vec{r}') \, T_{\vec{r}',\vec{r}} \, G_g(g\vec{r}) = T_{g\vec{r}', g\vec{r}}$

$$T_{ij} = \begin{pmatrix} \chi_{ij} & \Delta_{ij} \\ \Delta^*_{ij} & -\chi^*_{ij} \end{pmatrix}$$

$$\begin{cases} G_R^\dagger(j) \, T_{i,j} \, G_R(i) = T_{j,i} = T^\dagger_{i,j} . \\ G_\sigma^\dagger(i) \, T_{i,j} \, G_\sigma(j) = T_{i,j} \end{cases}$$