Entanglement, symmetries and their interplay in quantum spin chains

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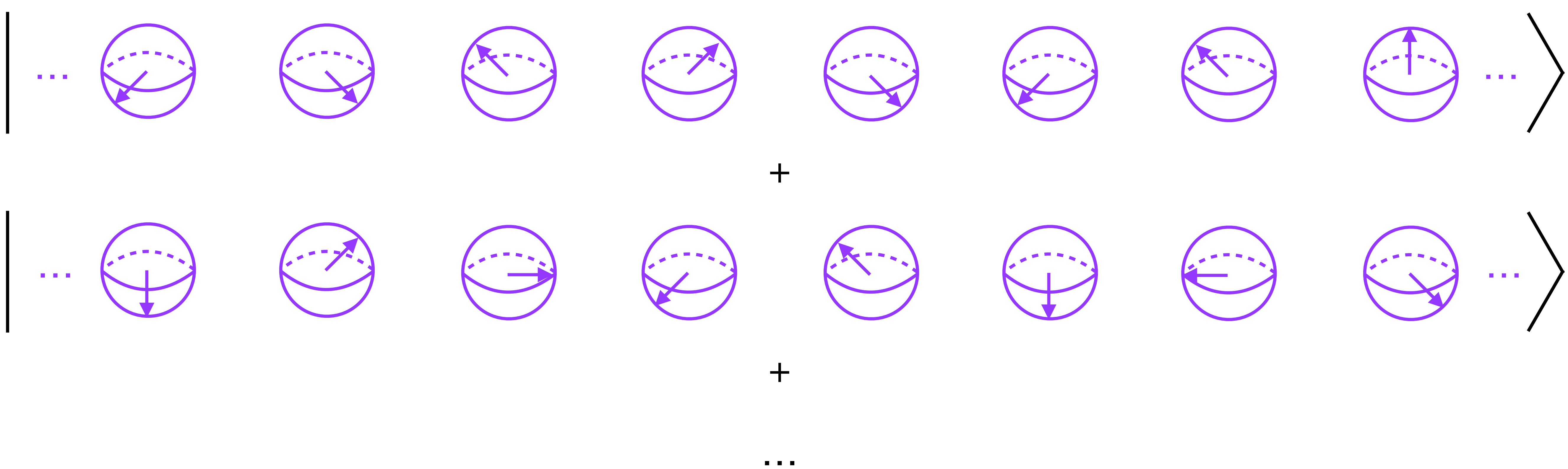
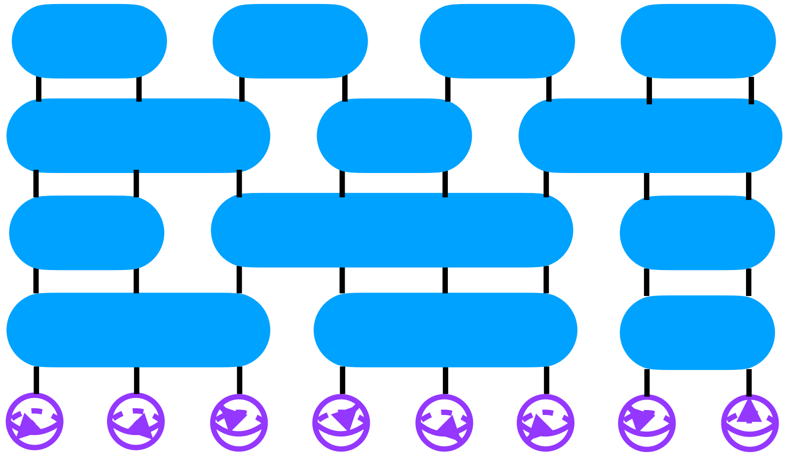
(Talk @ Advanced school and conference on quantum matter, ICTP, 12/8/2025)



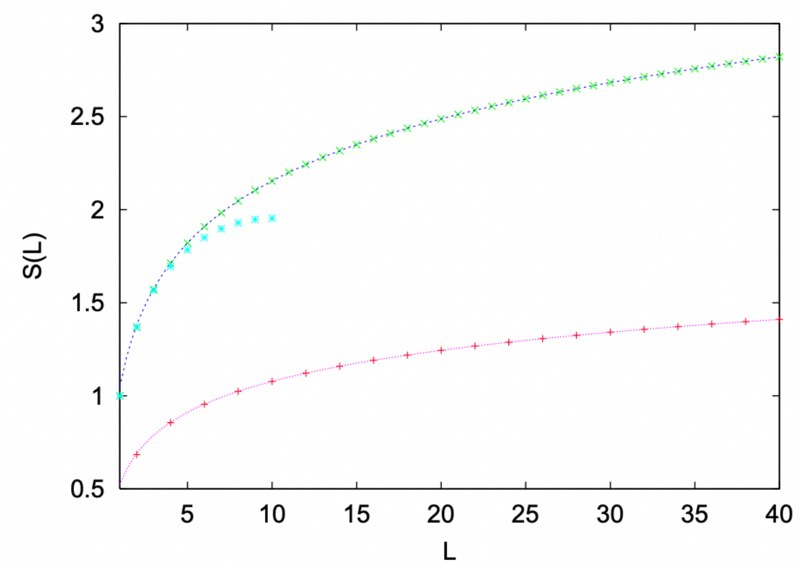
Motivation I:
Understand many-body quantum entanglement

Different types of quantum matter have different patterns of entanglement.

short-range entangled states

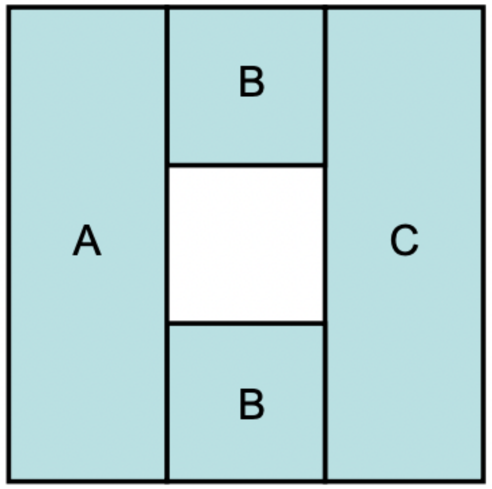


1D quantum criticality



Vidal, Latorre, Rico, Kitaev, quant-ph/0211074

2D topological order



Kitaev, Preskill, hep-th/0510092; Levin, Wen, cond-mat/0510613

Questions:

What are all possible patterns of all quantum many-body states?

How can these entanglement patterns be realized in physical systems?

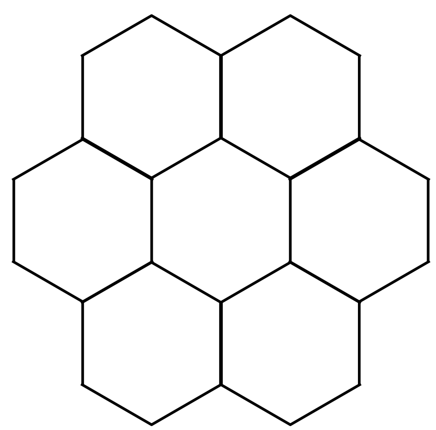
Motivation II:
Understand the consequences of symmetries

Symmetries have profound consequences in quantum systems.

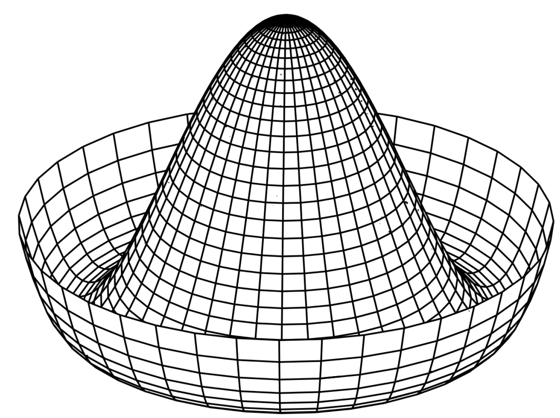
conservation laws

$$H = \begin{pmatrix} H_1 & & \\ & H_2 & \\ & & \ldots \end{pmatrix}$$

enforced degeneracy



spontaneous breaking



anomaly matching

$$\omega = \Omega \begin{cases} \text{phase 1} \\ \text{phase 2} \\ \ldots \end{cases}$$

Questions:

How should symmetries be characterized?

How do symmetries constrain the entanglement pattern?

Take-home message:
Anomalous symmetries guarantee nontrivial pattern of entanglement.

symmetry group G & symmetry action $\Rightarrow$ anomaly index $H^3_\varphi(G; U(1))$

Consequences of an anomalous symmetry:

- A state must either have long-range correlation or violate entanglement area law.
- A state must be long-range entangled.
- A symmetric "admissible" Hamiltonian cannot have a unique gapped ground state.
- …

Liu, Yi, Zhou, **Zou**, 2405.14929
Liu, Yi, **Zou**, 2510.06555

Outline

- Introduction
- Anomaly of a symmetry
- Implications of anomaly
- Discussion

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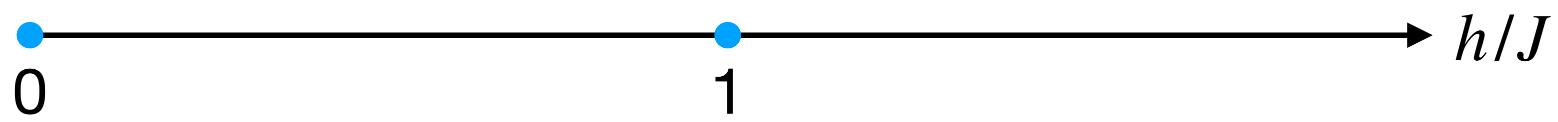
Entanglement can be used to characterize different quantum phases of matter.

ground state phase diagram of transverse field Ising model:

$$H = -J\sum_i Z_i Z_{i+1} - h\sum_i X_i$$

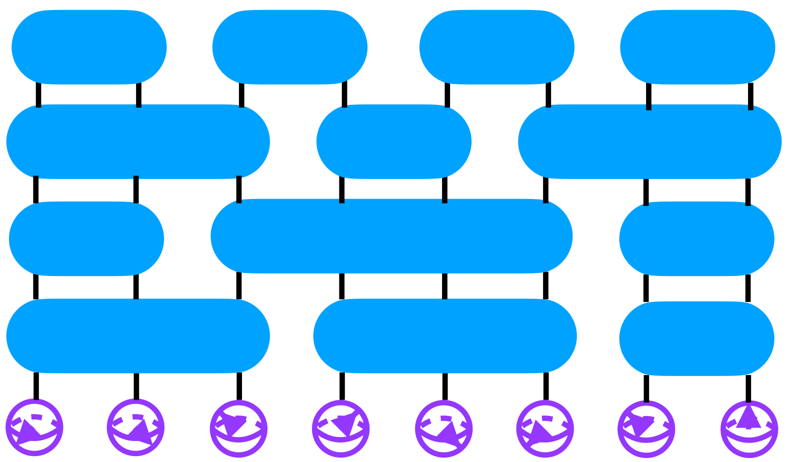


regime	phase or transition	area law?	correlation
$h/J > 1$	paramagnet	satisfied	short-range
$h/J = 1$	criticality	violated	quasi-long-range
$h/J < 1$	ferromagnet	satisfied	long-range

Short-range entangled states are in the same quantum phase as product states.

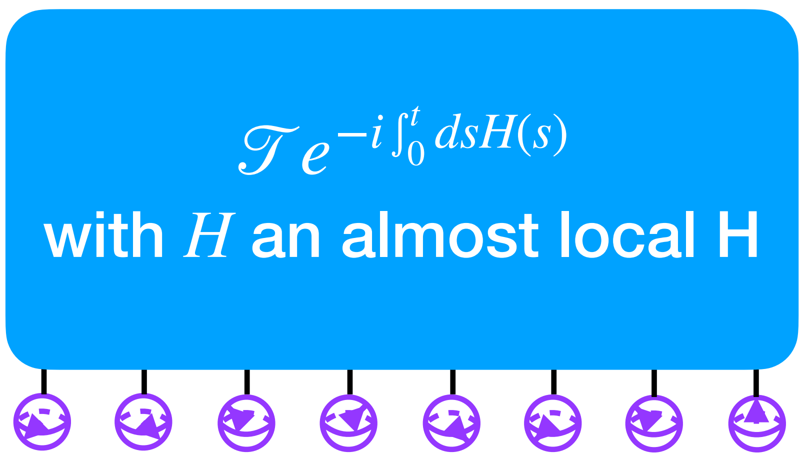
short-range entangled (SRE) states

finite-depth quantum circuit (FDQC)



accurate version relevant to quantum matter →

locally generated automorphism (LGA)



almost local Hamiltonian: $H = \sum_i H_i$ with $||H_i|| < (\mathrm{supp}(H_i))^{-\infty}$

Quasi-adiabatic continuation (automorphic equivalence):
Two gapped almost local Hamiltonians are adiabatically connected if and only if their ground states are connected by LGA.

Bachmann, et al,1102.0842, Kapustin, Sopenko, 2201.01327

SRE states have short-range correlations and satisfy the entanglement area law.

Superpolynomial clustering: $\langle AB \rangle - \langle A \rangle \langle B \rangle \sim O(r^{-\infty})$

Entanglement area law: $S_{[0,n]} < S_0$ (S_0 is independent of n)

Kapustin, Sopenko, Yang, 2012.15491, Liu, Yi, Zhou, **Zou**, 2405.14929

ground states of $H = -J\sum_i Z_i Z_{i+1} - h\sum_i X_i$

regime	phase or transition	area law?	correlation	SRE?
$h/J > 1$	paramagnet	satisfied	short-range	yes
$h/J = 1$	criticality	violated	quasi-long-range	no
$h/J < 1$	ferromagnet	satisfied	long-range	no

States that are not short-range entangled are call **long-range entangled (LRE) states**.

Remark: Beyond the definition, no sufficient and necessary condition for SRE is known.

Some symmetry conditions force states to be long-range entangled.

symmetry conditions:
SO(3) spin rotation & lattice translation,
spin-1/2 at each site

examples of symmetric states:
all long-range entangled

1. Ground state of anti-ferromagnetic Heisenberg chain: CFT

2. Ground state of Majumdar-Ghosh chain: long-range correlation

…

symmetry conditions:
SO(3) spin rotation & lattice translation,
spin-1 at each site

examples of symmetric states:
some are short-range entangled

1. Ground state of anti-ferromagnetic Heisenberg chain

2. Ground state of the AKLT chain

…

Questions:

Why are these two symmetry conditions so different?

Which symmetries force states to be LRE?

Previous proposal:
Anomaly-matching is a systematic framework to study the realizability of a theory in a quantum material.

A quantum material can realize a theory.

$\Updownarrow$

$\omega = \Omega$

ω: quantum anomaly of the quantum material

Ω: quantum anomaly of the theory

(previous literature: case-by-case & cannot answer many questions)

Refs: **Zou**, He, Wang, 2101.07805
Ye, Guo, He, Wang, **Zou**, 2111.12097
Ye, **Zou**, 2210.02444
Ye, **Zou**, 2309.15118

Preview:
All Lieb-Schultz-Mattis-like symmetry conditions force states to be long-range entangled.

Lieb-Schultz-Mattis-like symmetry conditions:

on-site internal symmetry + lattice translation + projective representation in a unit cell

Example 1: $SO(3) \times \mathbb{Z}$ with a spin-1/2 per unit cell
Example 2: $\mathbb{Z}_2^T \times \mathbb{Z}$ with a Kramers doublet per unit cell

Example of Hamiltonian: $H = \sum_{i,j} J_{ij} \overrightarrow{S}_i \cdot \overrightarrow{S}_j$

The symmetry-enforced long-range entanglement applies to **all** symmetric states (not only ground states of natural Hamiltonians)!

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Outline

- Introduction
- Anomaly of a symmetry
- Implications of anomaly
- Discussion

Symmetries are not just described by groups. They are also described by their actions on operators.

Same symmetry group: $SO(3) \times \mathbb{Z}$

Different operator contents

symmetry conditions: SO(3) spin rotation & lattice translation, spin-1/2 at each site	symmetry conditions: SO(3) spin rotation & lattice translation, spin-1 at each site

Same symmetry group: $\mathbb{Z}_2$

Different actions on operators

symmetry conditions: a qubit at each site, symmetry: $U = \prod_j e^{\frac{i\pi}{4} Z_j Z_{j+1}} \prod_k X_k$	symmetry conditions: a qubit at each site, symmetry: $U = \prod_k X_k$

What does it mean to define a quantum system?
What does it mean to define a symmetry action?

Data defining the quantum many-body system:
A set of local operators $a_1, a_2, \cdots,$ and relations $a_i a_j = \sum_k c_{ij,k} a_k$

Data defining the symmetry action β:
Given a local operator a, what is $\beta(a)$?

Example 1: a chain of qubits with spin rotation symmetry
Local operators: $\vec{\sigma}_i$ and their tensor products. Operator products: $\sigma_i^x \sigma_i^y = i\sigma_i^z$, etc
Action of spin rotation: $\vec{\sigma}_i$ rotates as a vector

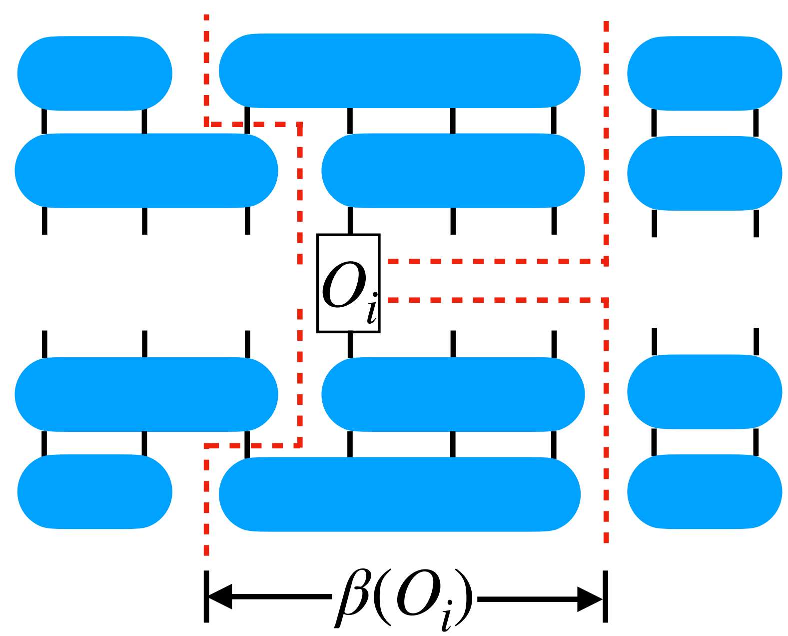
Example 2: a chain of qutrits with spin rotation symmetry
Local operators: $|x\rangle_i \langle y|_i$ and their tensor products $(x, y = 0,1,2)$. Operator products: …
Action of spin rotation: $\{ |x\rangle_i \langle y|_i \}$ transform in different representations of $SO(3)$

A broad class of symmetry actions are given by twisted quantum cellular automata (QCA).

Twisted QCA: A linear or anti-linear operation β that
1) preserves the operator algebra ($\beta(a+b) = \beta(a) + \beta(b), \beta(ab) = \beta(a)\beta(b), \beta(a^\dagger) = (\beta(a))^\dagger$).
2) maps any local operator to another local operator nearby.

Example 1: Finite-depth quantum circuit

$O_i \to \beta(O_i) =$



Example 2: Translation

$O_i \to \beta(O_i) = O_{i+1}$

Example 3: Complex conjugation

$O_i \to \beta(O_i) = K_i O_i K_i^{-1}$

Structure of twisted QCA: $\tilde{\mathscr{G}}^{\mathrm{QCA}} = \mathscr{G}^{\mathrm{QCA}} \rtimes \mathbb{Z}_2 = (\mathscr{G}^{\mathrm{circuit}} \rtimes \mathscr{G}^{\mathrm{translation}}) \rtimes \mathbb{Z}_2$
(Any twisted QCA can be decomposed into a circuit, a translation and a conjugation.)

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Each twisted QCA has a GNVW index.

GNVW index for ordinary linear QCA (measuring information flow):
$\mathrm{ind}(\beta^{\mathrm{circuit}}) = 0$, $\mathrm{ind}(\beta^{\mathrm{translation}}) \neq 0$, $\mathrm{ind}(\beta_1 \circ \beta_2) = \mathrm{ind}(\beta_1) + \mathrm{ind}(\beta_2)$

GNVW index for anti-linear QCA $\tilde{\beta} = \beta \circ K \Rightarrow \mathrm{ind}(\tilde{\beta}) = \mathrm{ind}(\beta)$
$\tilde{\beta}$: anti-linear QCA, β: linear QCA, K: complex conjugation

$\Rightarrow$

- $\mathrm{ind}(\tilde{\beta})$ is independent of the decomposition $\tilde{\beta} = \beta \circ K$
- $\mathrm{ind}(\tilde{\beta}_1 \circ \tilde{\beta}_2) = \mathrm{ind}(\tilde{\beta}_1) + \mathrm{ind}(\tilde{\beta}_2)$

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Each symmetry action has an anomaly index.
I. Unitary internal symmetry

symmetry action: $\alpha : G \to \tilde{\mathscr{G}}^{\mathrm{QCA}}$

anomaly index for
unitary internal symmetry

$\alpha : G \to \mathscr{G}^{QCA}$

↓ Decompose α into 3 pieces

$$\alpha(g) = \alpha_{<0}(g) \circ \alpha_0(g) \circ \alpha_{\geqslant 0}(g)$$

↓

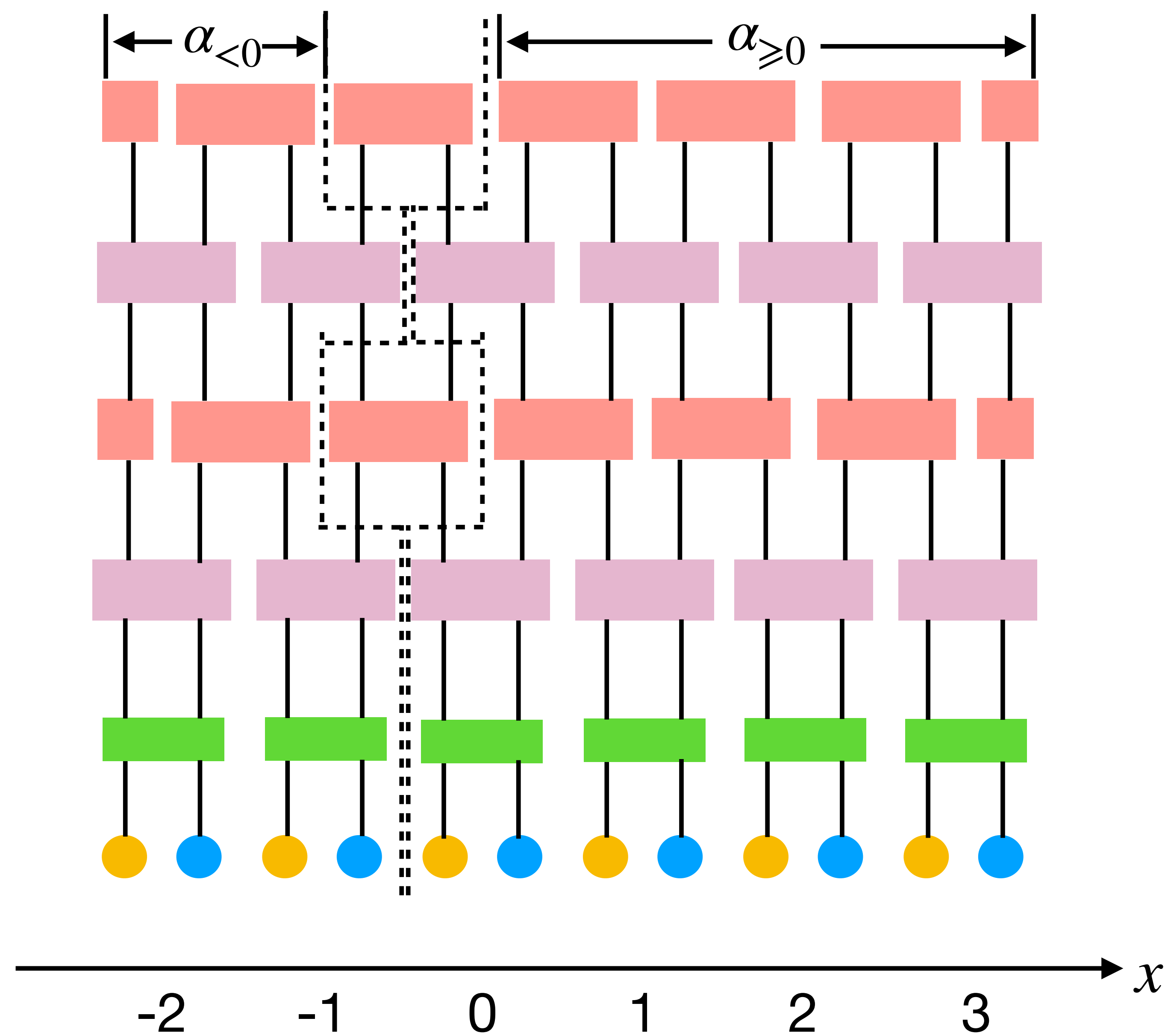
$$\mathrm{Ad}_{V(g,h)} = \alpha_{\geqslant 0}(g)\, \alpha_{\geqslant 0}(h)\, \alpha_{\geqslant 0}^{-1}(gh)$$

↓

$$\omega(g,h,k) = V(g,h)\, V(gh,k)\, V(g,hk)^{-1} \big(\alpha_{\geqslant 0}(g)\,(V(h,k))\big)^{-1} \in \mathrm{U}(1)$$

↓

$$[\omega] \in \mathrm{H}^3(G, \mathrm{U}(1))$$



Remark: $[\omega]$ is independent of the decomposition of α.

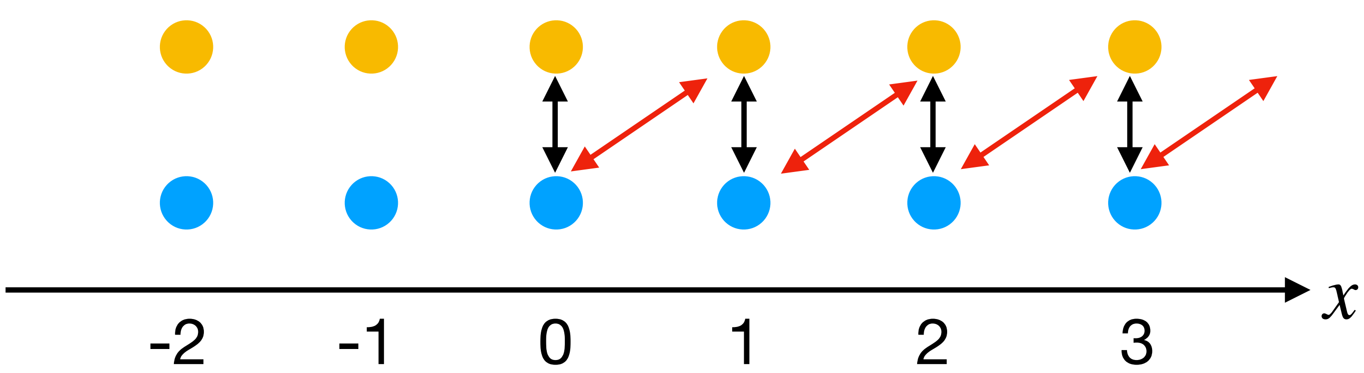
Else, Nayak, 1409.5436; Kapustin, Sopendo, 2401.02533; Liu, Yi, Zhou, **Zou**, 2405.14929; Liu, Yi, **Zou**, 2510.06555

Each symmetry action has an anomaly index.
II. Unitary translation symmetry

symmetry action: $\alpha : G \to \tilde{\mathscr{G}}^{\mathrm{QCA}}$

anomaly index for
unitary translation symmetry

1. Add another copy of the system
2. Define a new action with a vanishing GNVW index: $\alpha' = \alpha \otimes \tau^{-1}$
3. $[\omega]_\alpha = [\omega]_{\alpha'}$



Kapustin, Sopendo, 2401.02533; Liu, Yi, Zhou, **Zou**, 2405.14929; Liu, Yi, **Zou**, 2510.06555

Each symmetry action has an anomaly index.
III. Anti-unitary symmetry

symmetry action: $\alpha : G \to \tilde{\mathscr{G}}^{\mathrm{QCA}}$

anomaly index for anti-unitary internal symmetry

$\alpha : G \to \tilde{\mathscr{G}}^{QCA}$

↓ Decompose α into 3 pieces

$$\alpha(g) = \alpha_{<0}(g) \circ \alpha_0(g) \circ \alpha_{\geqslant 0}(g)$$

↓

$$\mathrm{Ad}_{V(g,h)} = \alpha_{\geqslant 0}(g)\, \alpha_{\geqslant 0}(h)\, \alpha_{\geqslant 0}^{-1}(gh)$$

↓

$$\omega(g,h,k) = V(g,h)\, V(gh,k)\, V(g,hk)^{-1} \big(\alpha_{\geqslant 0}(g)\,(V(h,k))\big)^{-1} \in \mathrm{U}(1)$$

↓

$$[\omega] \in \mathrm{H}^3_{\varphi}(G, \mathrm{U}(1))$$

anomaly index for anti-unitary translation symmetry

1. Add another copy of the system
2. Define a new action: $\alpha' = \alpha \otimes \tau^{-1}$
3. $[\omega]_\alpha = [\omega]_{\alpha'}$

Technical remarks for anti-unitary symmetry:

1. $[\omega]_\alpha$ is independent of the decomposition of α.
2. $\alpha_{\geqslant 0}$ and $\alpha_{<0}$ are neither linear nor anti-linear.

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Example 1:
A symmetry $G_{\text{int}} \times \mathbb{Z}$ is anomalous if and only if the degrees of freedom in a unit cell form a projective representation under G_{int}.

Lieb-Schultz-Mattis-like symmetry conditions:

on-site internal symmetry + lattice translation + projective representation in a unit cell

Example 1.1: $SO(3) \times \mathbb{Z}$ with a spin-1/2 per unit cell
Example 1.2: $\mathbb{Z}_2^T \times \mathbb{Z}$ with a Kramers doublet per unit cell

projective representation: $R_i(g_1)R_i(g_2) = \eta(g_1, g_2)R_i(g_1g_2), \forall g_{1,2} \in G_{\text{int}}$

anomaly cocycle: $\omega((g_1, n_1), (g_2, n_2), (g_3, n_3)) = \eta(g_1, g_2)^{n_3}$
$(g_i \in G_{\text{int}}, \quad n_i \in \mathbb{Z}, \quad [\eta] \in \mathrm{H}^2_\varphi(G; U(1)))$

$\Rightarrow$

anomalous if and only if $[\eta]$ is a nontrivial element in $\mathrm{H}^2_\varphi(G; U(1))$

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Example 2:
An anti-unitary translation
together with $\mathbb{Z}_2 \times \mathbb{Z}_2$ spin rotations
can be anomalous.

symmetry group: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}^T$

(example Hamiltonian: $H = \sum_{i\alpha} J_\alpha S_i^\alpha S_{i+1}^\alpha + \sum_{i\alpha\beta\gamma} (-1)^i K_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} S_i^\alpha S_{i+1}^\beta S_{i+2}^\gamma$ in Yao, Li, Oshikawa, Hsieh, 2307.09843)

symmetry generators:
x: spin rotation around the x-axis by π
z: spin rotation around the z-axis by π
t: composition of ordinary time reversal and translation

$\Rightarrow$

anomaly cocycle:
$\omega((x_1, z_1, t_1), (x_2, z_2, t_2), (x_3, z_3, t_3)) = 1$ for integer spins $\Rightarrow$ non-anomalous
$\omega((x_1, z_1, t_1), (x_2, z_2, t_2), (x_3, z_3, t_3)) = (-1)^{(x_1 z_2 + n_1 n_2) n_3}$ for half-odd-integer spins $\Rightarrow$ anomalous

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Any state with an anomalous symmetry must have long-range correlation or violate area law.

symmetry conditions:
SO(3) spin rotation & lattice translation,
spin-1/2 at each site

examples of symmetric states:
all long-range entangled

1. Ground state of anti-ferromagnetic Heisenberg chain: violate area law

2. Ground state of Majumdar-Ghosh chain: long-range correlation

…

intuition:

anomalous symmetries cannot be on-site

$\Rightarrow$

nontrivial correlation and entanglement

The proof is very abstract, which uses von Neumann algebra.

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An anomalous symmetry forces a state to be long-range entangled.

$\Uparrow$ previous theorem

SRE states have short-range correlation and obeys area law:

Superpolynomial clustering: $\langle AB \rangle - \langle A \rangle \langle B \rangle \sim O(r^{-\infty})$
Entanglement area law: $S_{[0,n]} < S_0$ (S_0 is independent of n)

Kapustin, Sopenko, Yang, 2012.15491, Liu, Yi, Zhou, **Zou**, 2405.14929

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Admissible Hamiltonians with an anomalous symmetry cannot have a unique gapped ground state.

$\Uparrow$ previous theorem

Admissible Hamiltonians allow any 2-body interactions decaying faster than $1/r^2$.

Unique gapped ground states of admissible Hamiltonians

1. obey the area law, Liu, Yi, Zhou, **Zou**, 2405.14929 (see also Kuwahara, Saito, 1908.11547, Ukai, 2407.12324)
2. have no long-range order (only quasi-long-range order). Hastings, Koma, cond-mat/0507008

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Take-home message:
Anomalous symmetries guarantee nontrivial pattern of entanglement.

symmetry group G & symmetry action $\Rightarrow$ anomaly index $H^3_\varphi(G; U(1))$

Consequences of an anomalous symmetry:

- The state must either have long-range correlation or violate entanglement area law.
- The state must be long-range entangled.
- Any symmetric admissible Hamiltonian cannot have a unique gapped ground state.
- …

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Open questions

- What equivalence classes of symmetry actions do the anomaly indices classify?
- How can these considerations be extended to higher dimensions, fermionic systems, point-group symmetries, and generalized symmetries?
- How can the general anomaly matching be rigorously established?

Previous proposal:
Anomaly-matching is a systematic framework to study the realizability of a theory in a quantum material.

A quantum material can realize a theory.

$$\Updownarrow$$

$$\omega = \Omega$$

ω: quantum anomaly of the quantum material

Ω: quantum anomaly of the theory

(previous literature: case-by-case & cannot answer many questions)

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