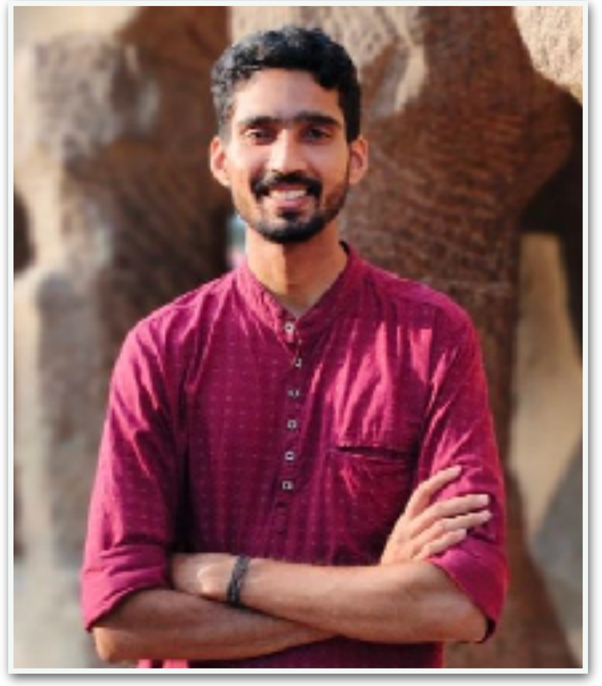
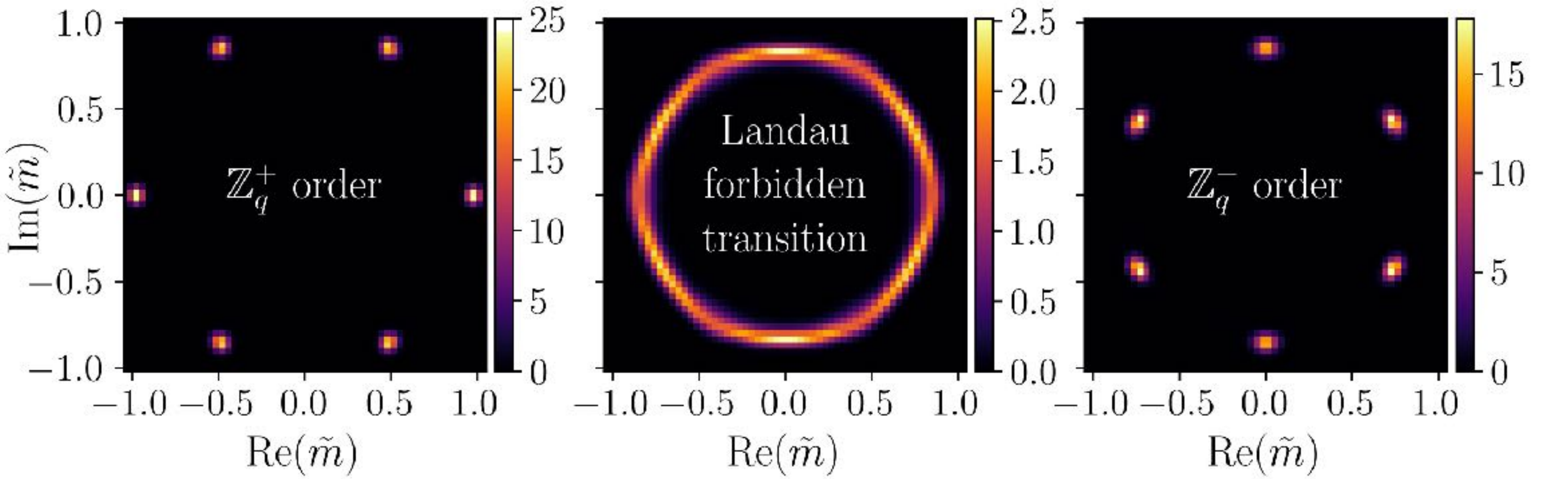
Emergence in a classical frustrated system: Symmetry, order, and '*deconfined classical*' criticality



Vishnu

Rajesh



Abhishodh

Advanced School and Conference on Quantum Matter
ICTP, Dec 1 - Dec 12 2025

Vishnu P. K., A. Prakash, R. Narayanan, **TC**
arXiv:2505.05194

Titas Chanda
Dept. of Physics, IIT Madras, India

Emergence

Sunrise in Marina Beach, Chennai, India

Emergence

The Principle of Emergence…

The collective behavior of interacting microscopic particles leads to **new phenomena at macroscopic scales**

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The Principle of Emergence…

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Will talk about two types

Emergence of order…
by spontaneous symmetry breaking
as described by Landau-Ginzburg theory

Microscopic theory
described by $\mathscr{L}$ or $\mathscr{H}$
respects certain global symmetry

SSB $\Rightarrow$

The ground state $|\psi_{GS}\rangle$
or the thermal state $\rho_{th}(T)$
violates the symmetry
develops (local) order

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accidental symmetries

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Low-energy description
invariant under the particular symmetry operation

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We are living in a era of **(analog) Quantum Simulations…**

are at the stage to **quantum simulate** even **fundamental theories** using **synthetic** quantum systems… e.g., cold-atoms, Rydberg arrays

Eur. Phys. J. D (2020) 74: 165
https://doi.org/10.1140/epjd/e2020-100571-8
THE EUROPEAN PHYSICAL JOURNAL D
Colloquium

Simulating lattice gauge theories within quantum technologies

Mari Carmen Bañuls[1,2], Rainer Blatt[3,4], Jacopo Catani[5,6,7], Alessio Celi[3,8], Juan Ignacio Cirac[1,2], Marcello Dalmonte[9,10], Leonardo Fallani[5,6,7], Karl Jansen[11], Maciej Lewenstein[8,12,13], Simone Montangero[14,15,a], Christine A. Muschik[3], Benni Reznik[16], Enrique Rico[17,18], Luca Tagliacozzo[19], Karel Van Acoleyen[20], Frank Verstraete[20,21], Uwe-Jens Wiese[22], Matthew Wingate[23], Jakub Zakrzewski[24,25], and Peter Zoller[3]

PHILOSOPHICAL TRANSACTIONS A
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Review
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nature > nature physics > perspectives > article

Perspective | Published: 01 February 2018

Symmetry and emergence

Edward Witten

Nature Physics 14, 116–119 (2018) | Cite this article

"…in the early days of particle physics, the global symmetries or conservation laws were considered fundamental…"

"…in a modern understanding of particle physics, global symmetries are approximate"… and *"…are in significant part an **"accidental"** consequence of the gauge symmetries of the Standard Model…"*

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Emergence in Rydberg Ladders

PHYSICAL REVIEW B **105**, 155159 (2022)

Emergent $\mathbb{Z}_2 \Rightarrow$ Emergent SUSY

Phase diagram of Rydberg-dressed atoms on two-leg square ladders: Coupling supersymmetric conformal field theories on the lattice

Mikheil Tsitsishvili,[1,2,*] Titas Chanda,[1,†] Matteo Votto,[1] Pierre Fromholz,[1,2,‡] Marcello Dalmonte,[1,2] and Alexander Nersesyan[1,3,4]

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Emergent SU(2)

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Common Folklore…

RG **irrelevant** perturbations can lead to enhanced symmetry in the coarse-grained theory

Exact action: $S = S^{Sym}_{CFT} + g \int dtd\mathbf{r}\ O(t, \mathbf{r})$

Under $(t, \mathbf{r}) \rightarrow (\lambda t, \lambda \mathbf{r})$

$O(\lambda t, \lambda \mathbf{r}) = \lambda^{-\Delta} O(t, \mathbf{r})$

$S = S^{Sym}_{CFT} + g\lambda^{d-\Delta} \int d(t)d(\mathbf{r})\ O(t, \mathbf{r})$

Becomes **irrelevant** in the infrared limit for $\Delta > d$

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Becomes **irrelevant** in the infrared limit for $\Delta > d$

We are going to contradict this conventional wisdom on emergence… in a **classical many-body system**

Frustration + relevant perturbation $\longrightarrow$ emergent symmetry $\xrightarrow{\text{SSB}}$ emergent order

Classical Many-Body Systems
not quantum...

Classical many-body systems @ IIT Madras Campus

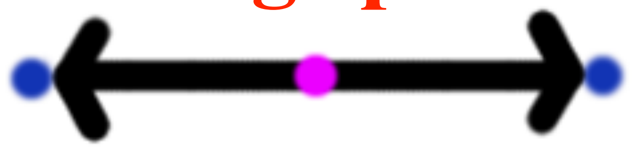
Classical Many-Body Systems

Example: 2D Ising model

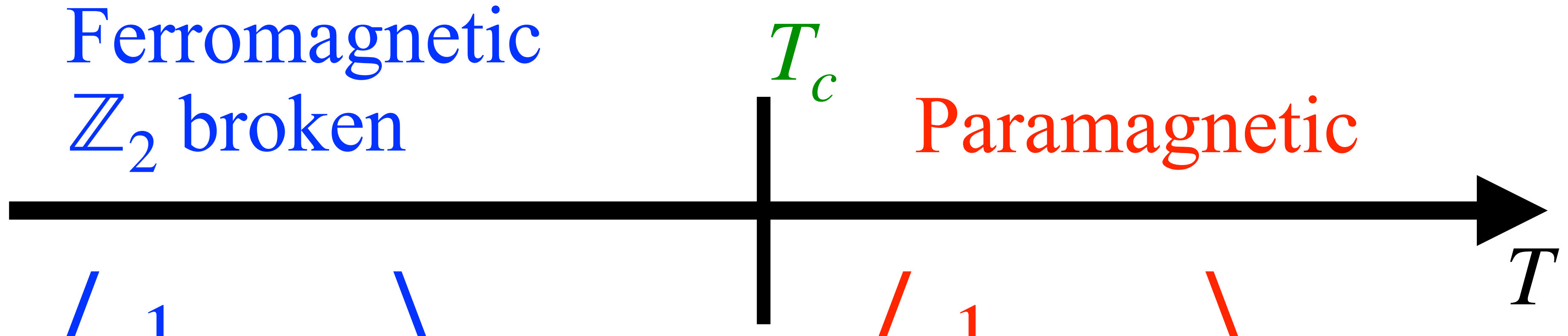
$$H = -J\sum_{\langle ij\rangle} s_i s_j \qquad s = \pm 1$$

$$= -J\sum_{\langle ij\rangle} \cos(\theta_i - \theta_j) \qquad \theta = 0,\ \pi$$

Ising spins



$\mathbb{Z}_2$ symmetry: $\begin{matrix} s_i \to \pm s_i \\ \theta_i \to \theta_i + n\pi \end{matrix} \quad \forall i$



$$\left\langle \frac{1}{L^2}\sum_i s_i \right\rangle = \pm 1 \qquad \left\langle \frac{1}{L^2}\sum_i s_i \right\rangle = 0$$

Textbook examples

Classical Many-Body Systems

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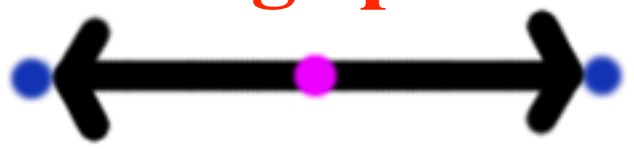

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$\mathbb{Z}_2$ broken

T_c

Paramagnetic

T

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Example: 2D XY model

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XY spins

$O(2)$ symmetry: $\begin{array}{l} SO(2):\ \theta_i \to \theta_i + \alpha \\ \mathscr{R}\ :\ \theta_i \to -\theta_i \end{array} \quad \ldots\ \forall i$

In 2D… continuous symmetry **cannot be broken** (Mermin-Wagner theorem)

But… still has a phase transition…
Berezinskii-Kosterlitz-Thouless (BKT) Transition

T_{BKT}

Quasi-long range order (QLRO)

Paramagnetic

T

$$\langle \cos(\theta_i - \theta_{i+R}) \rangle \sim 1/R^\eta \qquad \langle \cos(\theta_i - \theta_{i+R}) \rangle \sim \exp(-R/\xi)$$

$$\left\langle \frac{1}{L^2}\sum_j \exp(i\theta_j) \right\rangle = 0$$

Classical Many-Body Systems

Example: 2D q-state clock model… somewhere between Ising and XY

$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \qquad \theta = \frac{2n\pi}{q}; \quad n = 0,1,\ldots,q-1$$

$$D_q \text{ symmetry: } \begin{array}{l} \mathbb{Z}_q : \;\; \theta_i \to \theta_i + \dfrac{2m\pi}{q} \;\; \ldots \forall i \\ \mathscr{R} \; : \;\; \theta_i \to -\theta_i \end{array}$$

Ising spins

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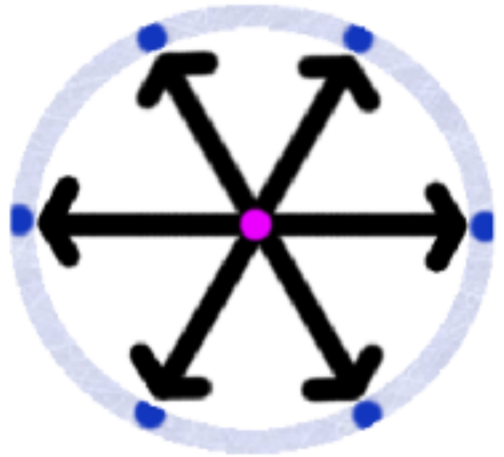

6-state clock spins

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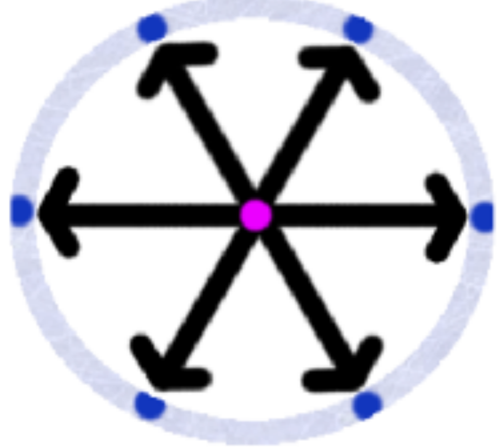

6-state clock spins

Interesting scenario happens for $q \geq 5$

at intermediate T…
there is an emergent O(2) symmetry
that cannot be broken… gives a QLRO phase

Ferromagnetic
$\mathbb{Z}_q$ broken

$$\left\langle \frac{1}{L^2} \sum_j \exp(i\theta_j) \right\rangle \neq 0$$

$T_{BKT}^{(1)}$

QLRO
Emergent O(2) symmetry

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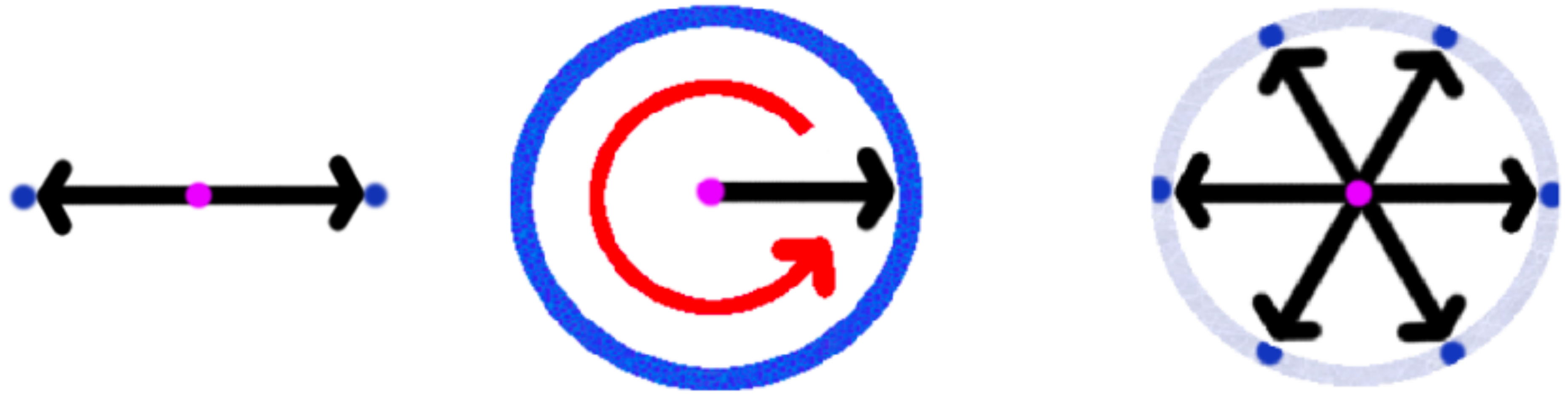

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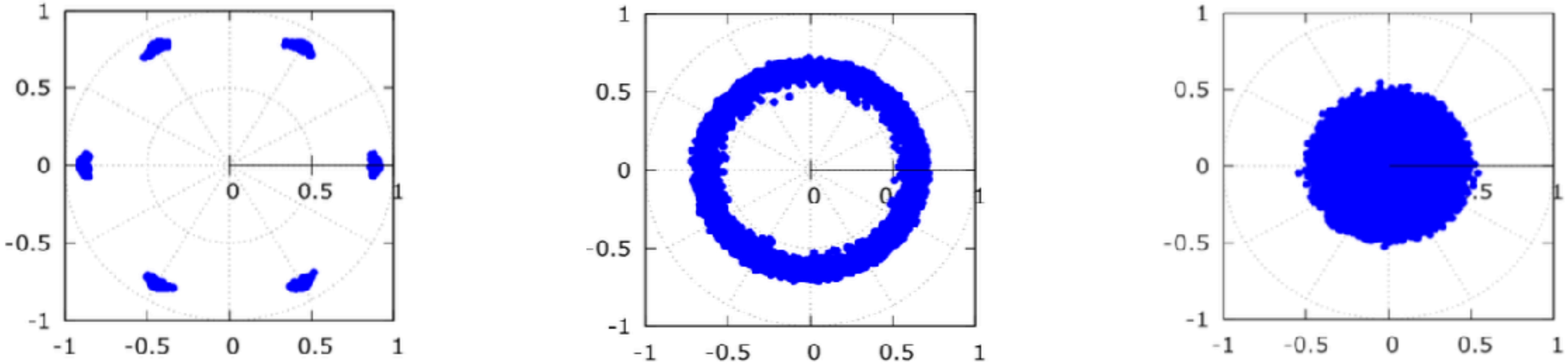
T

Polar plot of complex order parameter:

$$m = \left\langle \frac{1}{L^2}\sum_j \exp(i\theta_j)\right\rangle$$

from different Monte Carlo snapshots



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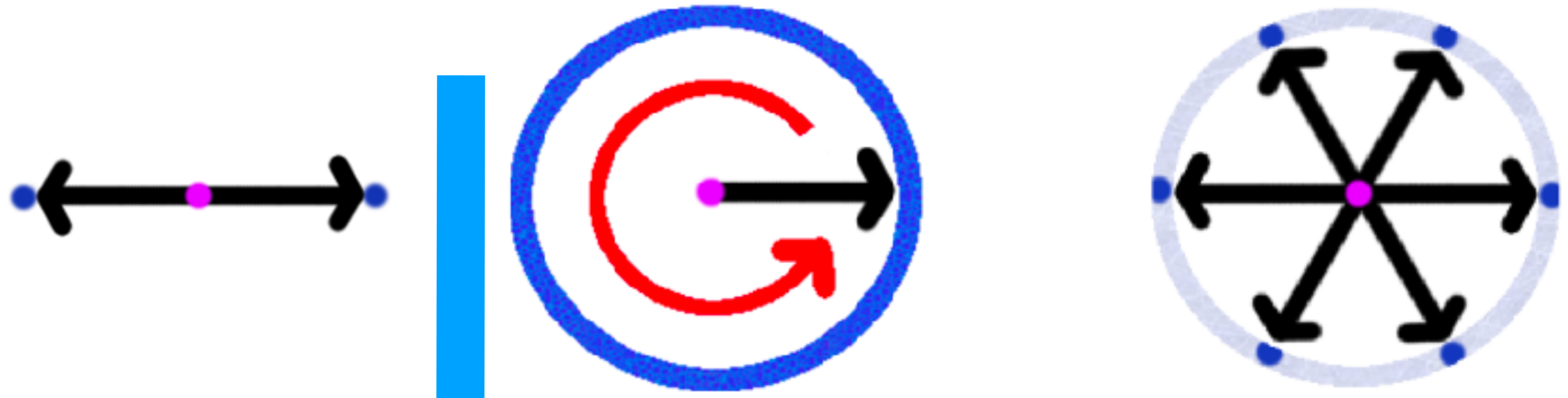

Ising spins | XY spins | **6-state clock spins**

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Soften the 'clock' restriction
by putting energy penalty

$$H_{soft} = -J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j) - h_1\sum_i \cos(q\theta_i) \qquad 0 \leq \theta < 2\pi,\ h_1 > 0$$

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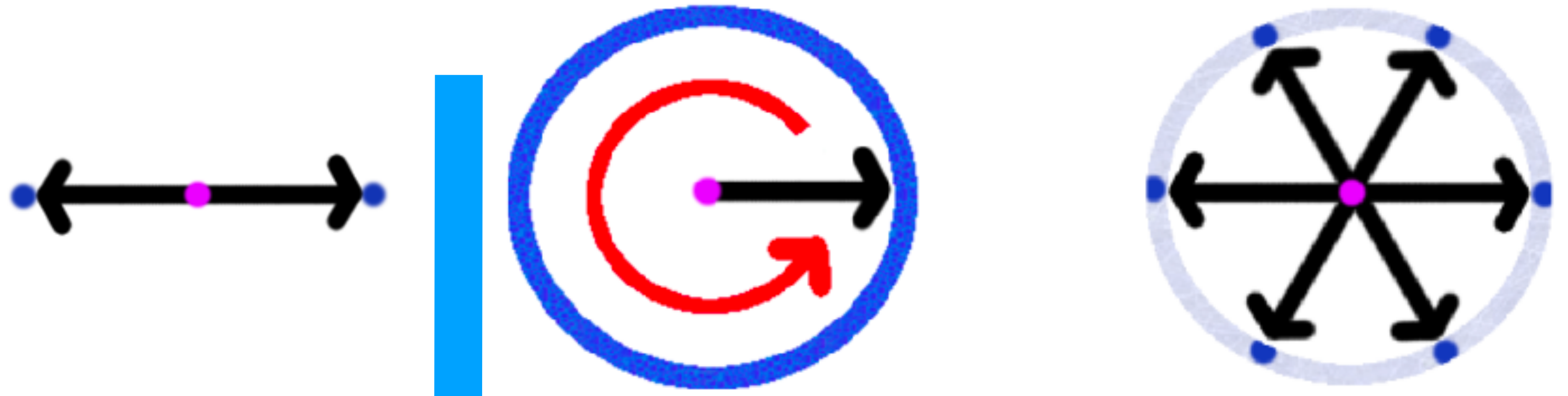

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RG relevant
i.e., $h_1 \to \infty$ at infrared

Ferromagnetic, $\mathbb{Z}_q$ broken	$T^{(1)}_{BKT}$	QLRO, Emergent O(2) symmetry	$T^{(2)}_{BKT}$	Paramagnetic
$\left\langle \frac{1}{L^2}\sum_j \exp(i\theta_j)\right\rangle \neq 0$		$\langle\cos(\theta_i-\theta_{i+R})\rangle \sim 1/R^{\eta}$		$\langle\cos(\theta_i-\theta_{i+R})\rangle \sim \exp(-R/\xi)$

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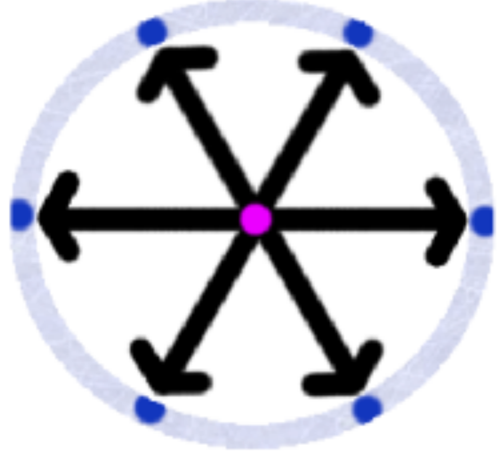
Ising spins

XY spins



6-state clock spins

Interesting scenario happens for $q \geq 5$

at intermediate T…
there is an emergent O(2) symmetry
that cannot be broken… gives a QLRO phase

Soften the 'clock' restriction by putting energy penalty

Ferromagnetic
$\mathbb{Z}_q$ broken

$$\left\langle \frac{1}{L^2}\sum_j \exp(i\theta_j)\right\rangle \neq 0$$

$T^{(1)}_{BKT}$

QLRO
Emergent O(2) symmetry

$$\langle\cos(\theta_i-\theta_{i+R})\rangle \sim 1/R^{\eta}$$

$T^{(2)}_{BKT}$

Paramagnetic

$$\langle\cos(\theta_i-\theta_{i+R})\rangle \sim \exp(-R/\xi)$$

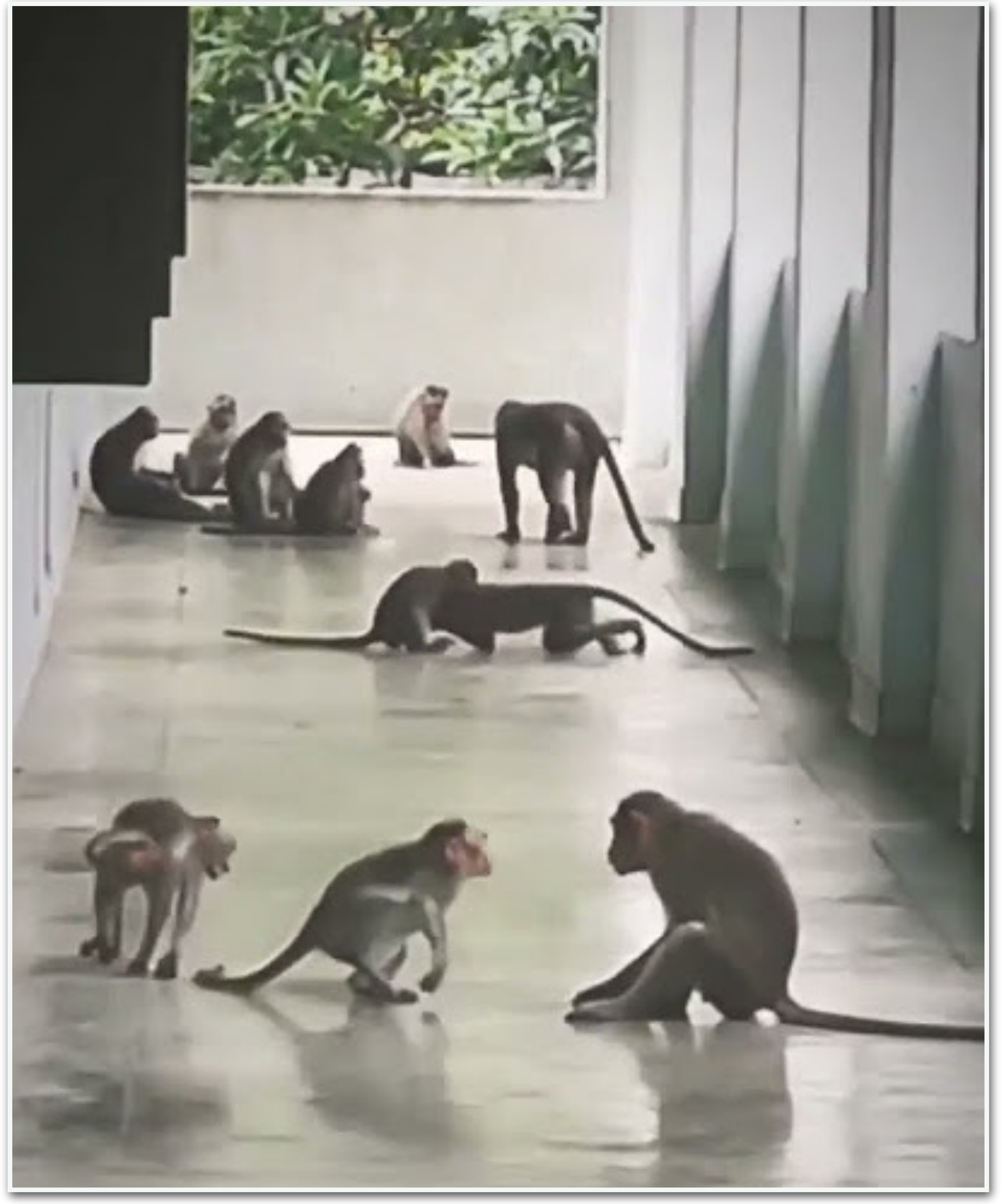
T

RG relevant
i.e., $h_1 \to \infty$ at infrared

Becomes RG **irrelevant**
i.e., $h_1 \to 0$ at infrared
Description becomes XY model

$$H_{soft} = -J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j) - h_1\sum_i \cos(q\theta_i) \qquad 0 \leq \theta < 2\pi,\ h_1 > 0$$

Frustrated Classical Many-Body Systems



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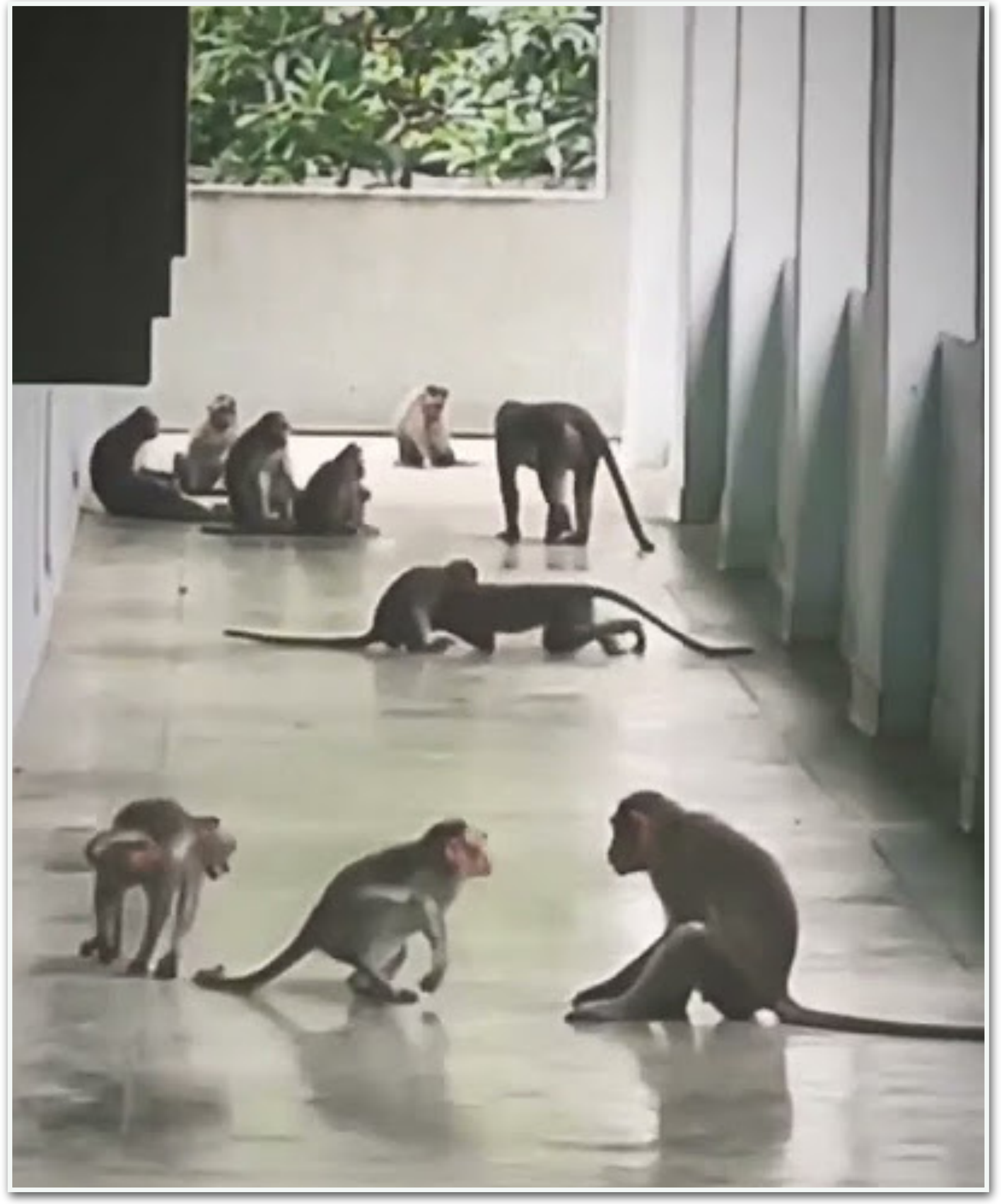
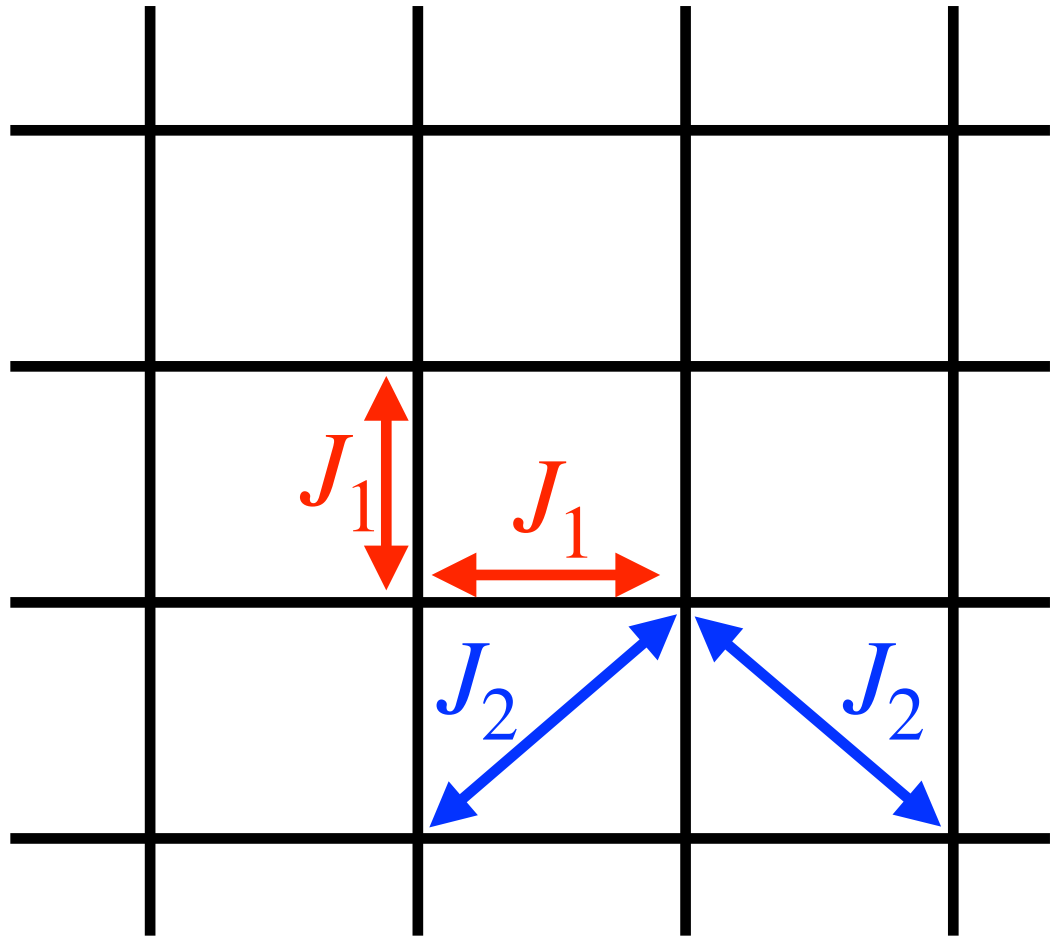


$$H = -J_1 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) + J_2 \sum_{\langle\langle ij \rangle\rangle} \cos(\theta_i - \theta_j)$$

Nearest-neighbor
ferromagnetic interaction

Next-nearest neighbor
anti-ferromagnetic interaction



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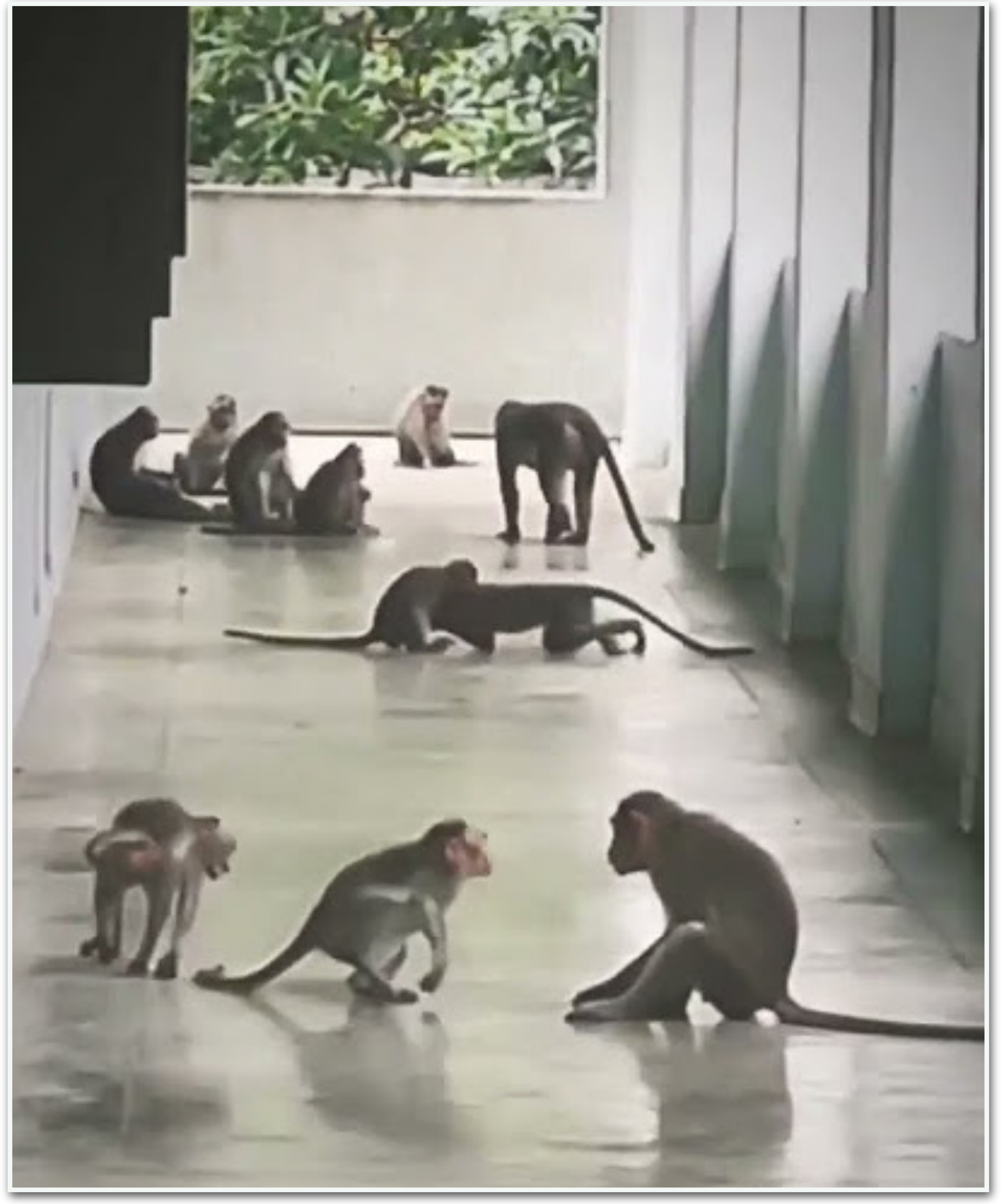

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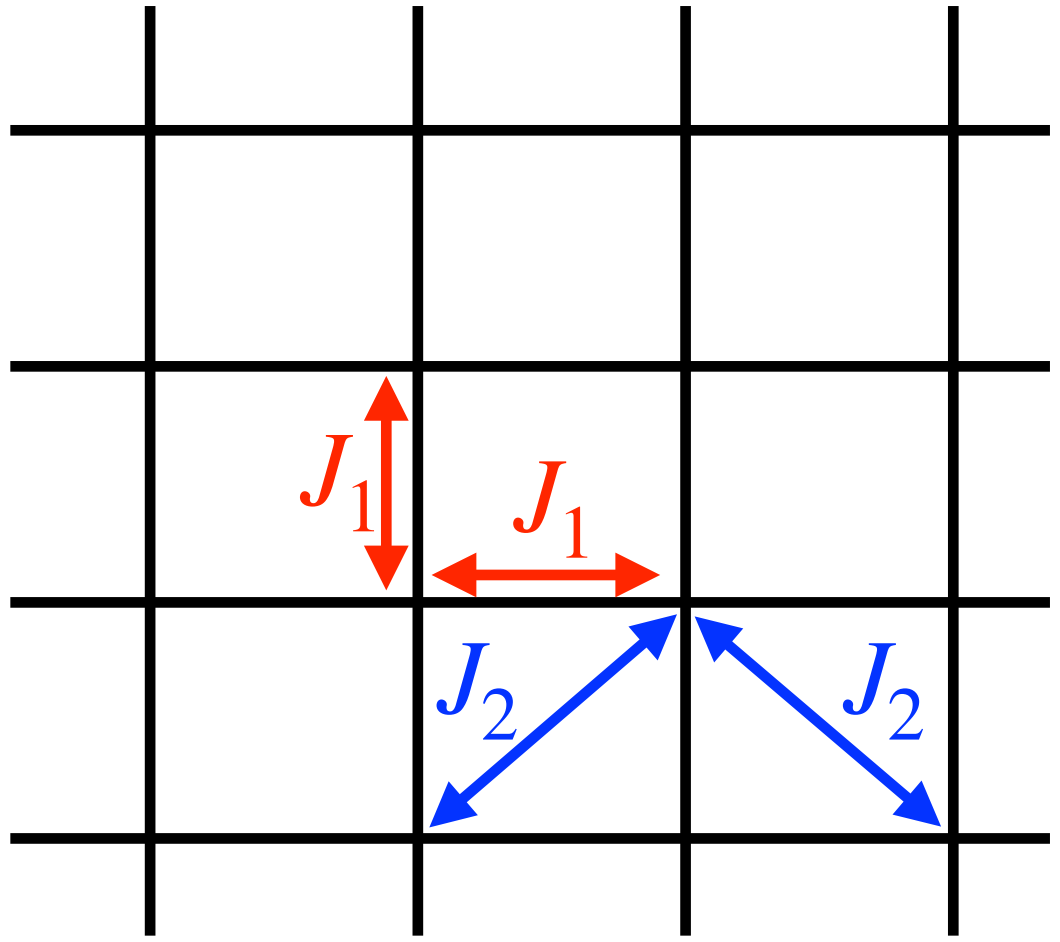
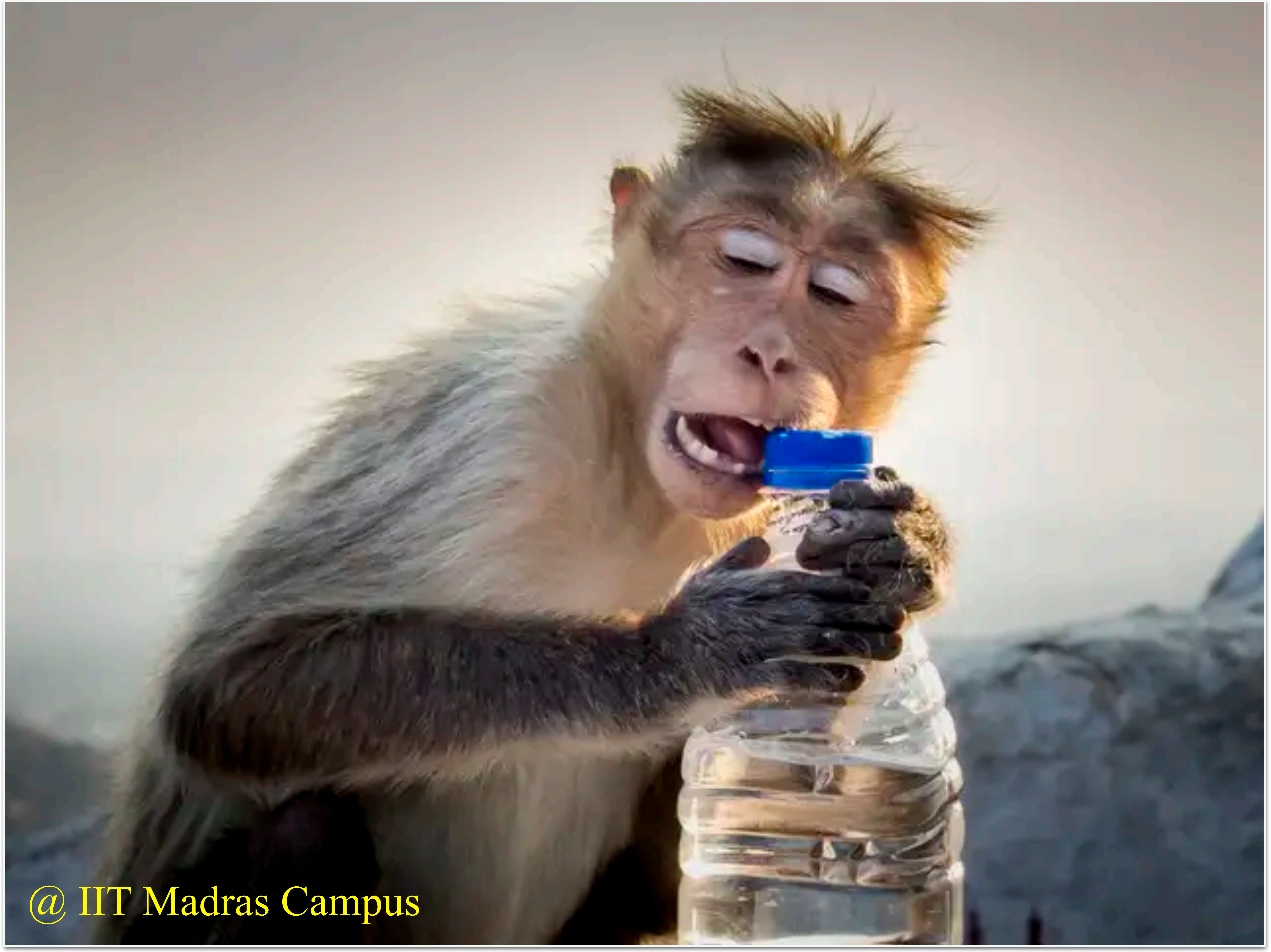


$\theta = \dfrac{2n\pi}{q}$; $\quad n = 0,1,\ldots, q-1$…. with even $q \geq 6$

Note: Even the **Ising** (Gangat, PRB 2024, Chatelain, PRE 2025), and **XY** (Song et. al., PRB 2025) limits are still debated and under active considerations

How to crack these systems



How to crack these systems

Option 1: Traditional **Classical Monte Carlo (CMC)…**

Frustration ⇒ Single spin-flip updates ⇒ large autocorrelation time / critical slowing down

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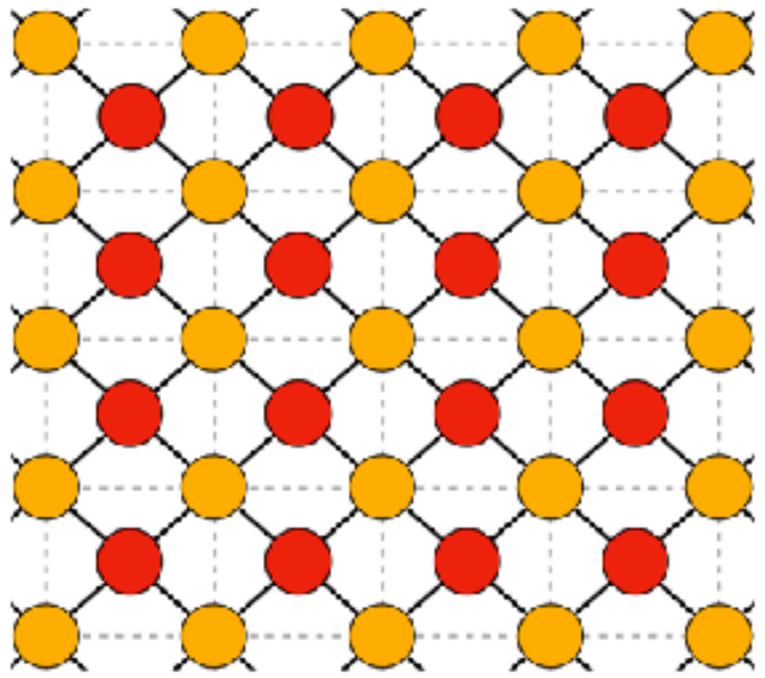
Corner transfer matrix renormalization group (CTMRG)… see the review by Okunishi, Nishino, Ueda (2022)

Partition function $Z = \sum_{\{s\}} e^{-\beta H(\{s\})} = \mathrm{Tr}\left[\quad\right]$



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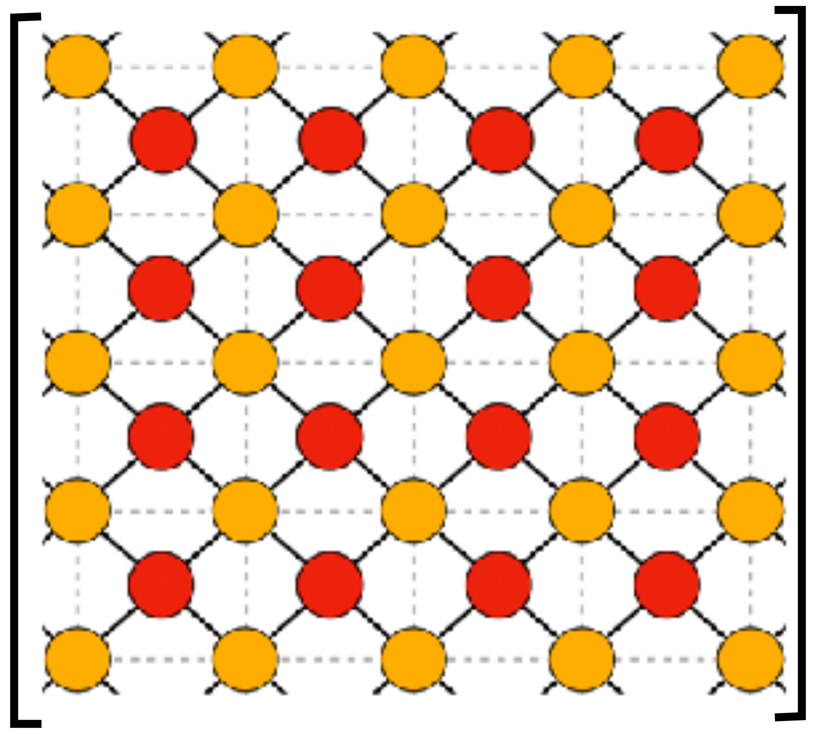
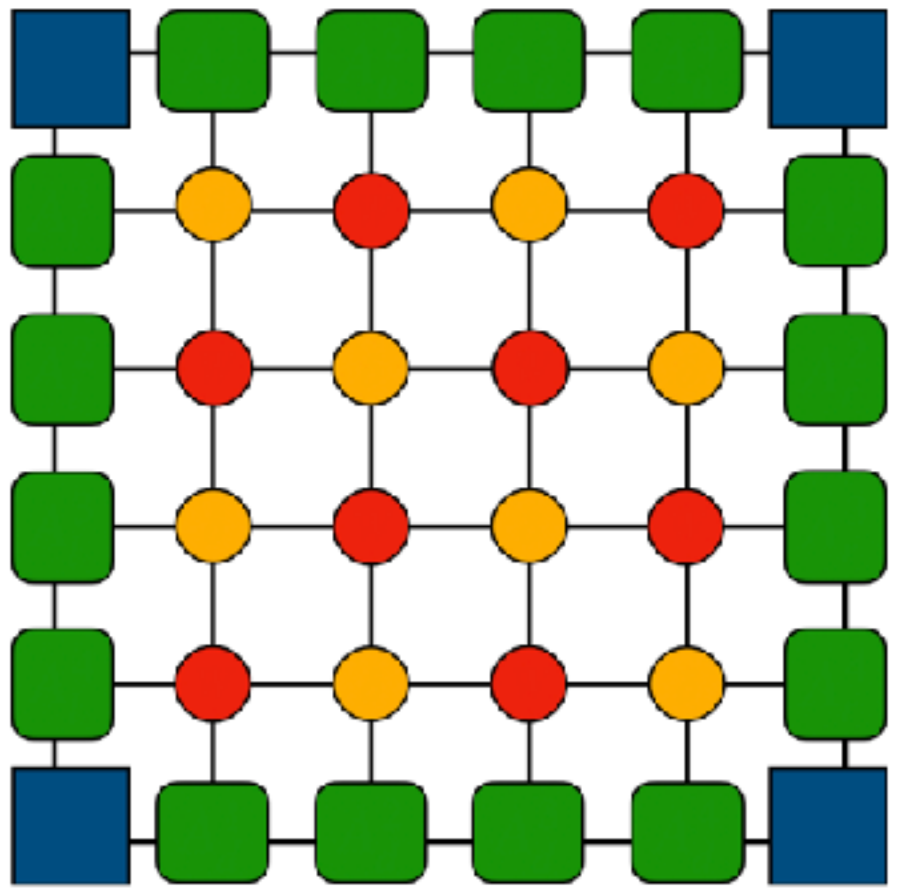


CTMRG



Corner transfer matrices

Boundary MPS tensors

MPS: Matrix Product states

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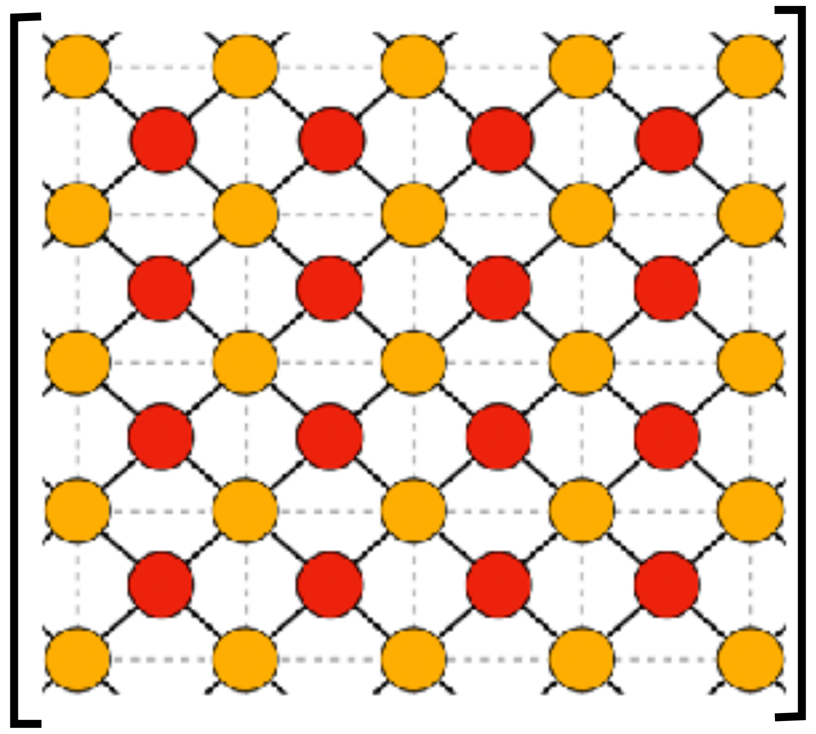
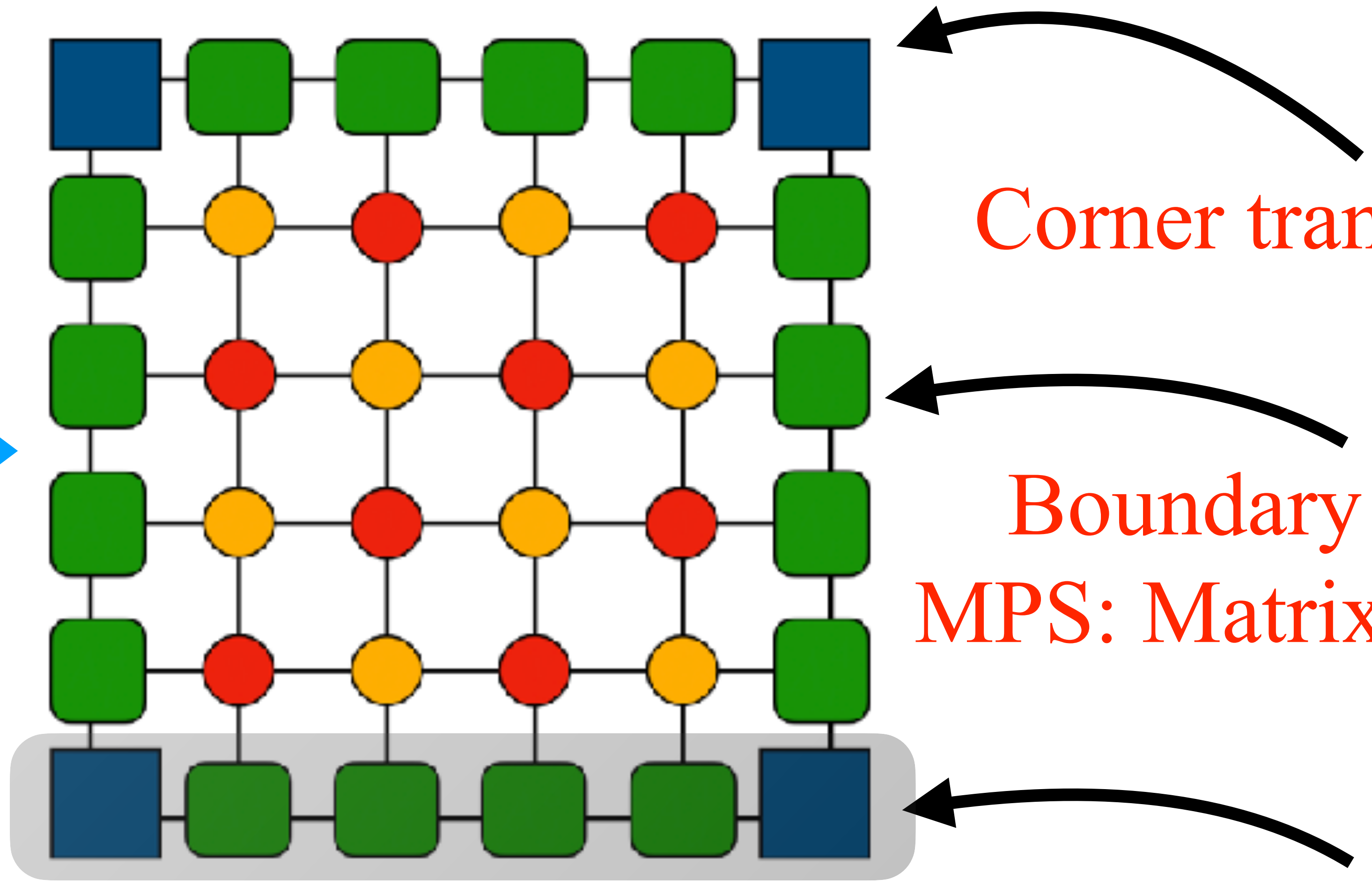
Partition function $Z = \sum_{\{s\}} e^{-\beta H(\{s\})} = \mathrm{Tr}$ 

CTMRG ⇒



“Classical-Quantum correspondence in action”

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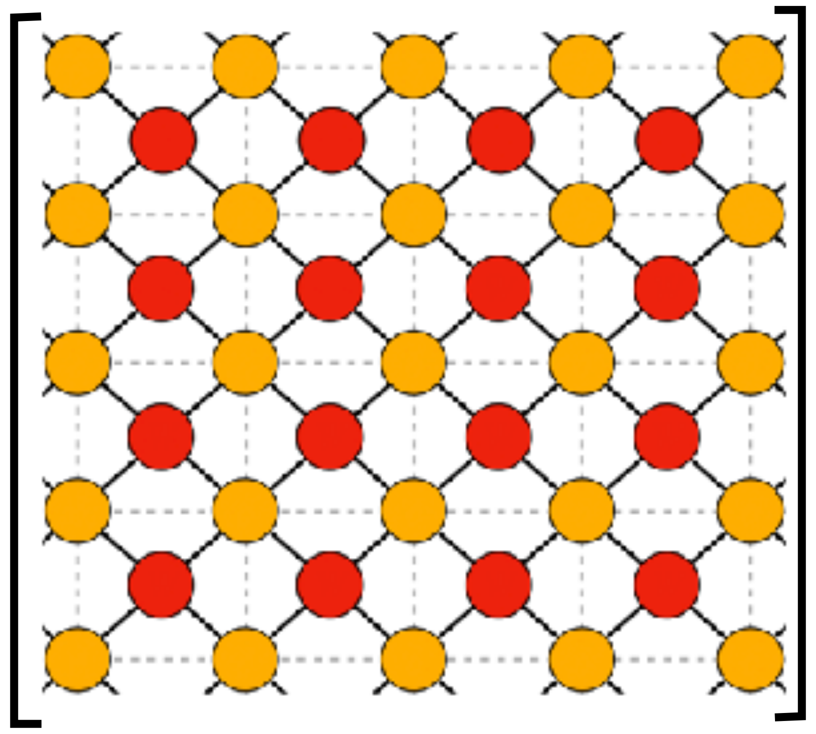
Partition function $Z = \sum_{\{s\}} e^{-\beta H(\{s\})} = \mathrm{Tr}$ 

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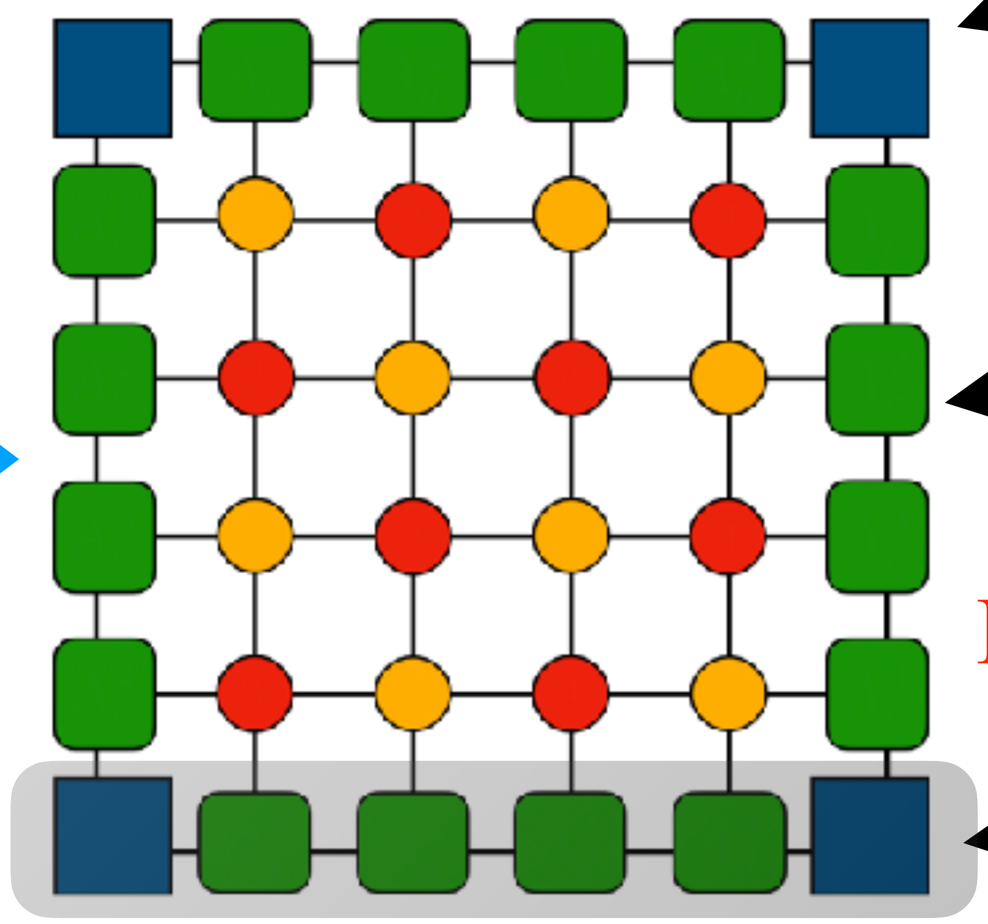

Corner transfer matrices

Boundary MPS tensors

MPS: Matrix Product states

Boundary (infinite) MPS

"Classical-Quantum correspondence in action"

- Efficient, elegant, deterministic
- Calculation of observables, correlation functions.. as easy as it can get.. exactly similar to MPS methods for 1D quantum systems
- von Neumann Entanglement Entropy (EE) 'for a classical system'… (almost) no extra computational cost…
- Calabrese-Cardy scaling of EE to get the central charge and the universality class… without much trouble…

How to crack these systems

Option 1: Traditional **Classical Monte Carlo (CMC)…**

CMC still a powerful and useful tool

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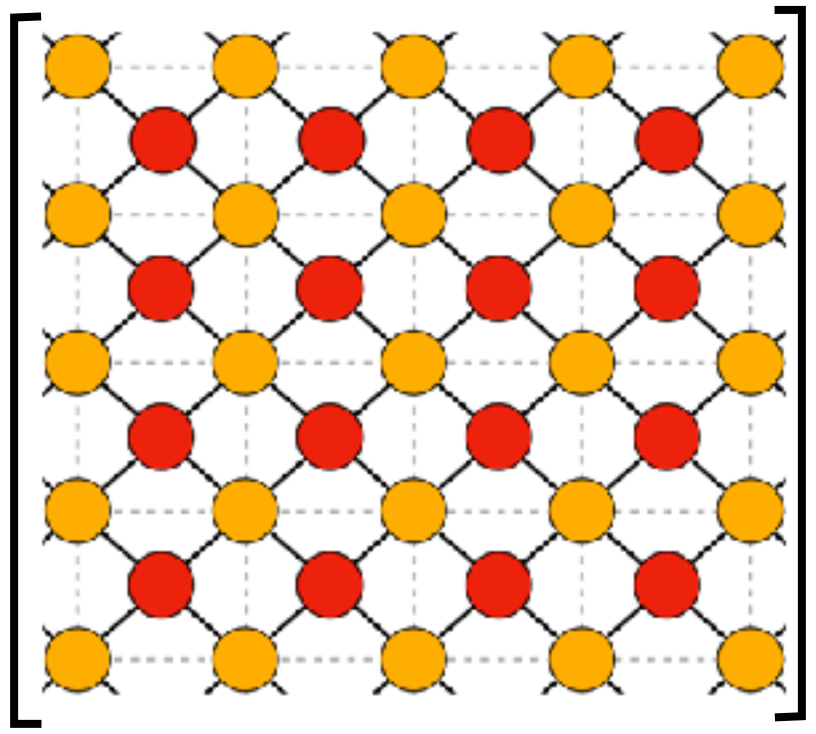

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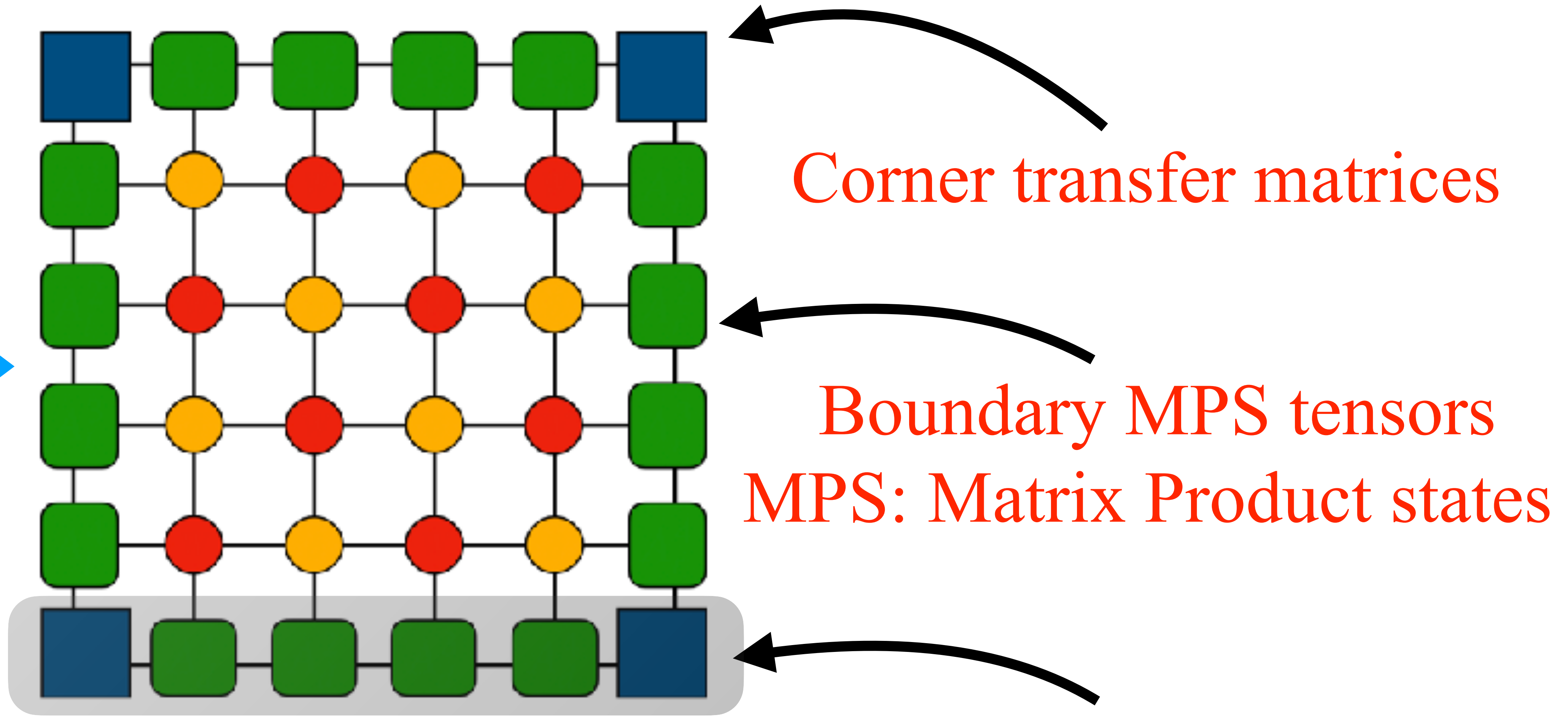

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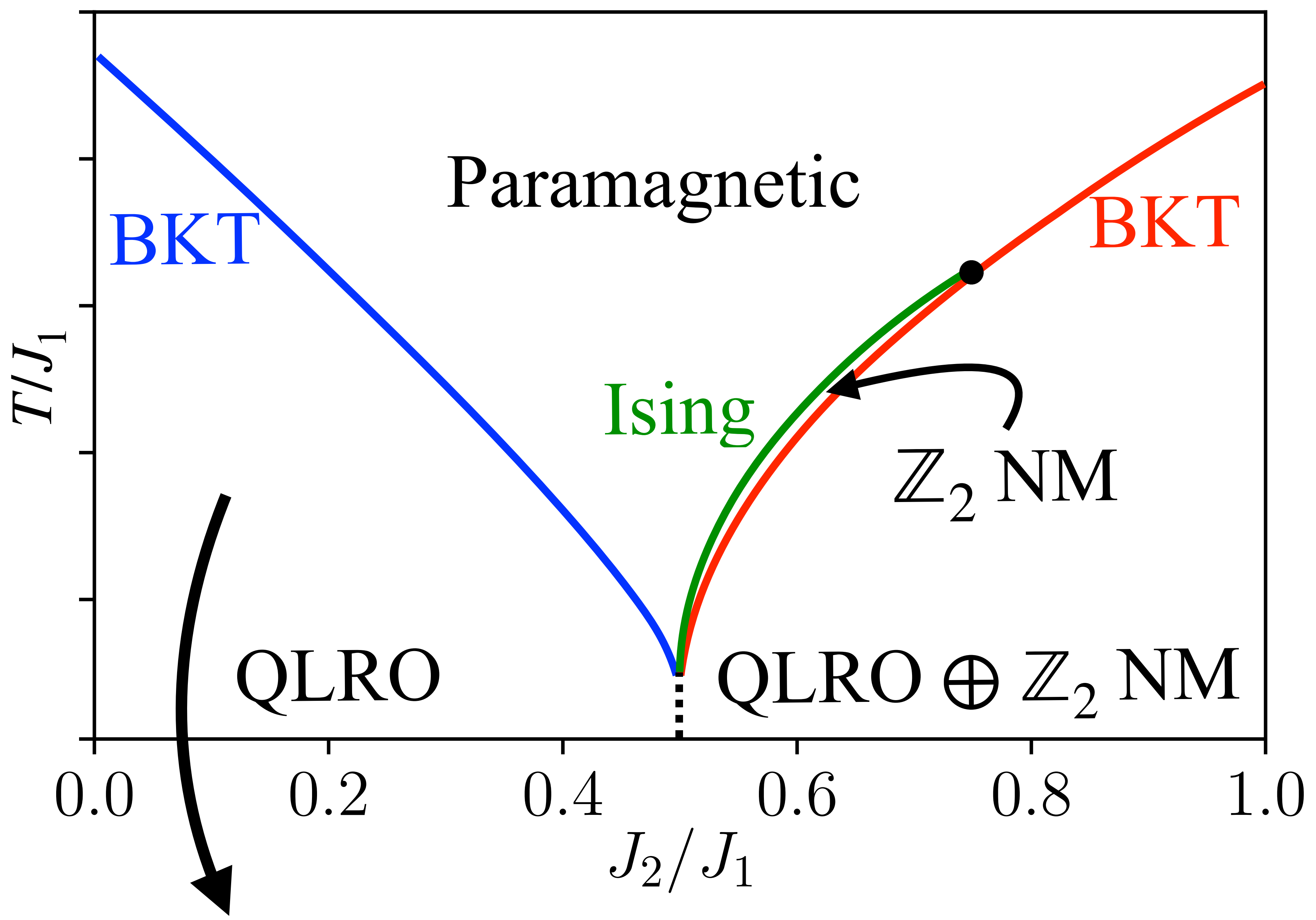
Results

J_1-J_2 Model… In the XY limit

$$H = -J_1 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) + J_2 \sum_{\langle\langle ij \rangle\rangle} \cos(\theta_i - \theta_j)$$



Quasi-long-range-order phase
as $U(1)$ symmetry cannot break

$$\left\langle \cos(\theta_i - \theta_{i+R}) \right\rangle \sim 1/R^{\eta}$$

Loison et. al., PRB 2000
Song et. al., PRB 2025

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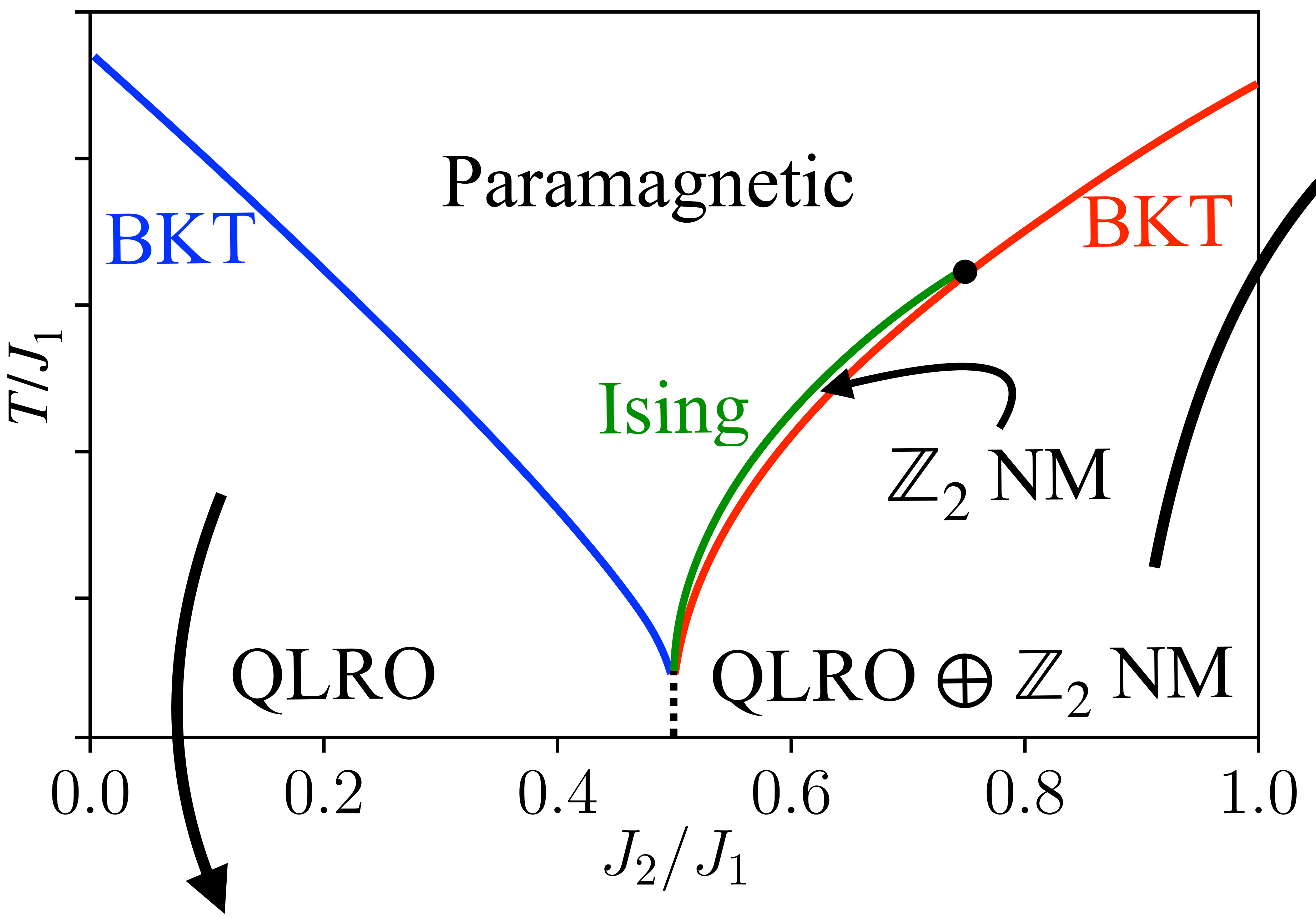

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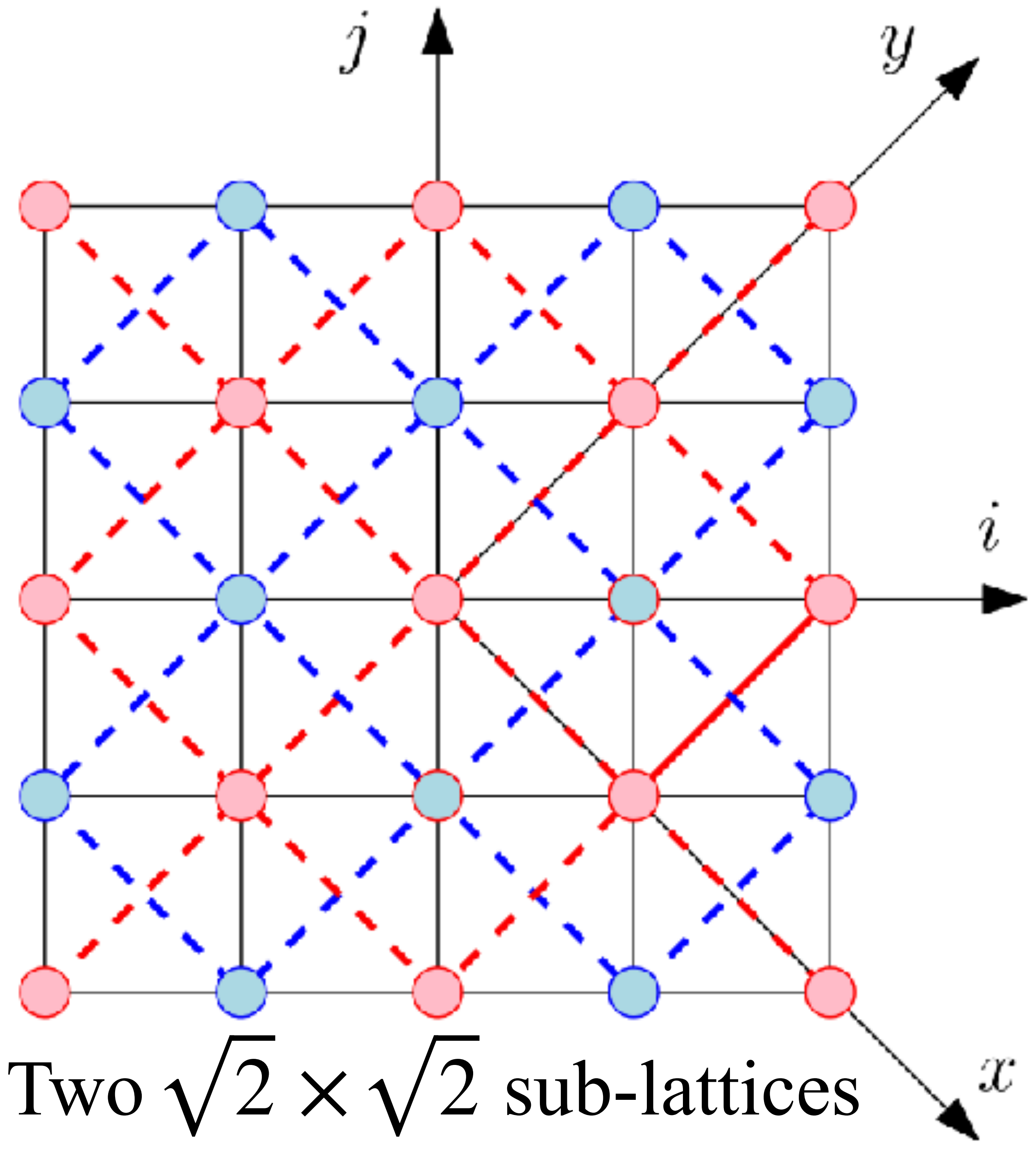

Two $\sqrt{2} \times \sqrt{2}$ sub-lattices $\Rightarrow$ red and blue sub-lattices

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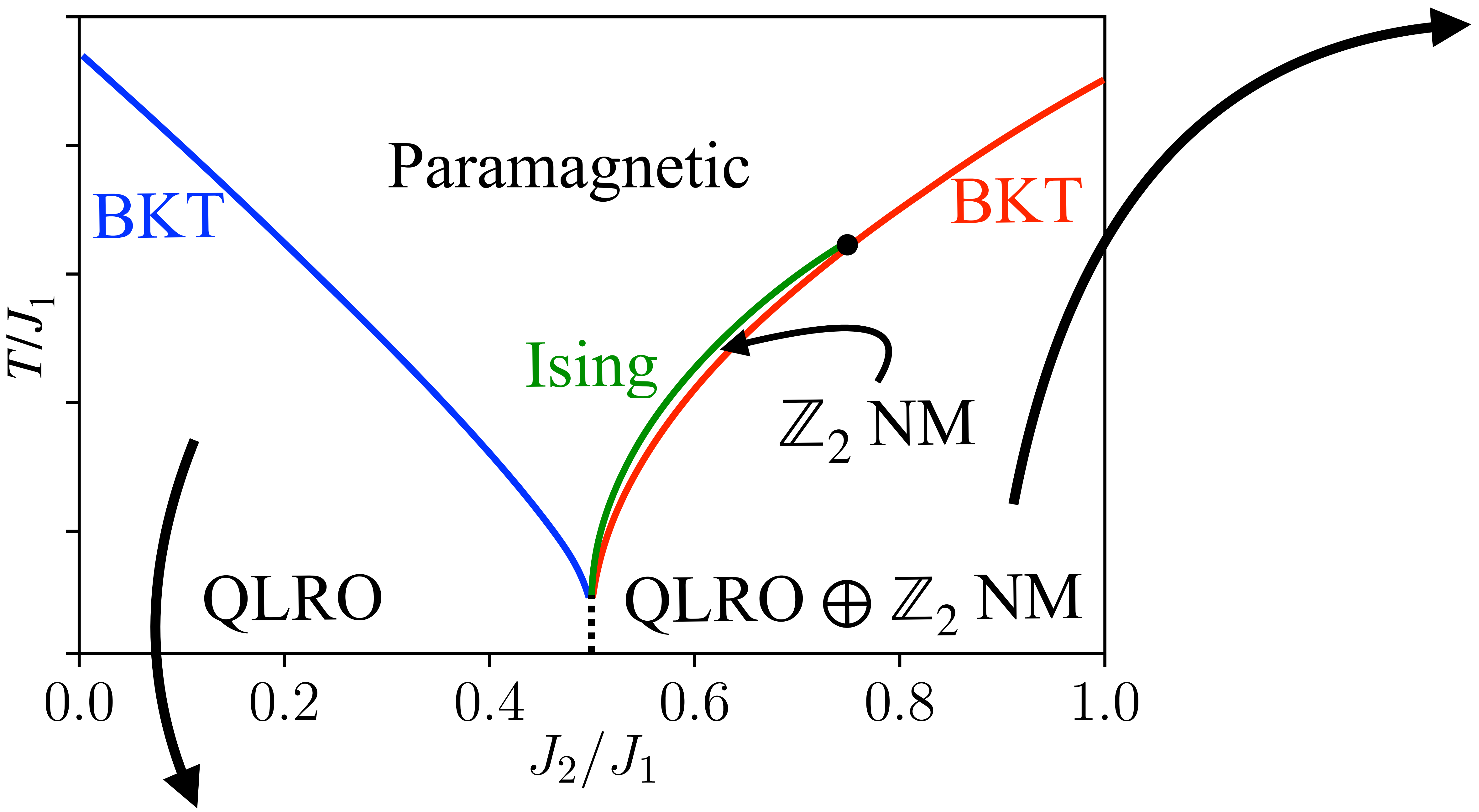

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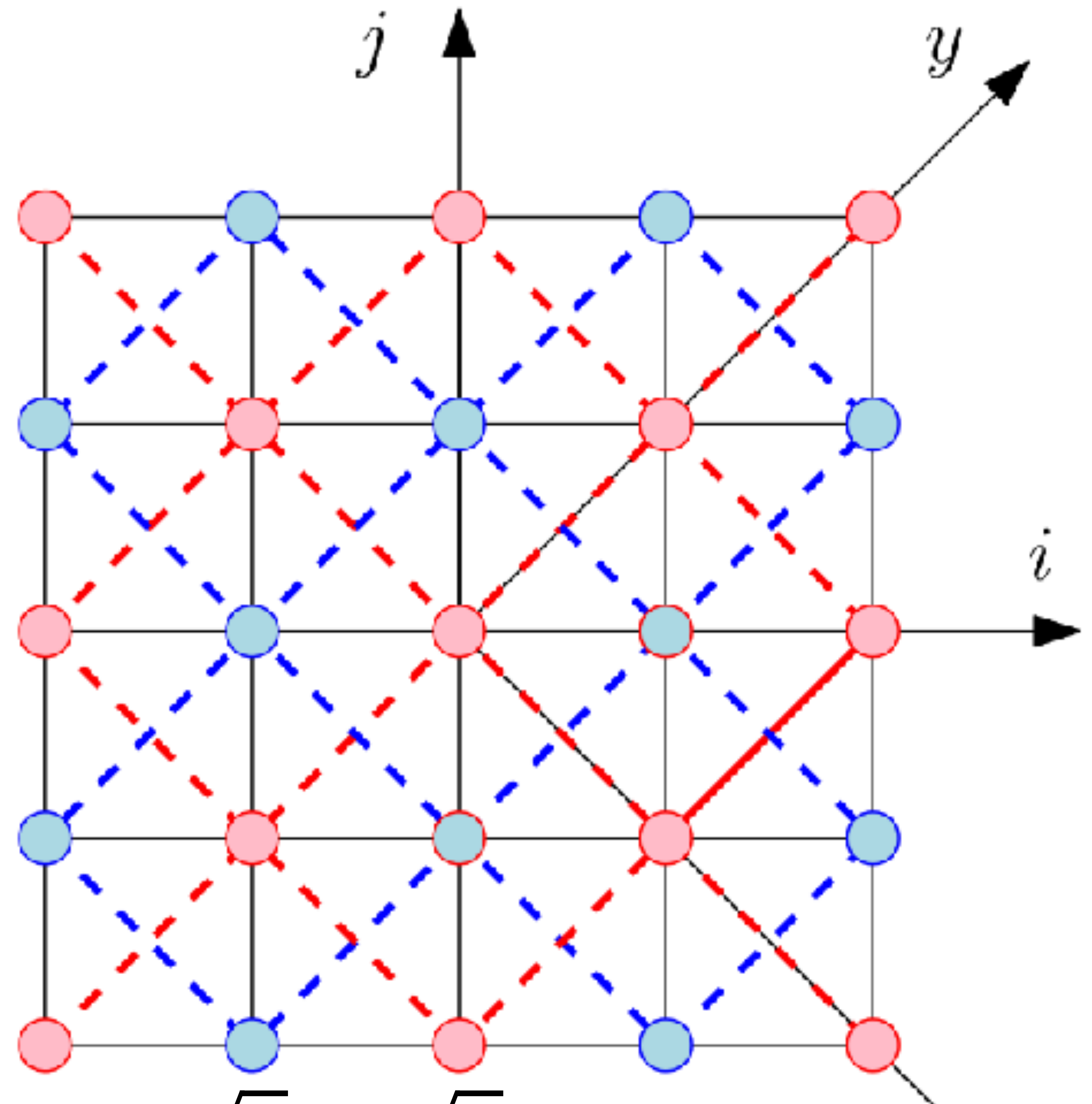

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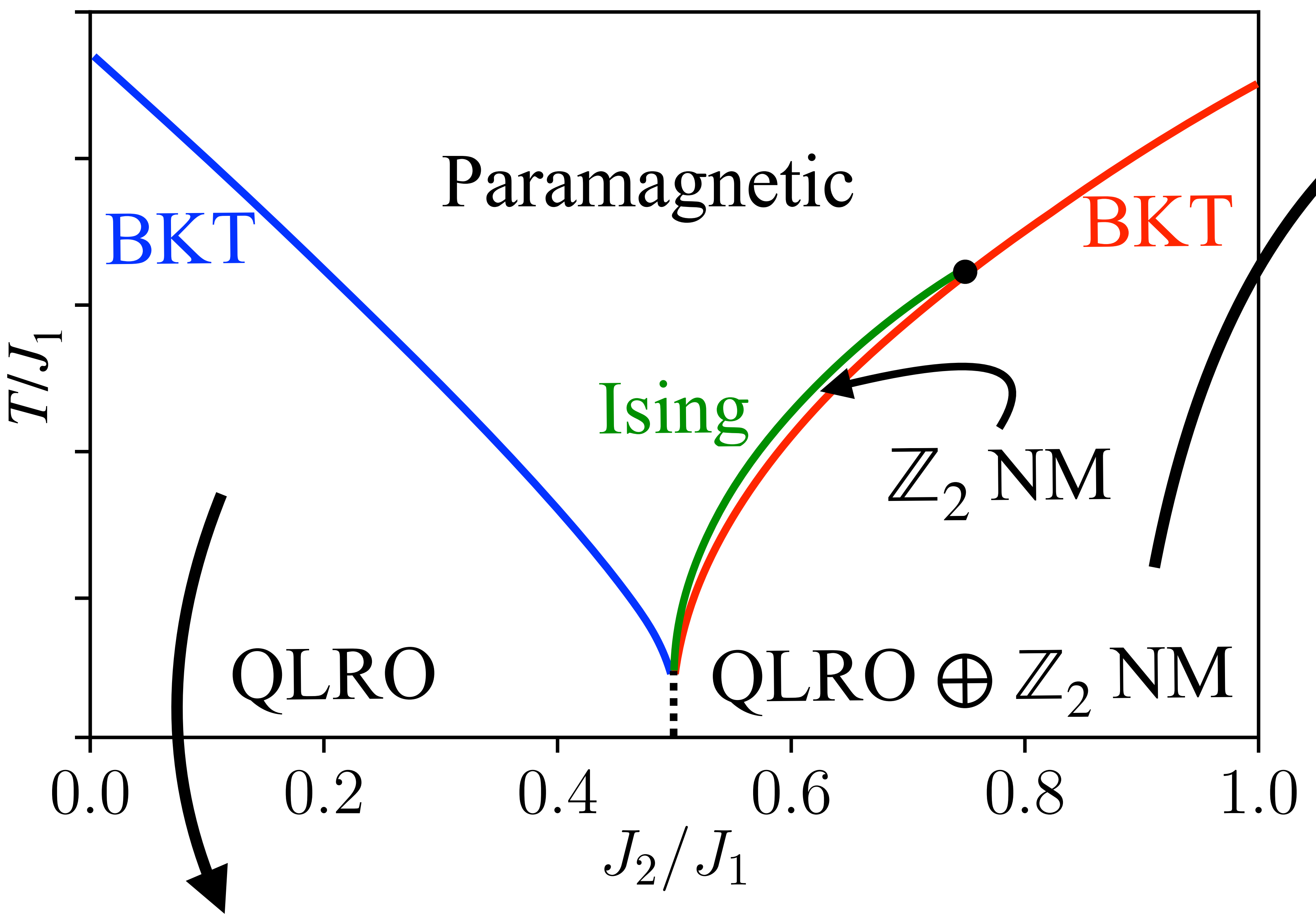

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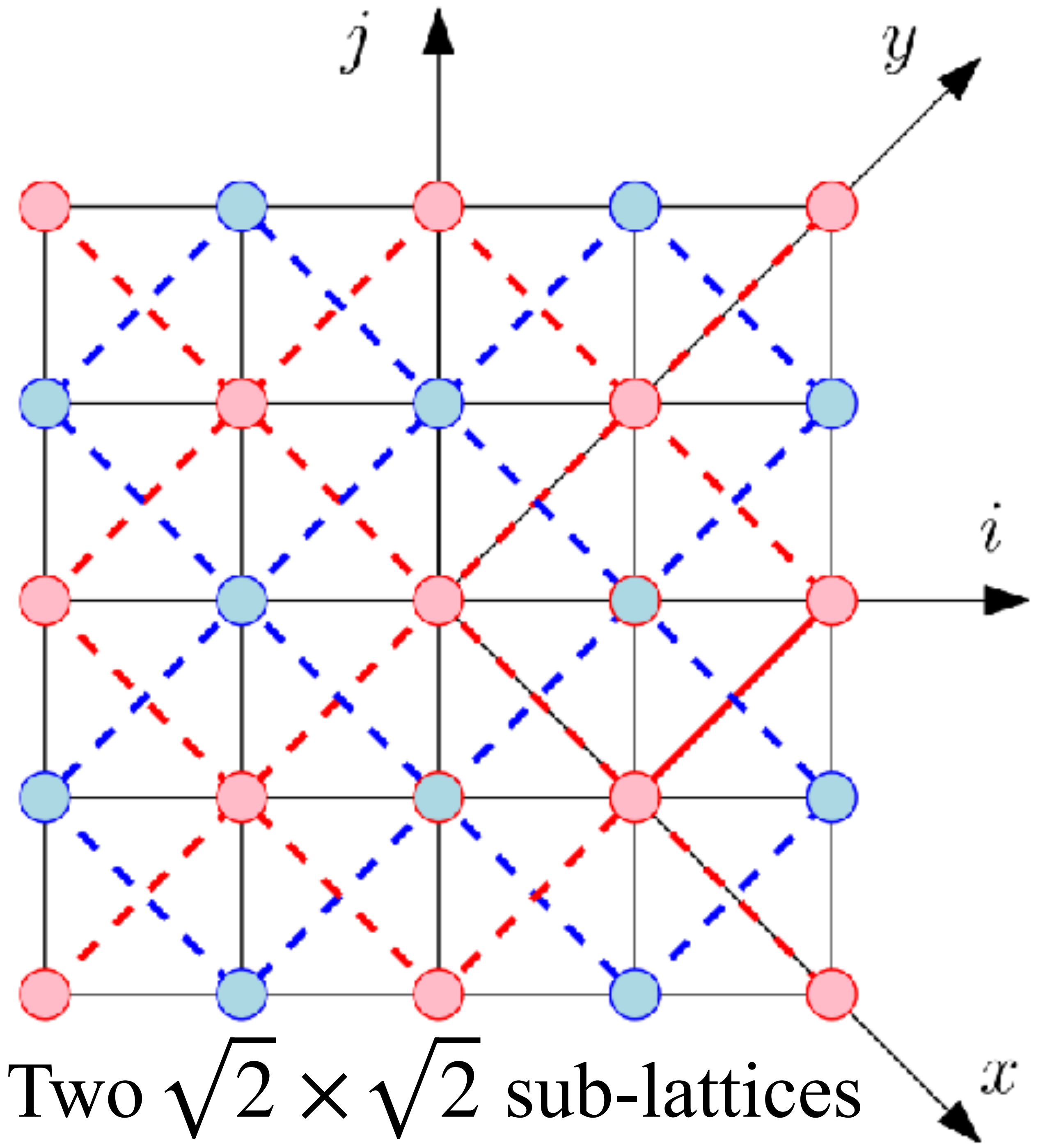
But at finite T "order-by-disorder" kicks in … locking the relative angle to either 0 or π
symmetry reduces to $U(1) \times \mathbb{Z}_2$
forming a $\mathbb{Z}_2$-broken **Nematic ($\mathbb{Z}_2$ NM) order**…



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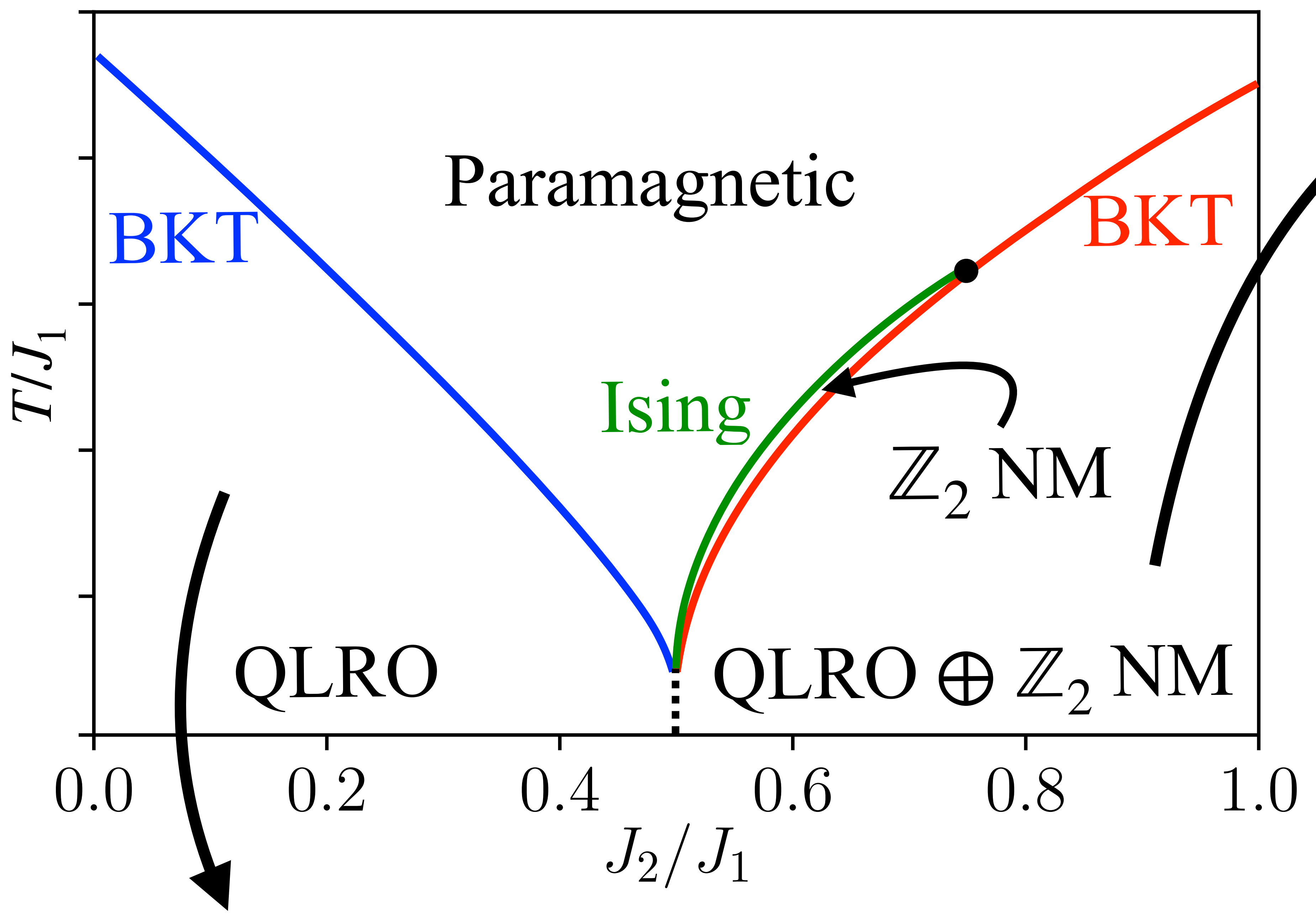

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forming a $\mathbb{Z}_2$-broken **Nematic ($\mathbb{Z}_2$ NM) order**…

Spins within each sublatice tries to order anti-ferromagnetically…

fails due to Mermin-Wagner… QLRO

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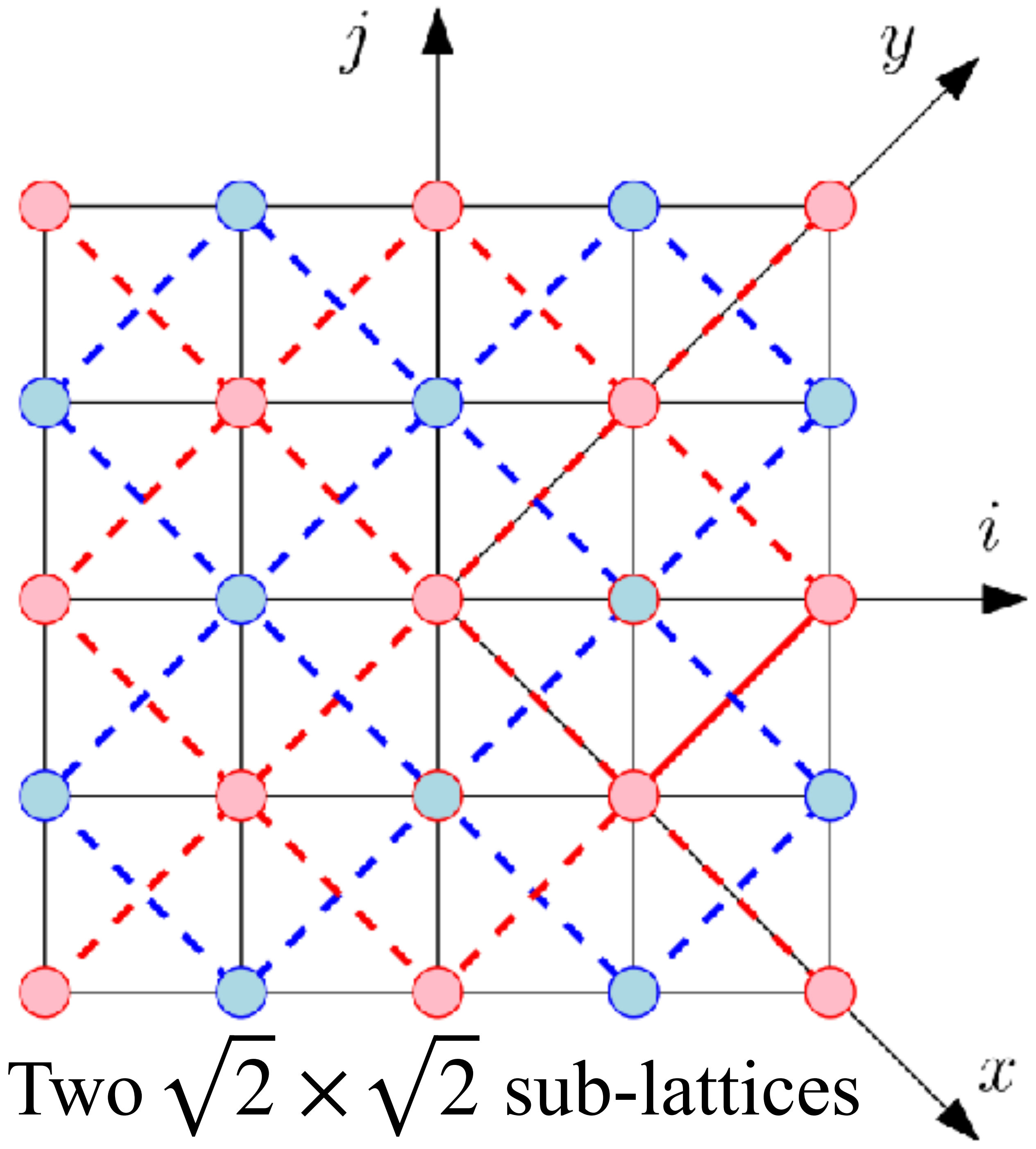
Nematic order parameter $\kappa \sim \pm 1$

$$\kappa = \frac{1}{4} \langle \cos(\theta_{\mathbf{i}} - \theta_{\mathbf{i}+\hat{x}}) + \cos(\theta_{\mathbf{i}+\hat{y}} - \theta_{\mathbf{i}+\hat{x}+\hat{y}}) - \cos(\theta_{\mathbf{i}} - \theta_{\mathbf{i}+\hat{y}}) - \cos(\theta_{\mathbf{i}+\hat{x}} - \theta_{\mathbf{i}+\hat{x}+\hat{y}}) \rangle.$$



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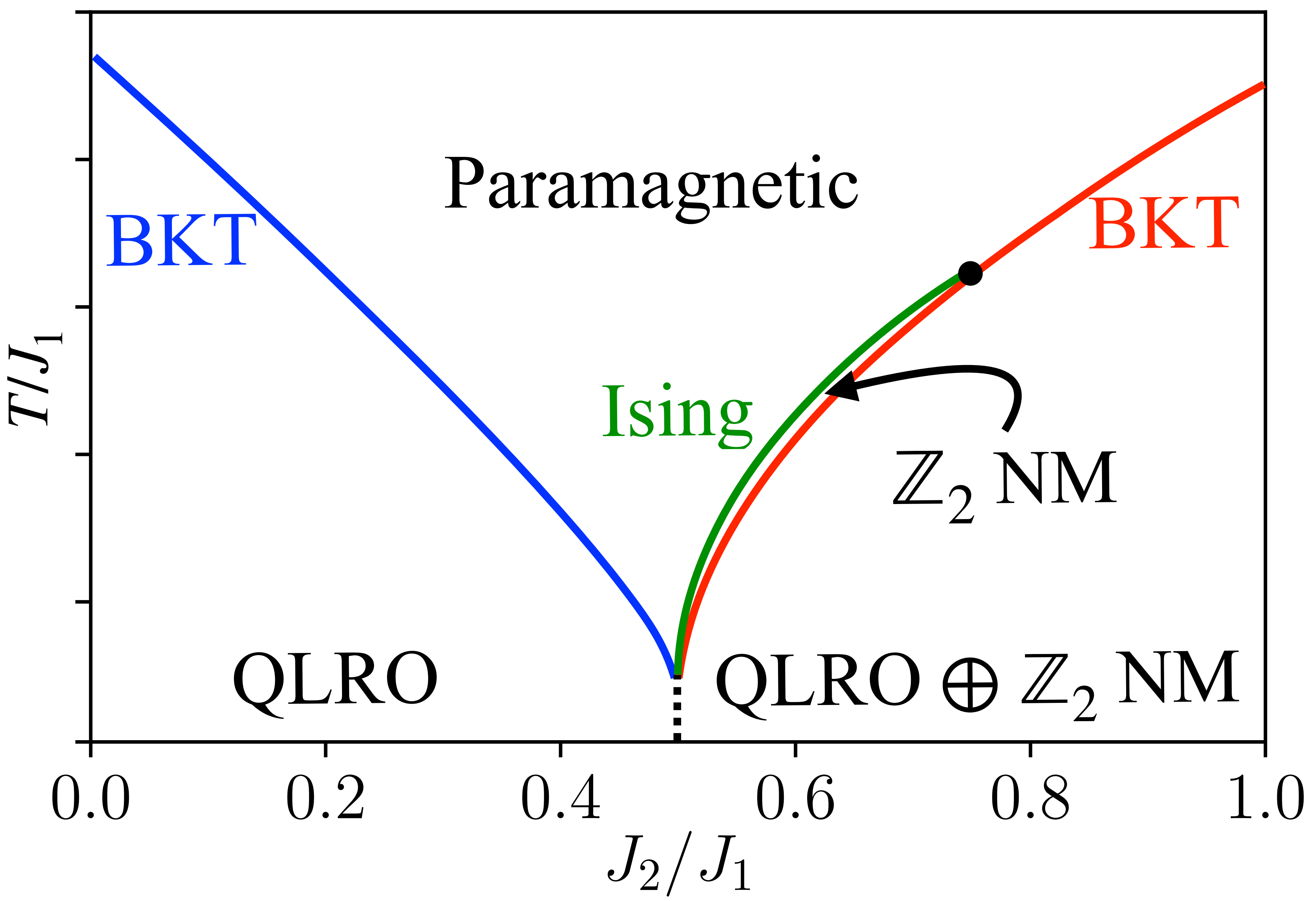

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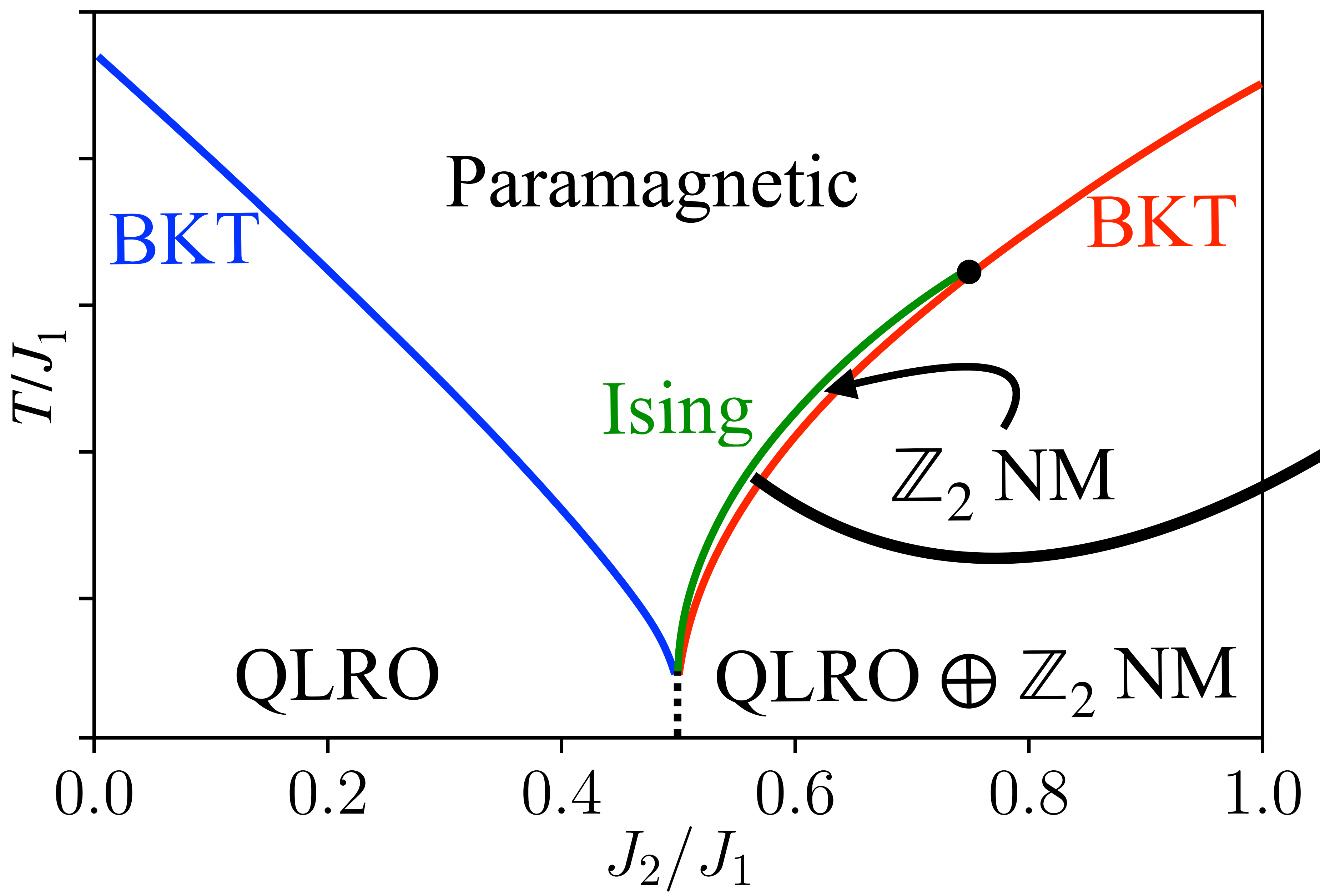

At small $J_2/J_1 > 1/2$… after BKT transition…

QLRO $\oplus$ $\mathbb{Z}_2$ NM phase goes to a phase… with no QLRO…
$\langle \cos(\theta_i - \theta_{i+R}) \rangle \sim \exp(-R/\xi)$

but **non-zero nematic order** $\kappa \neq 0$…
$\mathbb{Z}_2$ **NM phase**… two-fold degeneracy

This $\mathbb{Z}_2$ NM phase is connected to the paramagnetic phase by **Ising transition**

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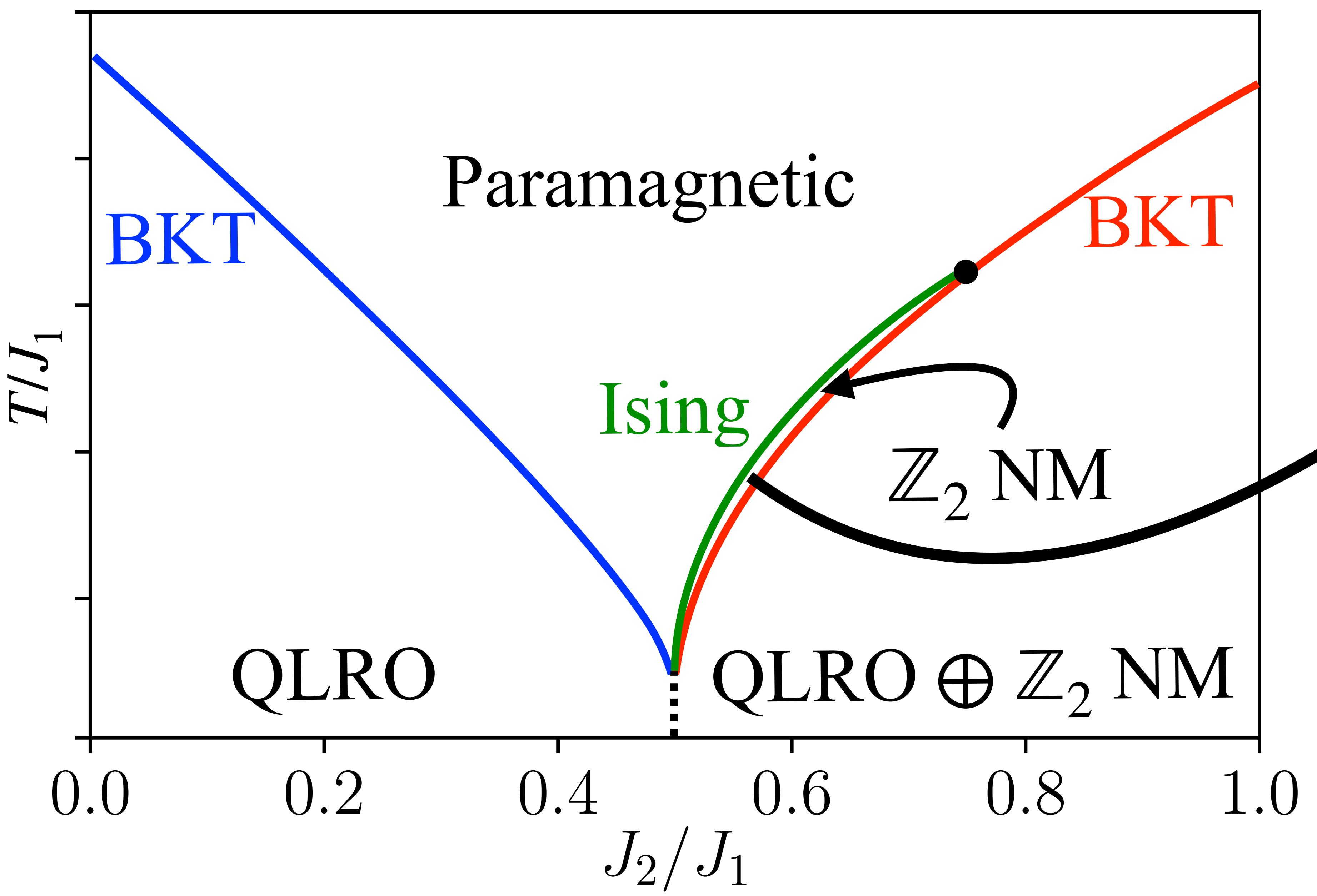

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At higher $J_2/J_1 > 1/2$… QLRO $\oplus$ $\mathbb{Z}_2$ NM phase **directly** goes paramagnetic phase via BKT

Simultaneous destruction of $\mathbb{Z}_2$ NM order and QLRO is **still not understood**

These results were hypothesized in two decades old CMC study: Loison et. al., PRB 2000
Only confirmed rigorously using TN: Song et. al., PRB 2025

Our work

@ IIT Madras Campus

J_1-J_2 Clock Model

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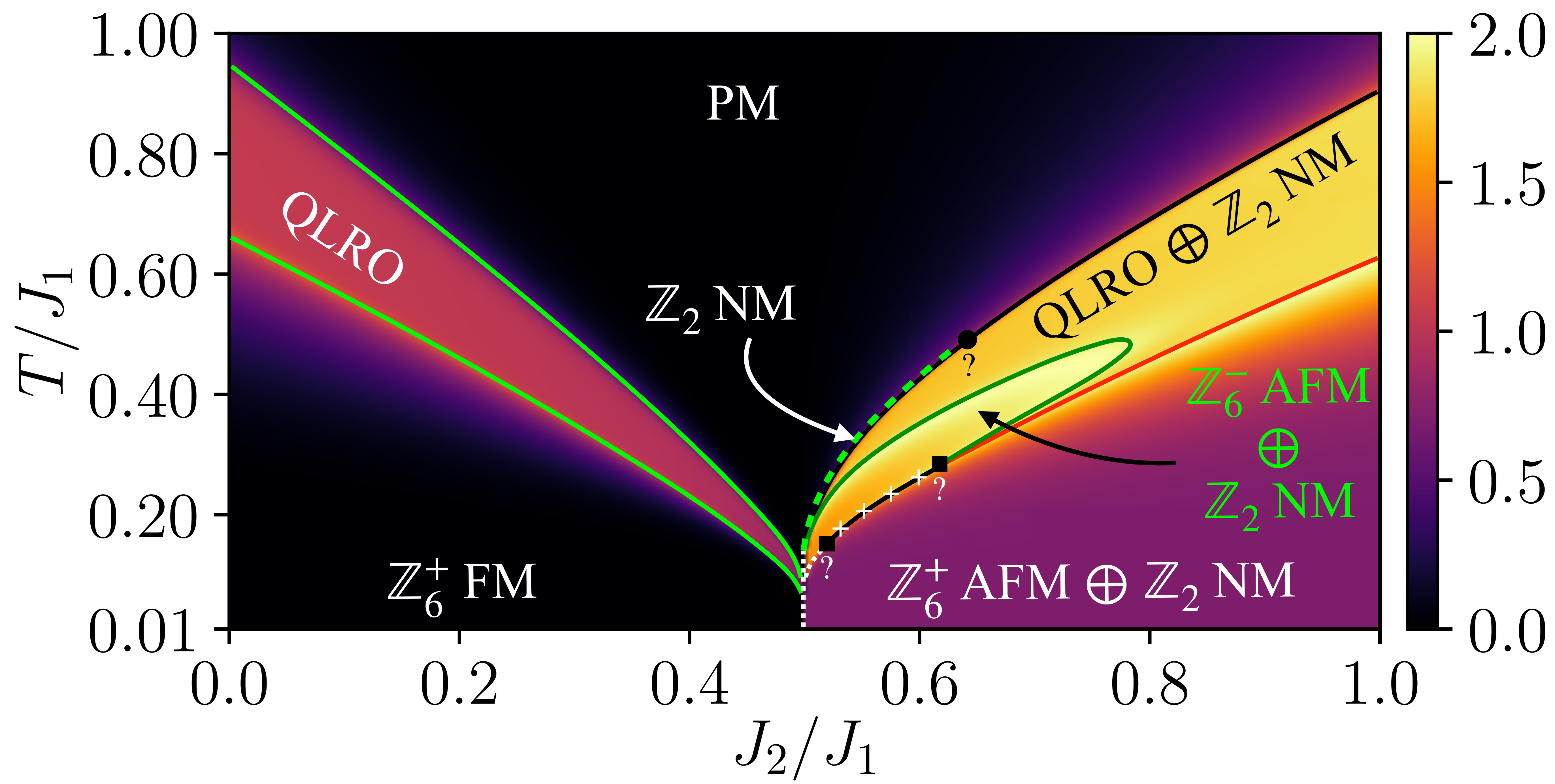
I will show results for $q = 6\ldots$
but results remain valid for any even $q \geq 6$



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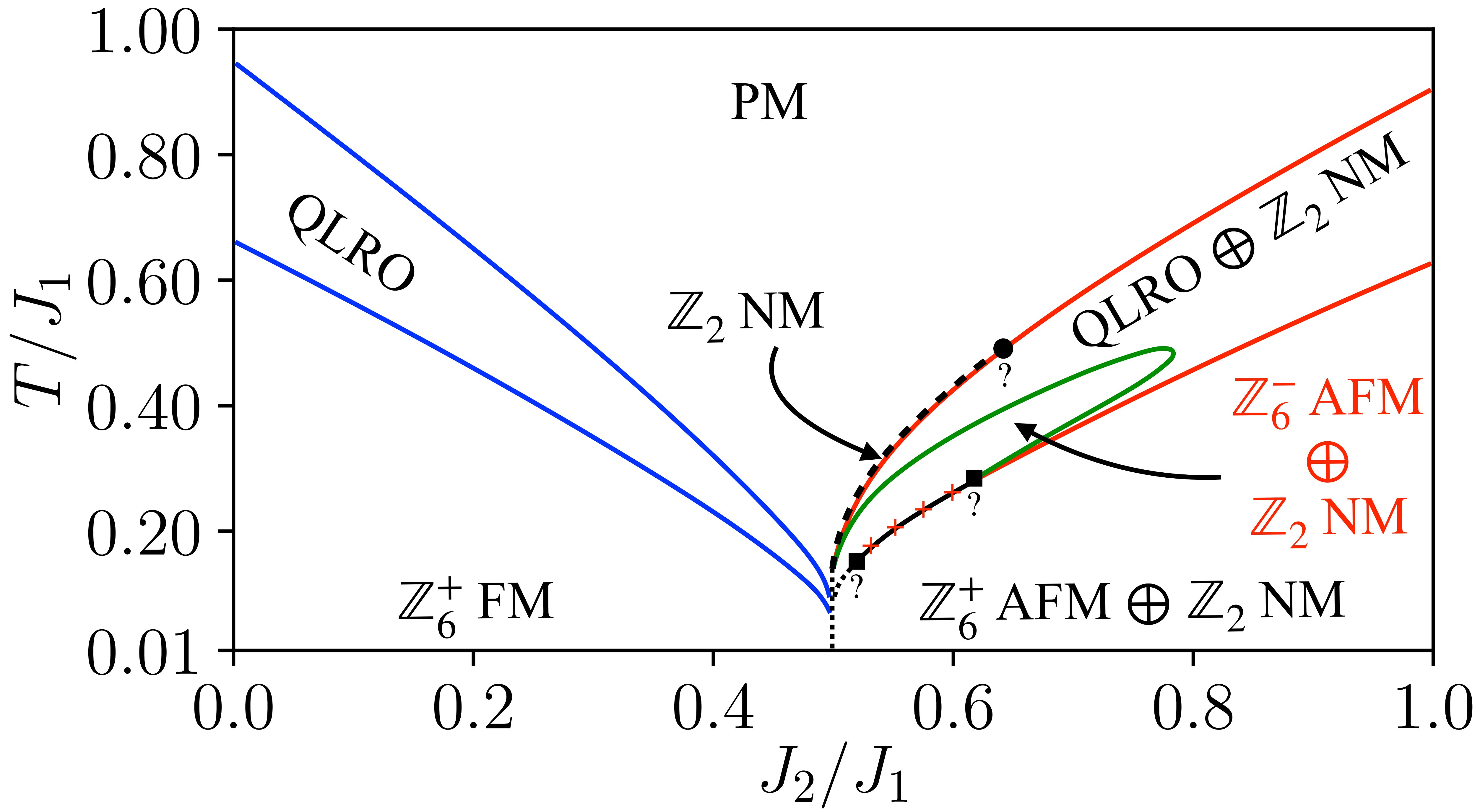

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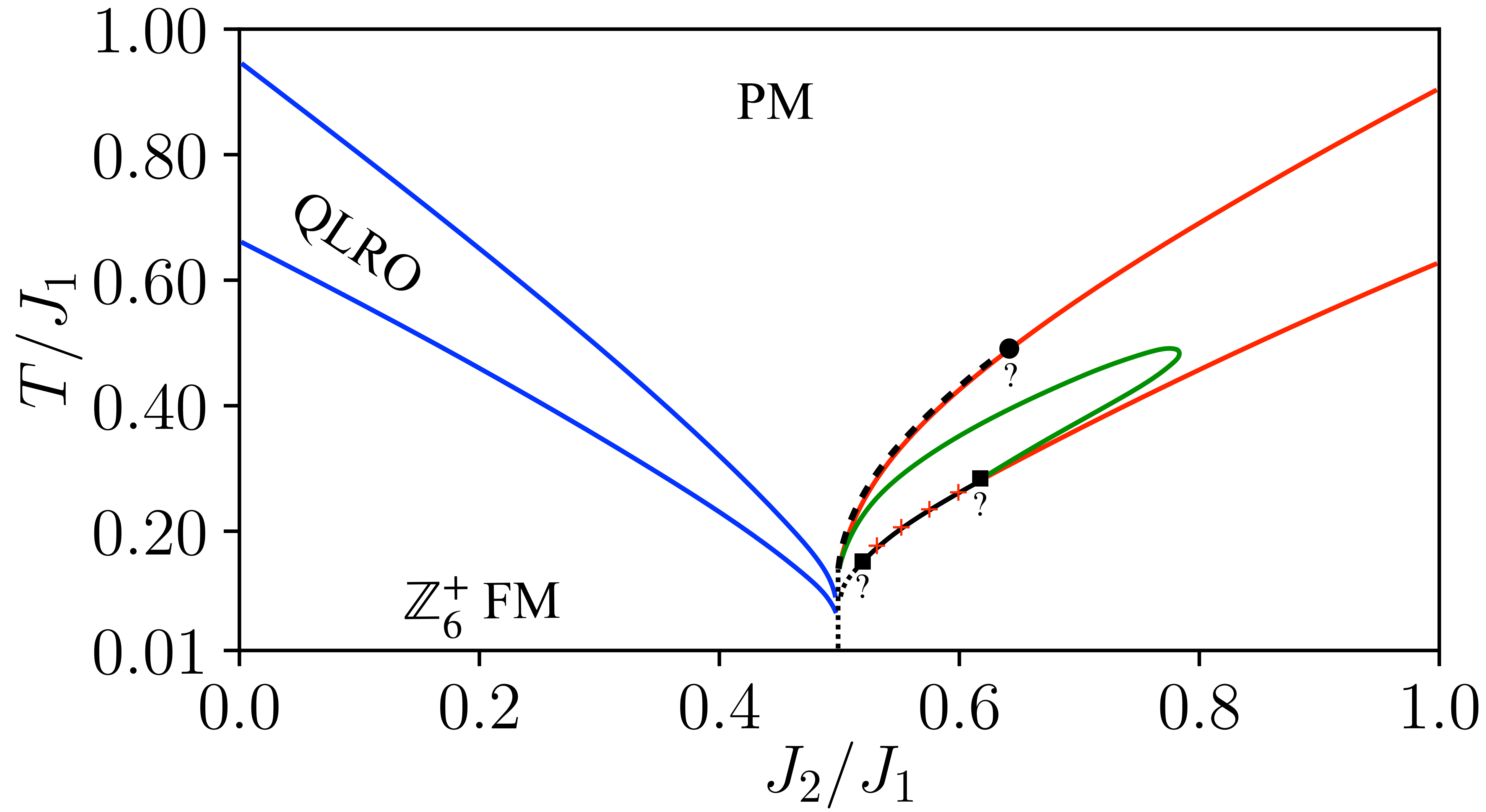

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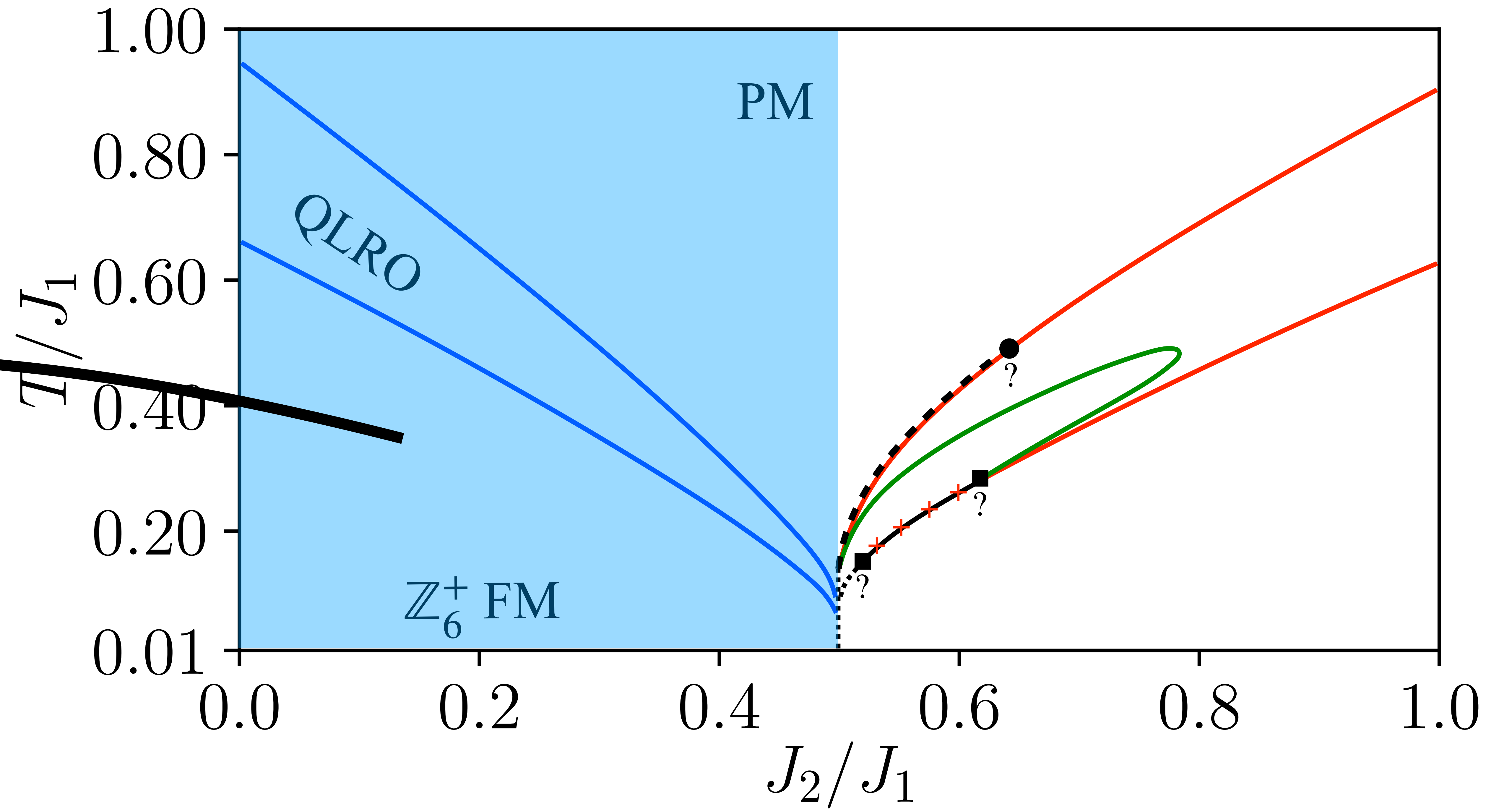
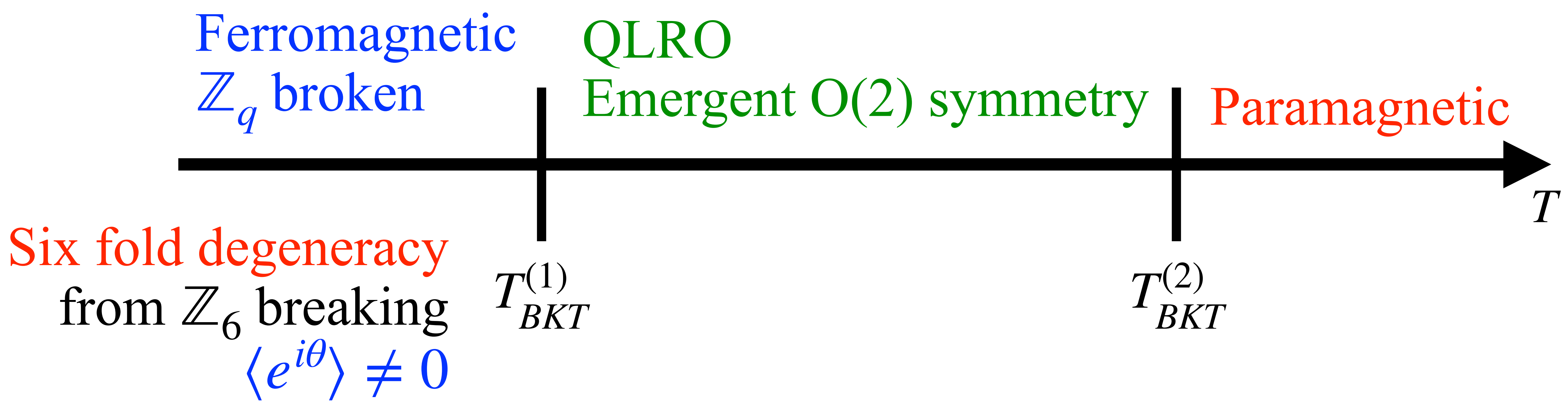


Unfrustrated regime:
$J_2 < J_1/2$



Our notation

$\mathbb{Z}_q^+ \Rightarrow$ effective (coarse grained) clock angles are **even** multiples of π/q, i.e., $2n\pi/q\ldots$ just like the microscopic theory

J_1-J_2 Clock Model

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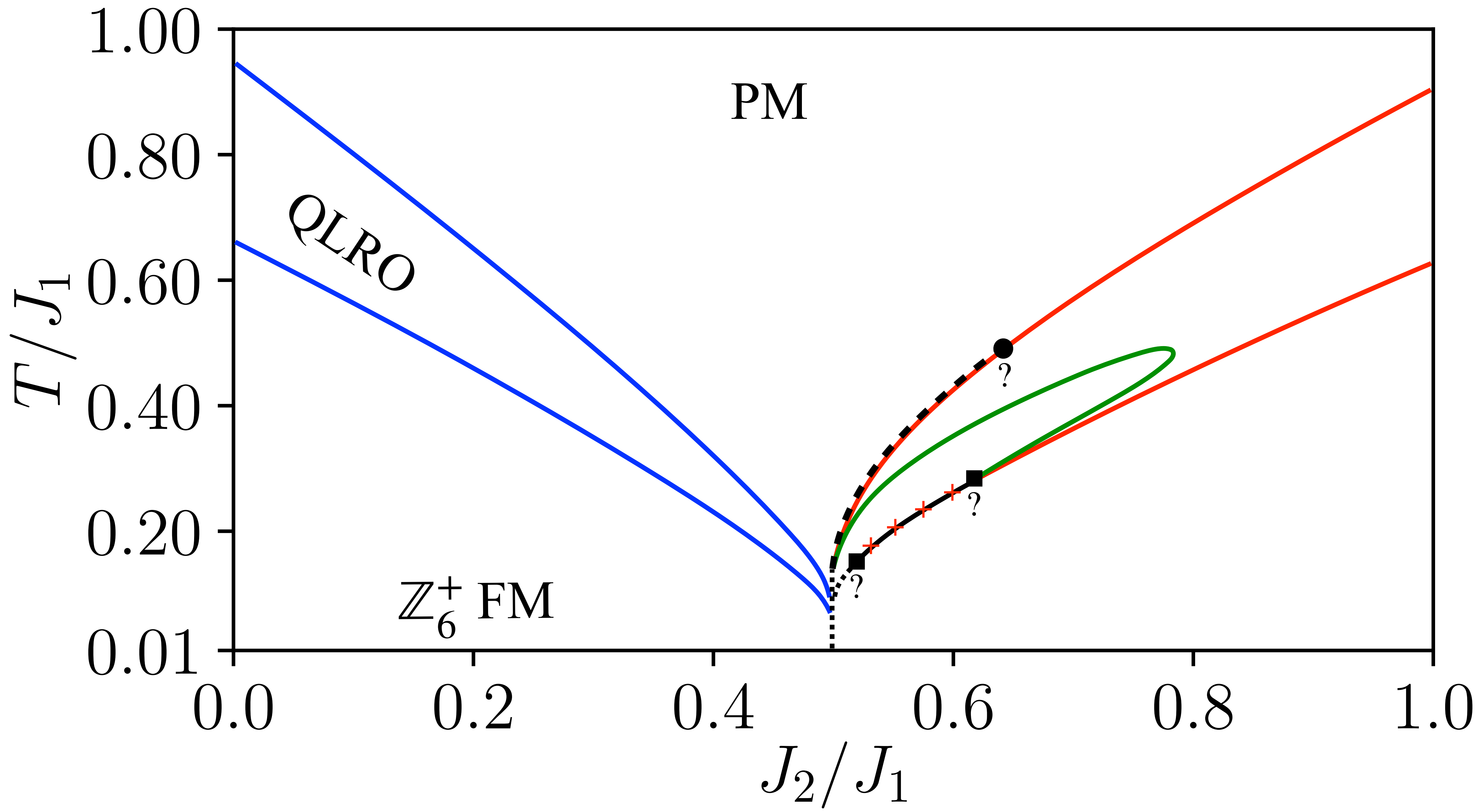

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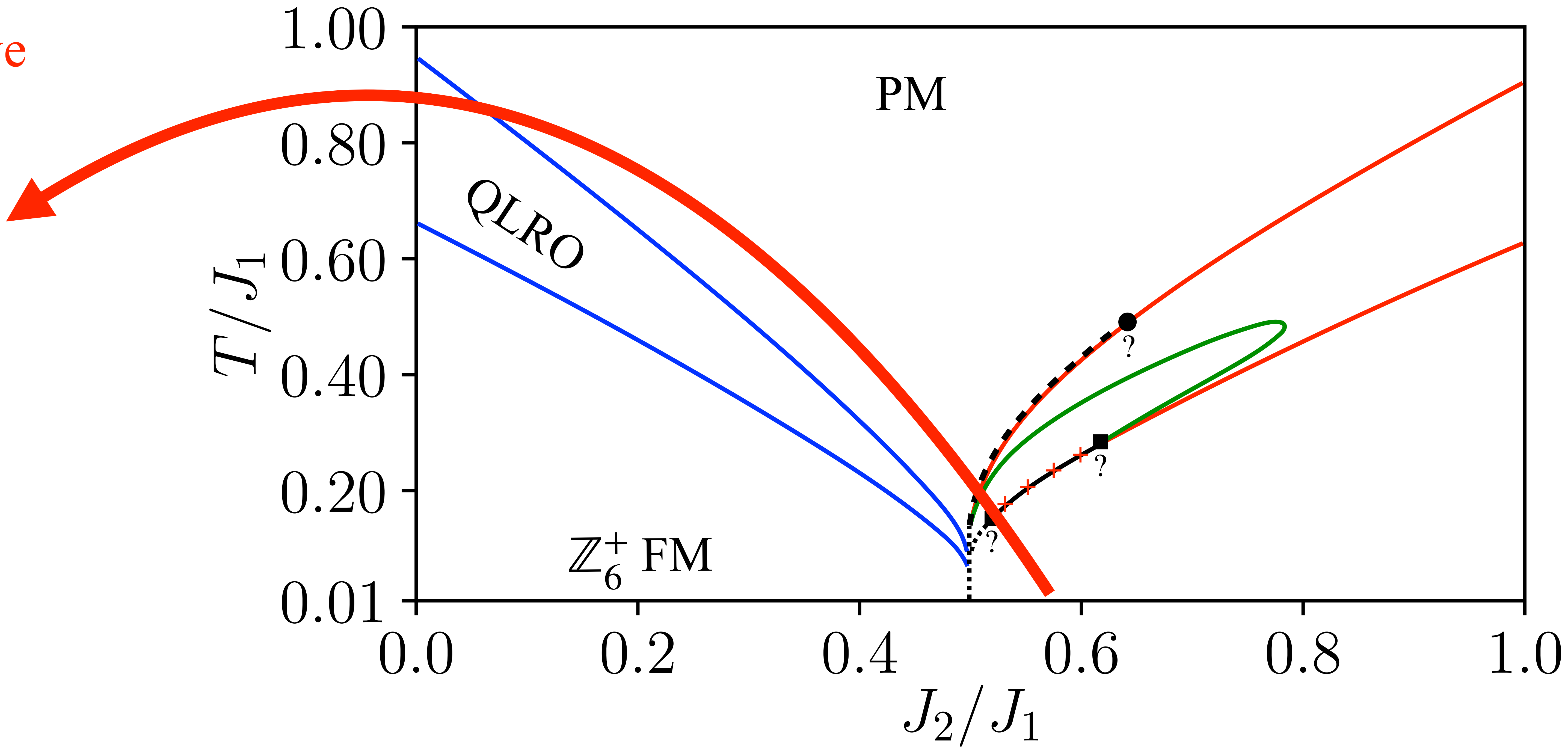

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At $T = 0$… the energy is independent of the relative angle between two $\sqrt{2} \times \sqrt{2}$ sub-lattices…

Giving $\mathbb{Z}_6 \times \mathbb{Z}_6$ degeneracy

But at finite T "order-by-disorder" kicks in …
locking the relative angle to either 0 or π
symmetry reduces to $\mathbb{Z}_6 \times \mathbb{Z}_2$
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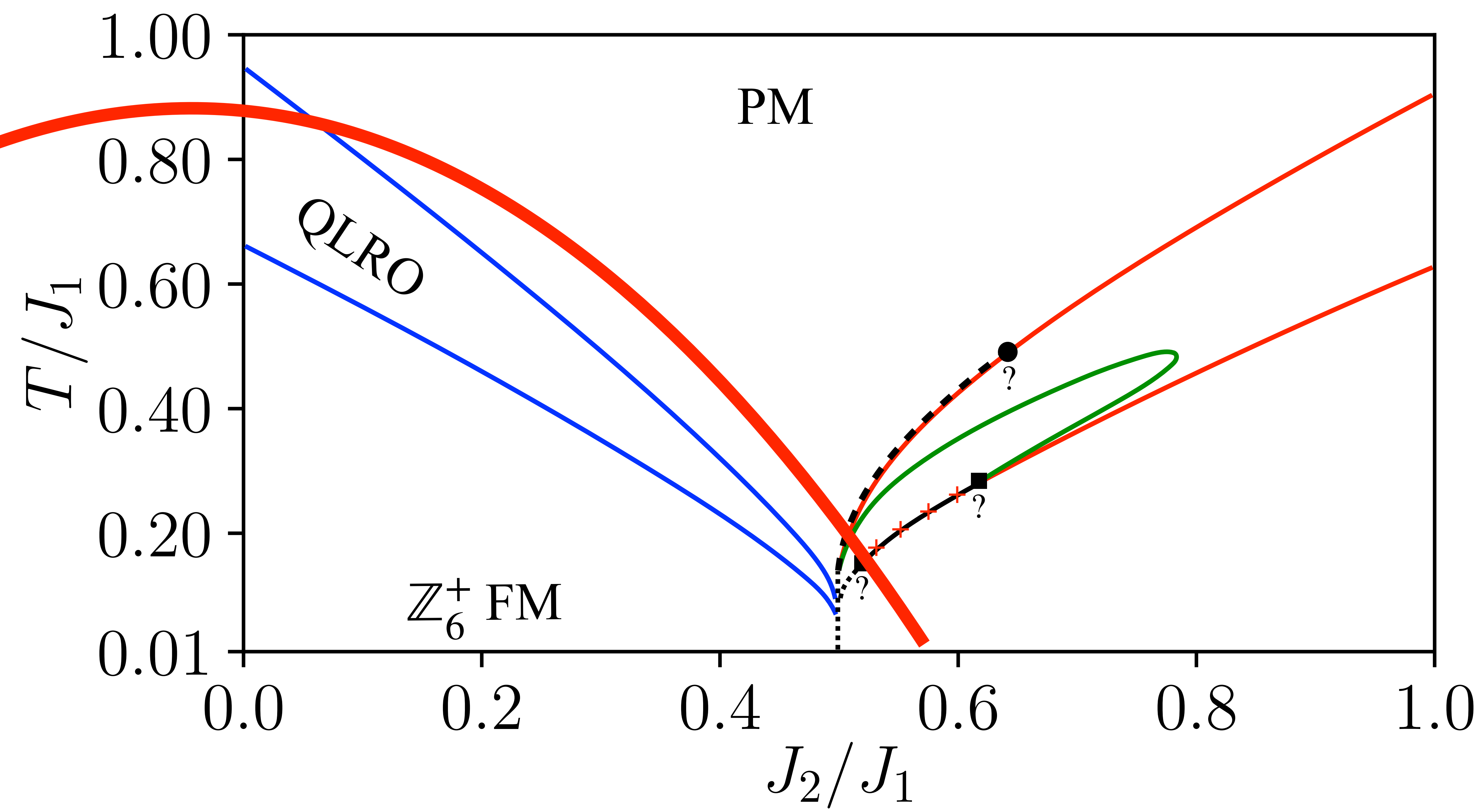

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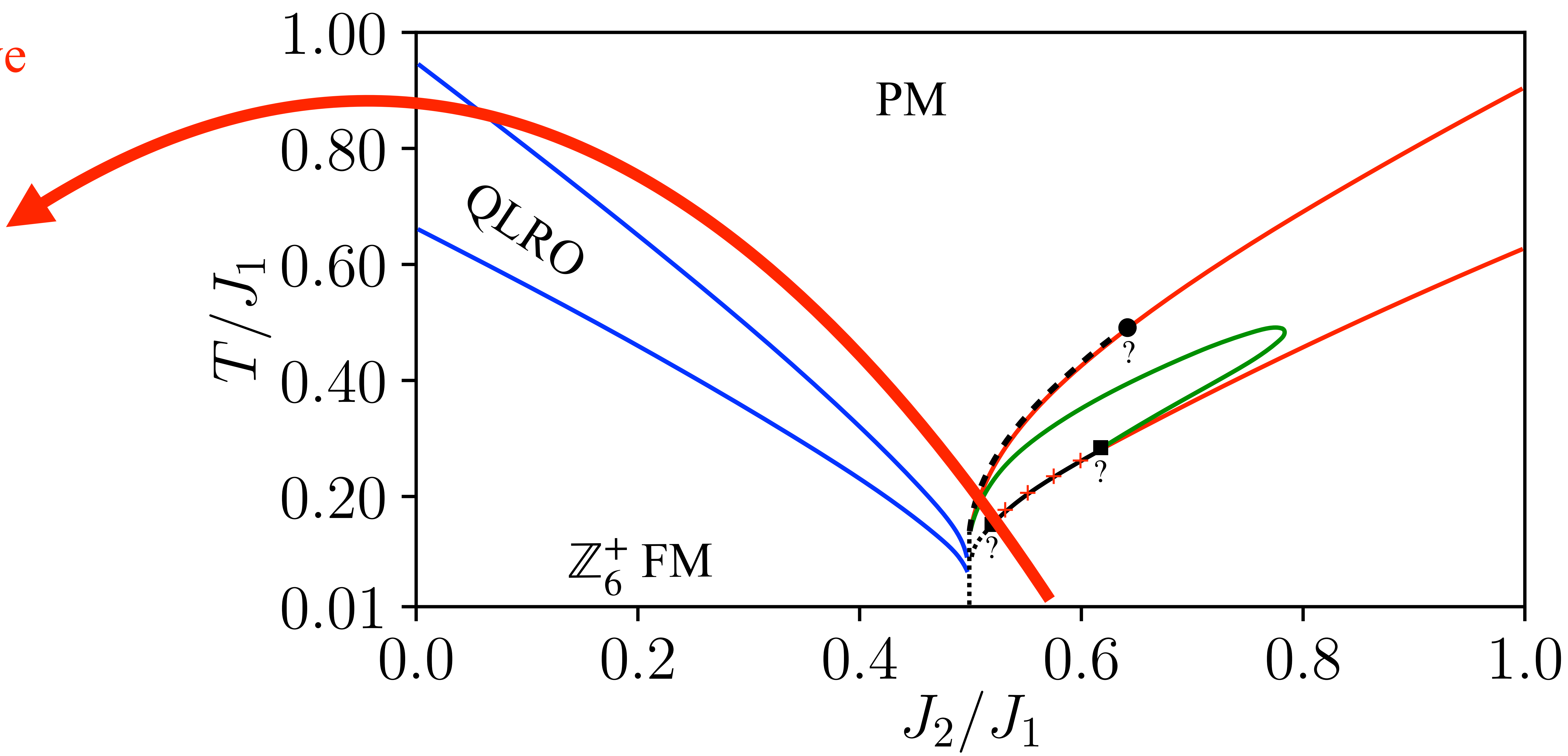

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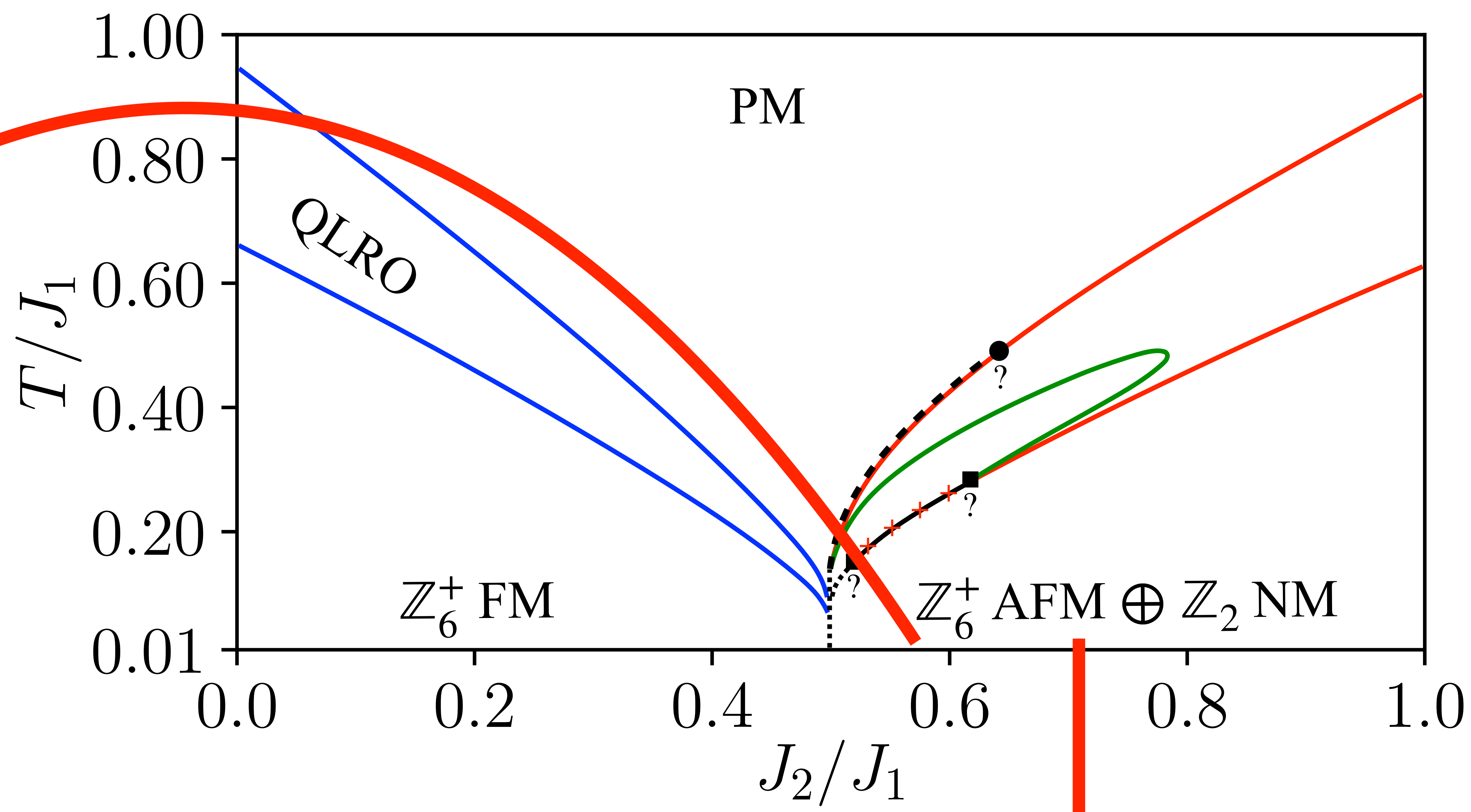
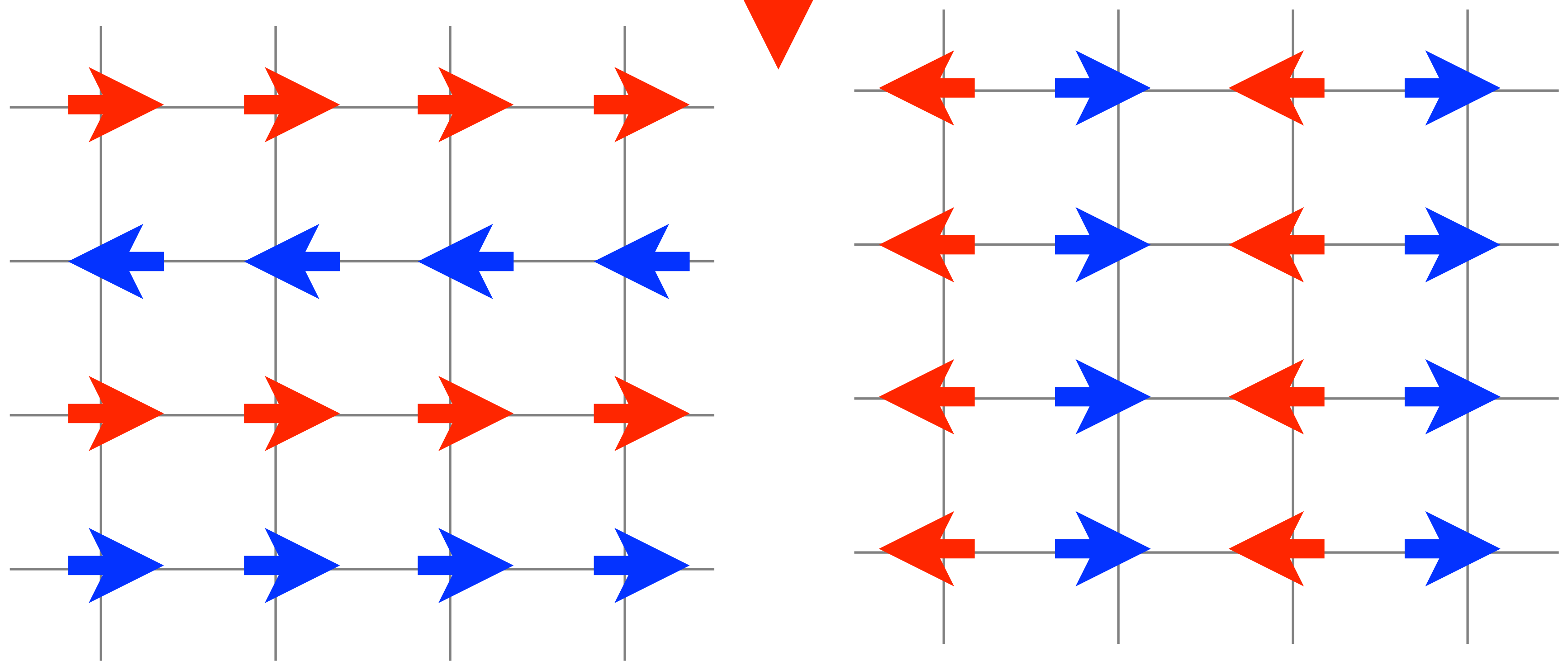


12 fold degeneracy from $\mathbb{Z}_6 \times \mathbb{Z}_2$ breaking



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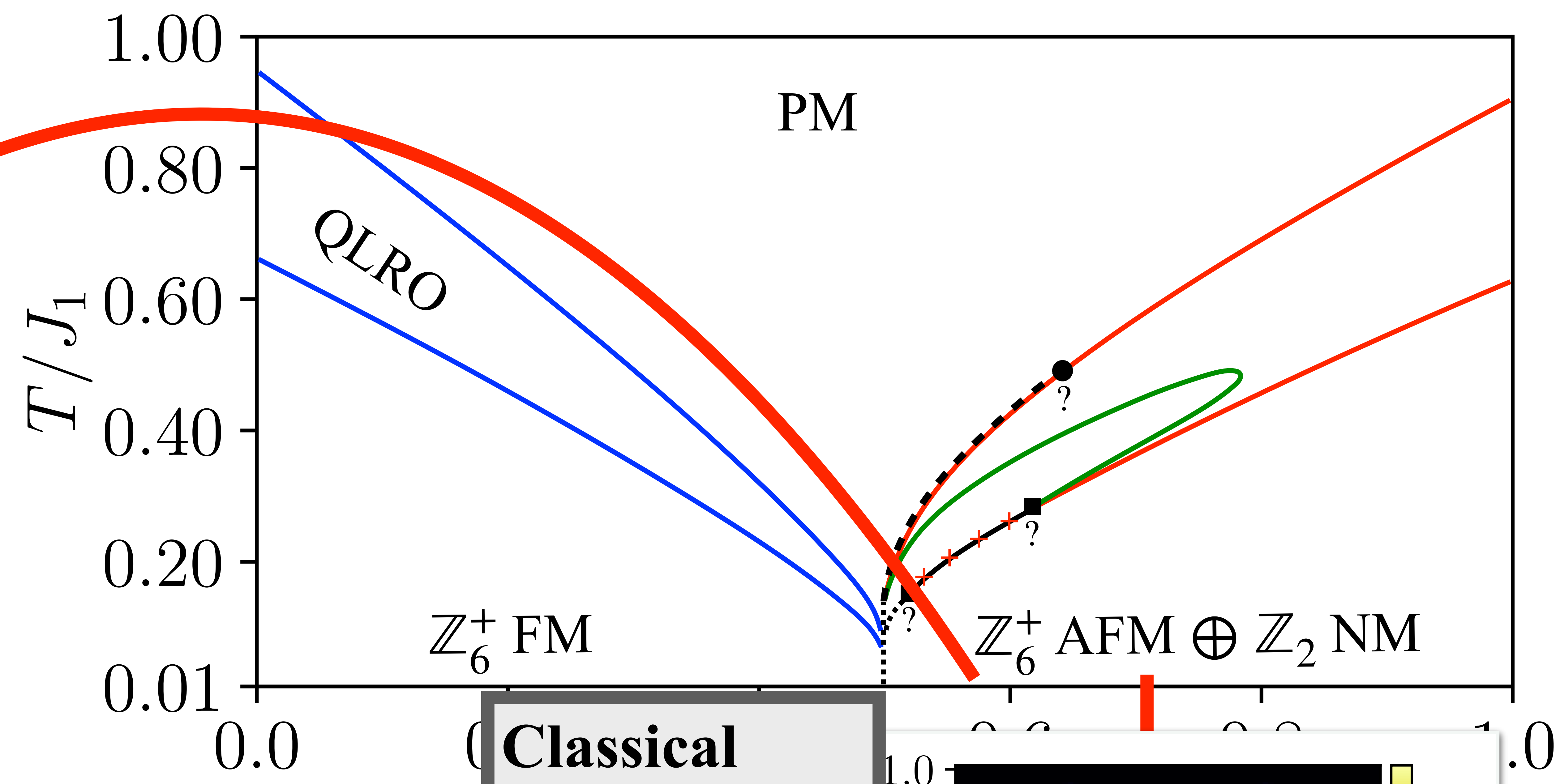
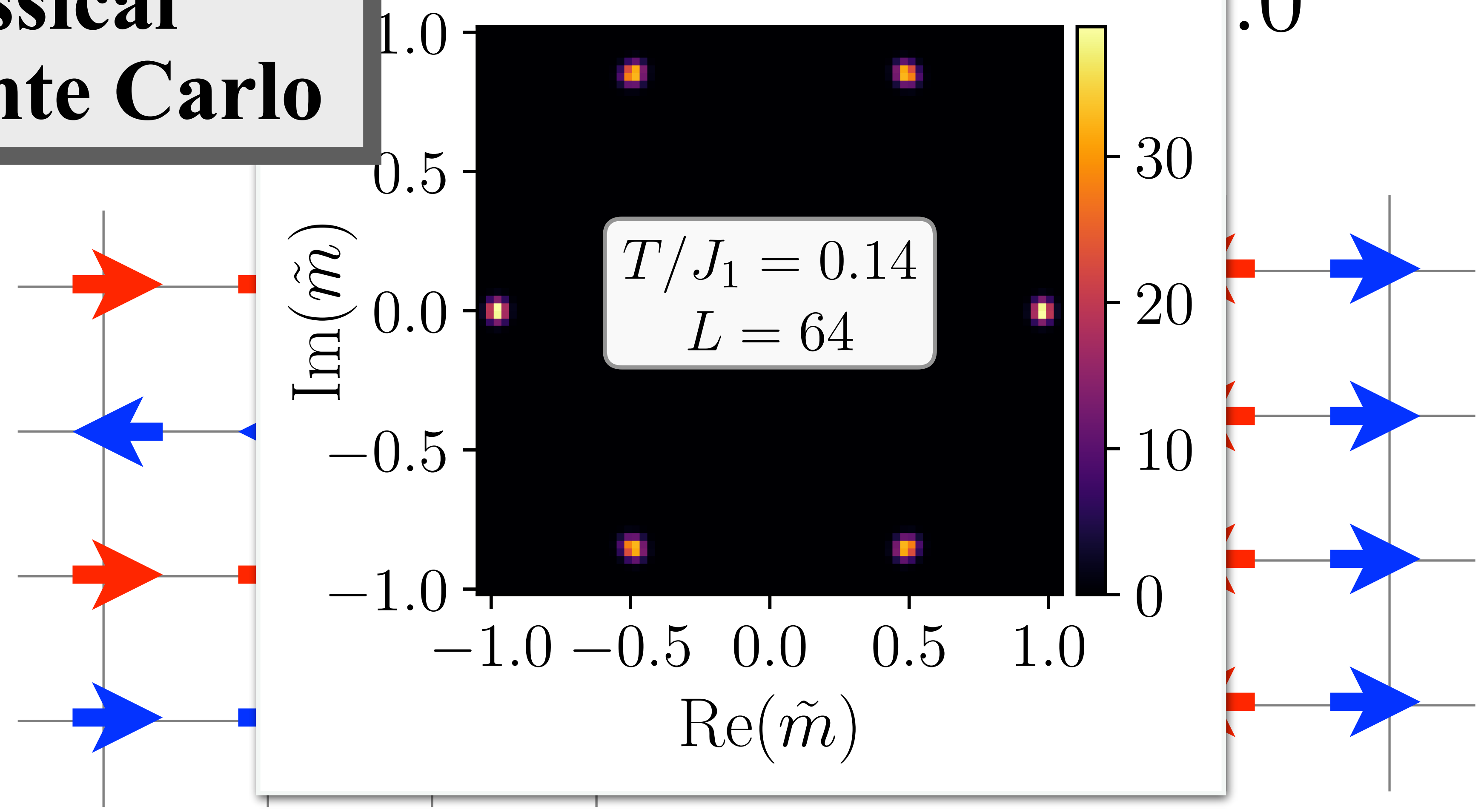


Classical Monte Carlo



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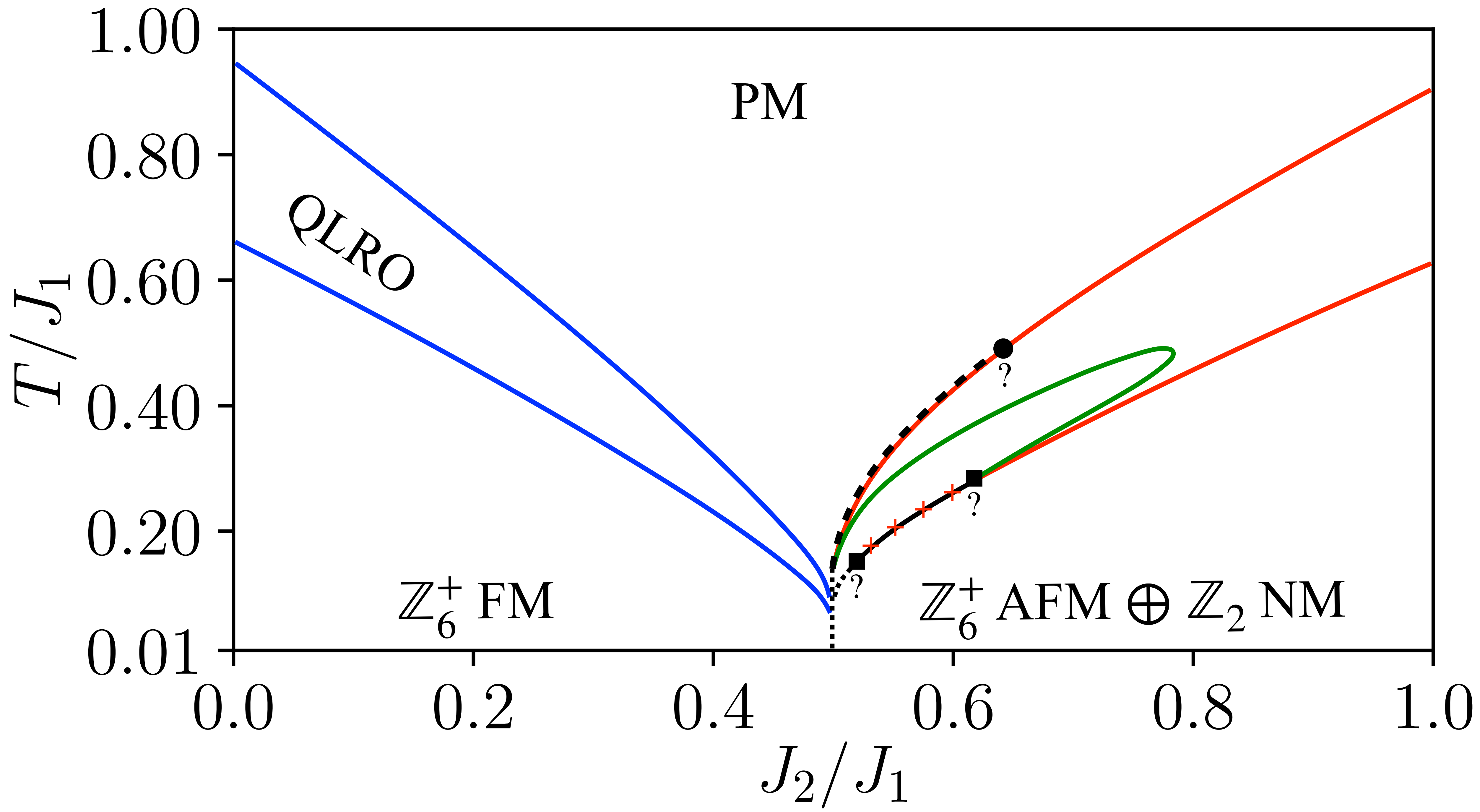
The J_1-J_2 XY physics



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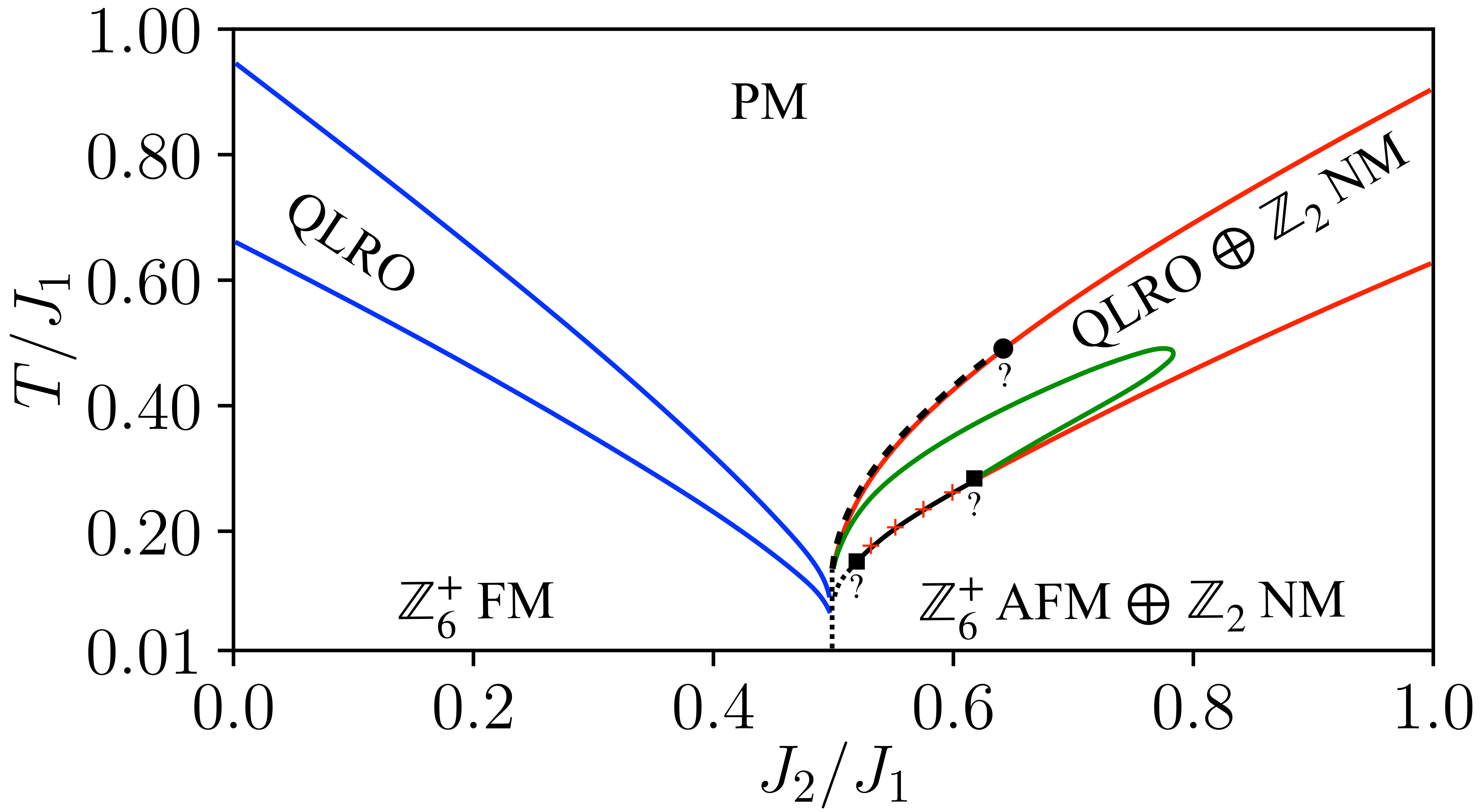
With increasing T… O(2) **symmetry emerges**…
that cannot hold AFM order…. **QLRO**

Nematic $\mathbb{Z}_2$ remains **broken**…

altogether… **QLRO $\oplus$ $\mathbb{Z}_2$ NM**…
like the low-T phase of XY model…



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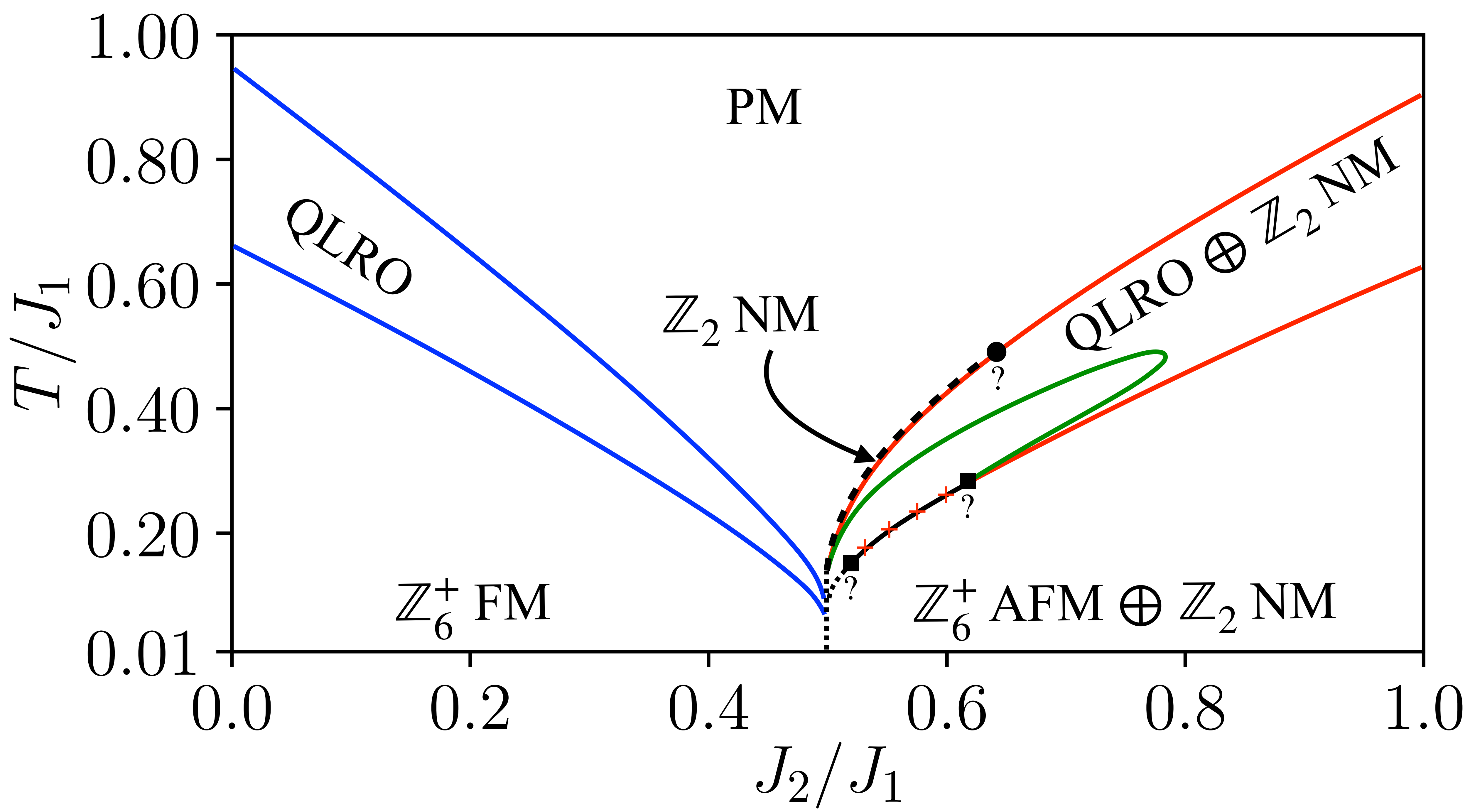

Again like XY model… **QLRO $\oplus$ $\mathbb{Z}_2$ NM** goes to (via BKT)
another **Nematic** phase… with no QLRO….

This $\mathbb{Z}_2$ **NM** phase… then goes to paramagnetic phase… via Ising transition…

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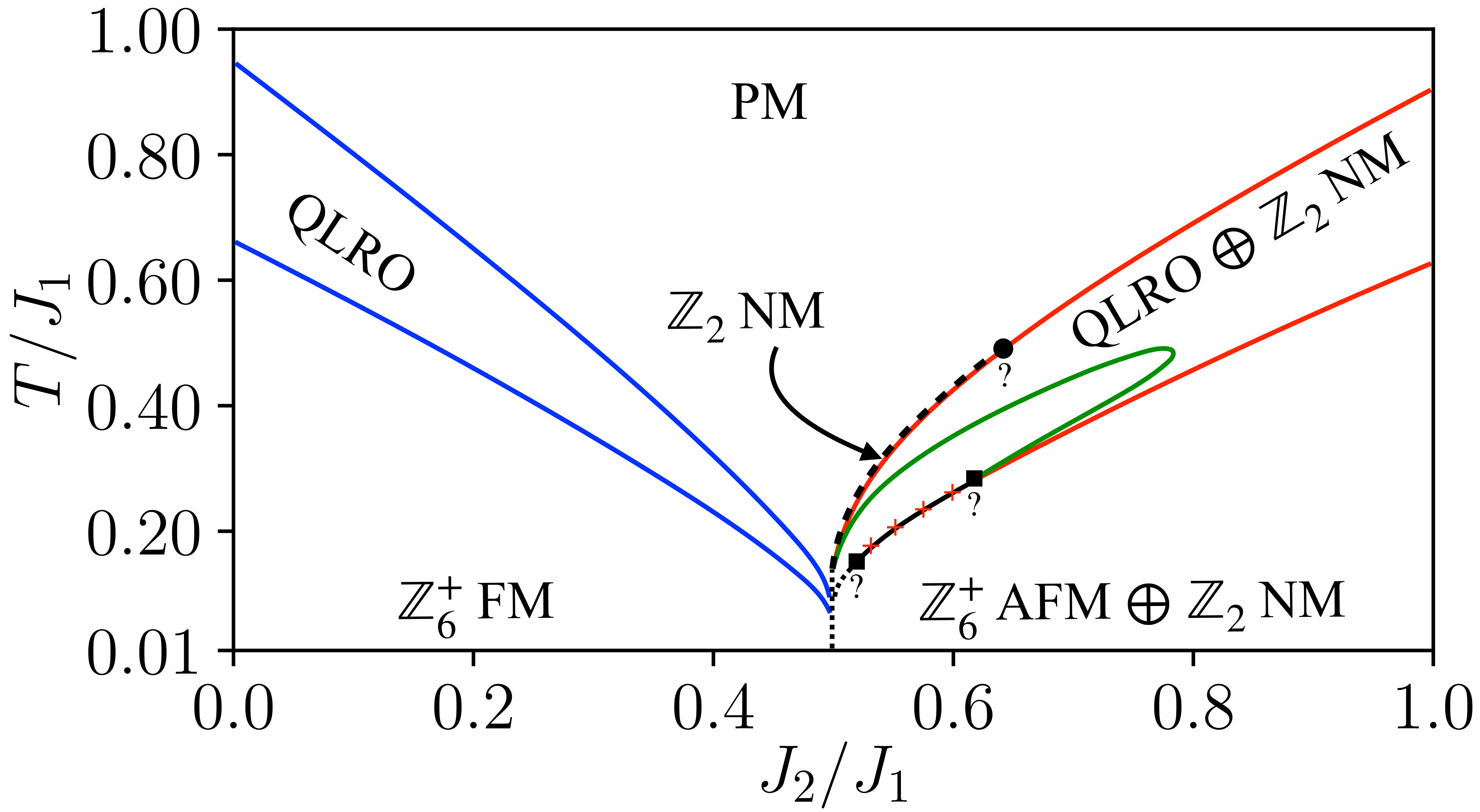

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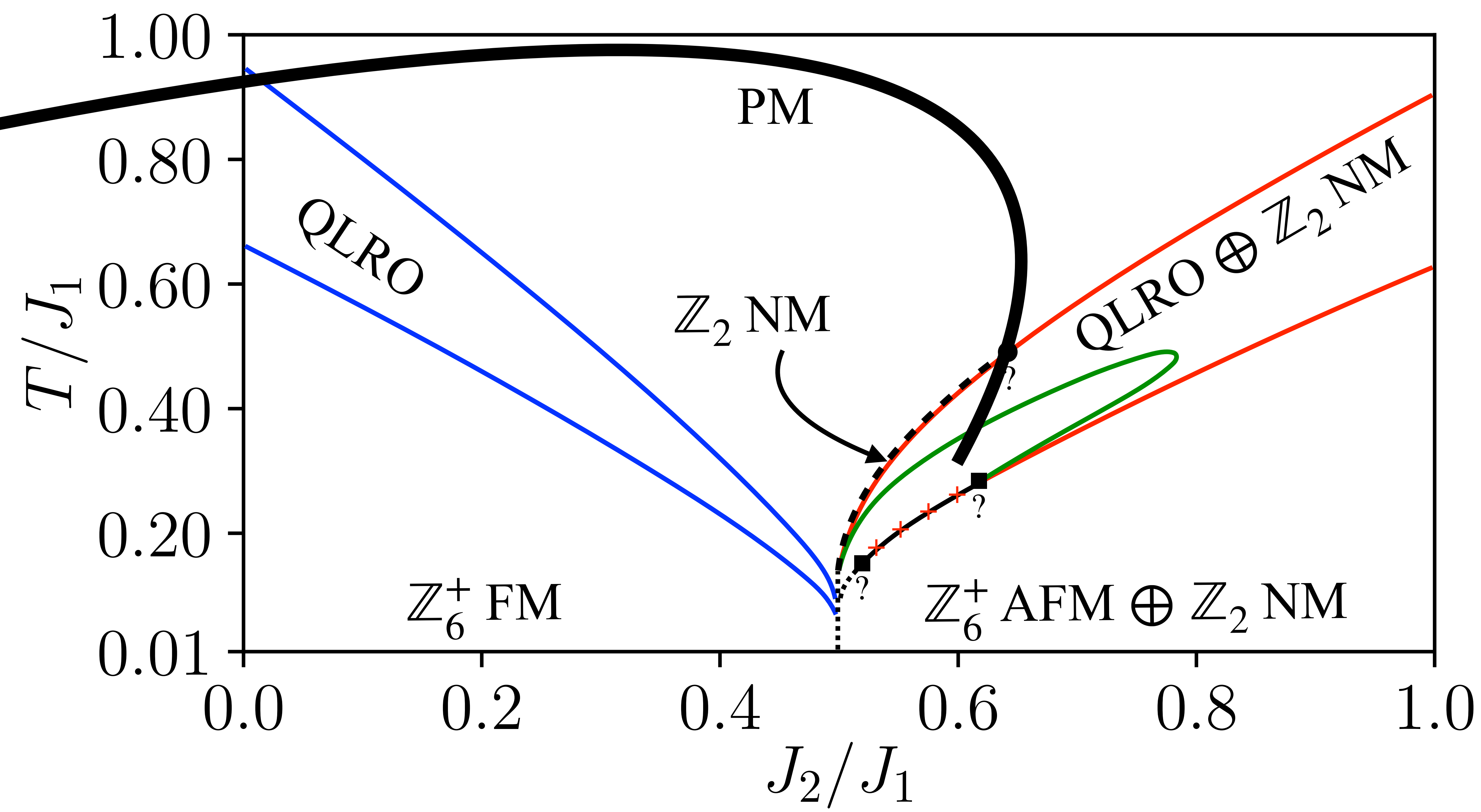

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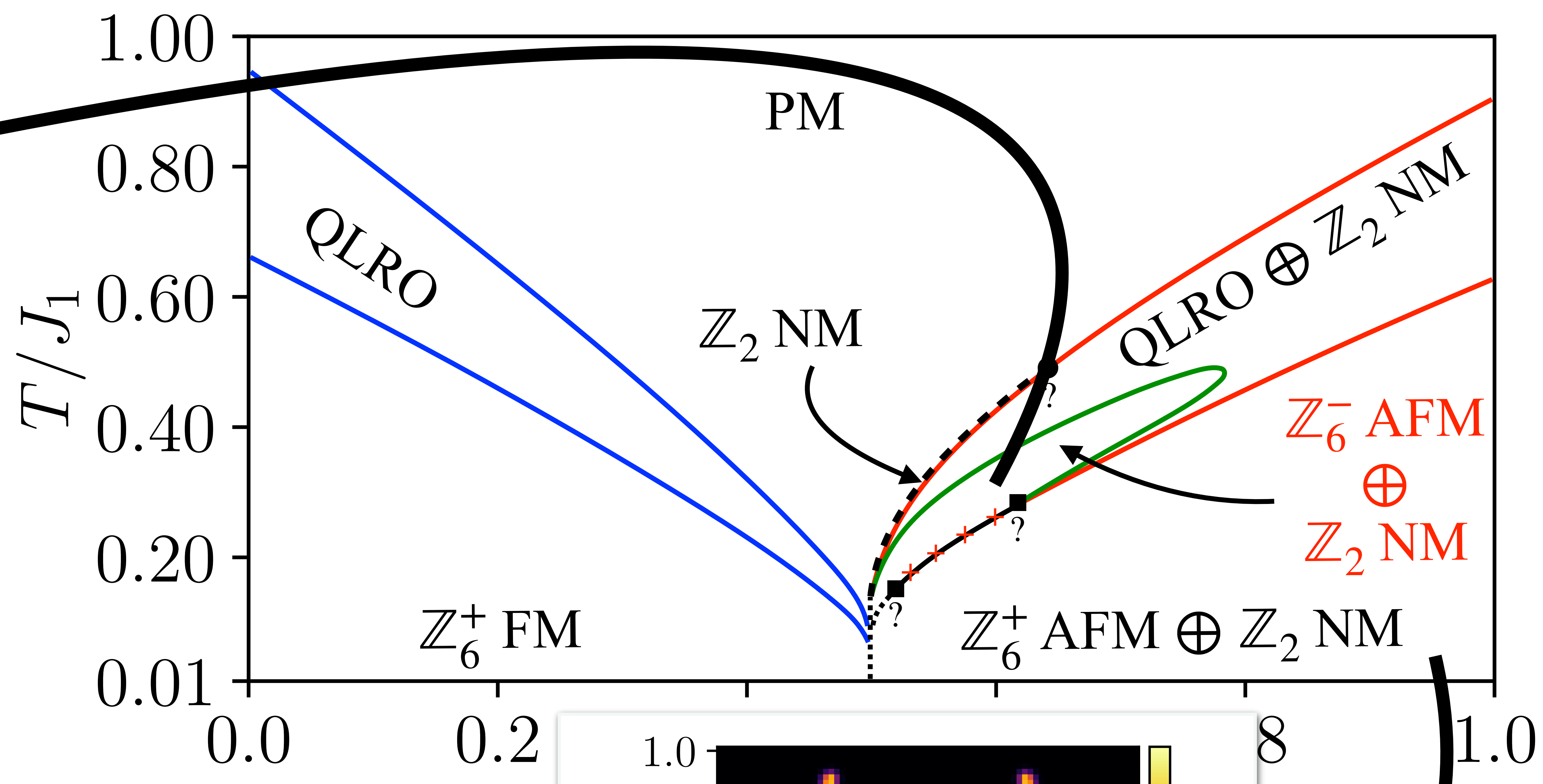

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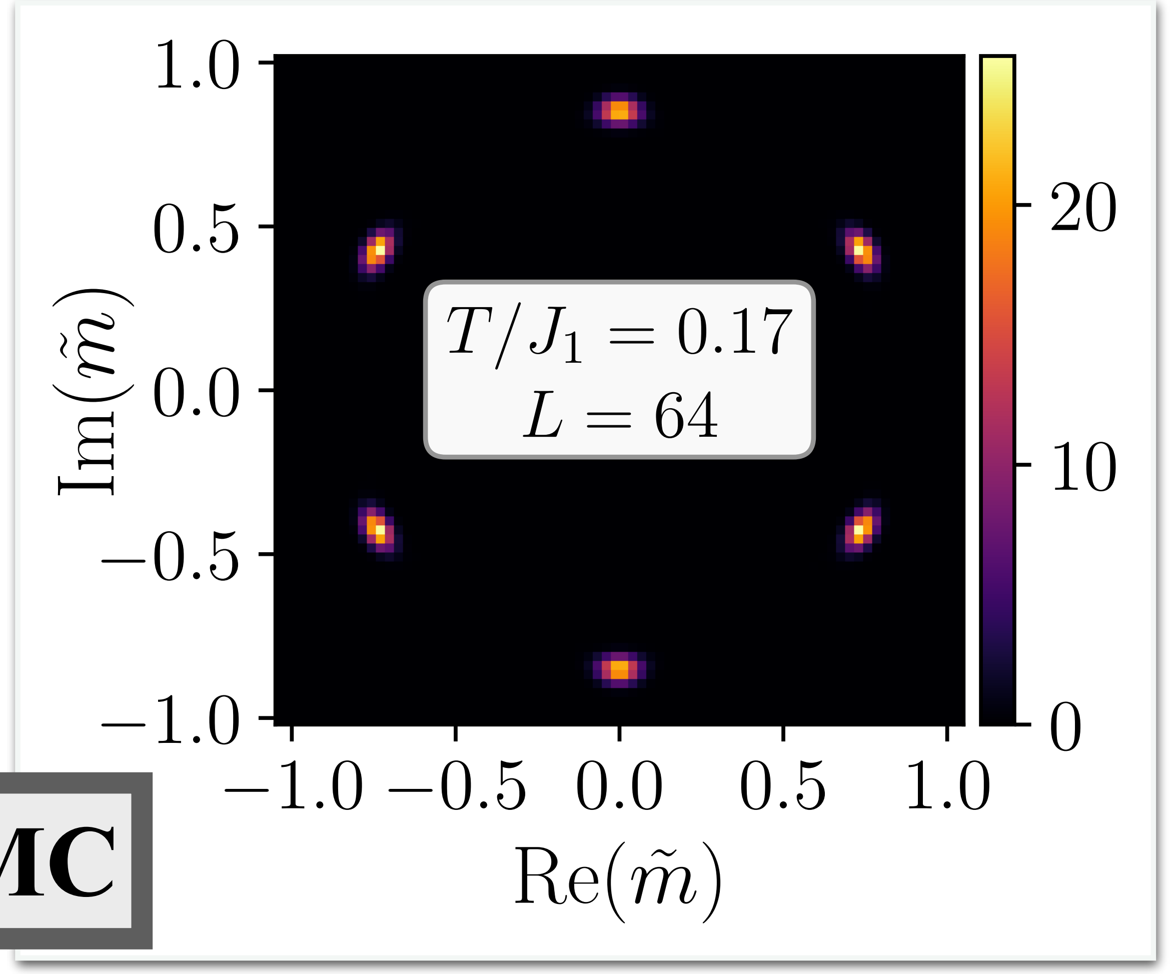
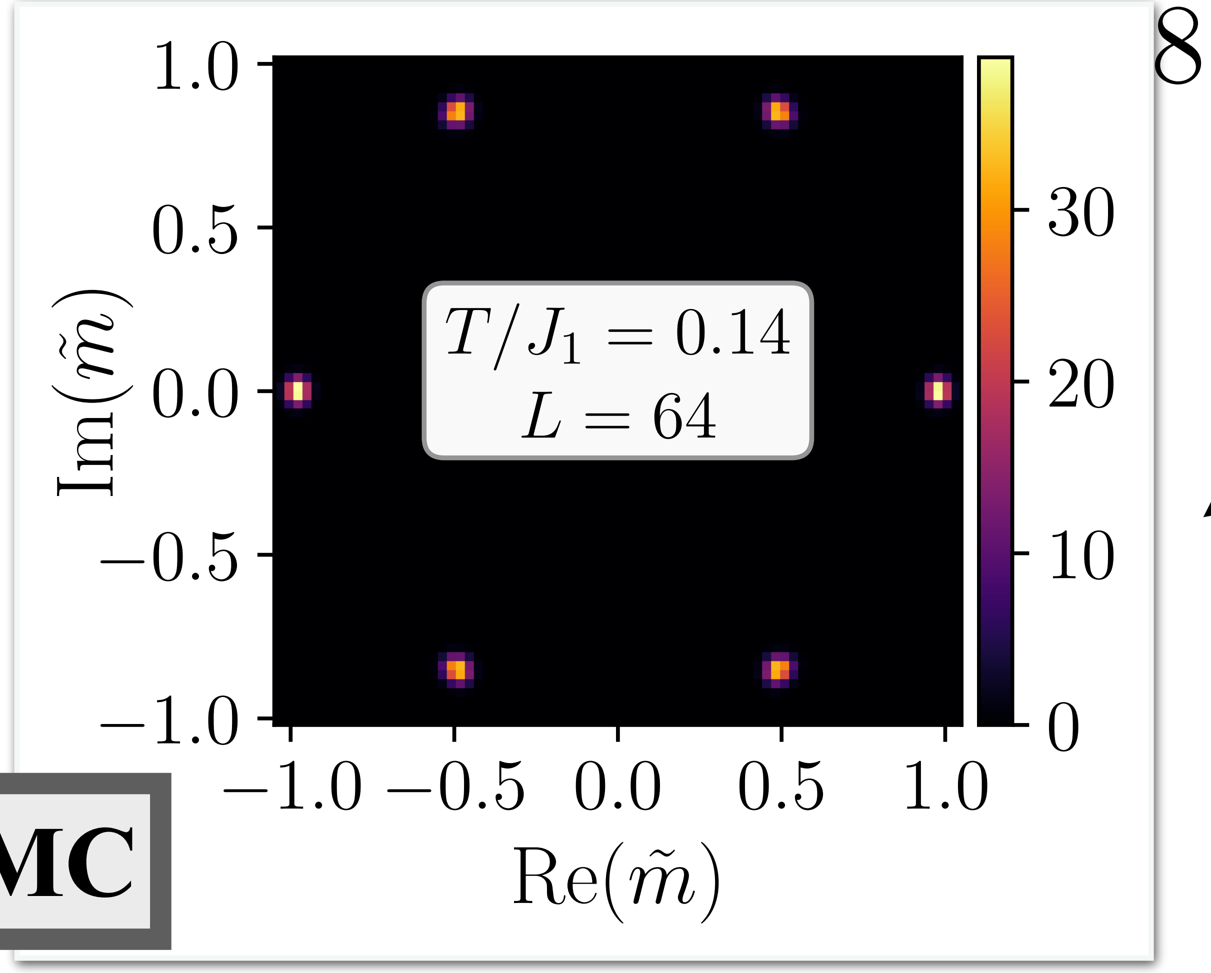


CMC



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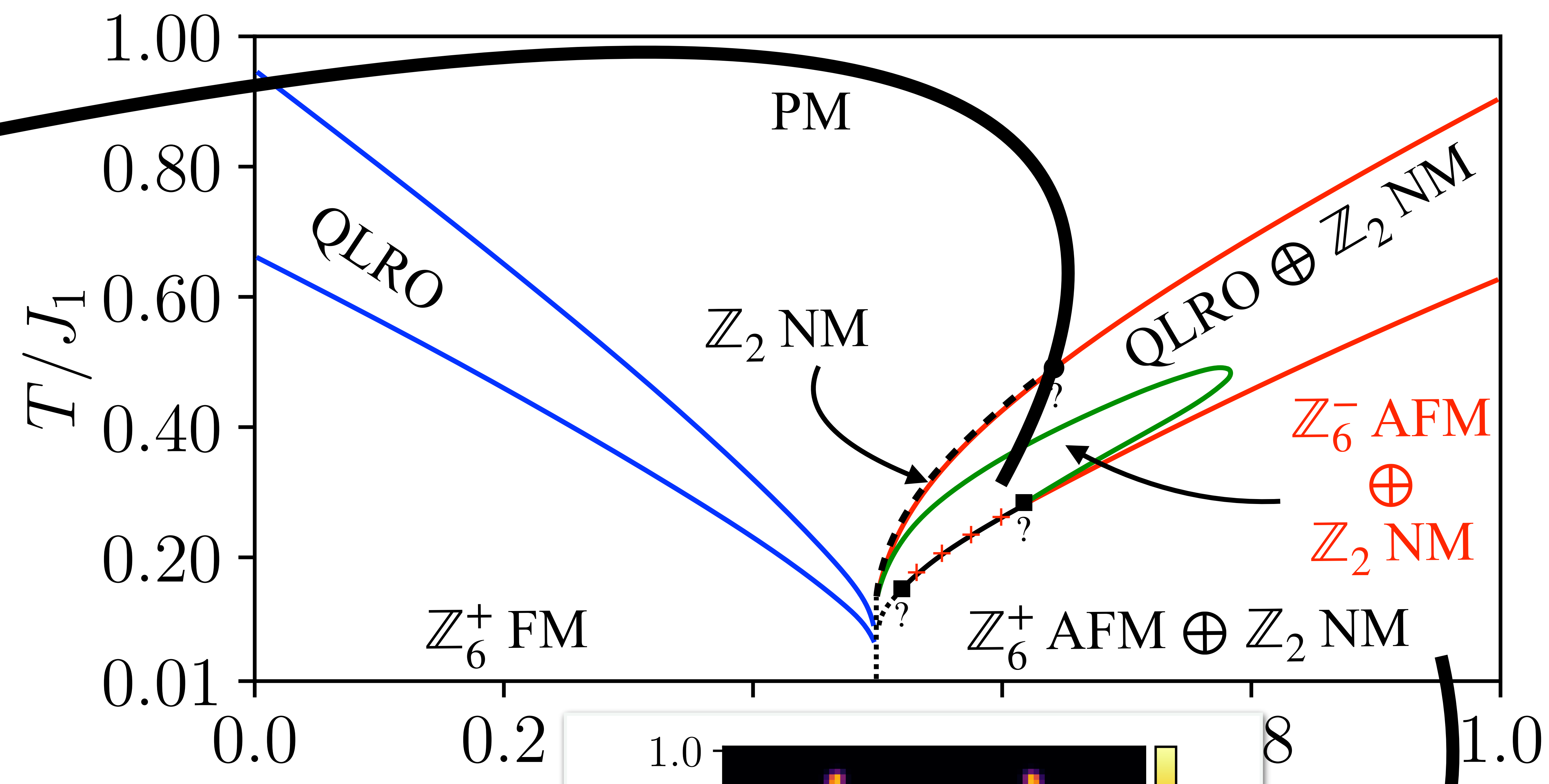

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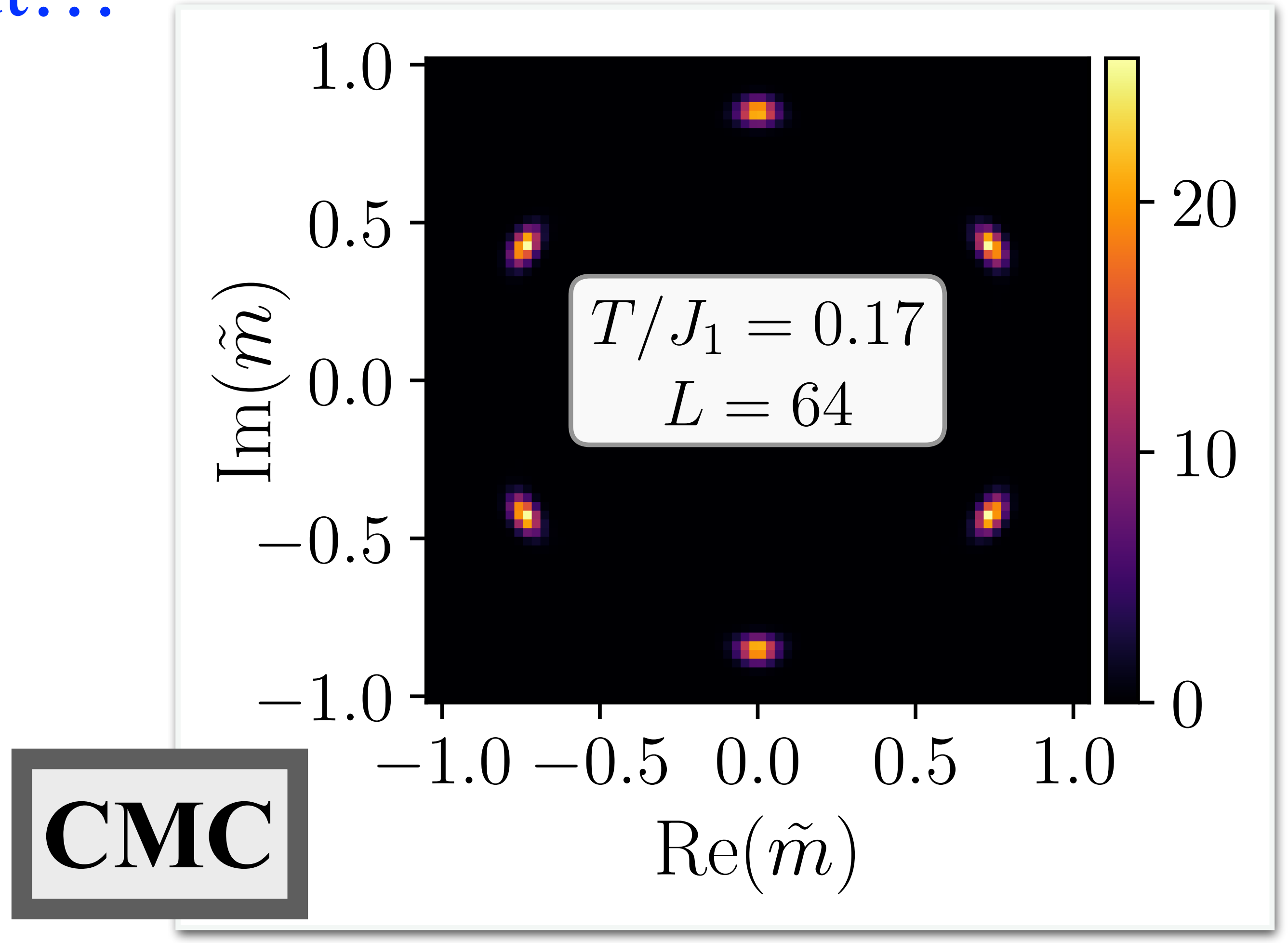
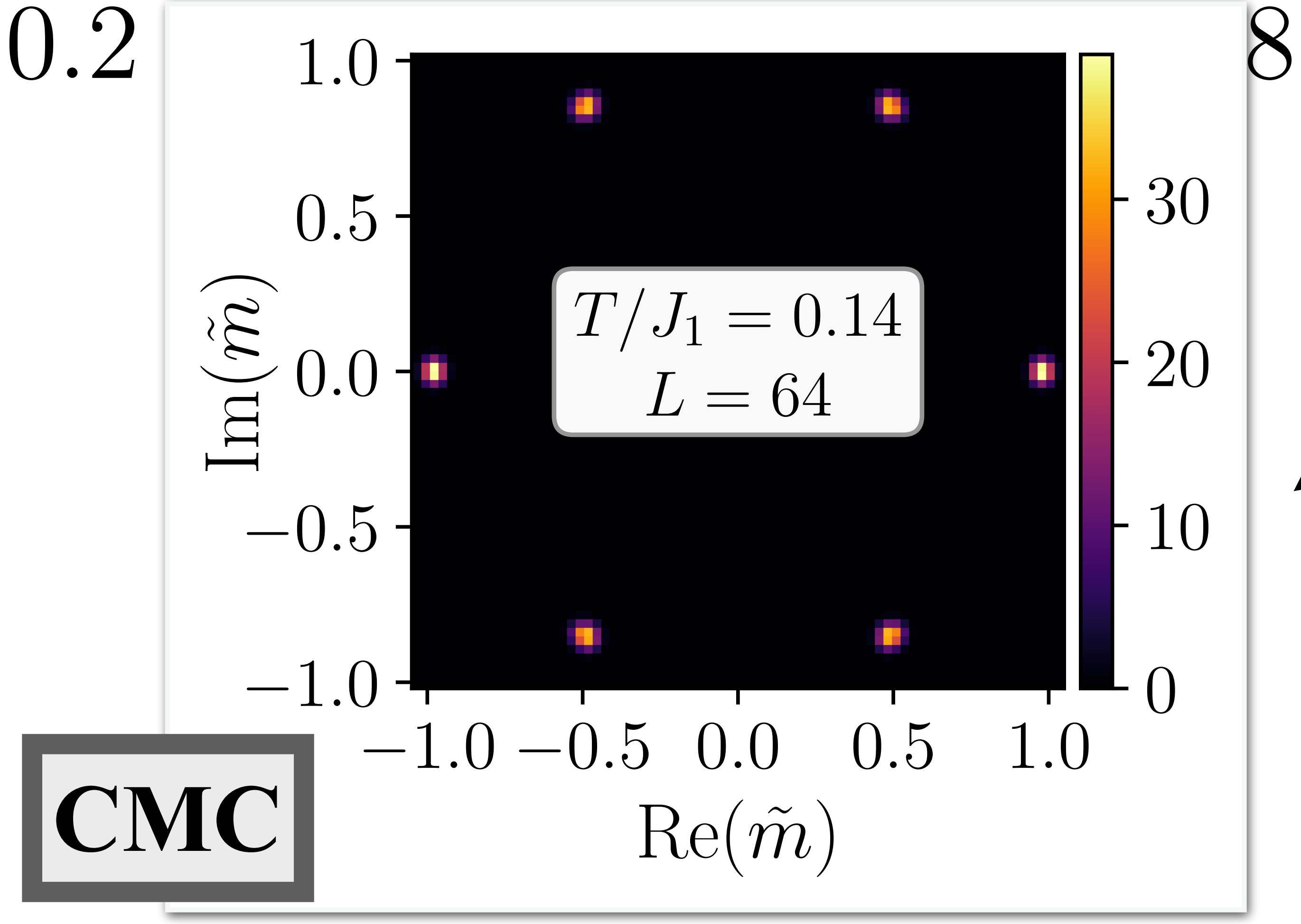


New **emergent degrees-of-freedom**

Not allowed in the microscopic theory

Comes from an emergent $\mathbb{Z}_q$ symmetry different from the microscopic theory



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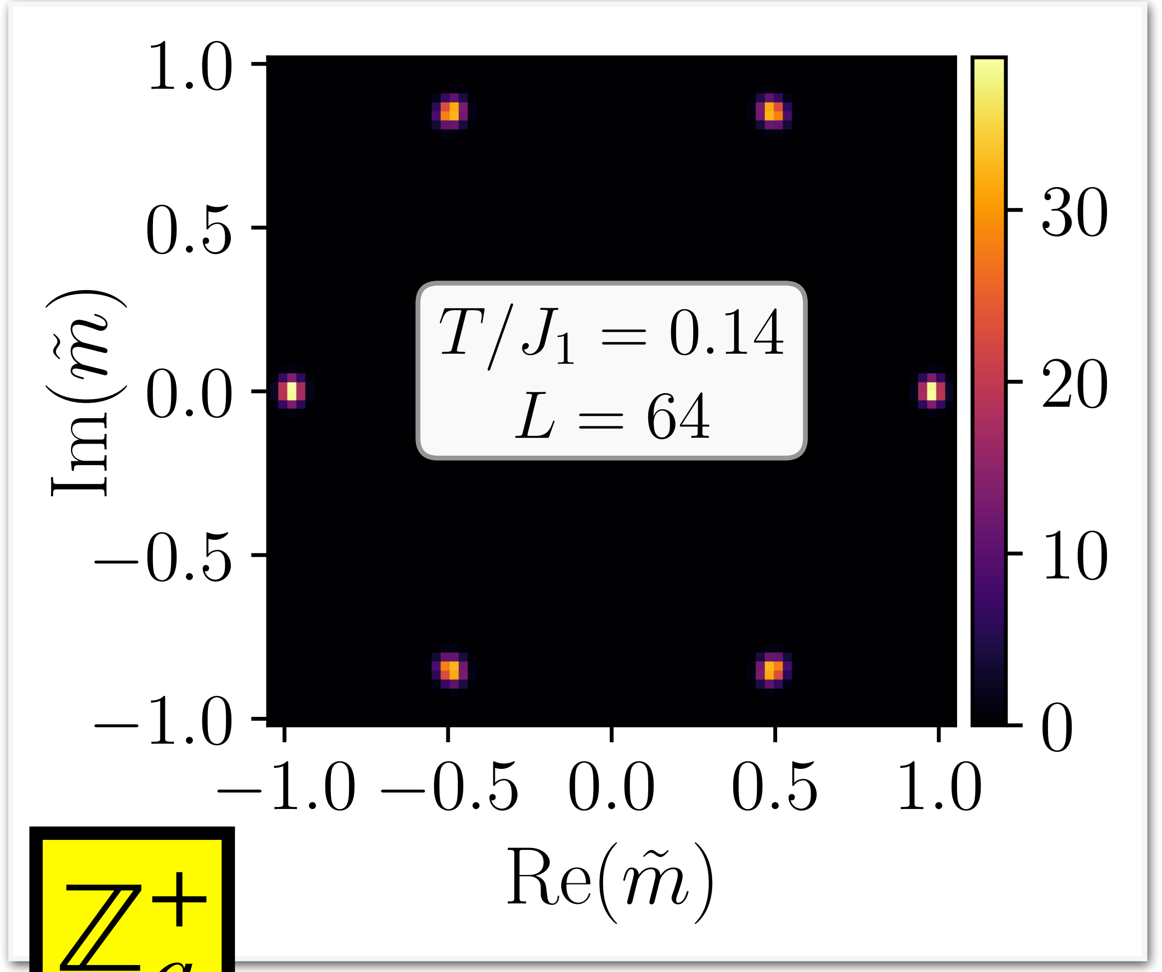
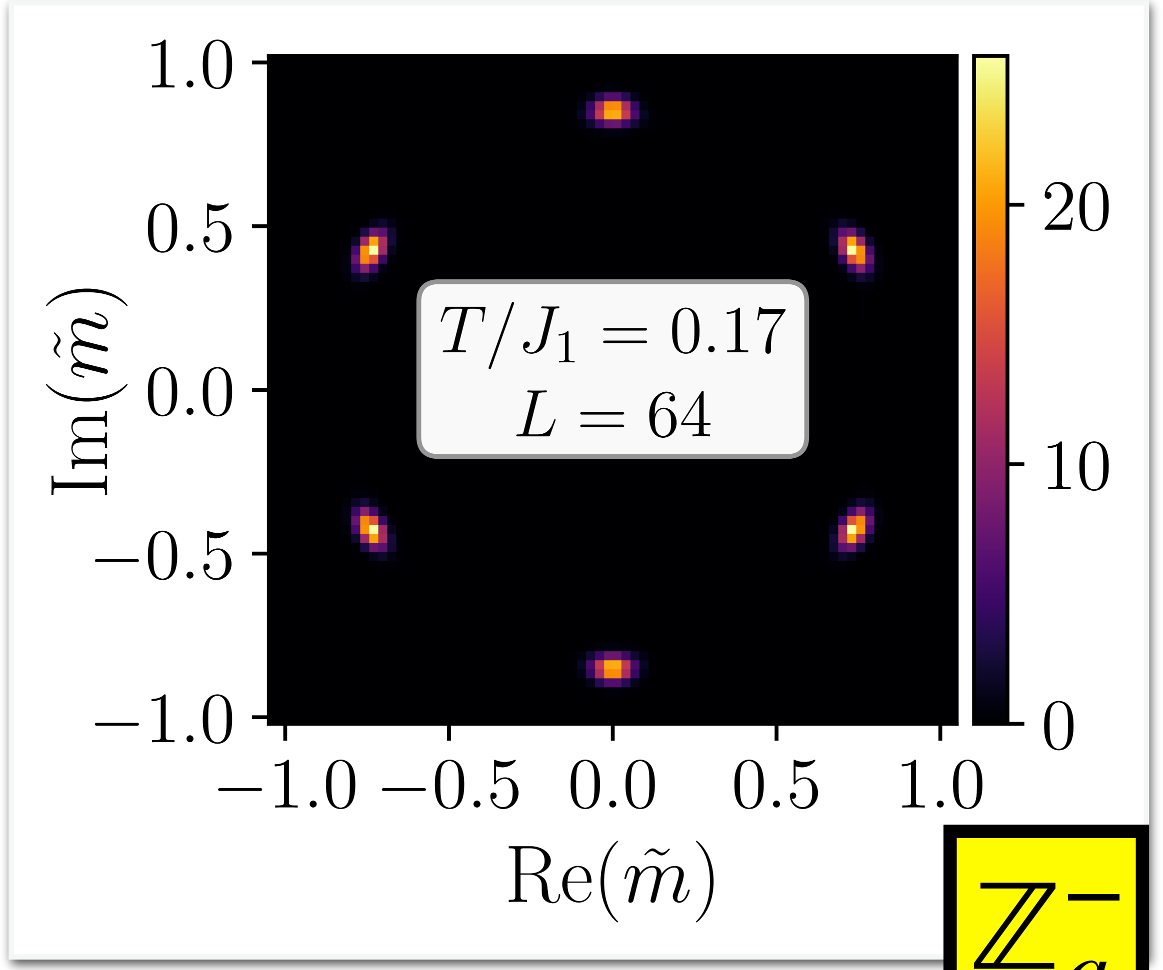


$\mathbb{Z}_q^+$

Transition ??



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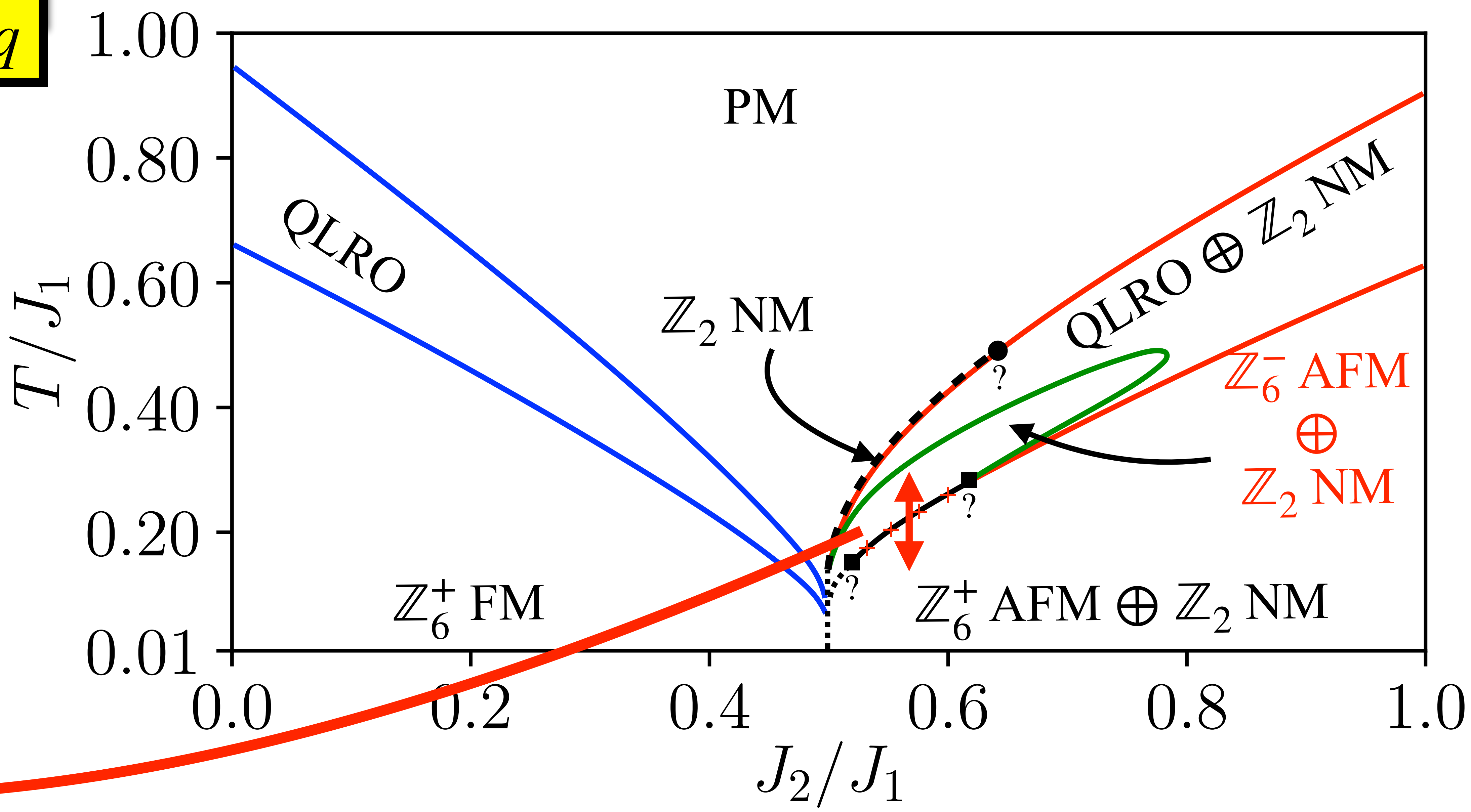

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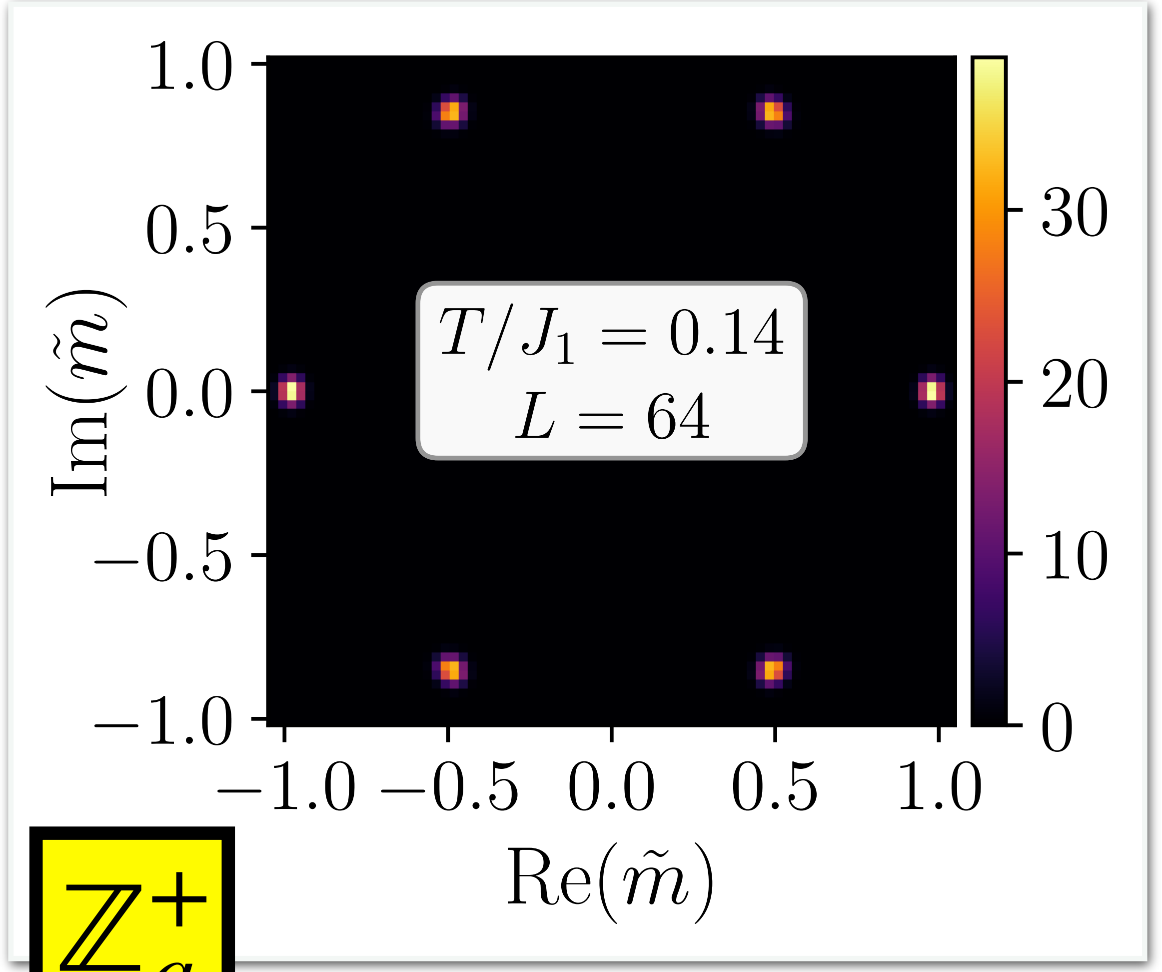

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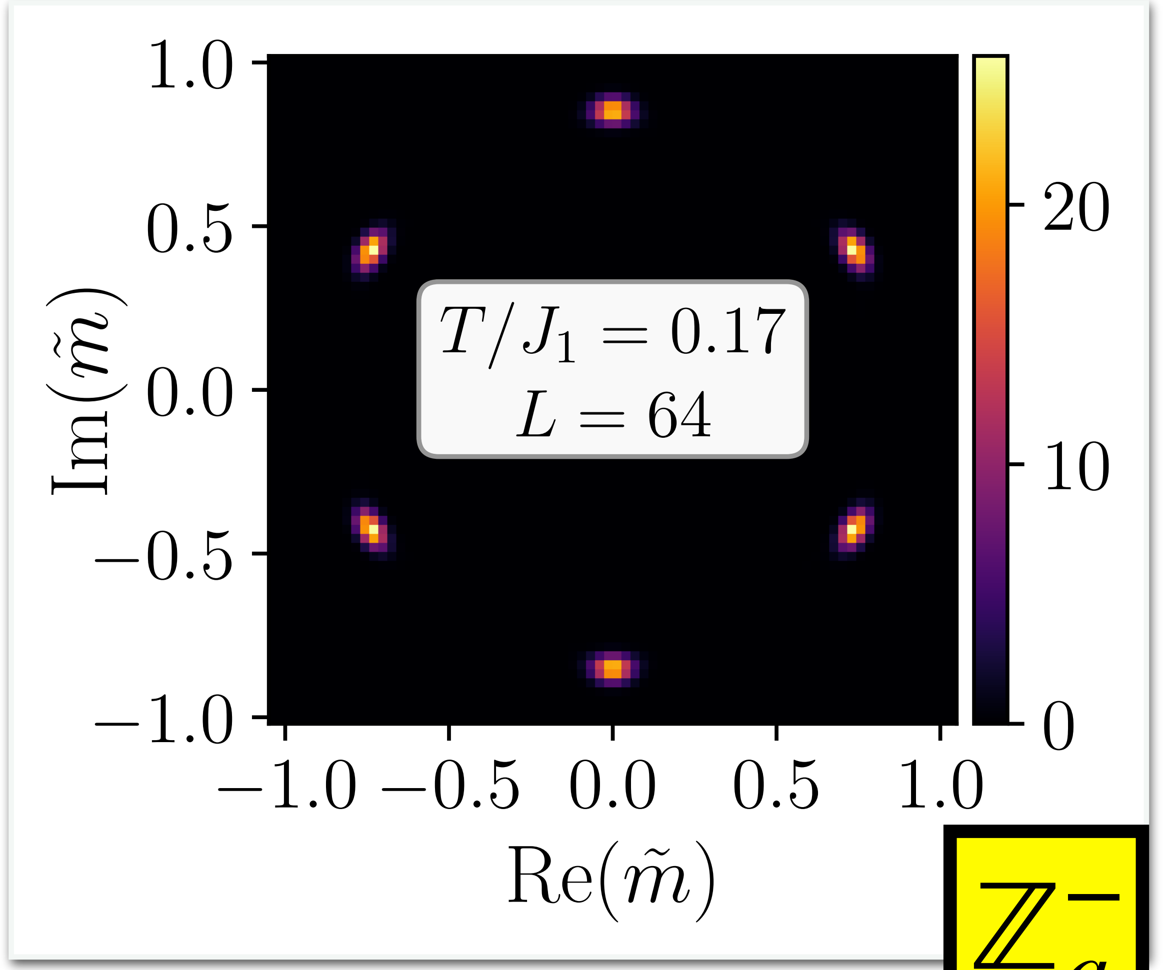

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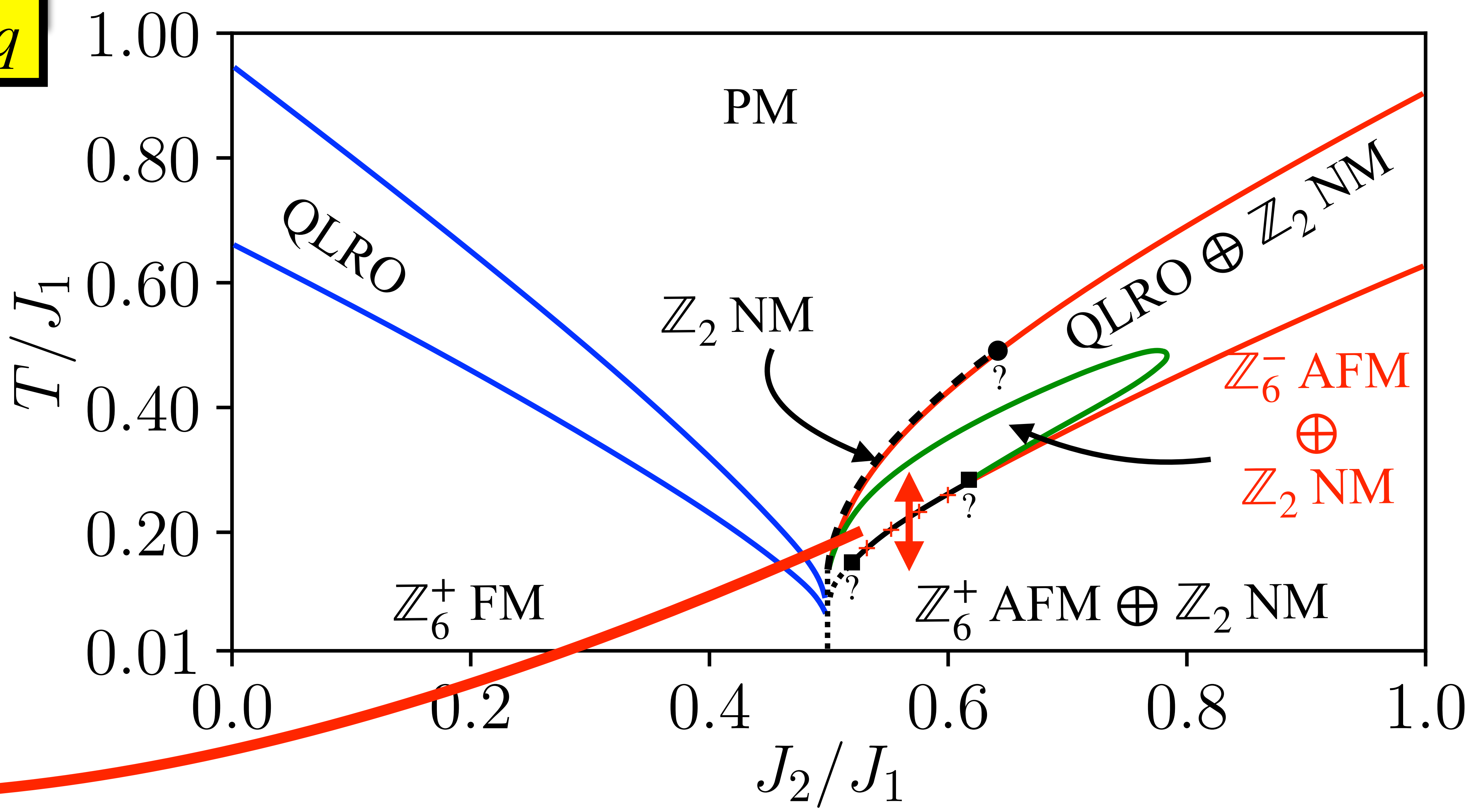

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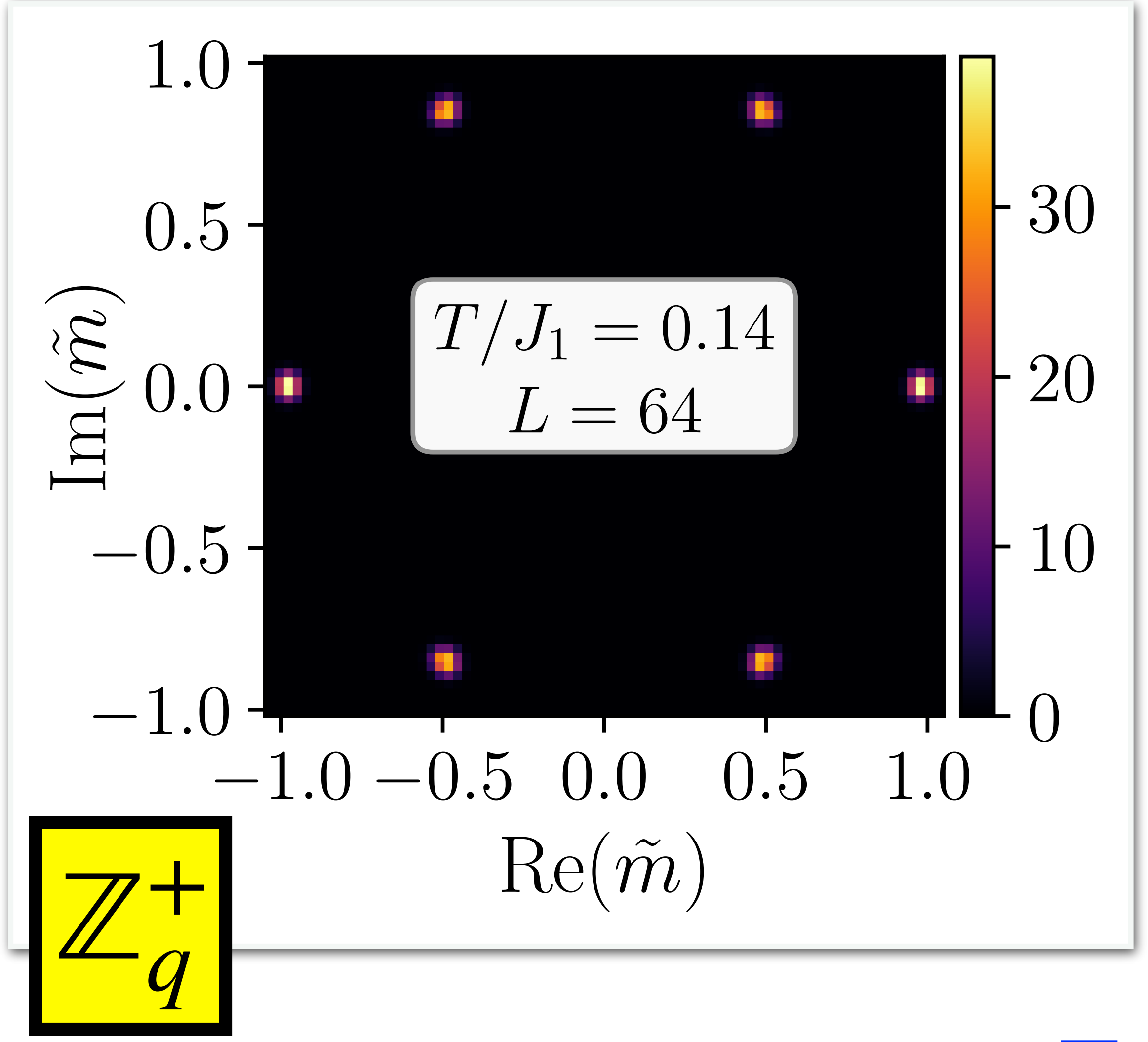

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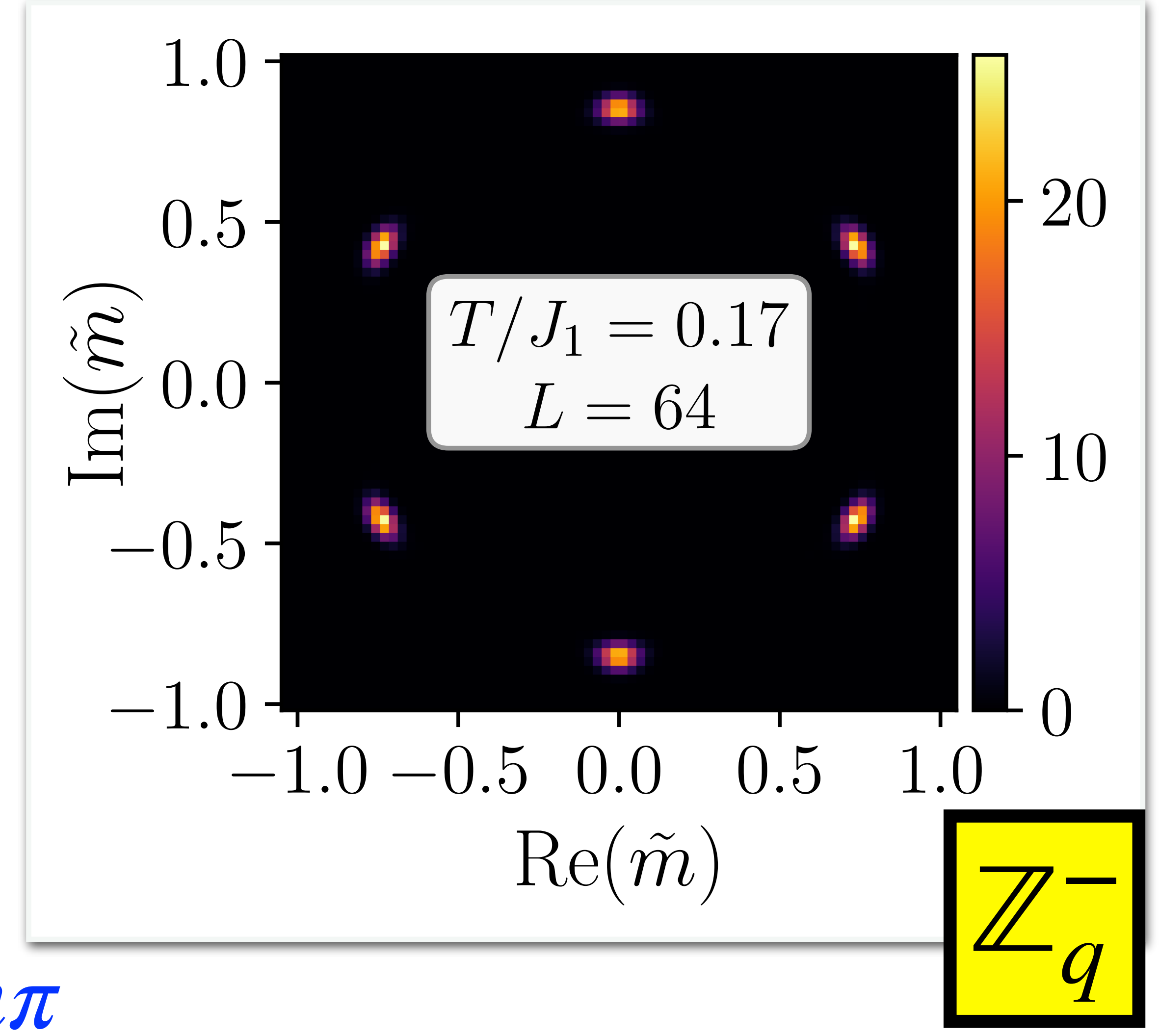

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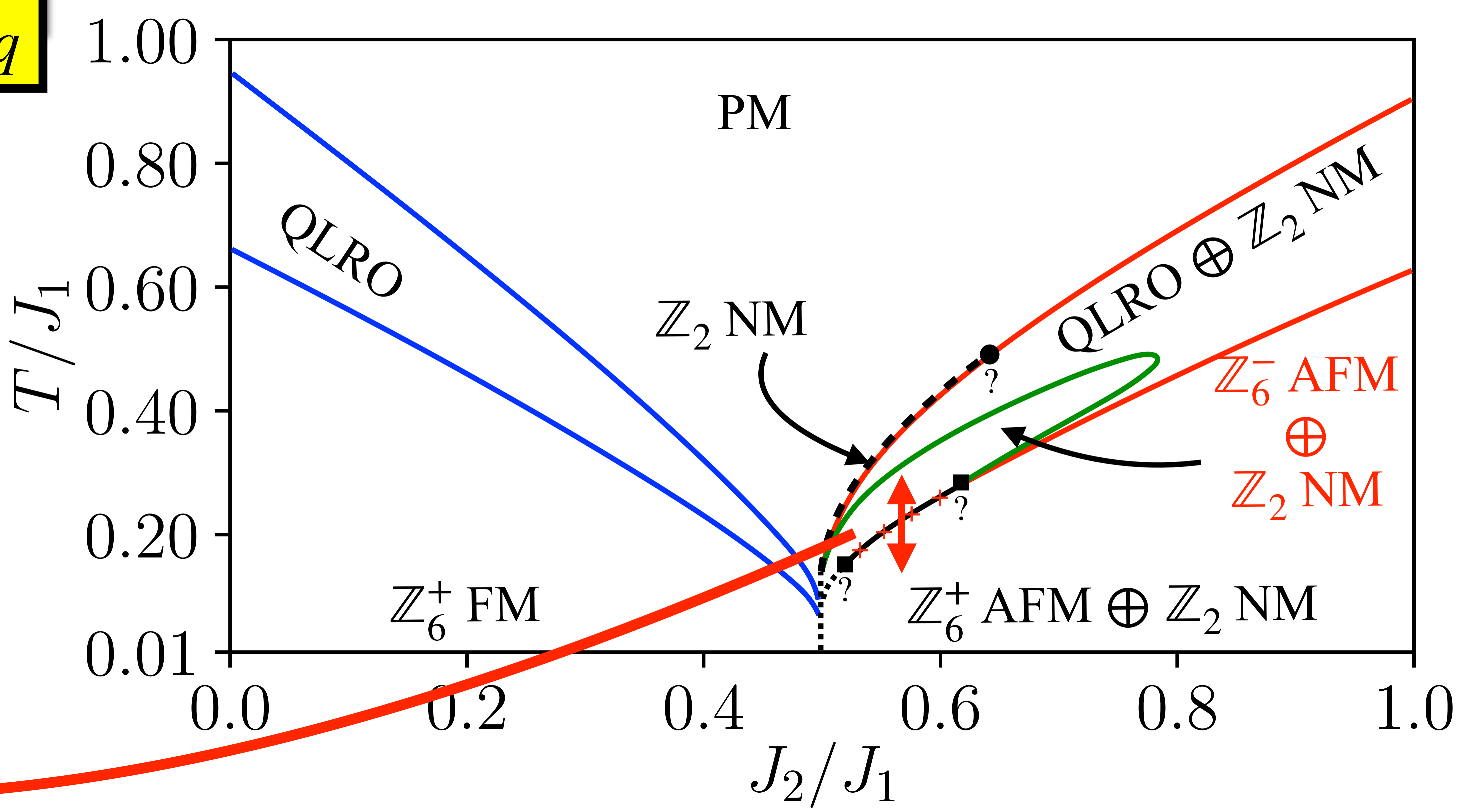

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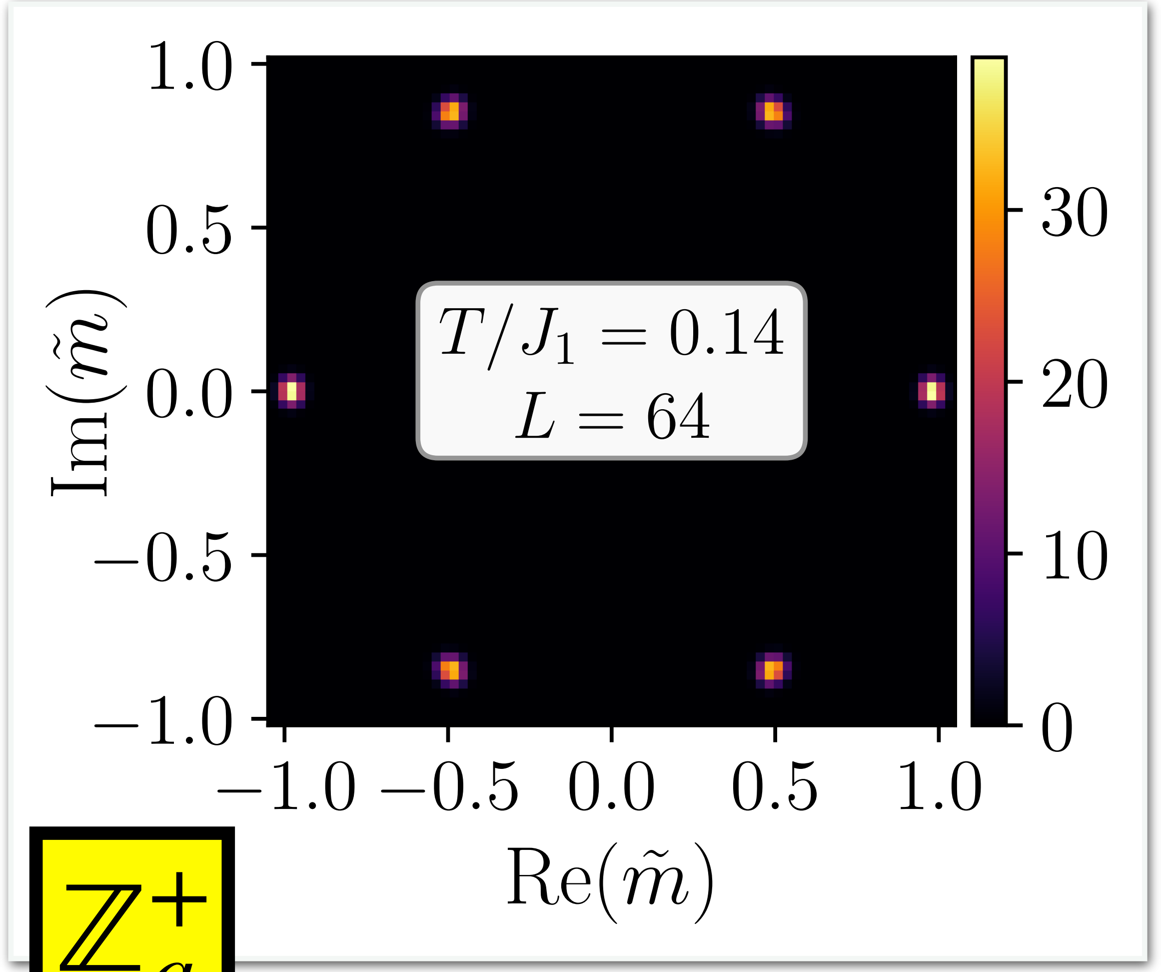

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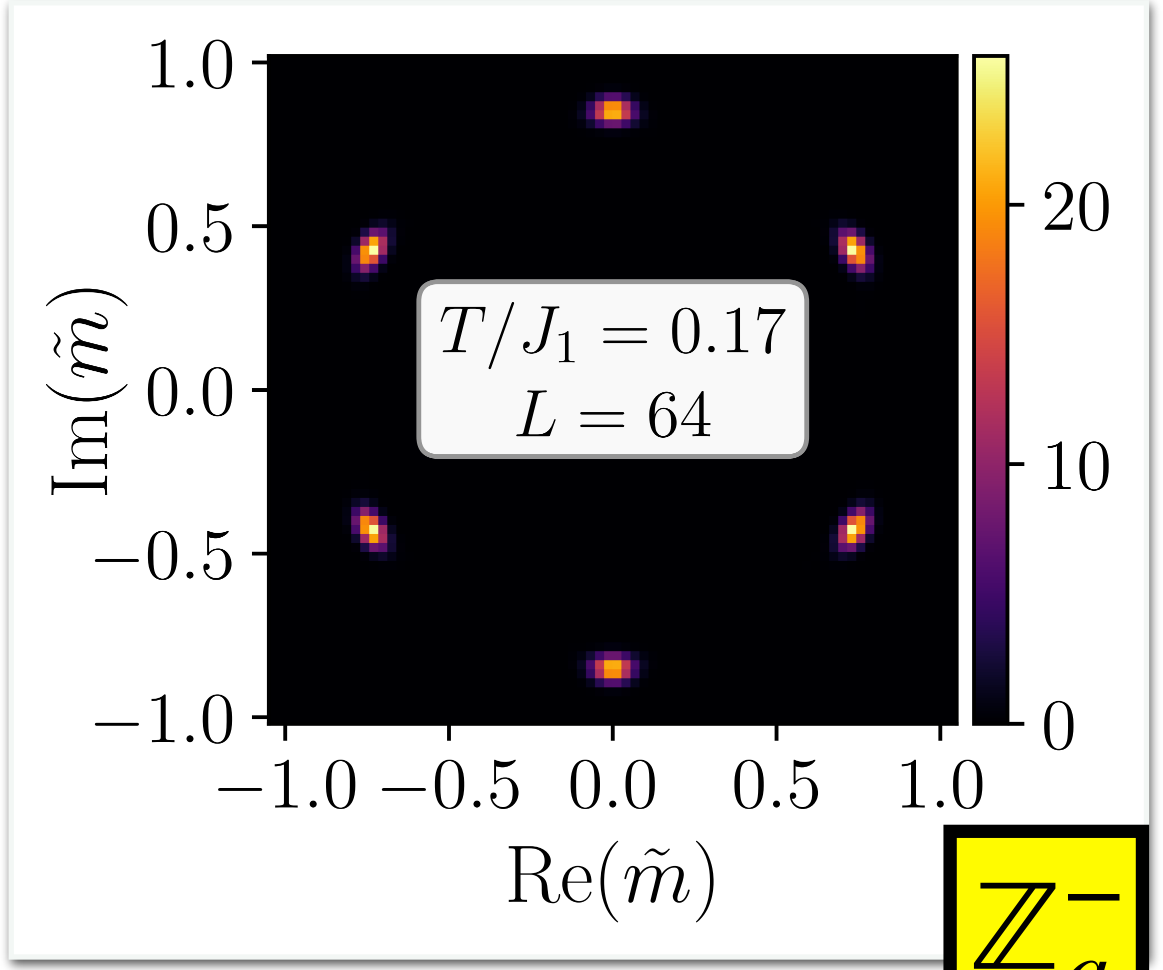
Transition ??



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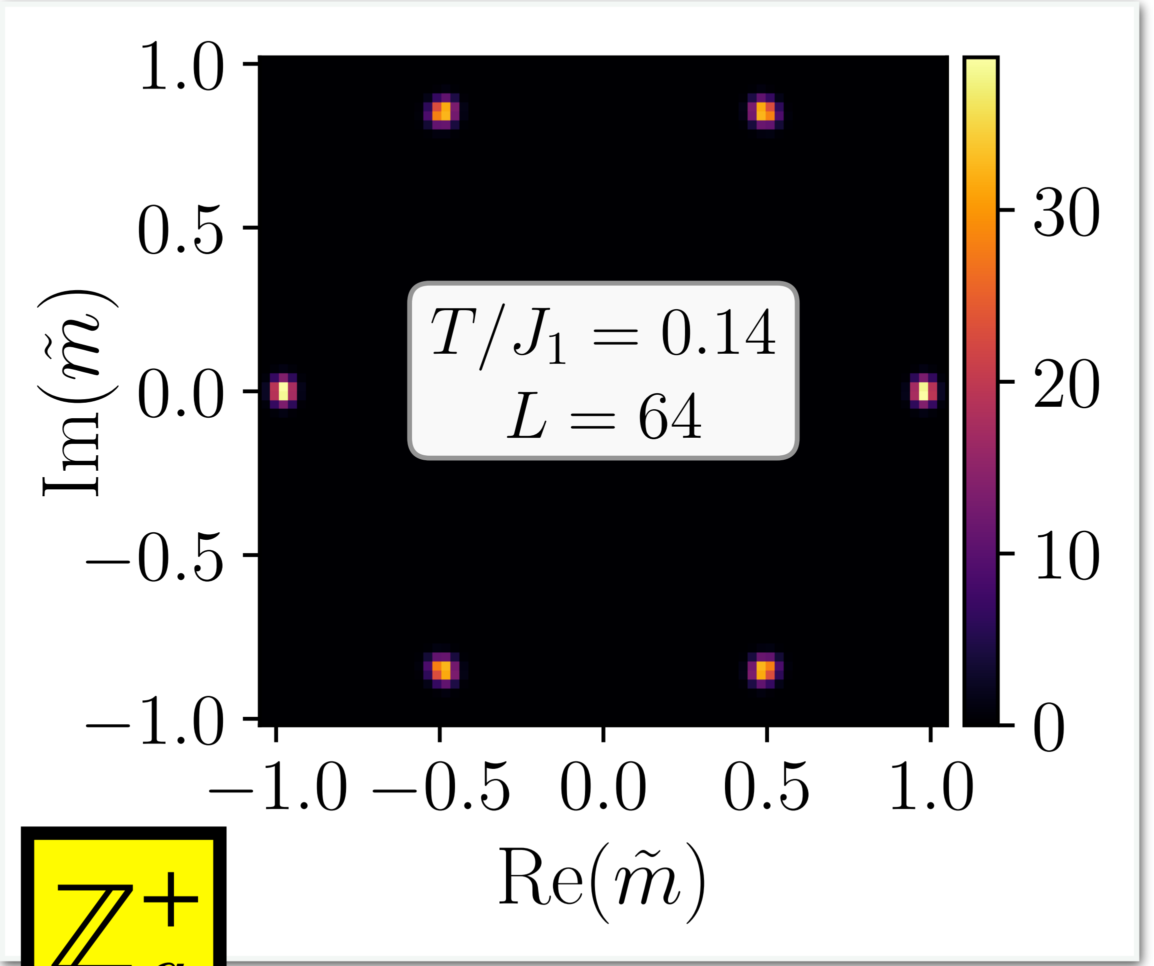

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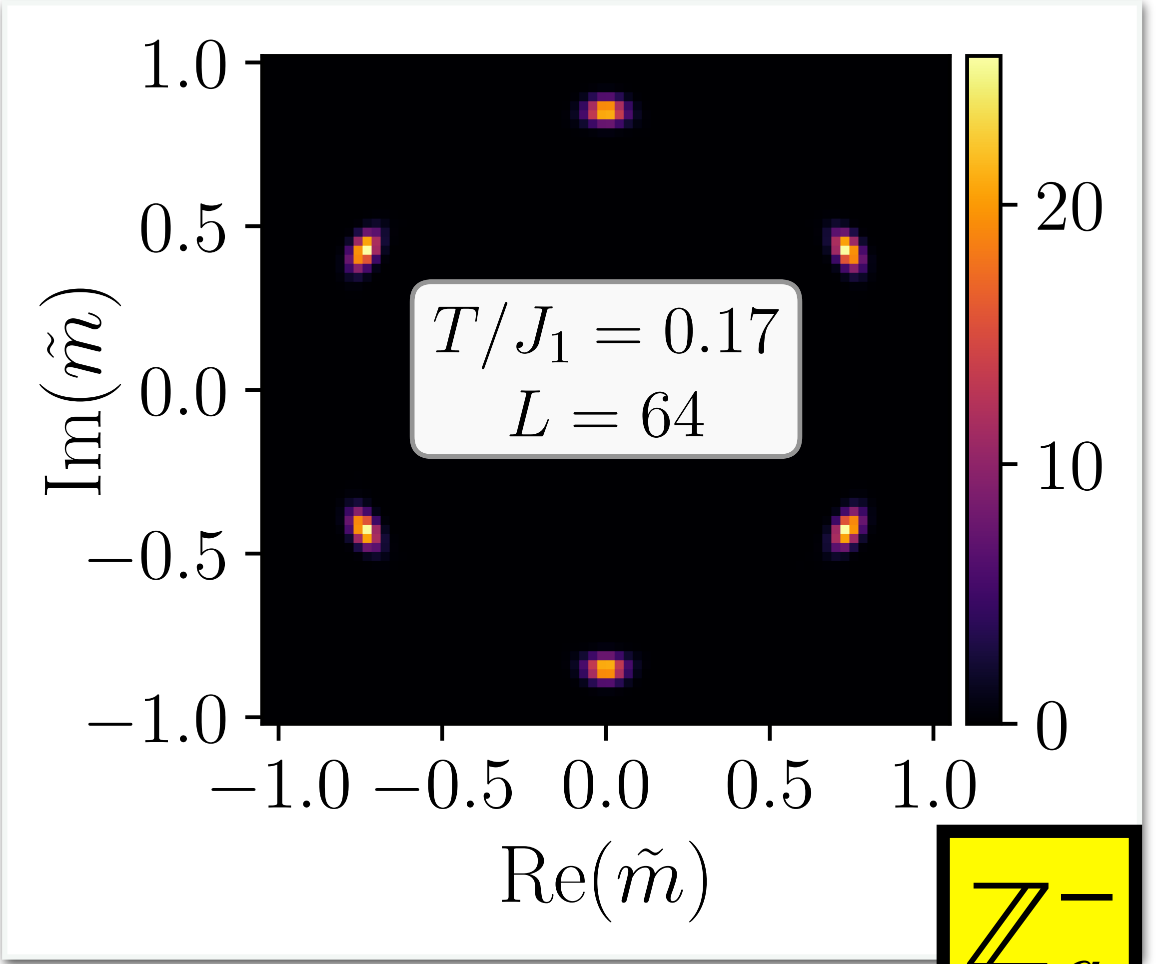

$\mathbb{Z}_q^-$

The $\mathbb{Z}_q^\pm$ phases breaks **incompatible** $\mathbb{Z}_q$ symmetries

Incompatible in the sense of Landau paradigm…
Two orders **cannot coexist**…

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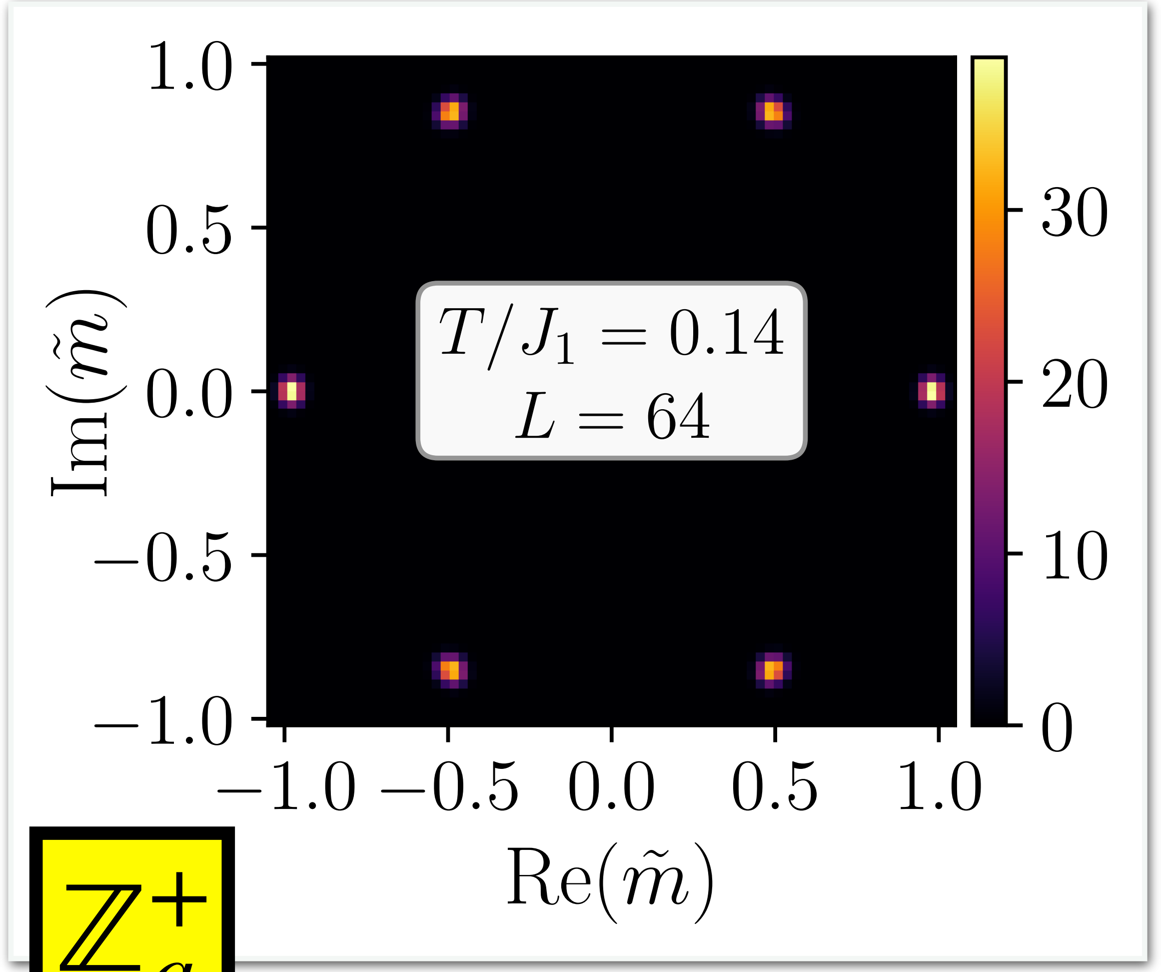
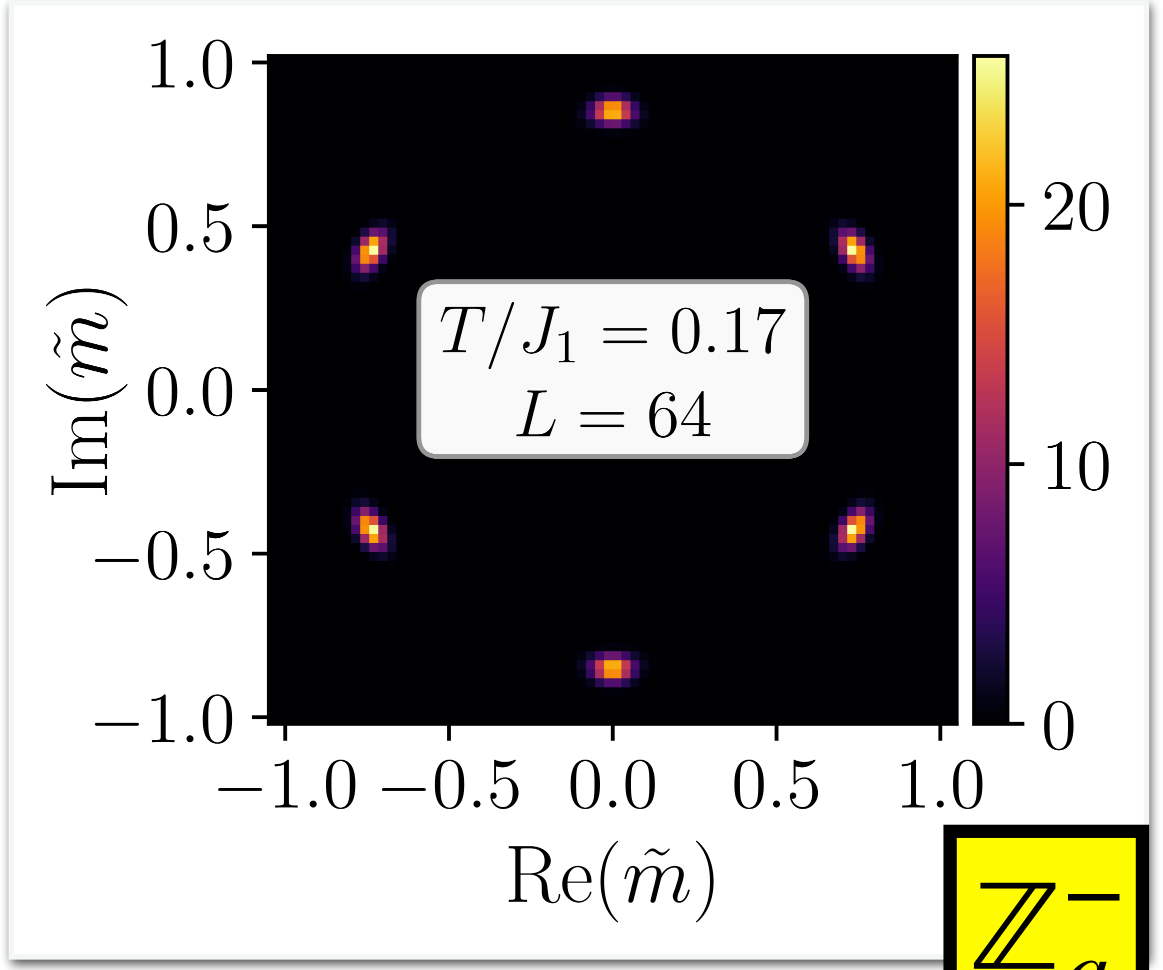


$\mathbb{Z}_q^+$

Transition ??



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Two orders **cannot coexist**…

According to Landau paradigm….
these two phases can be separated either by…

(a) a first-order direct transition (present in our case)
or, by
(b) a third intervening phase (also present)

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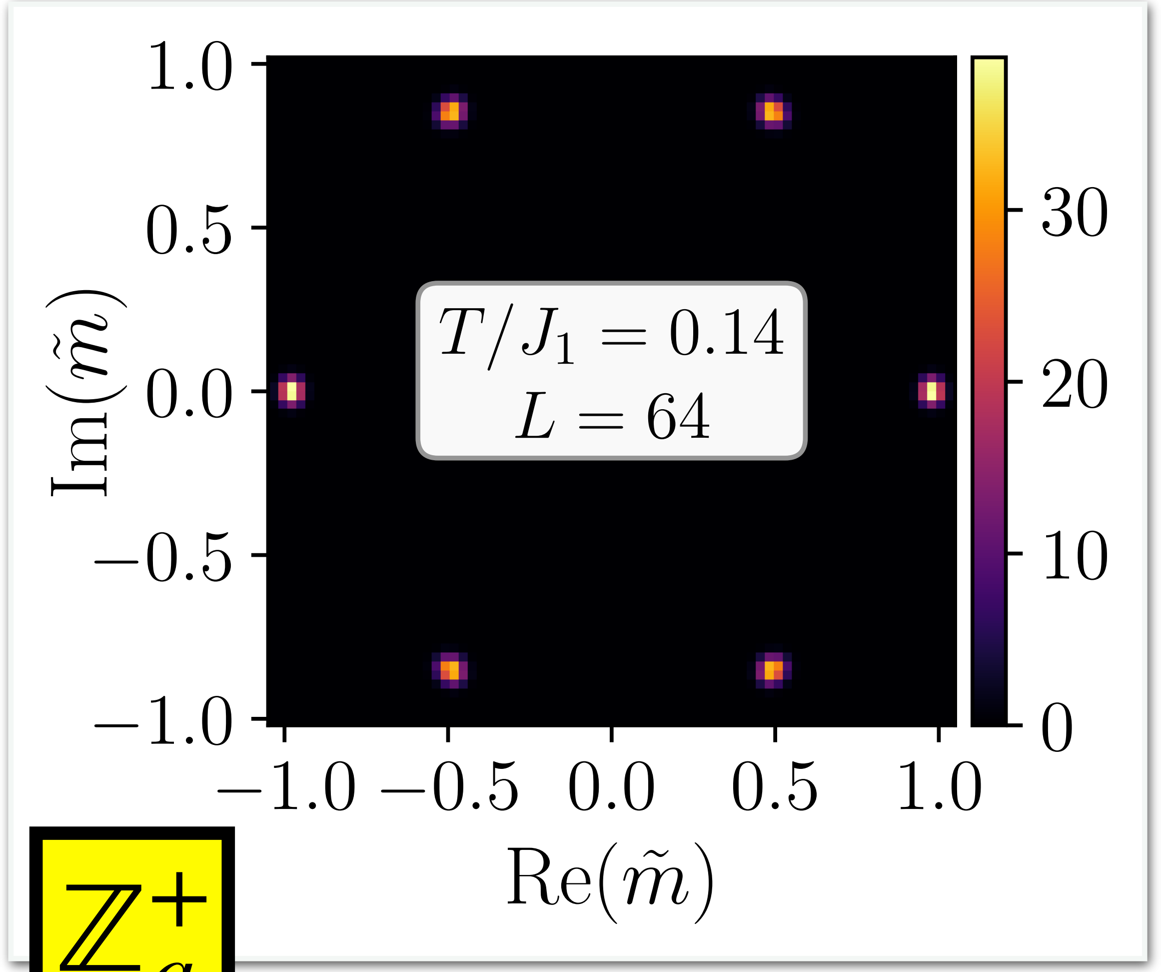
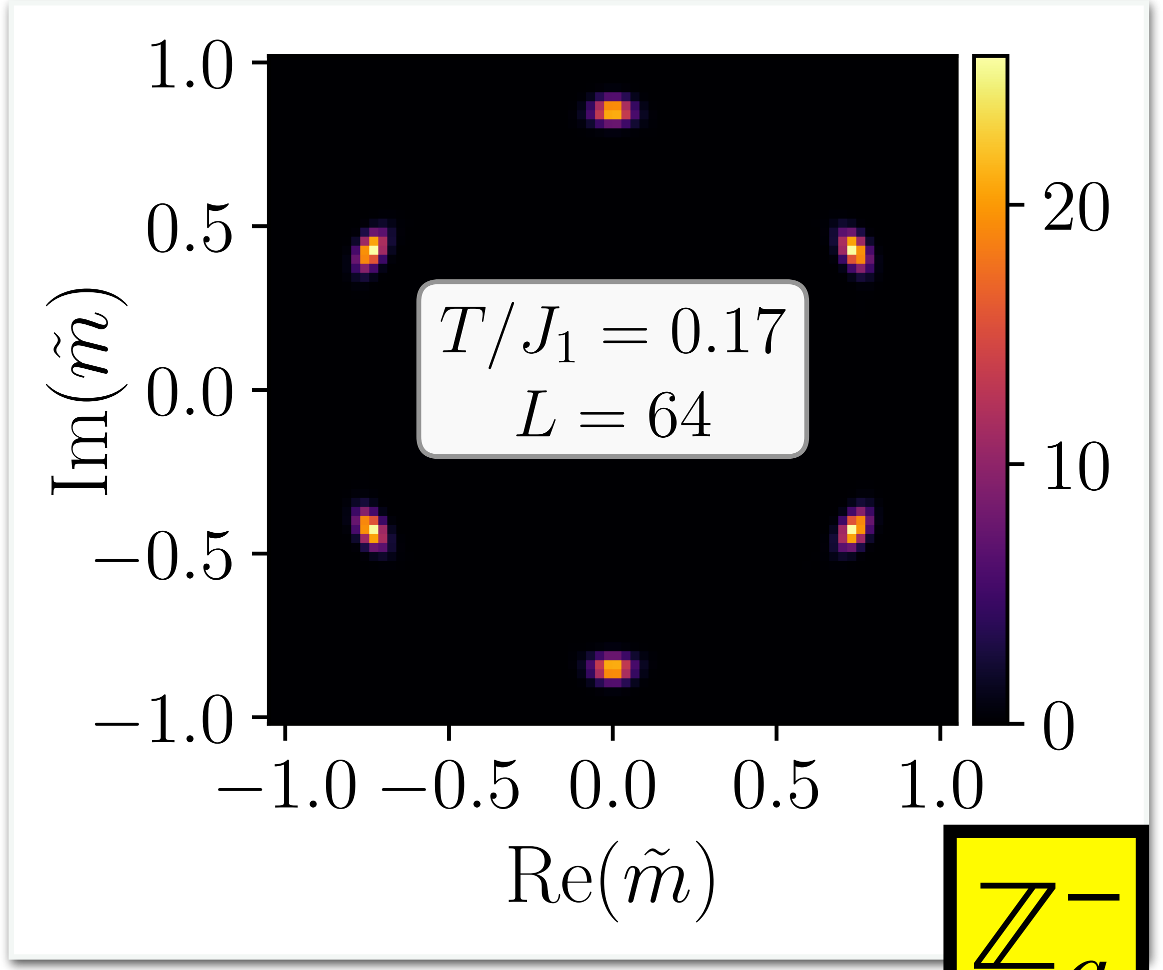
$$H = -J_1 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) + J_2 \sum_{\langle\langle ij \rangle\rangle} \cos(\theta_i - \theta_j) \qquad \theta = \frac{2n\pi}{q}; \quad n = 0,1,\ldots,q-1$$



$\mathbb{Z}_q^+$

Transition ??



$\mathbb{Z}_q^-$

Breaks $\mathbb{Z}_q : \ \theta_i \to \theta_i + \dfrac{2m\pi}{q} \ldots \forall i$

$\mathscr{R} \ : \ \theta_i \to -\theta_i \ldots \forall i$
is still a **residual symmetry**
Two vacua remain invariant

$\mathscr{R} \ : \ \theta_i \to -\theta_i \ldots \forall i$
not a residual symmetry

The $\mathbb{Z}_q^{\pm}$ phases breaks **incompatible** $\mathbb{Z}_q$ symmetries

Incompatible in the sense of Landau paradigm…
Two orders **cannot coexist**…

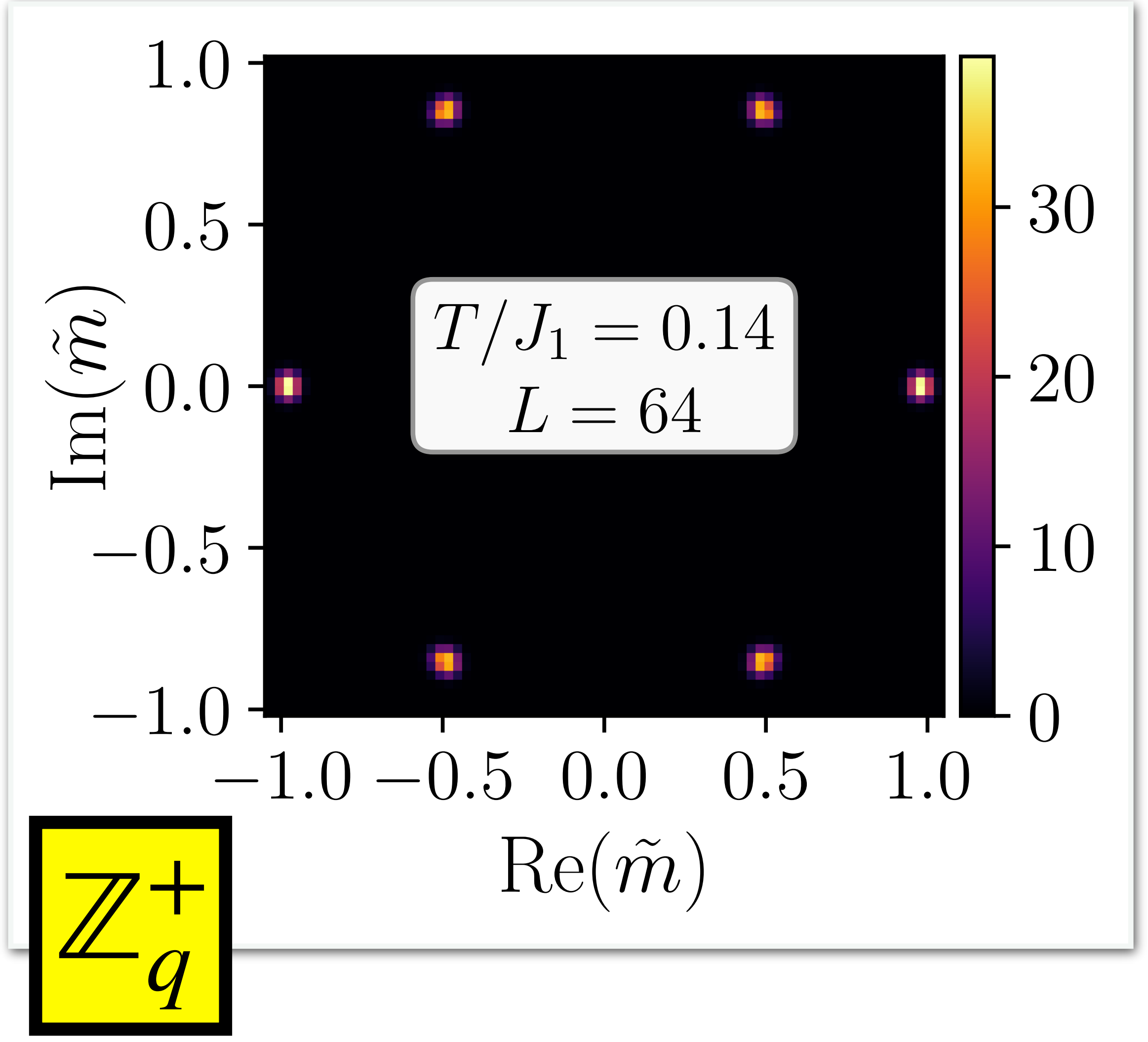
According to Landau paradigm….
these two phases can be separated either by…

(a) a first-order direct transition (present in our case)
or, by
(b) a third intervening phase (also present)

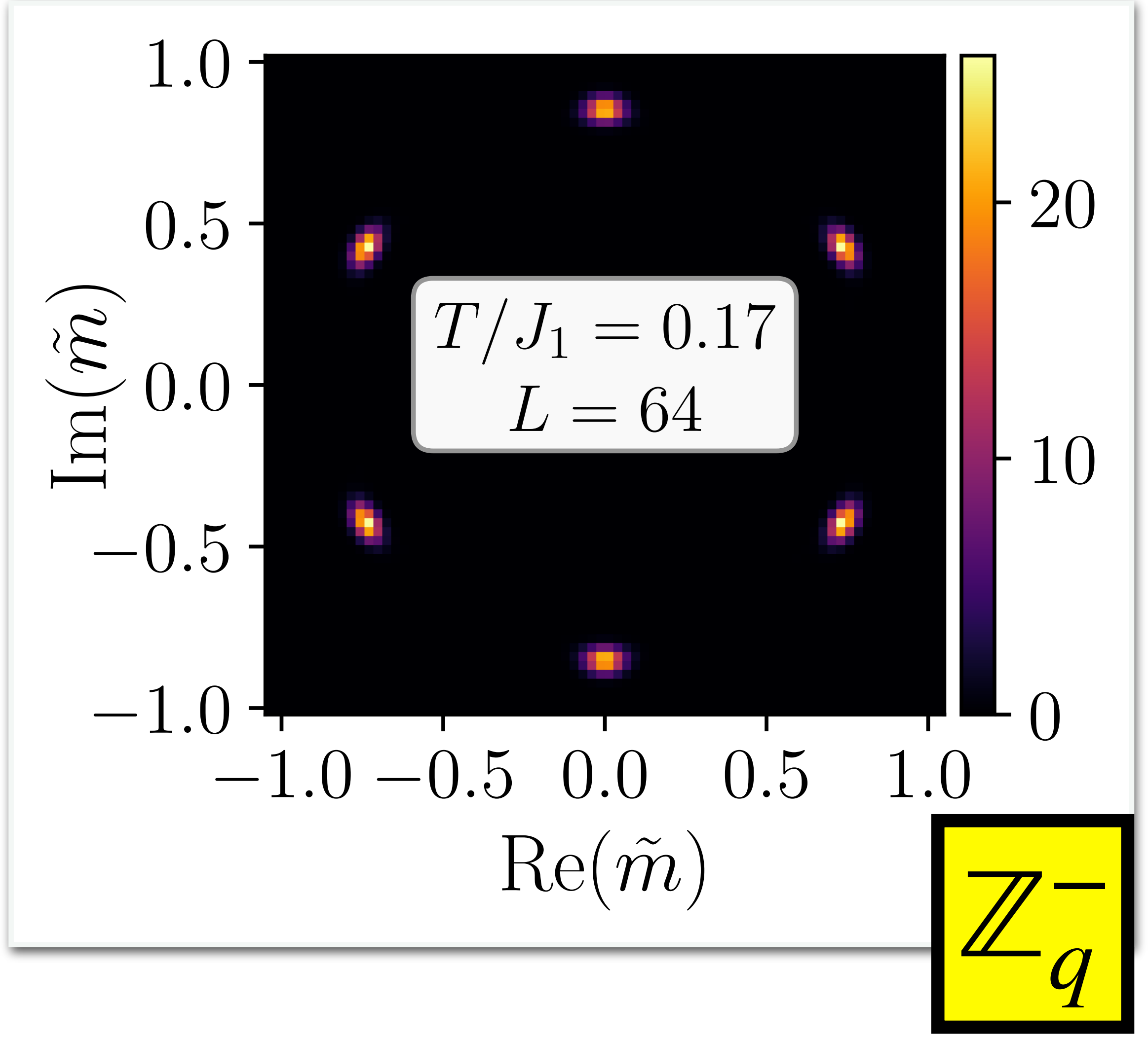
Any **direct continuous transition** between these two phases… is therefore… **Landau-incompatible**…

Classical analogue of **Deconfined Quantum Criticality**…

J_1-J_2 Clock Model



Transition ??



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Classical Origins of Landau-Incompatible Transitions, Abhishodh Prakash and Nick G. Jones, PRL (2025)

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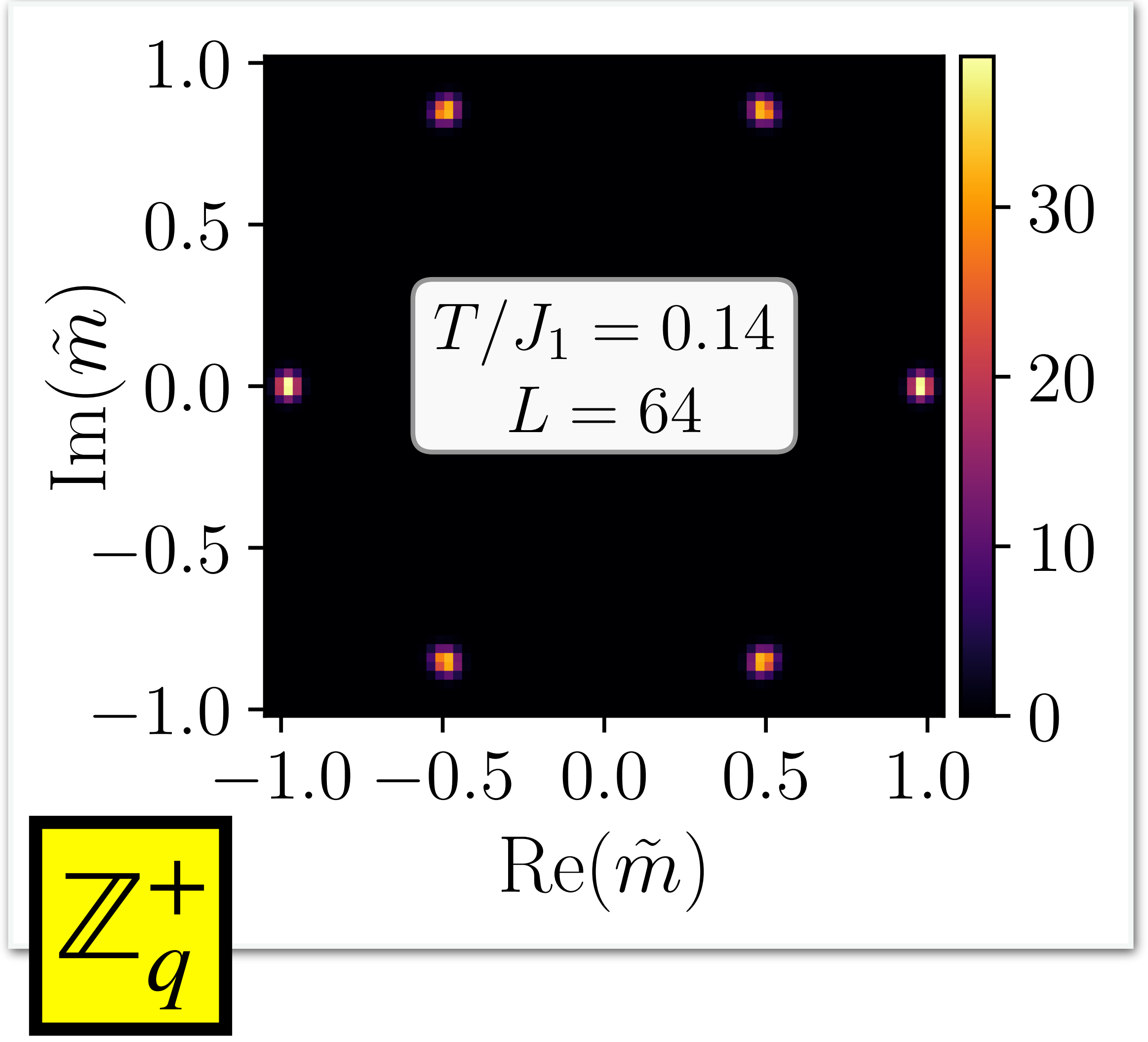
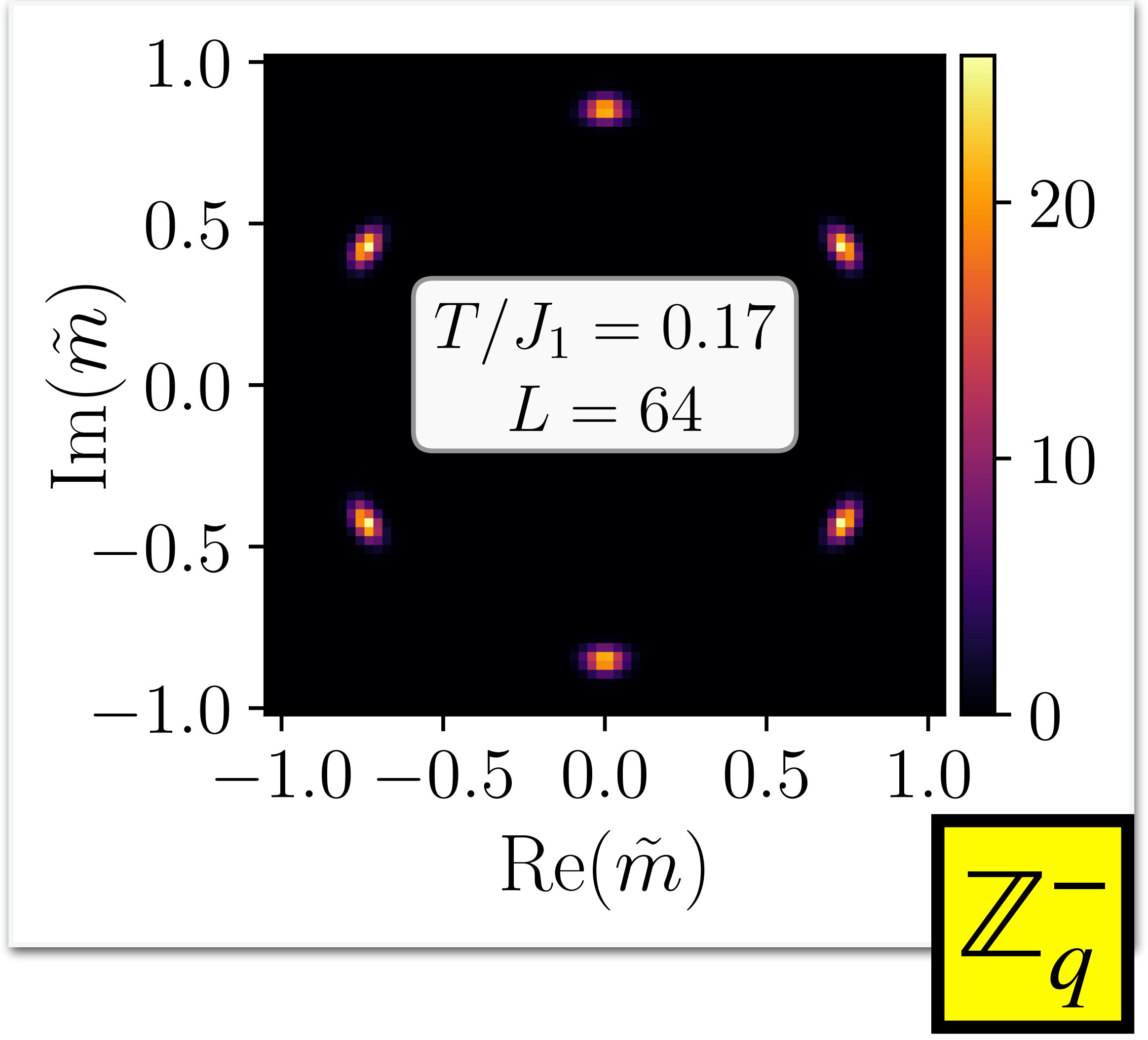


Transition ??



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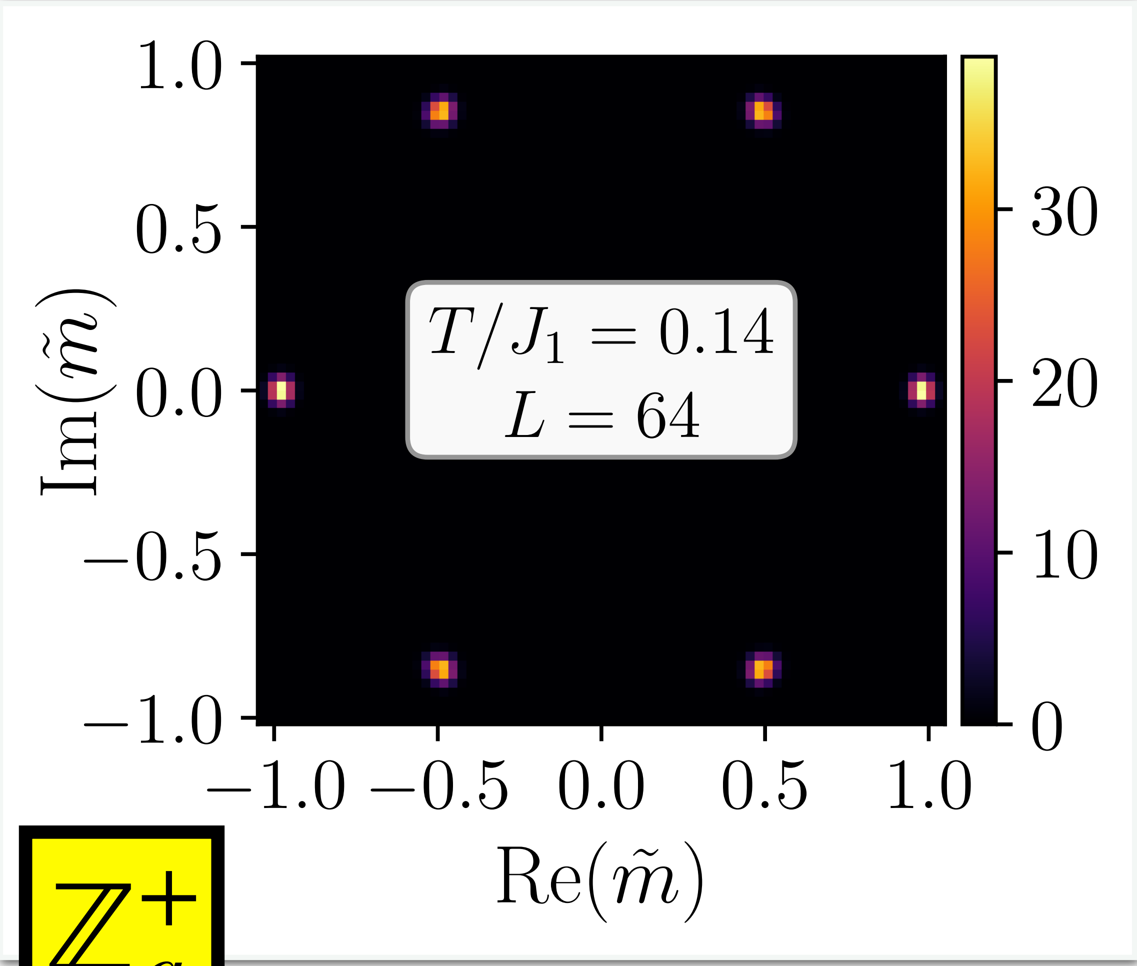
Condensed Matter > Strongly Correlated Electrons

[Submitted on 22 Jun 2023 (v1), last revised 6 Jul 2023 (this version, v2)]

Deconfined quantum critical points: a review

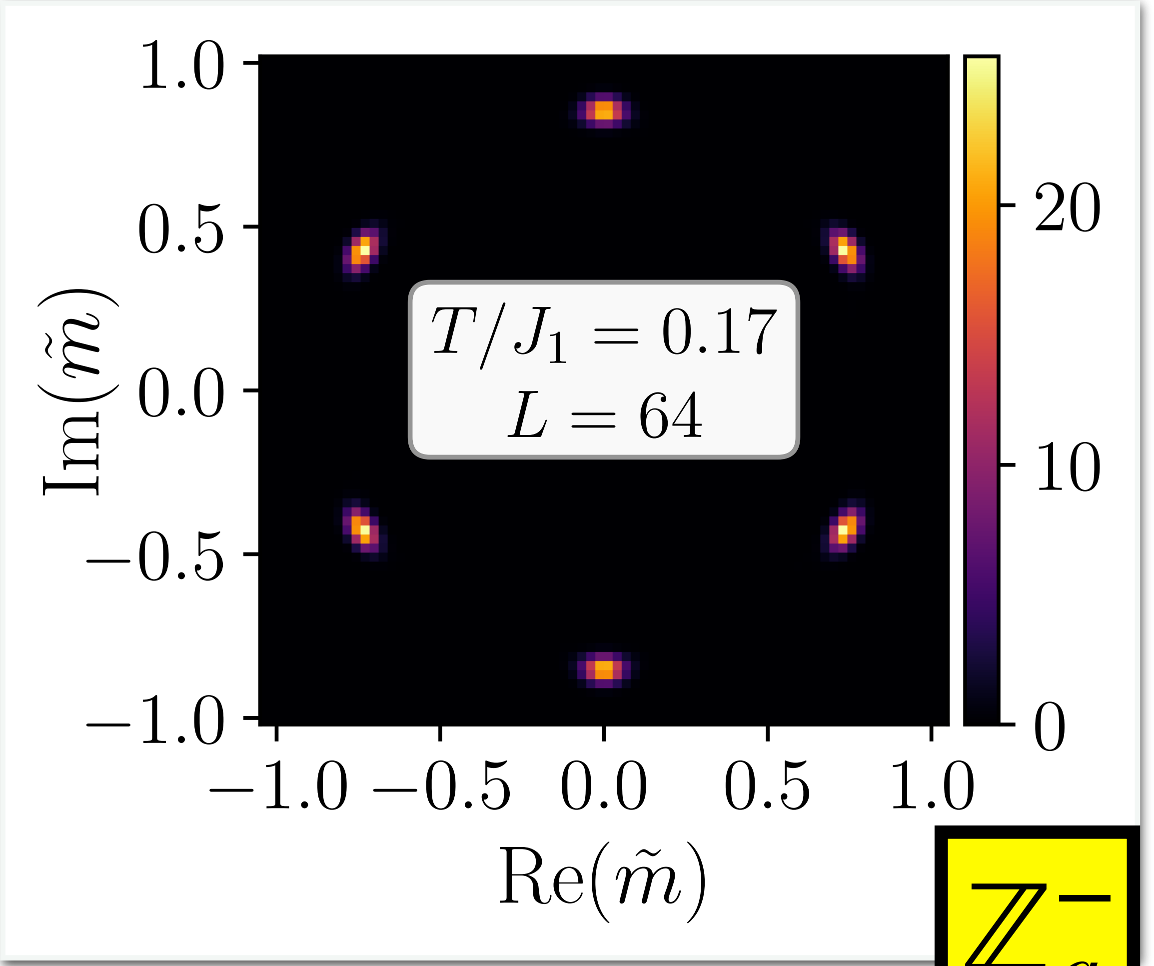
T. Senthil

J_1-J_2 Clock Model



$\mathbb{Z}_q^+$

Transition ??



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enlarged symmetry (will come back to this)… and…
topological defects of one order trap the charged degrees of freedom of the other, whose condensation drives the transition

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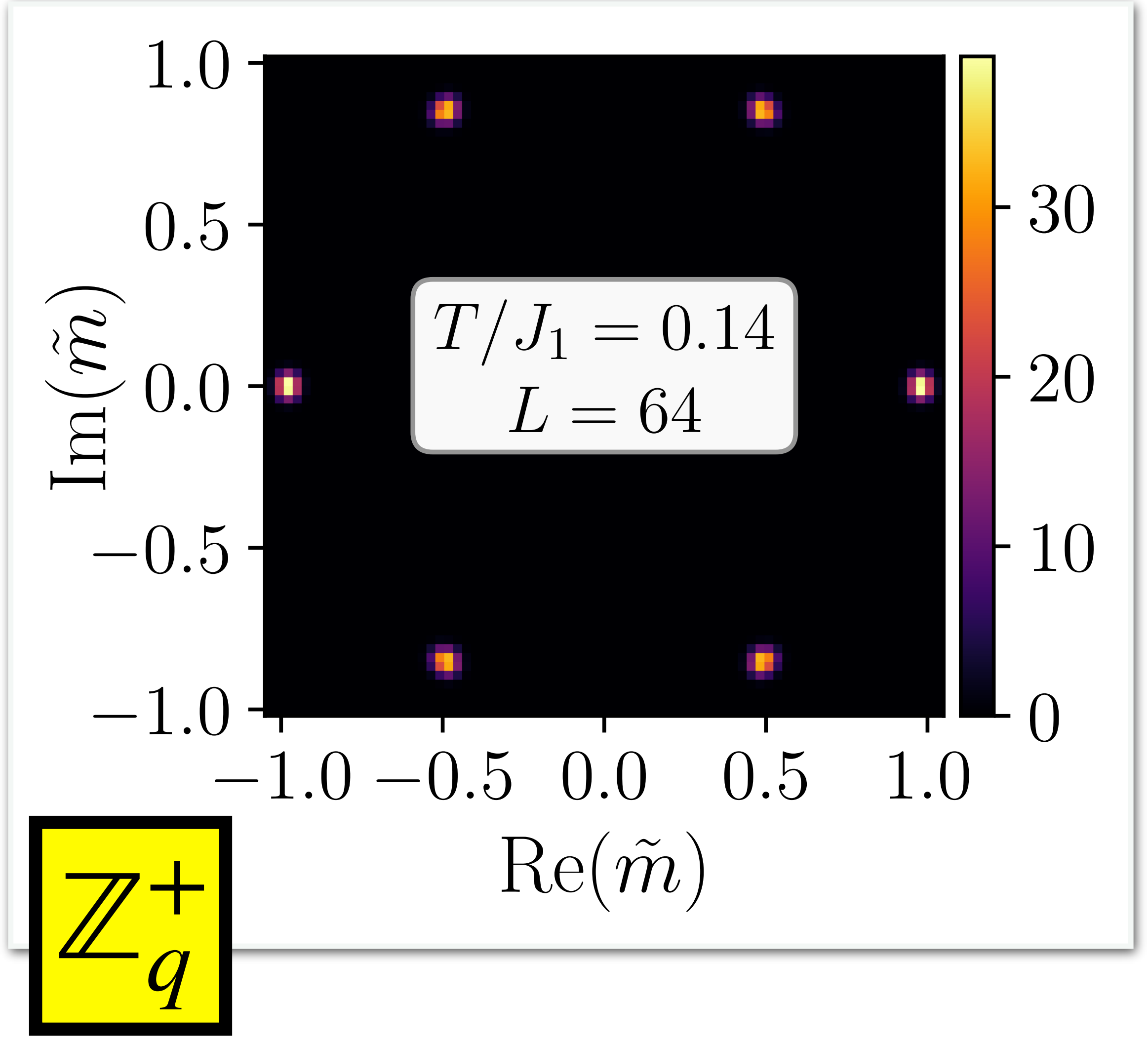
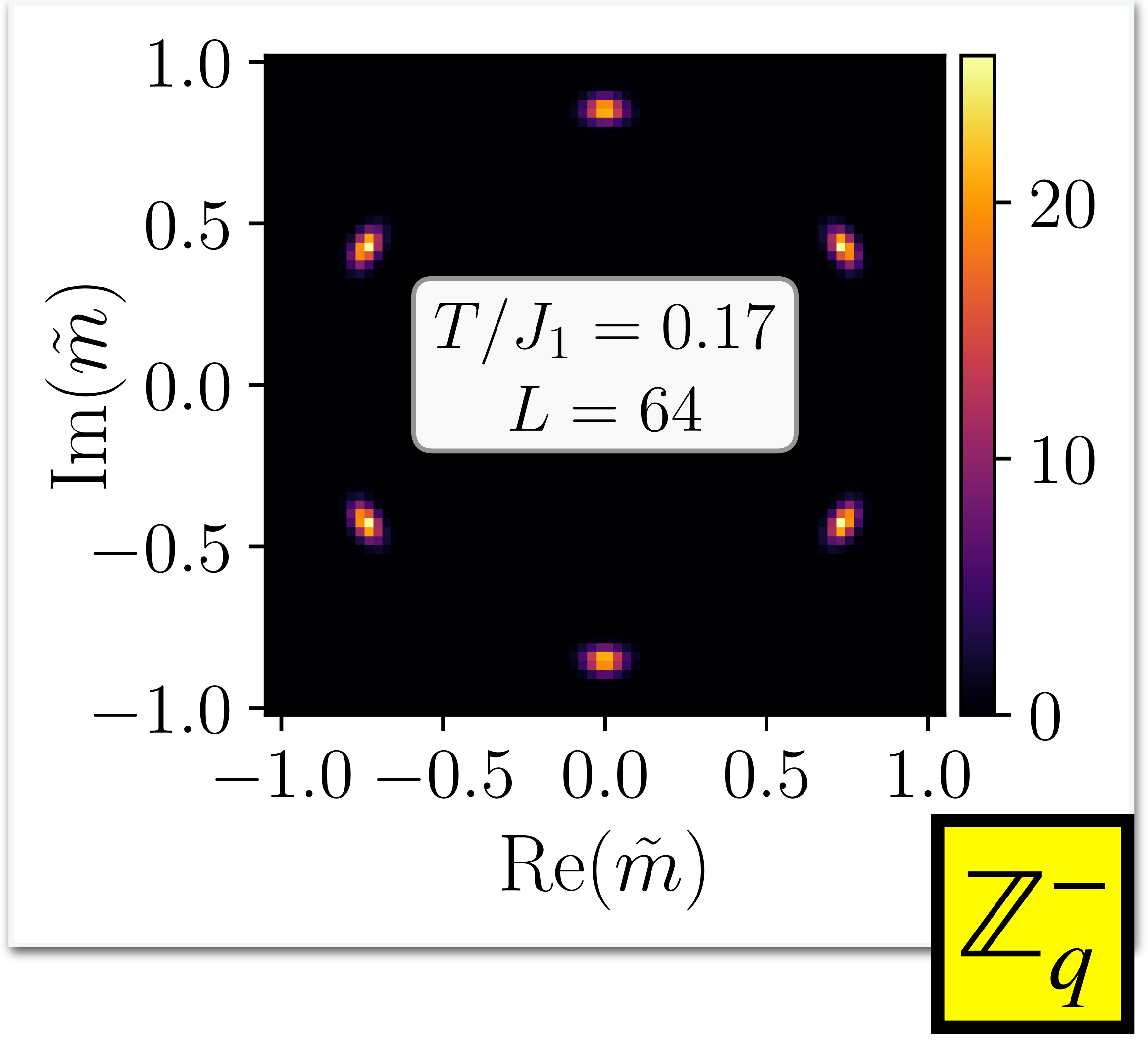


Transition ??



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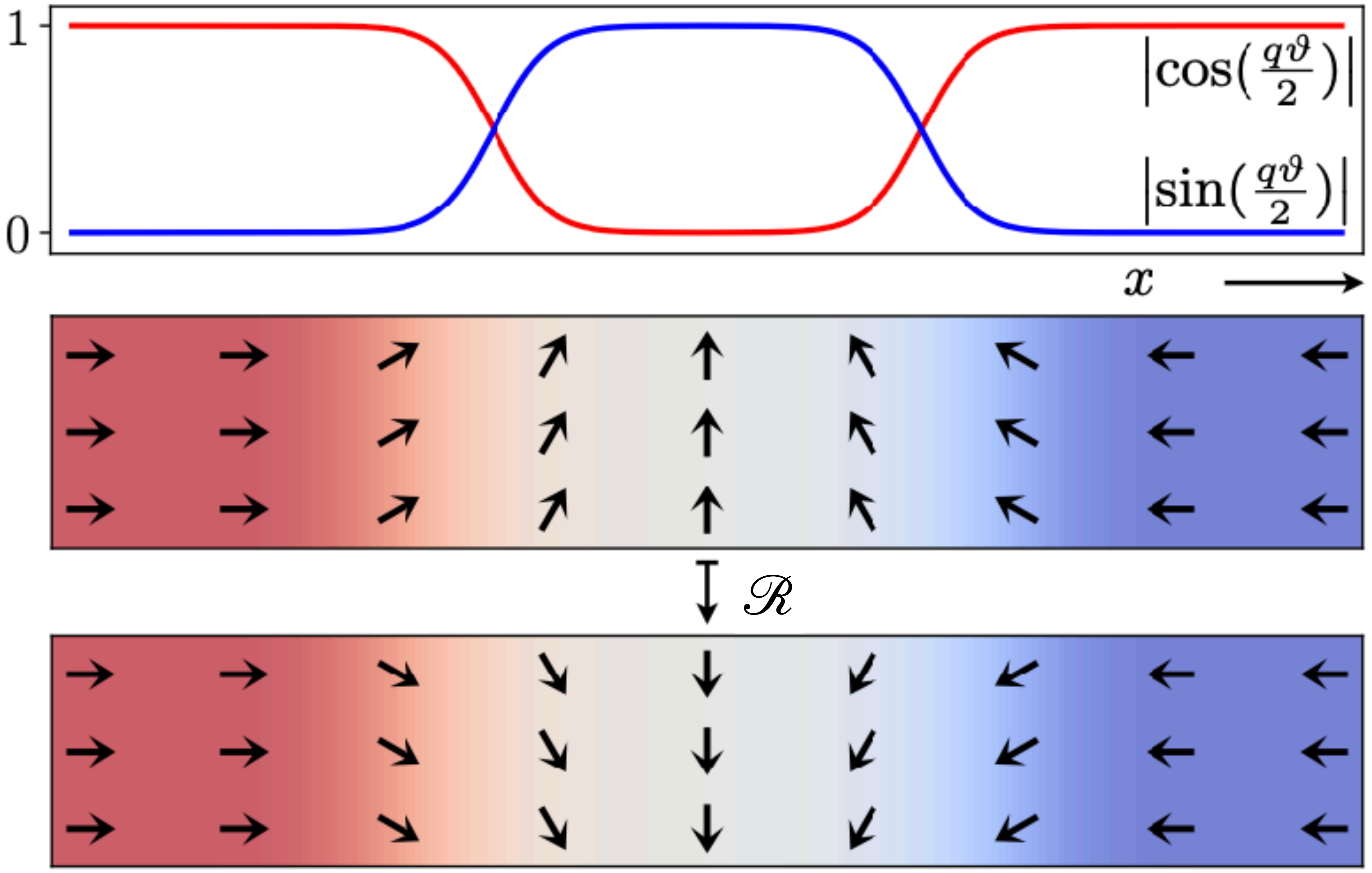

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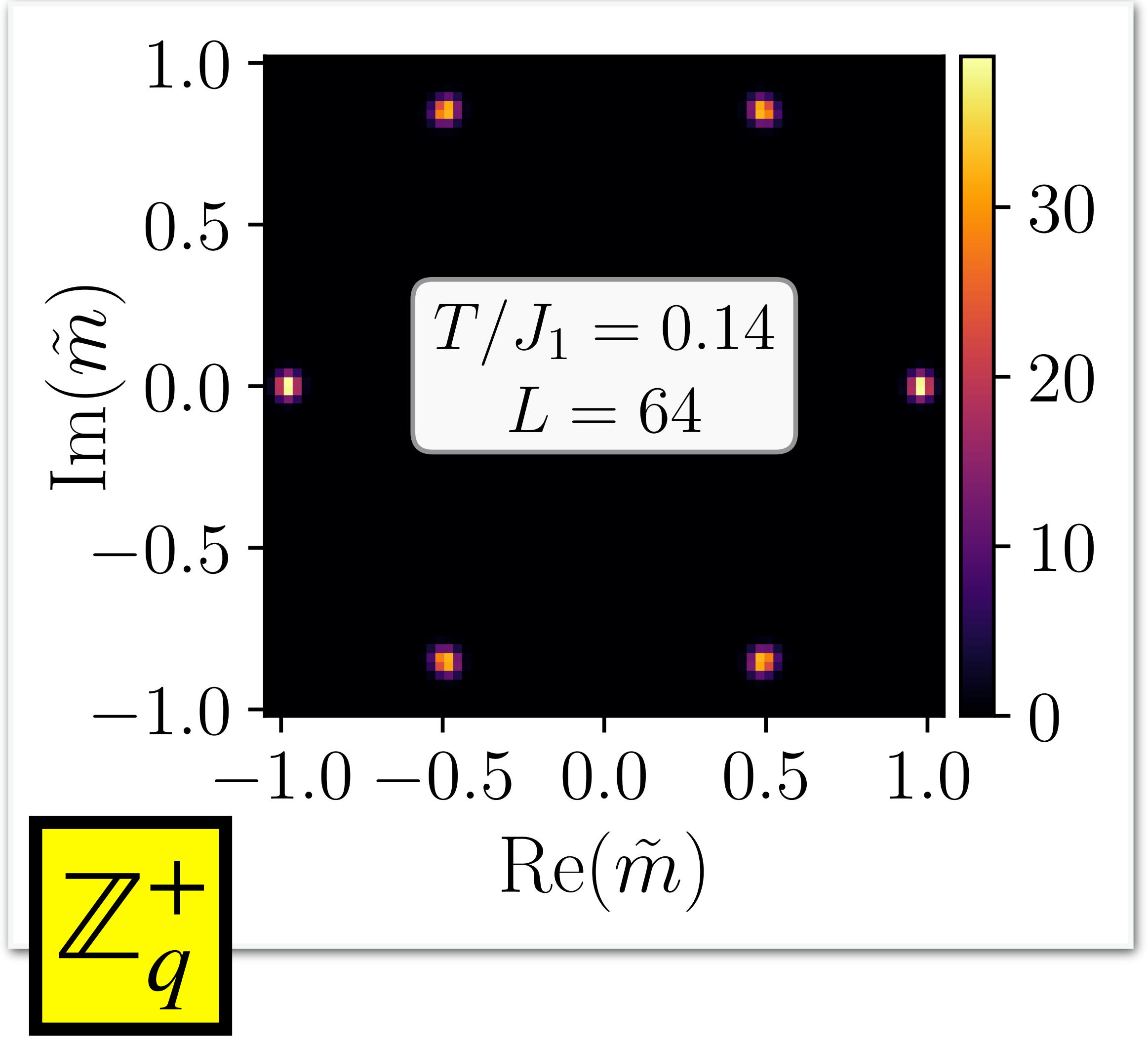
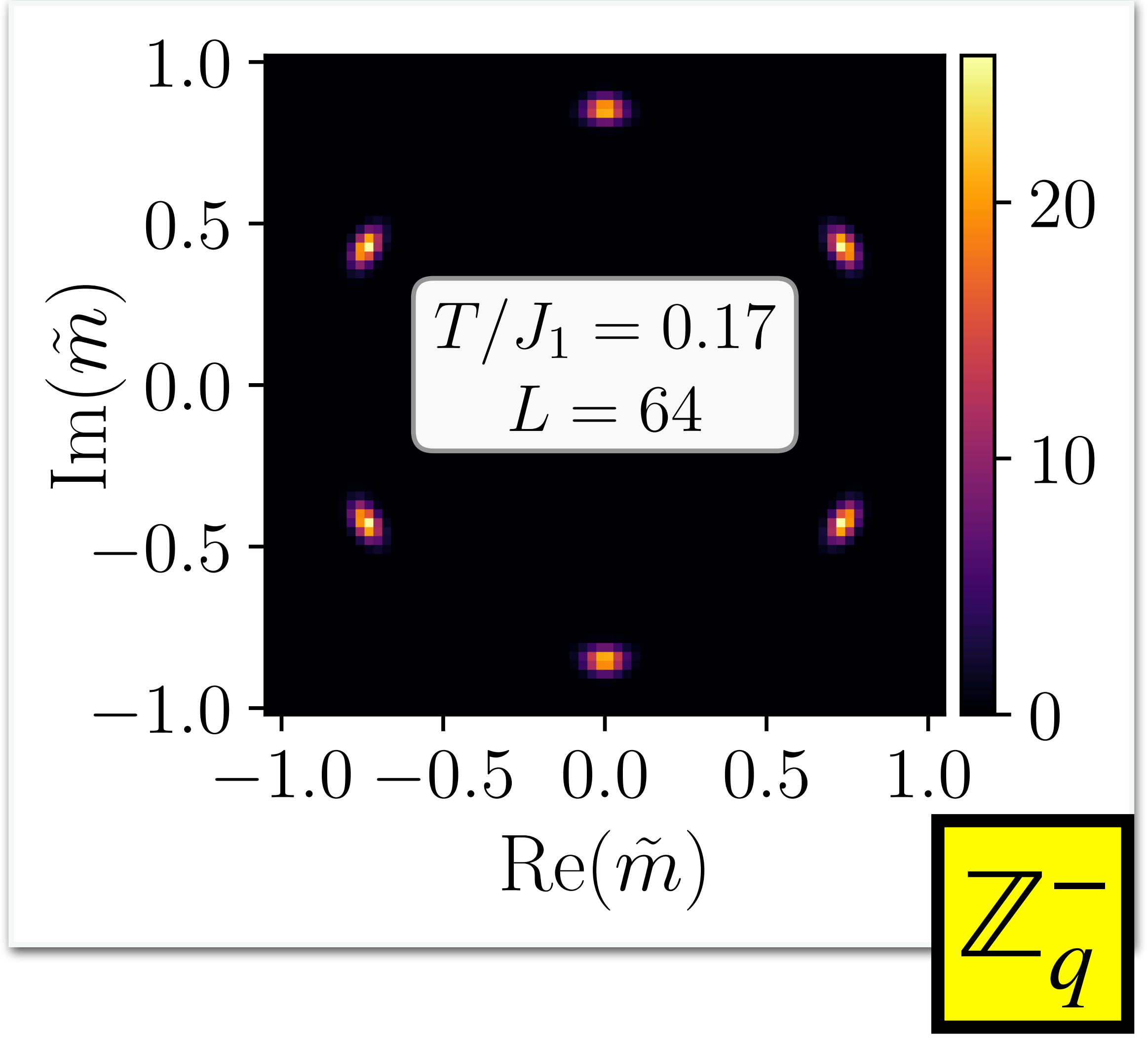


Transition ??



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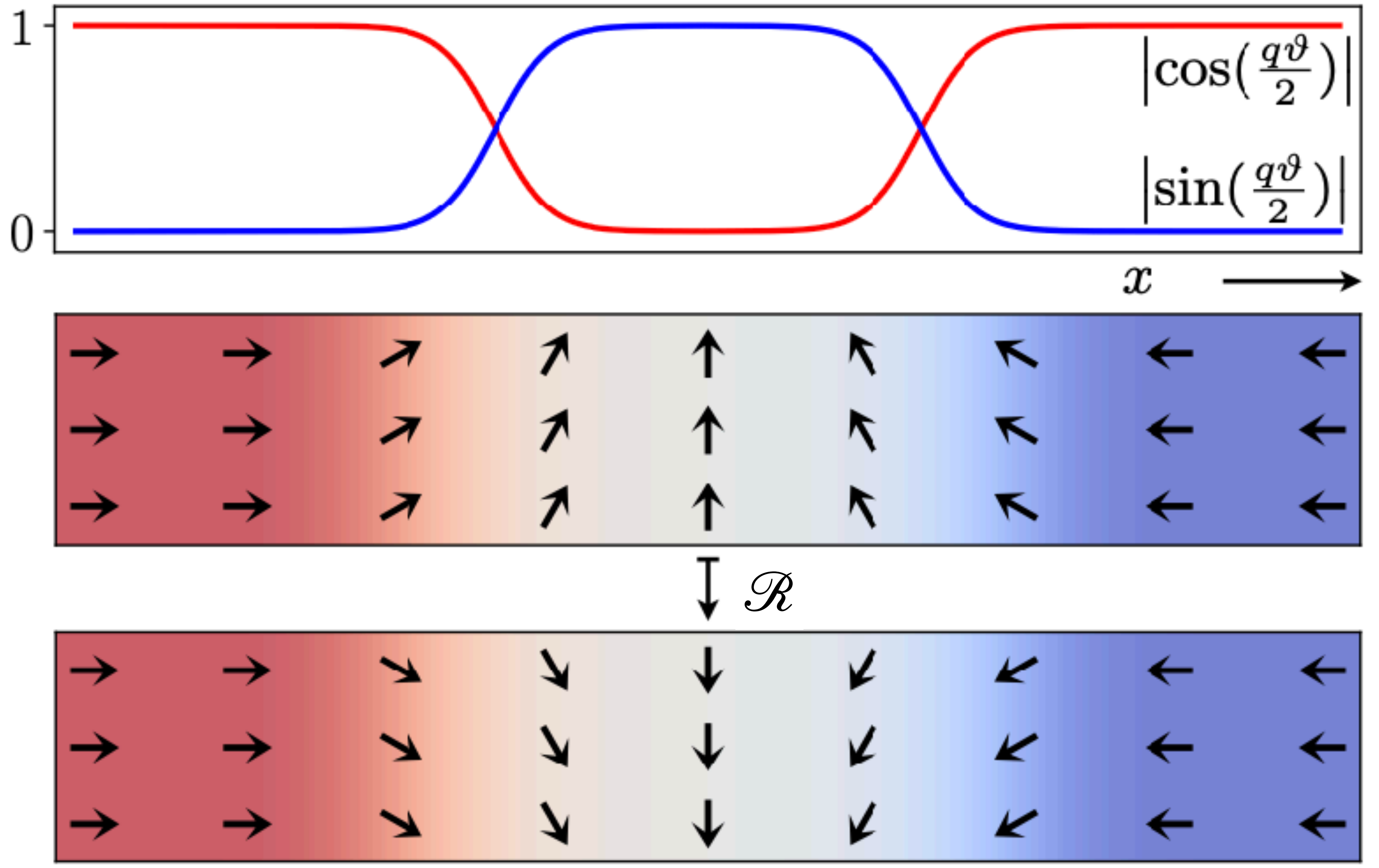

enlarged symmetry (will come back to this)… and…
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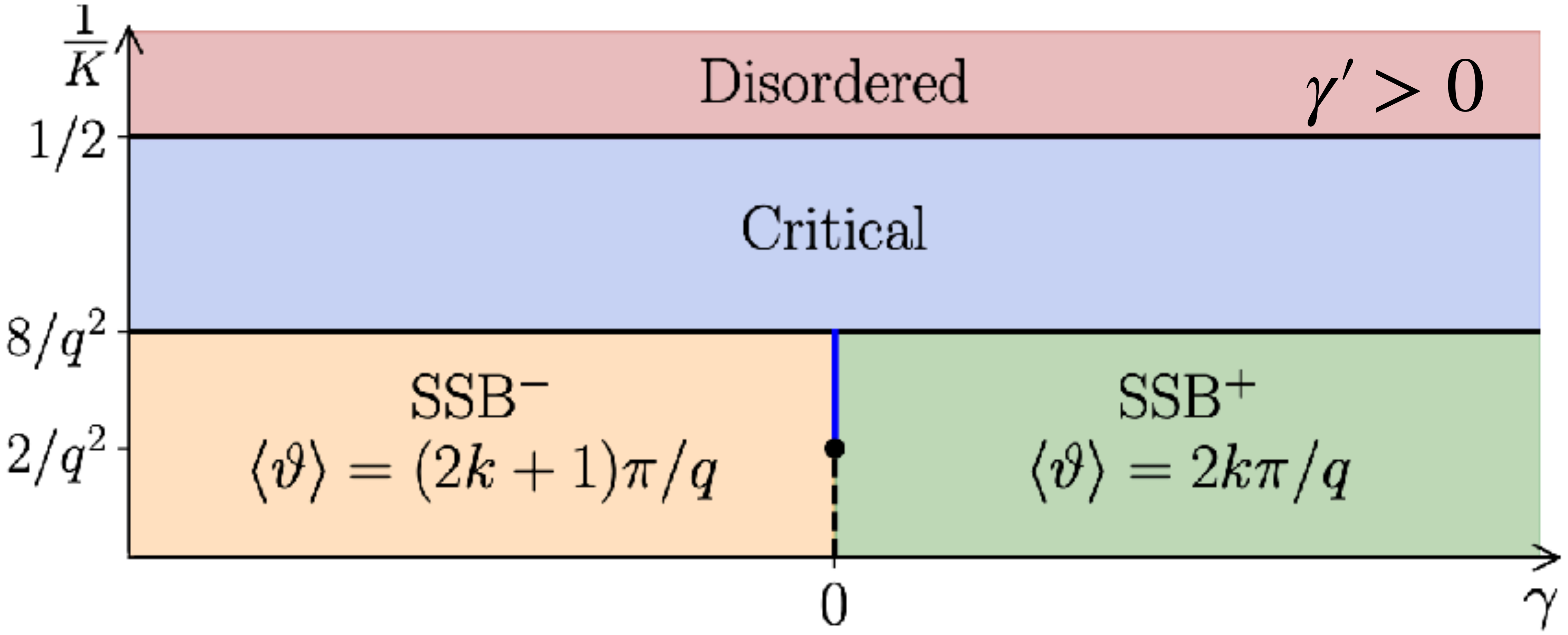
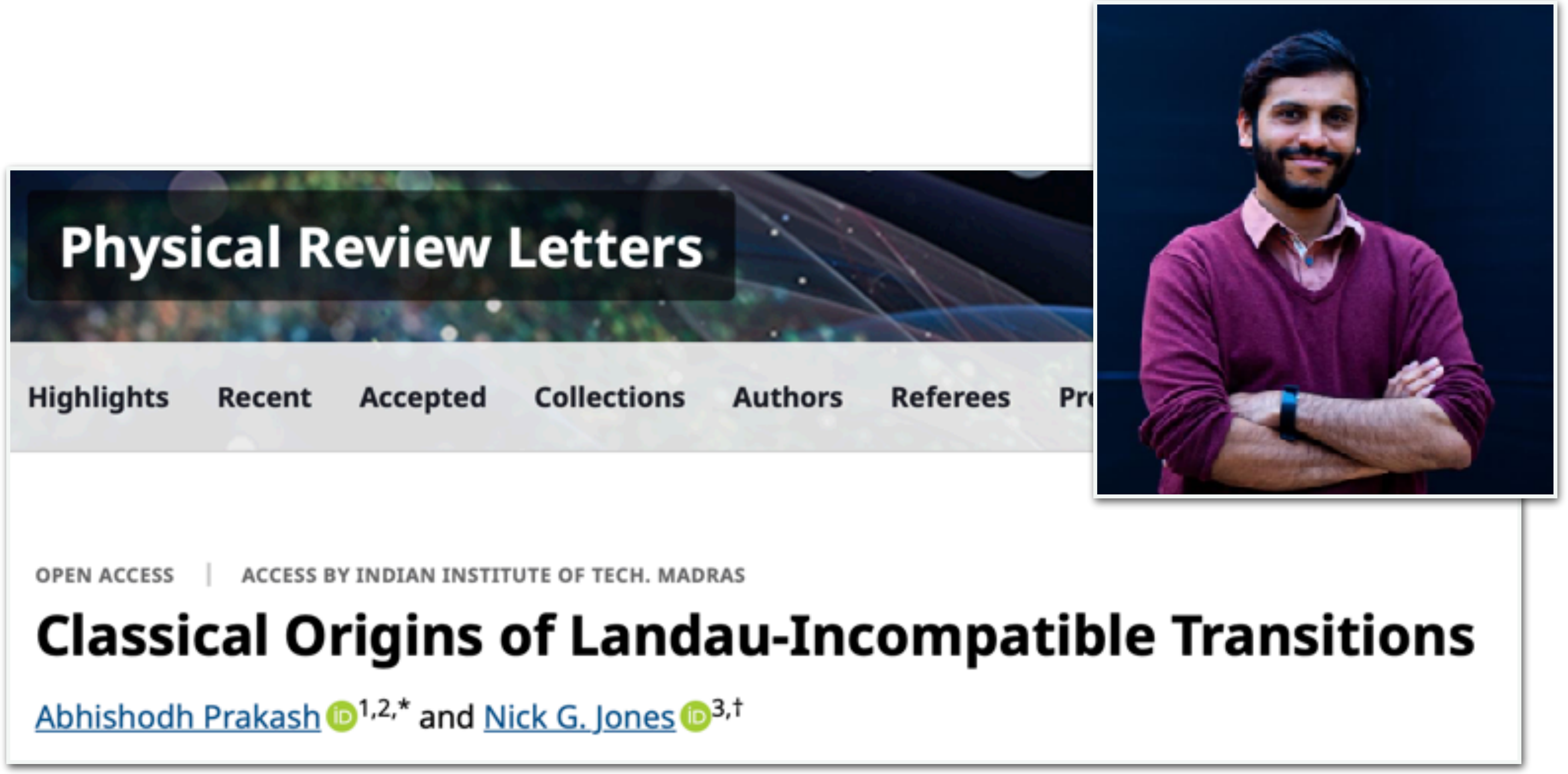
Do we also see this here ??
Let us first discuss this from field theoretic arguments

Phenomenological Field Theory

Postulated field theory *atop* the $\mathbb{Z}_2$ NM order

$$S \approx \frac{K}{2\pi}\int d^2x\,(\nabla\vartheta)^2 - h\int d^2x\cos(\phi) - \gamma\int d^2x\cos(q\vartheta) - \gamma'\int d^2x\cos(2q\vartheta) + \ldots,$$





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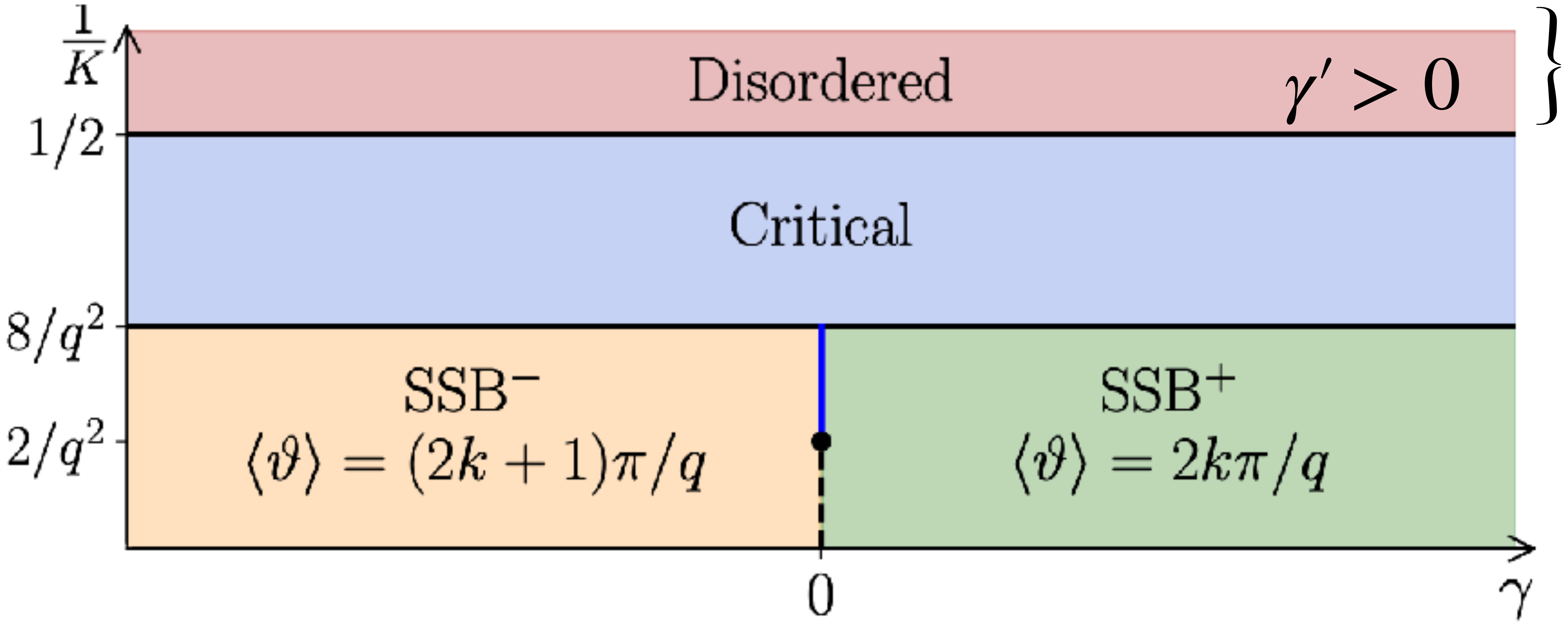
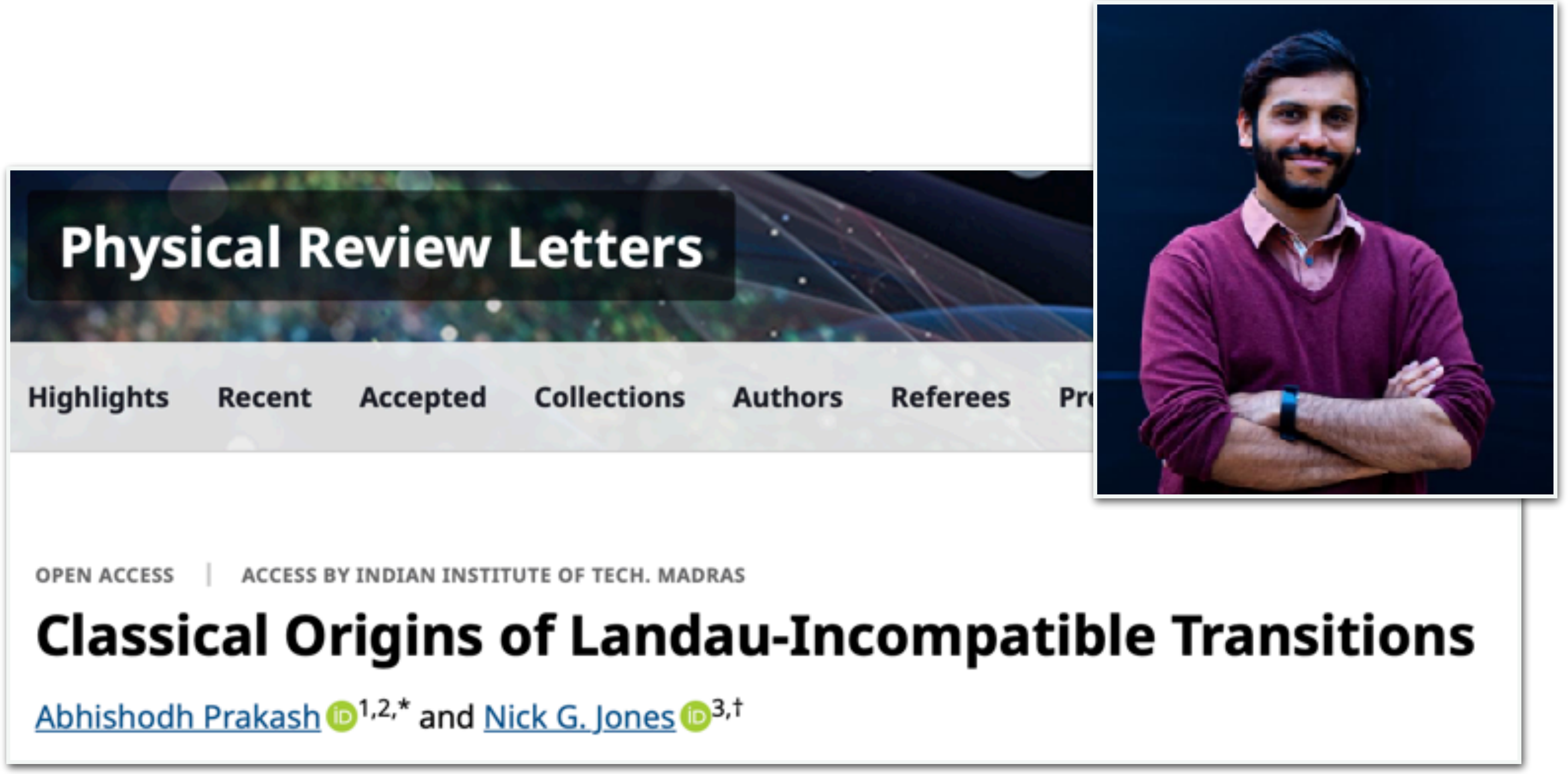
Relevant when $K < 2$

$$S \approx \frac{K}{2\pi}\int d^2x\,(\nabla\vartheta)^2 - h\int d^2x\cos(\phi) - \gamma\int d^2x\cos(q\vartheta) - \gamma'\int d^2x\cos(2q\vartheta) + \ldots,$$



$\}\ \cos(\phi)$



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Classical Origins of Landau-Incompatible Transitions

Abhishodh Prakash[1,2,*] and Nick G. Jones[3,†]

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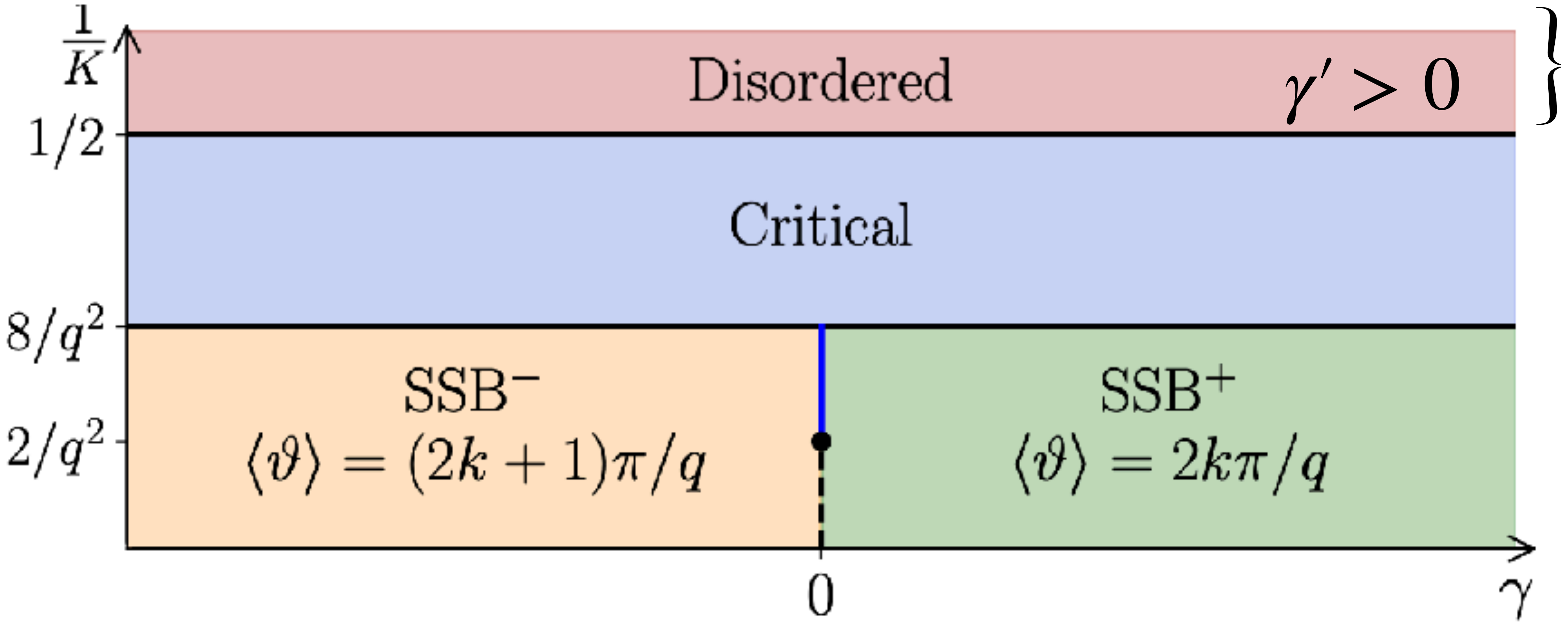
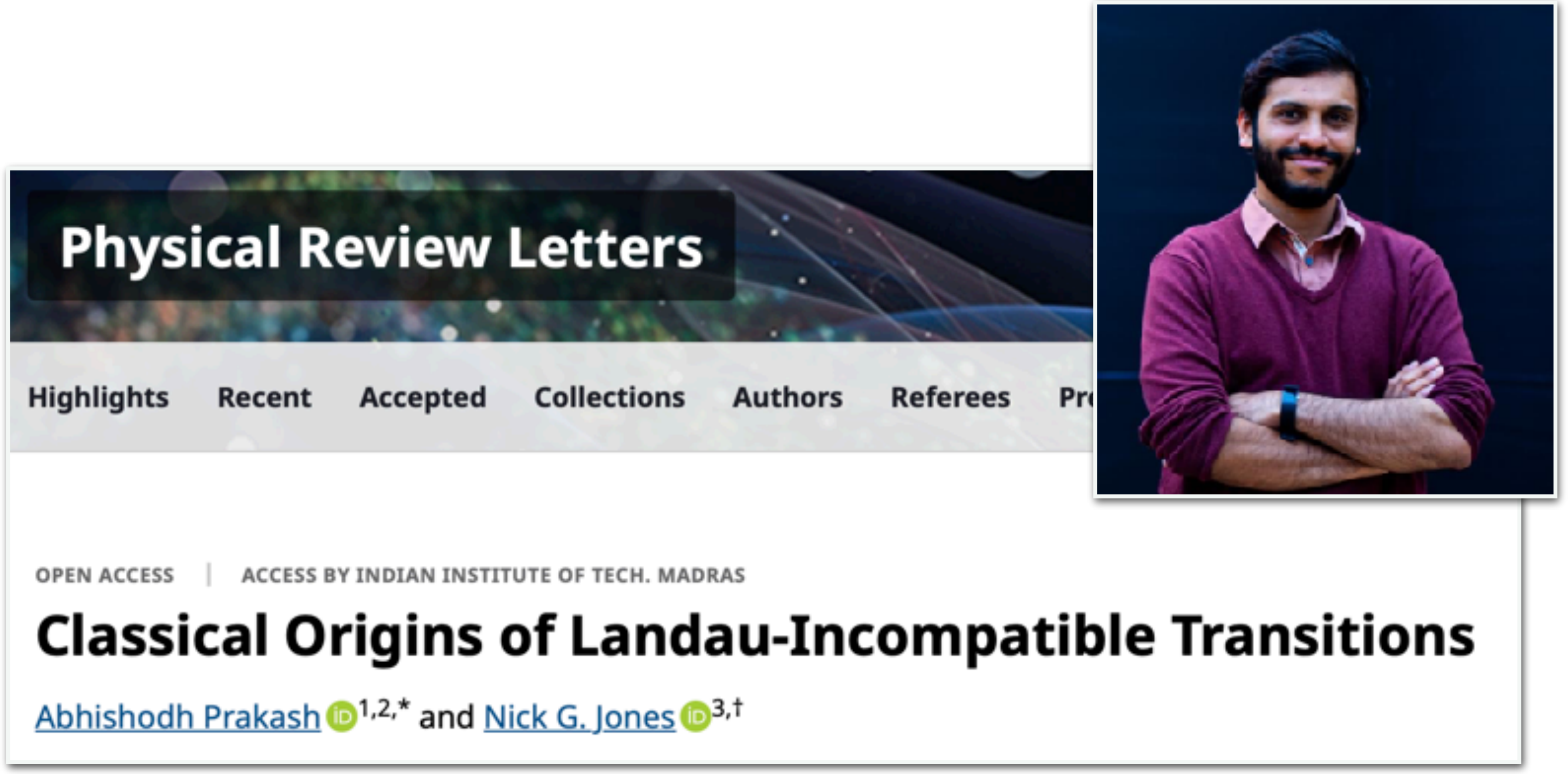
Relevant when $K > q^2/2$



$\cos(\phi)$

$\cos(q\vartheta)$



Responsible for emergent $\mathbb{Z}_q$ symmetry and order

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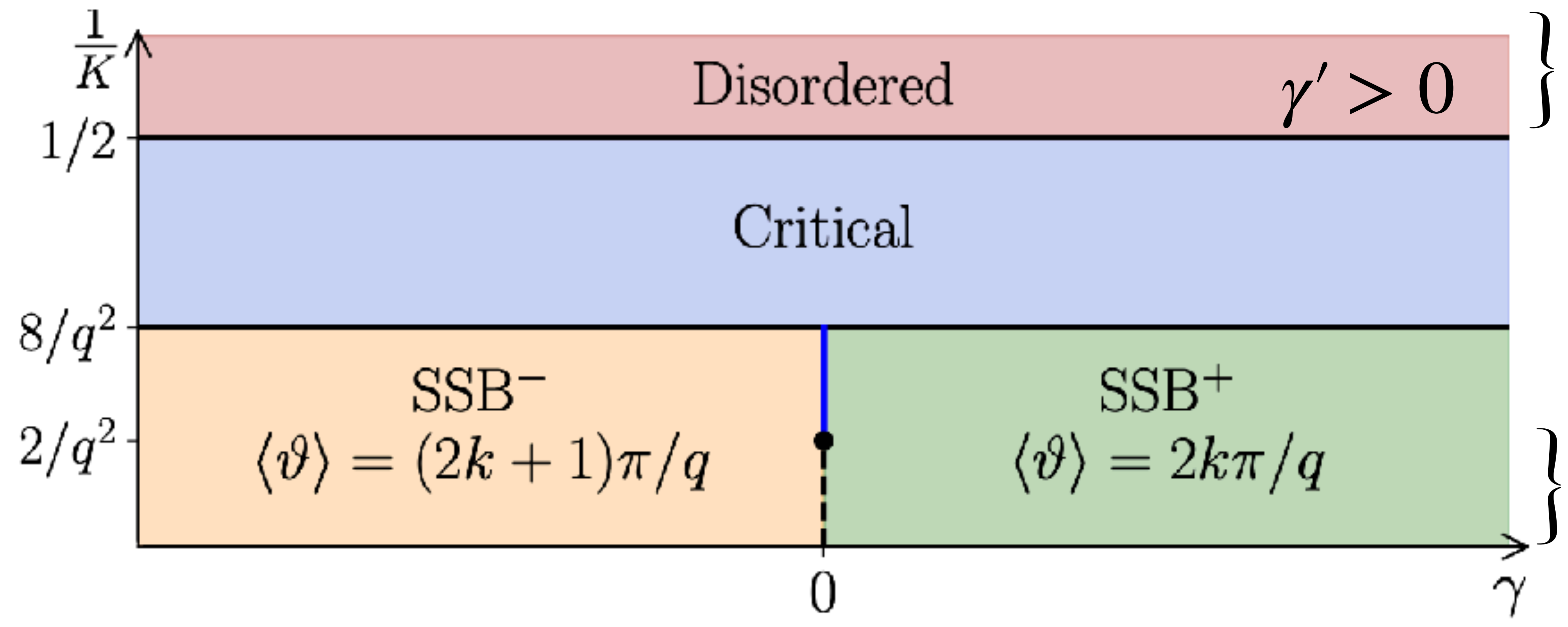
Postulated field theory *atop* the $\mathbb{Z}_2$ NM order

Relevant when $K < 2$

$$S \approx \frac{K}{2\pi}\int d^2x\,(\nabla\vartheta)^2 - h\int d^2x\cos(\phi)$$
$$-\gamma\int d^2x\cos(q\vartheta) - \gamma'\int d^2x\cos(2q\vartheta) + \ldots,$$

Relevant when $K > q^2/2$ Relevant when $K > q^2/8$



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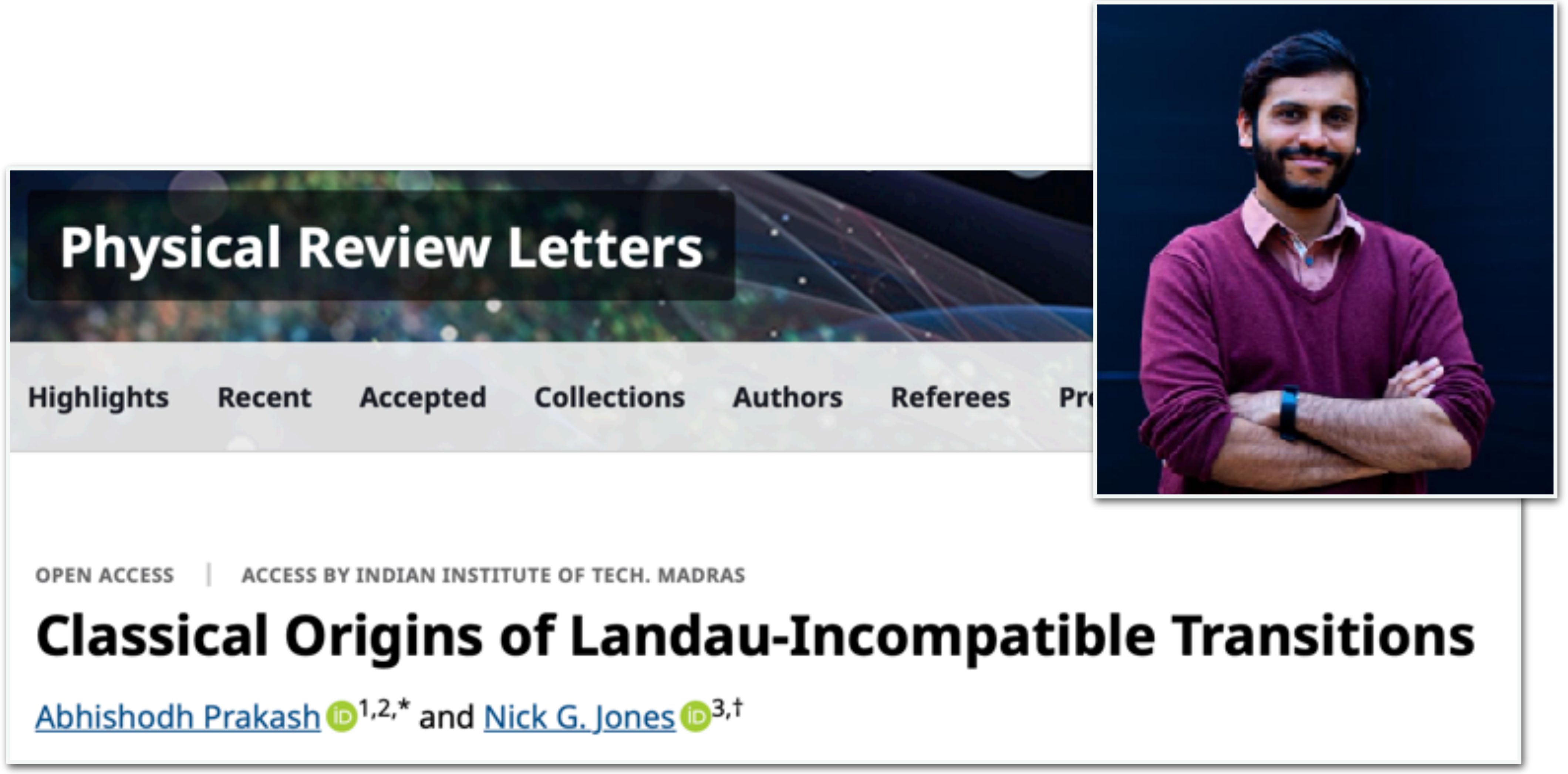
$\cos(2q\vartheta)$

$\cos(q\vartheta)$



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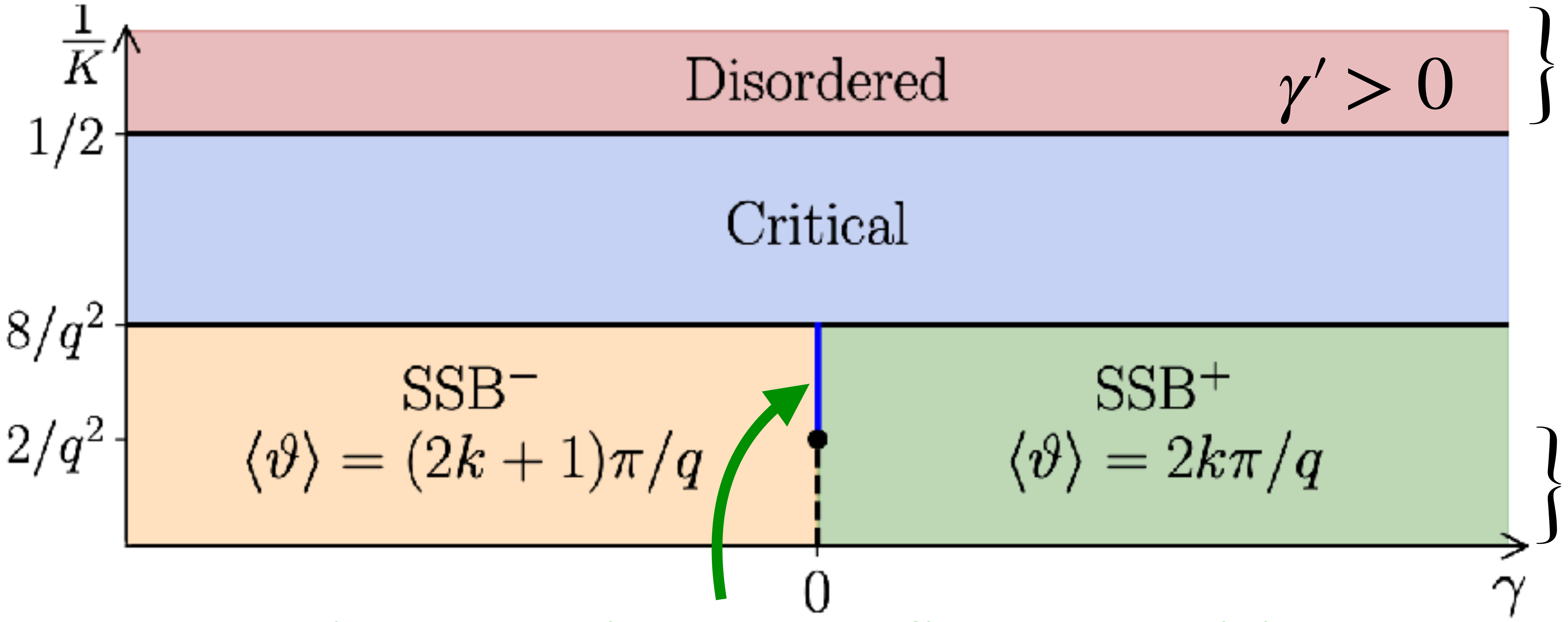

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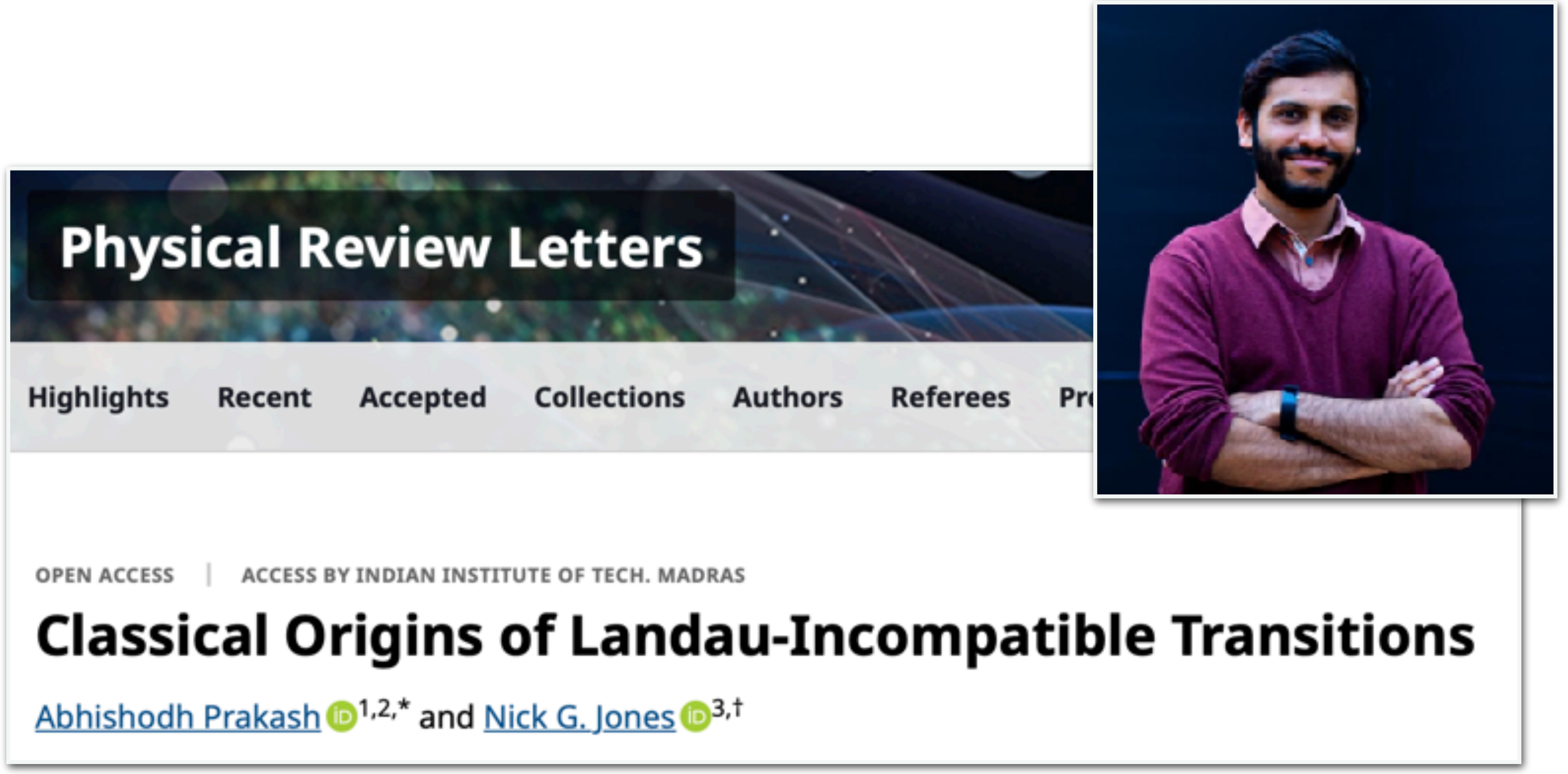
$\cos(q\vartheta)$

$\cos(2q\vartheta)$

Landau-incompatible deconfined transition

A critical **O(2) symmetric** Gaussian field theory
continuous transition as γ changes sign



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Classical Origins of Landau-Incompatible Transitions

Abhishodh Prakash[1,2,*] and Nick G. Jones[3,†]

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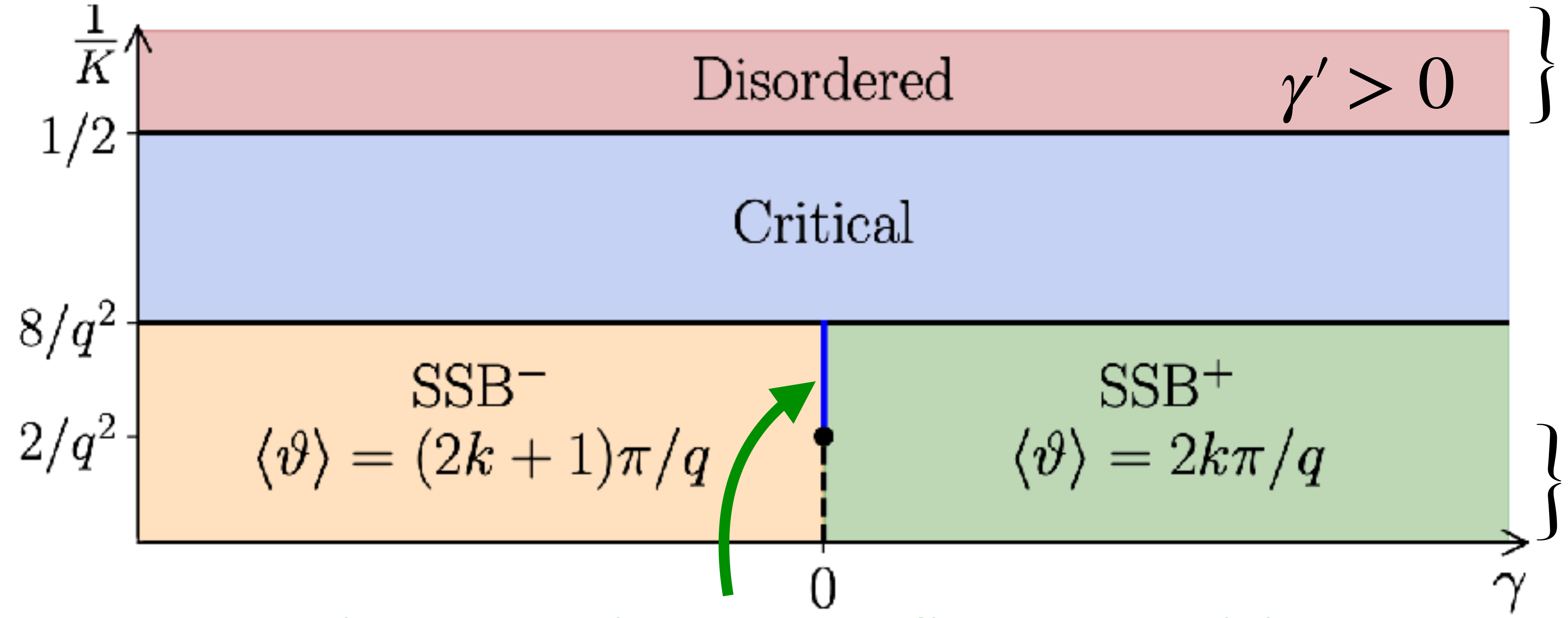
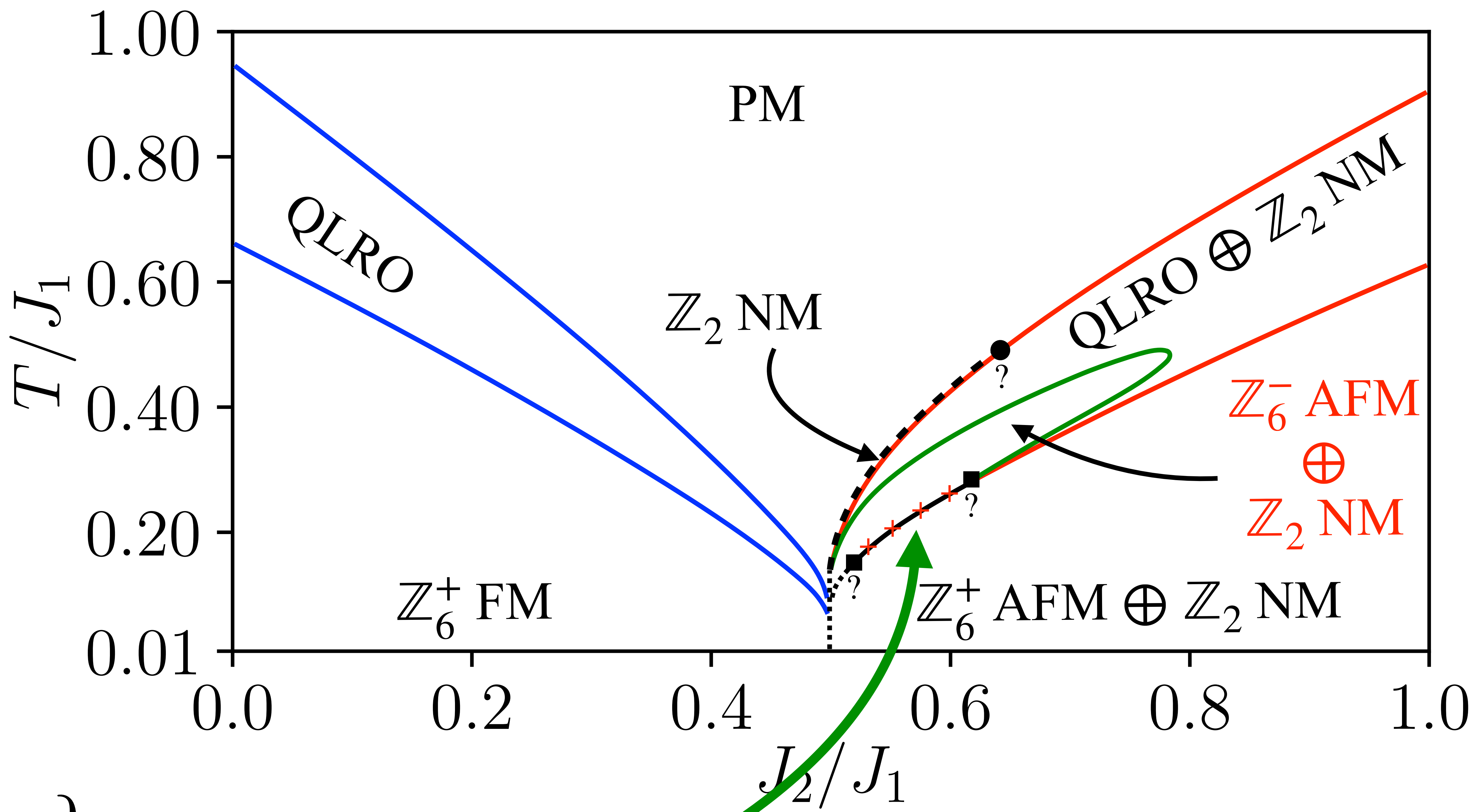


$\cos(\phi)$

$\cos(2q\vartheta)$

$\cos(q\vartheta)$



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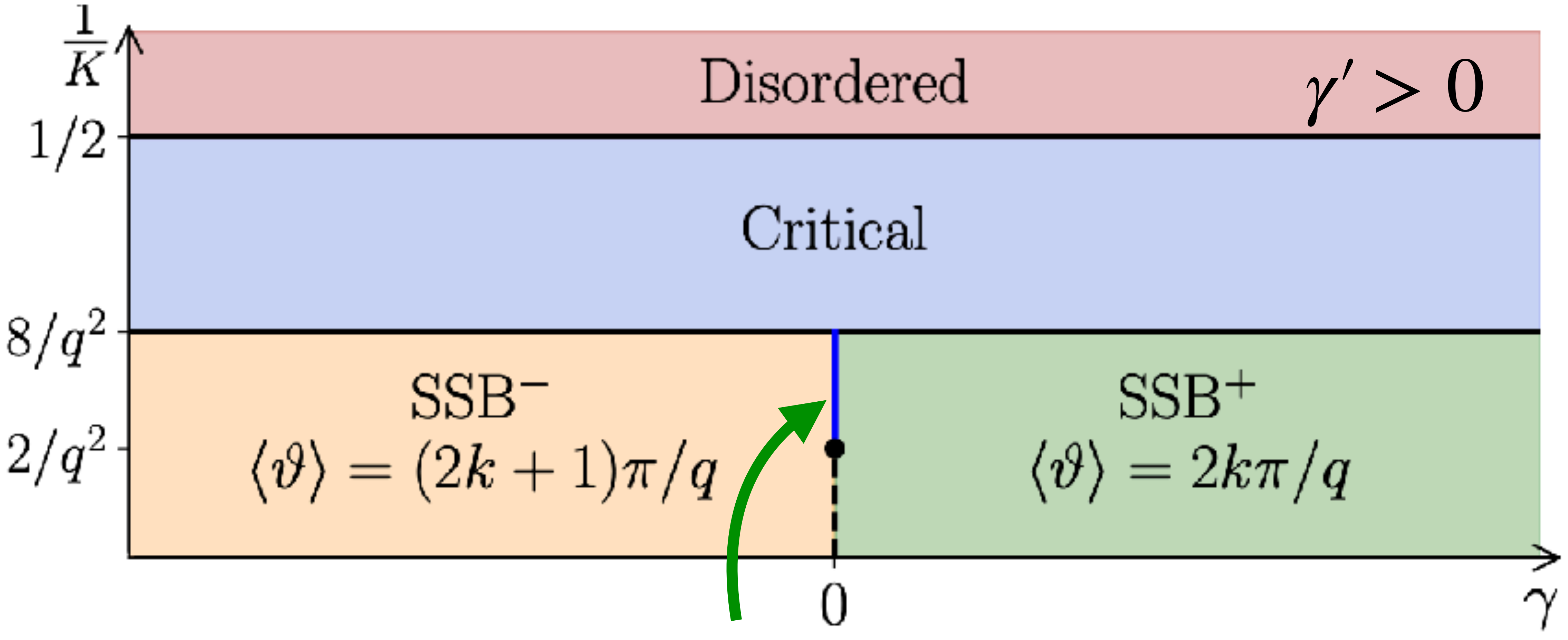
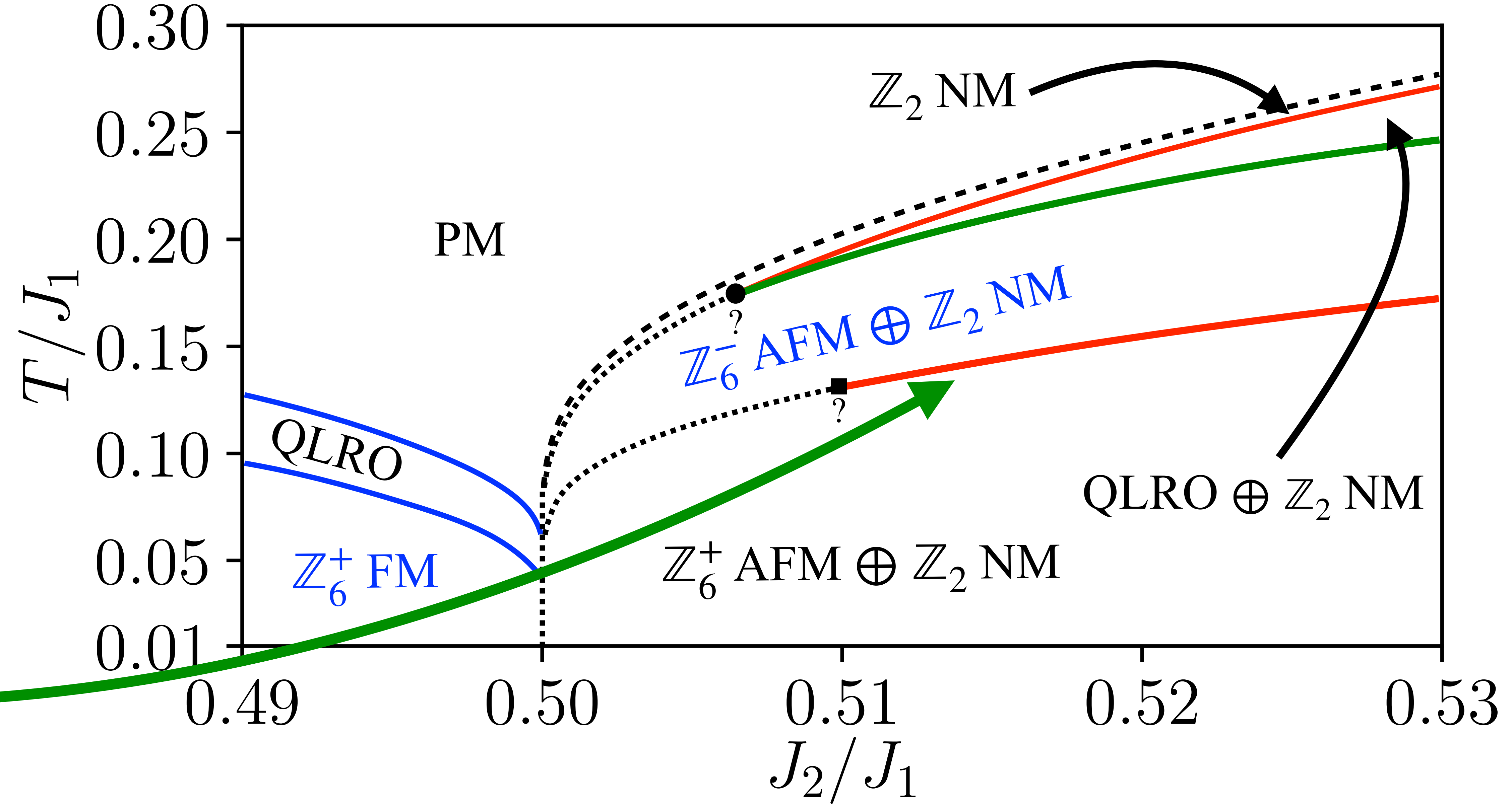


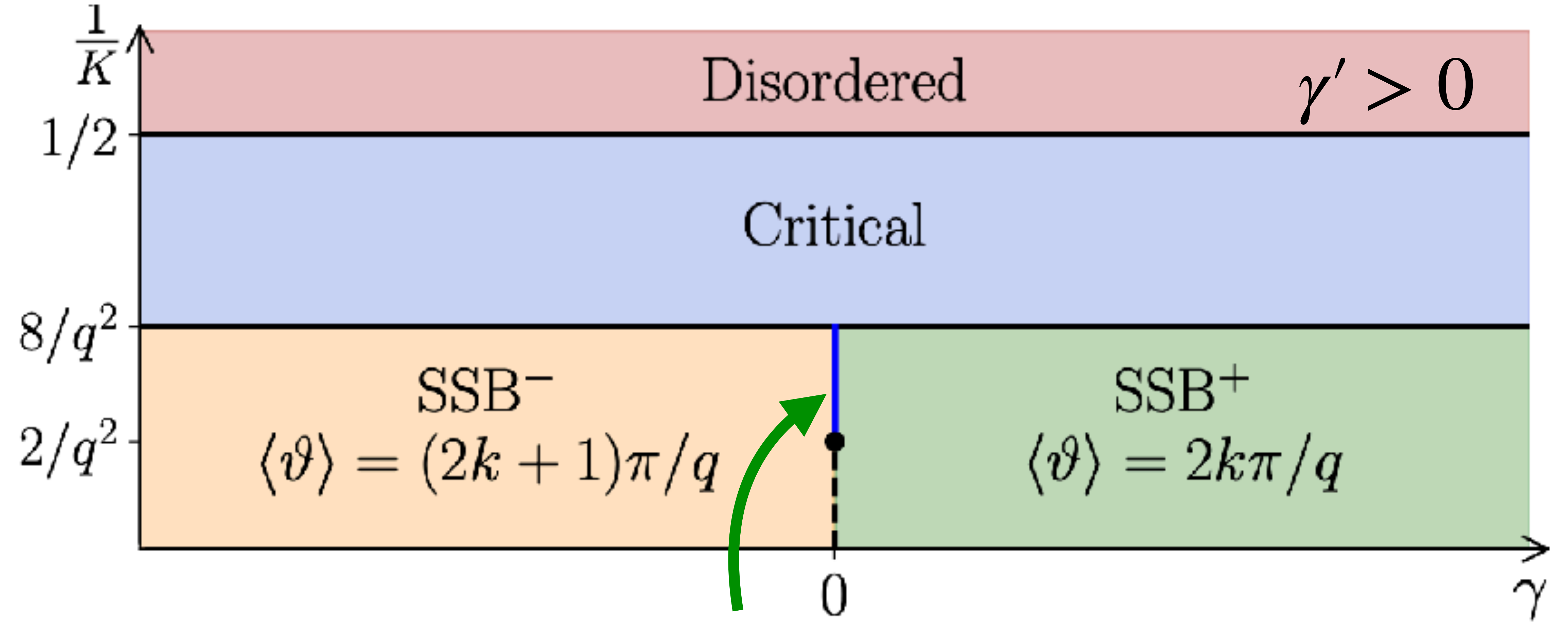


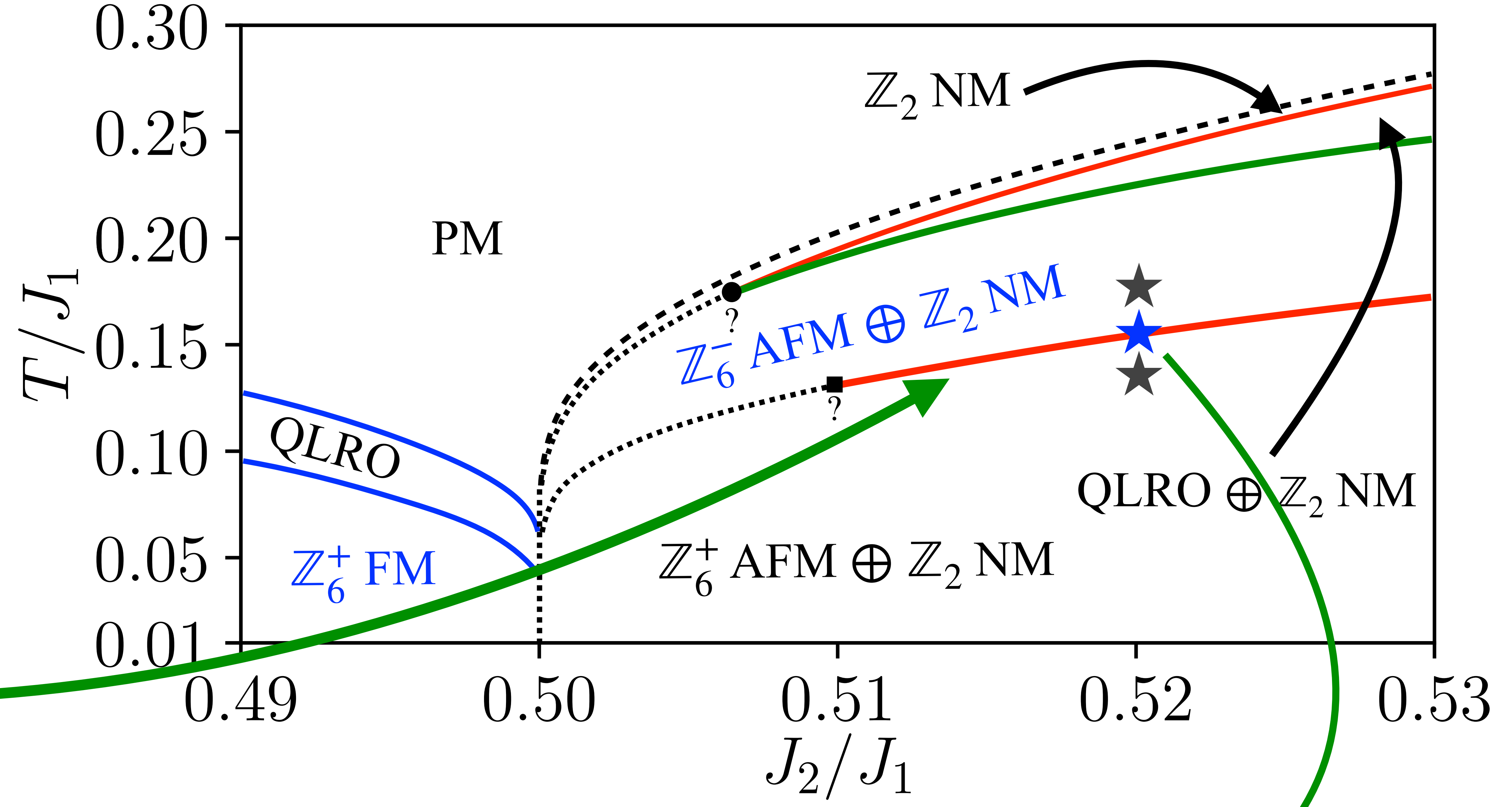
Landau-incompatible deconfined transition

Phenomenological Field Theory

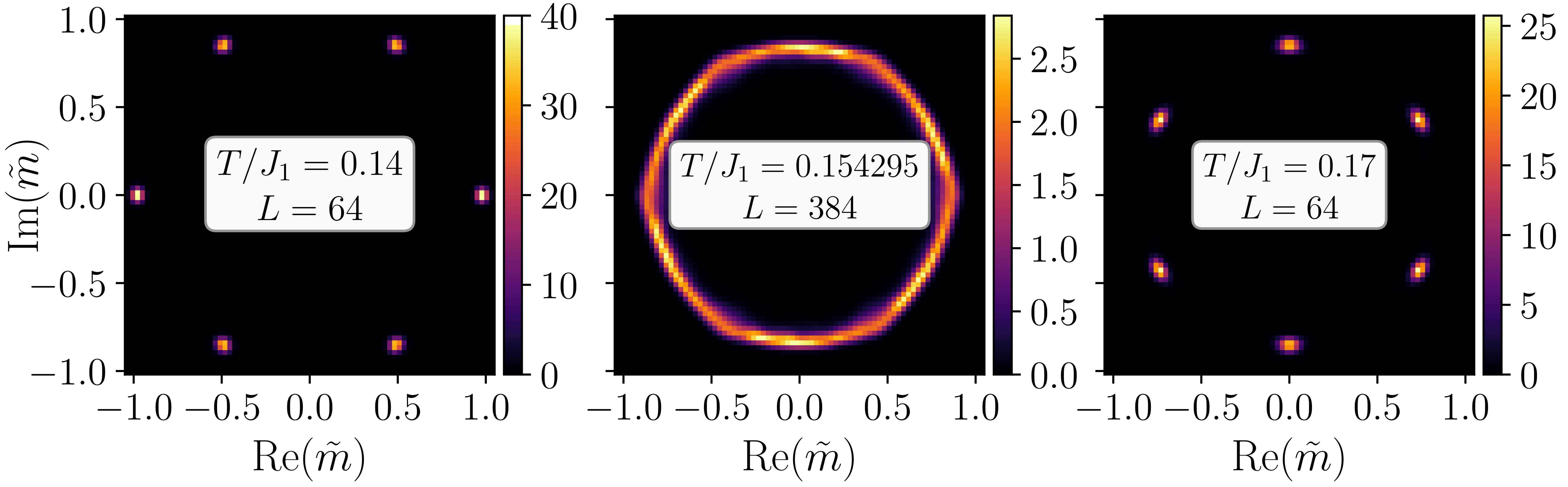
Calabrese-Cardy scaling: $\mathcal{S} = \dfrac{c}{6}\log\xi + b'$

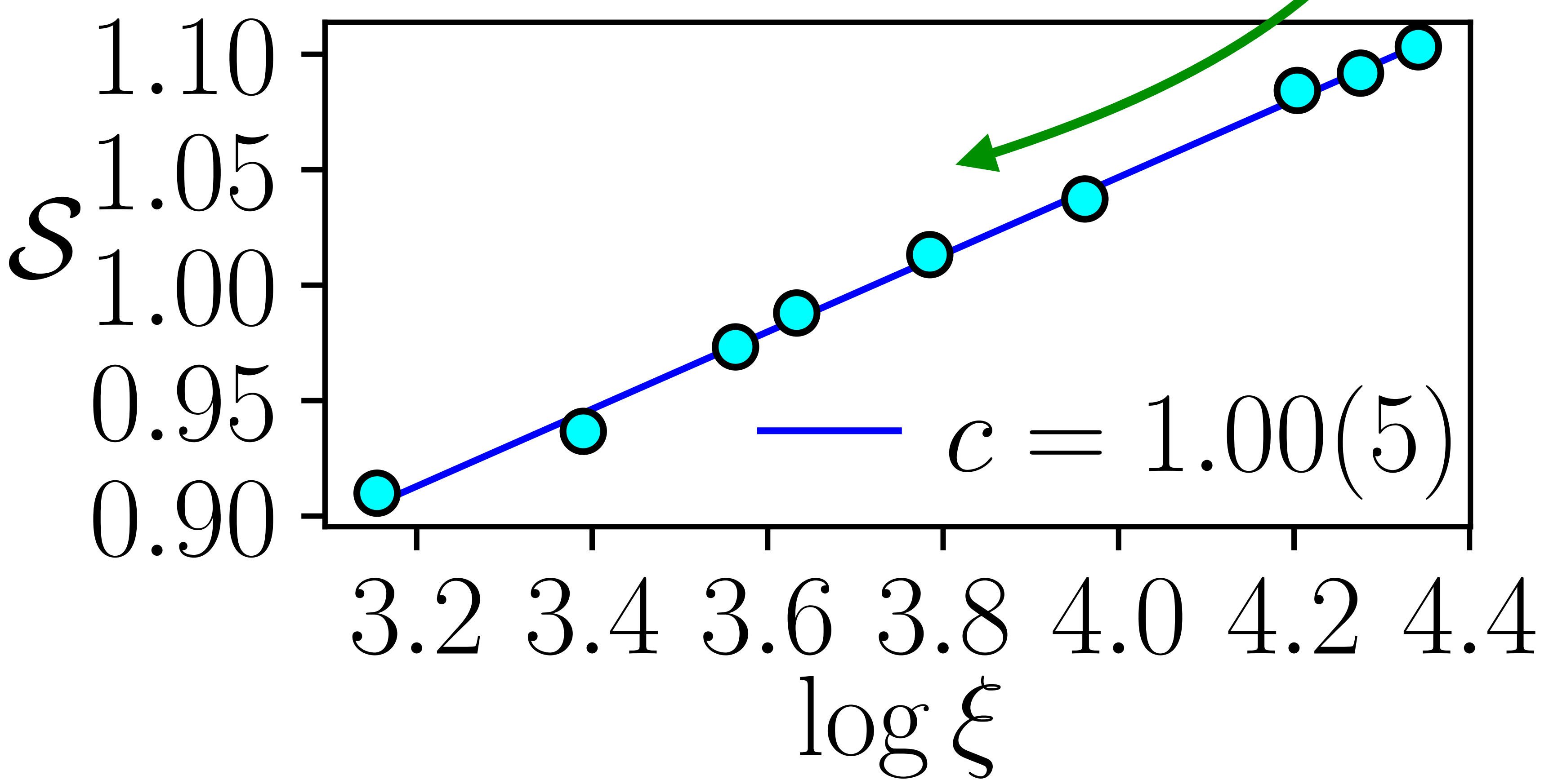
Phenomenological Field Theory





Landau-incompatible deconfined transition





Emergent O(2) symmetry at the deconfined criticality

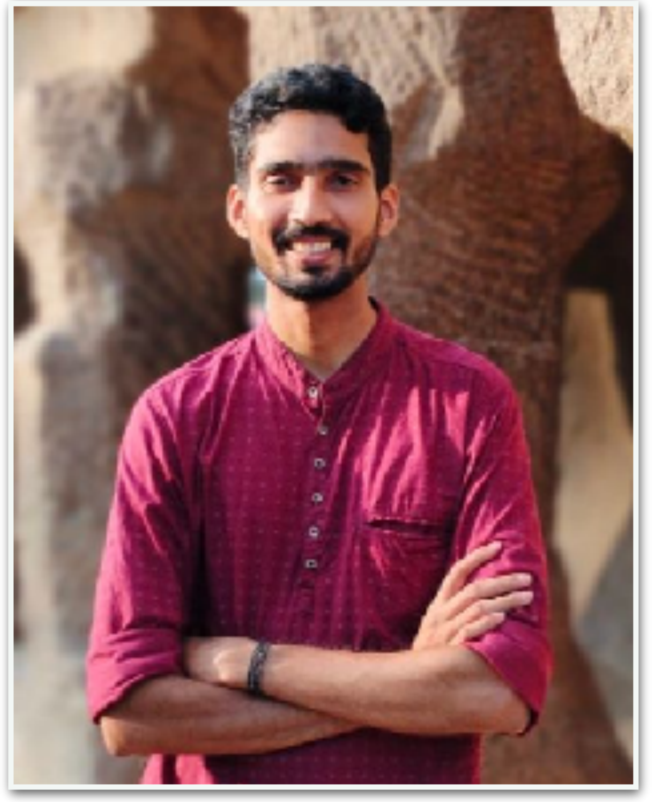
Calabrese-Cardy scaling: $\mathcal{S} = \dfrac{c}{6}\log\xi + b'$

Summary

1. **For the first time,** we map out the phase diagram of 2D frustrated classical J_1-J_2 clock model.

2. We show **a unique form of emergence**, where coarse-grained degrees of freedom are not allowed by microscopic theory, and realized **not by an RG irrelevant operator**… but **by frustration and an RG relevant operator**.

3. We show the **classical analogue of deconfined transitions**… between two symmetry-broken phase with incompatible symmetries… with being the emergent one…

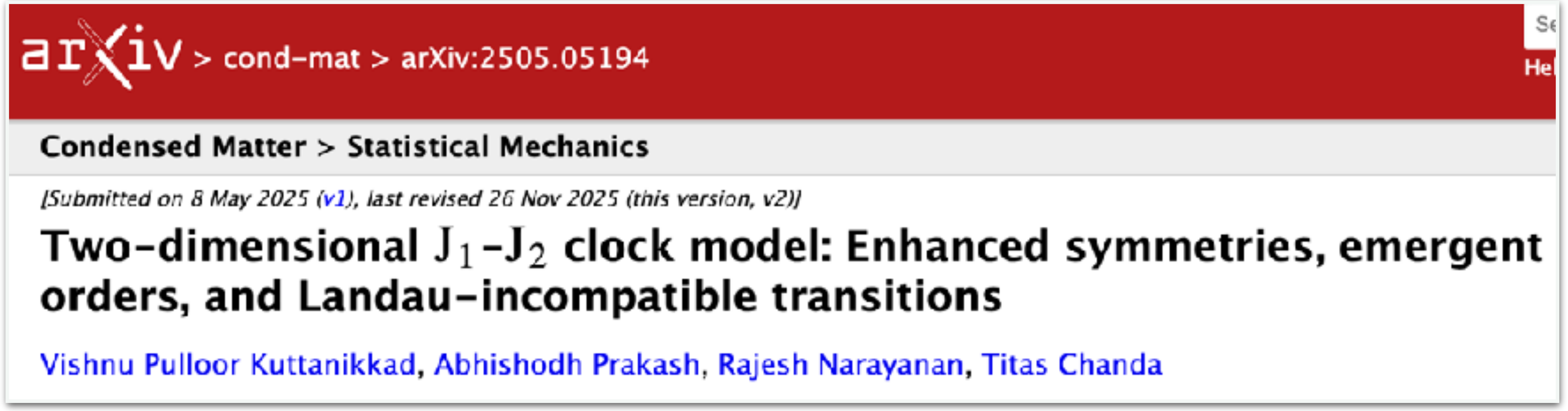


Vishnu Pulloor Kuttanikkad

Rajesh Narayanan

Abhishodh Prakash

Phases	$\mathbb{Z}_q$ **symm.**	**is critical ?**	$\mathbb{Z}_2$ **symm. (spatial)**
$\mathbb{Z}_q^+$ FM	Broken	No	Unbroken
QLRO	–	Yes	Unbroken
$\mathbb{Z}_q^+$ AFM $\oplus$ $\mathbb{Z}_2$ NM	Broken	No	Broken
QLRO $\oplus$ $\mathbb{Z}_2$ NM	–	Yes	Broken
$\mathbb{Z}_2$ NM	–	No	Broken
$\mathbb{Z}_q^-$ AFM $\oplus$ $\mathbb{Z}_2$ NM	Broken (!)	No	Broken
PM	Unbroken	No	Unbroken



arXiv > cond-mat > arXiv:2505.05194

Condensed Matter > Statistical Mechanics

[Submitted on 8 May 2025 (v1), last revised 26 Nov 2025 (this version, v2)]

Two-dimensional J_1-J_2 clock model: Enhanced symmetries, emergent orders, and Landau-incompatible transitions

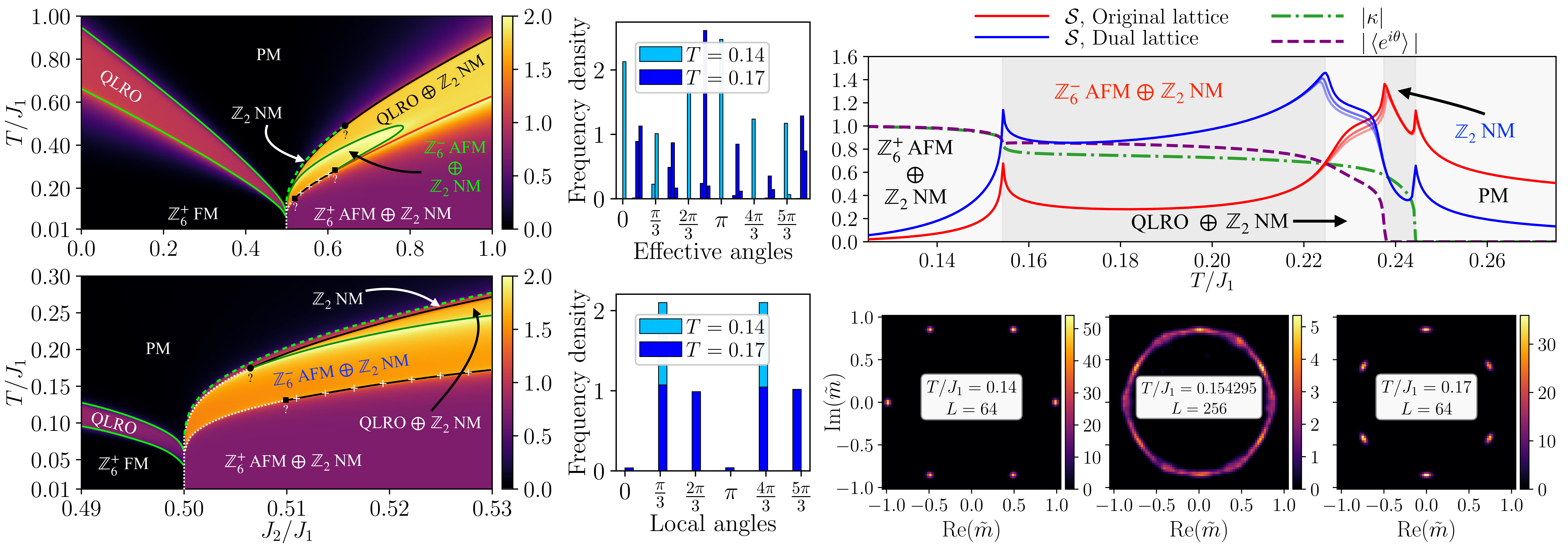
Vishnu Pulloor Kuttanikkad, Abhishodh Prakash, Rajesh Narayanan, Titas Chanda

To appear in PRL…

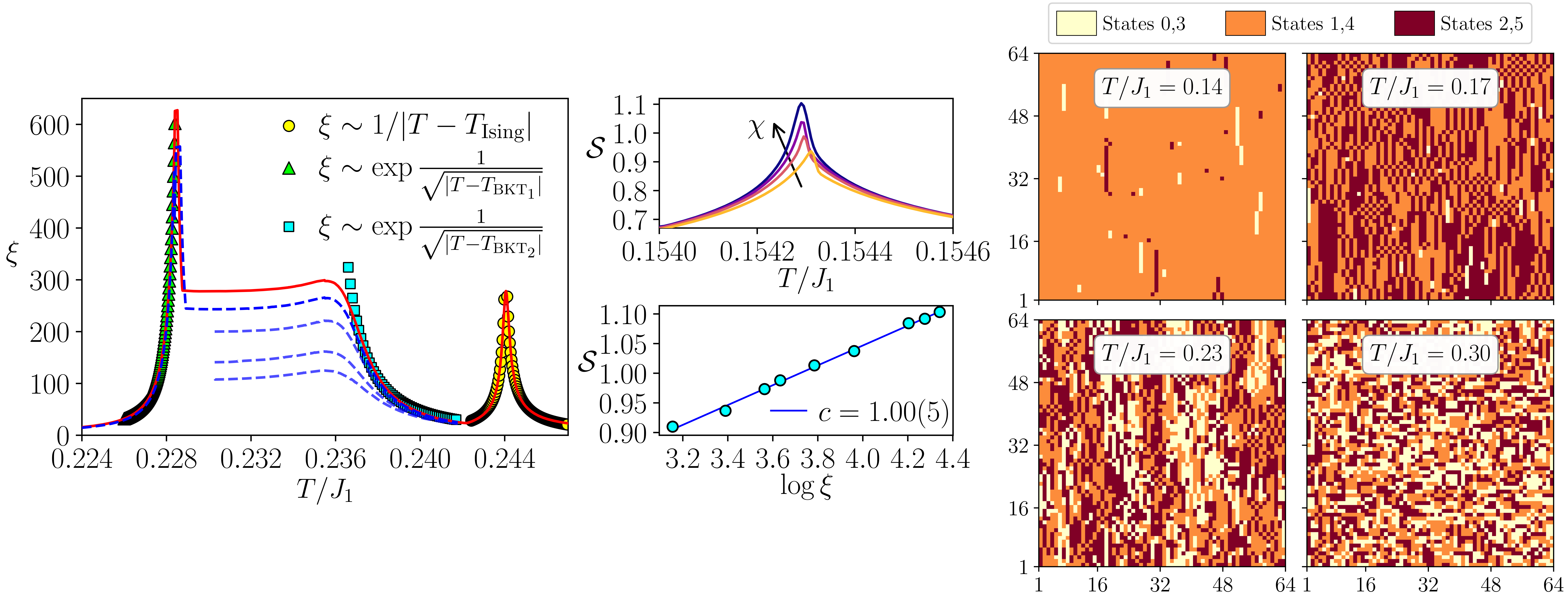
That would be the end…

Please visit us @ IIT Madras

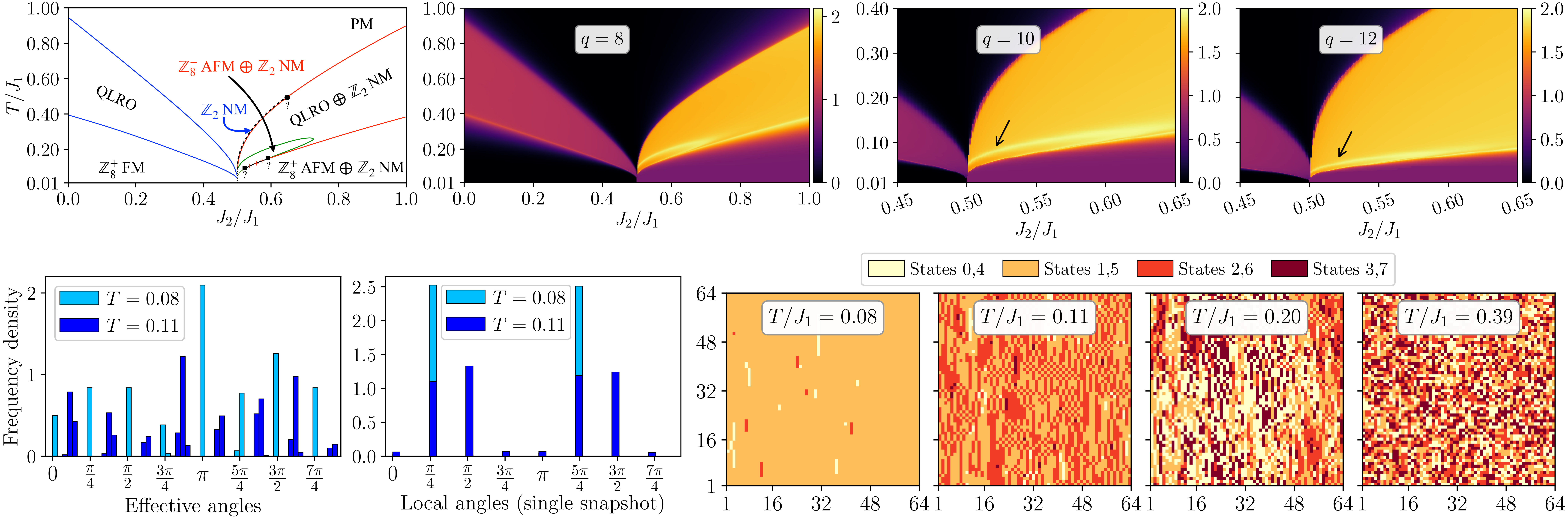
Thank you !!! …
questions??



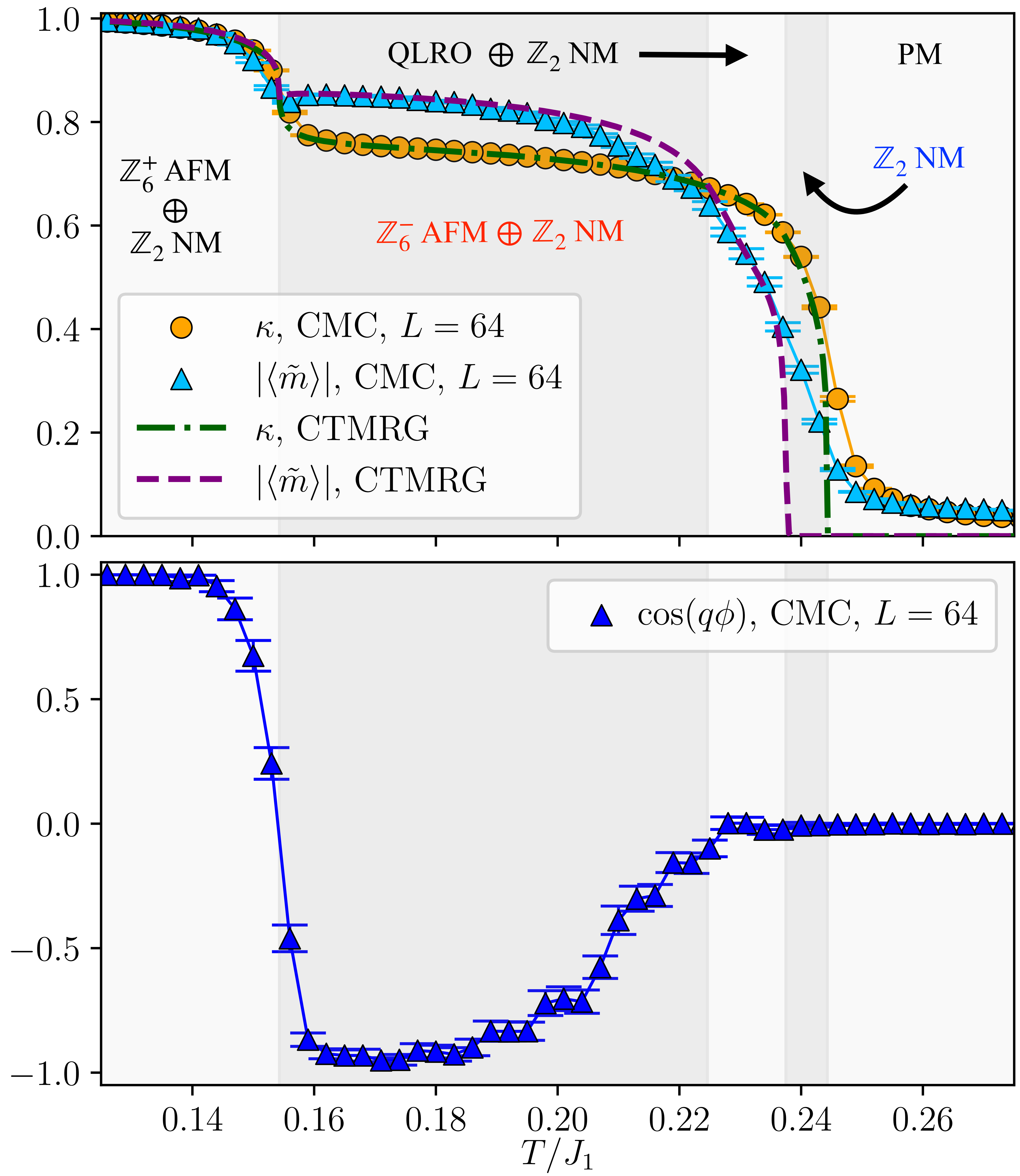
PM
QLRO
$\mathbb{Z}_2$ NM
QLRO $\oplus$ $\mathbb{Z}_2$ NM
$\mathbb{Z}_6^-$ AFM $\oplus$ $\mathbb{Z}_2$ NM
$\mathbb{Z}_6^+$ FM
$\mathbb{Z}_6^+$ AFM $\oplus$ $\mathbb{Z}_2$ NM
T/J_1
J_2/J_1
Frequency density
$T = 0.14$
$T = 0.17$
Effective angles
Local angles
$\mathcal{S}$, Original lattice
$\mathcal{S}$, Dual lattice
$|\kappa|$
$|\langle e^{i\theta}\rangle|$
$\mathbb{Z}_6^-$ AFM $\oplus$ $\mathbb{Z}_2$ NM
$\mathbb{Z}_6^+$ AFM $\oplus$ $\mathbb{Z}_2$ NM
$\mathbb{Z}_2$ NM
PM
QLRO $\oplus$ $\mathbb{Z}_2$ NM
T/J_1
$\mathrm{Im}(\tilde{m})$
$\mathrm{Re}(\tilde{m})$
$T/J_1 = 0.14$ $L = 64$
$T/J_1 = 0.154295$ $L = 256$
$T/J_1 = 0.17$ $L = 64$



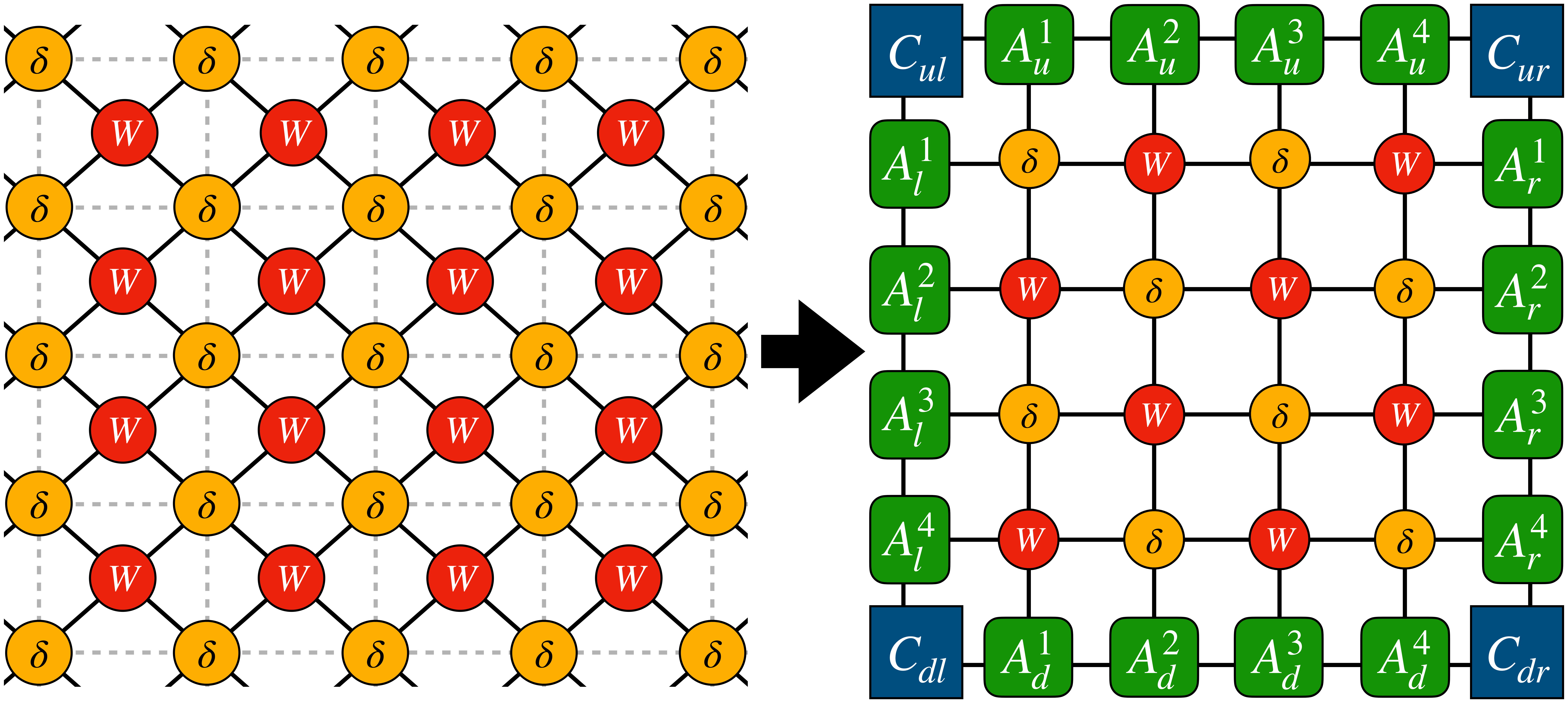
$\xi \sim 1/|T - T_{\mathrm{Ising}}|$
$\xi \sim \exp \frac{1}{\sqrt{|T-T_{\mathrm{BKT}_1}|}}$
$\xi \sim \exp \frac{1}{\sqrt{|T-T_{\mathrm{BKT}_2}|}}$
ξ
T/J_1
$\mathcal{S}$
χ
$c = 1.00(5)$
$\log \xi$
States 0,3
States 1,4
States 2,5
$T/J_1 = 0.14$
$T/J_1 = 0.17$
$T/J_1 = 0.23$
$T/J_1 = 0.30$



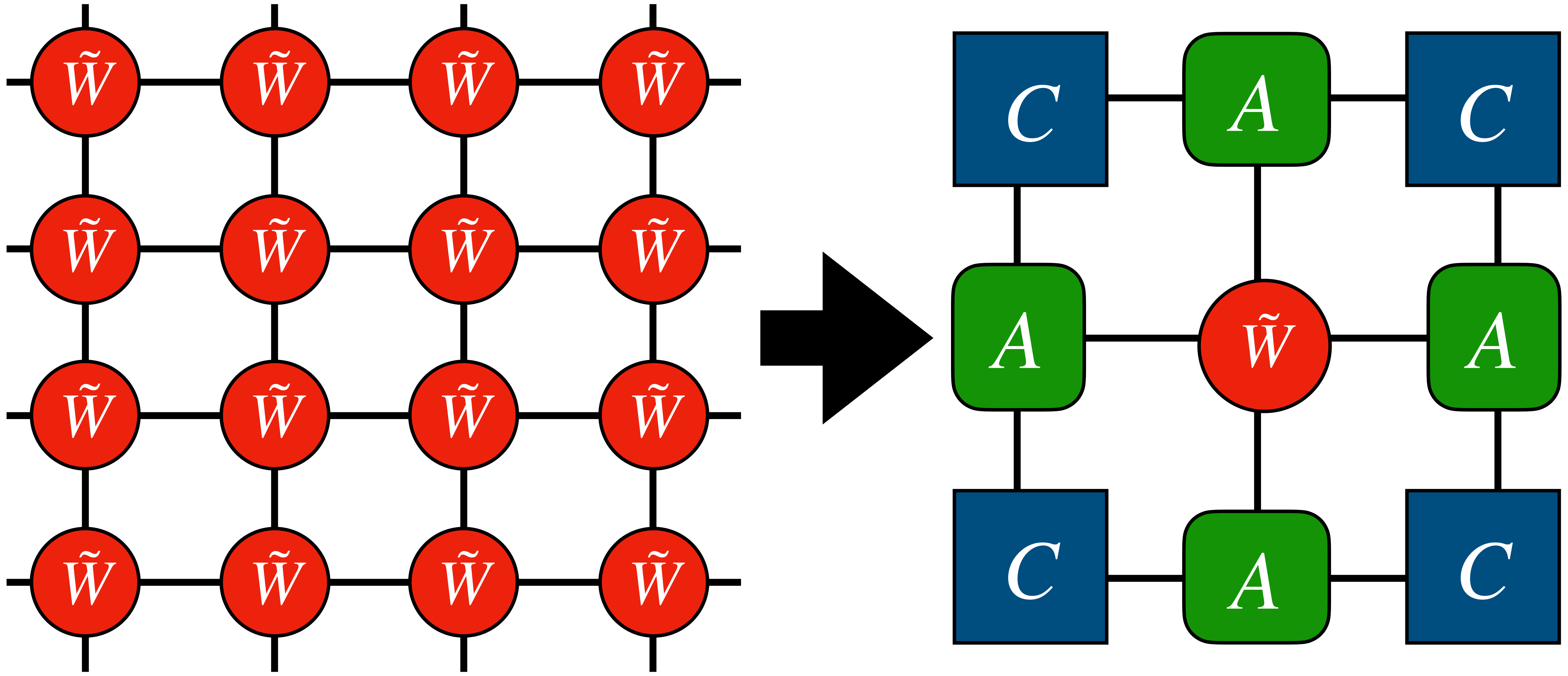
PM
QLRO
$\mathbb{Z}_8^-$ AFM $\oplus$ $\mathbb{Z}_2$ NM
$\mathbb{Z}_2$ NM
QLRO $\oplus$ $\mathbb{Z}_2$ NM
$\mathbb{Z}_8^+$ FM
$\mathbb{Z}_8^+$ AFM $\oplus$ $\mathbb{Z}_2$ NM
T/J_1
J_2/J_1
$q = 8$
$q = 10$
$q = 12$
States 0,4
States 1,5
States 2,6
States 3,7
$T/J_1 = 0.08$
$T/J_1 = 0.11$
$T/J_1 = 0.20$
$T/J_1 = 0.39$
Frequency density
$T = 0.08$
$T = 0.11$
Effective angles
Local angles (single snapshot)



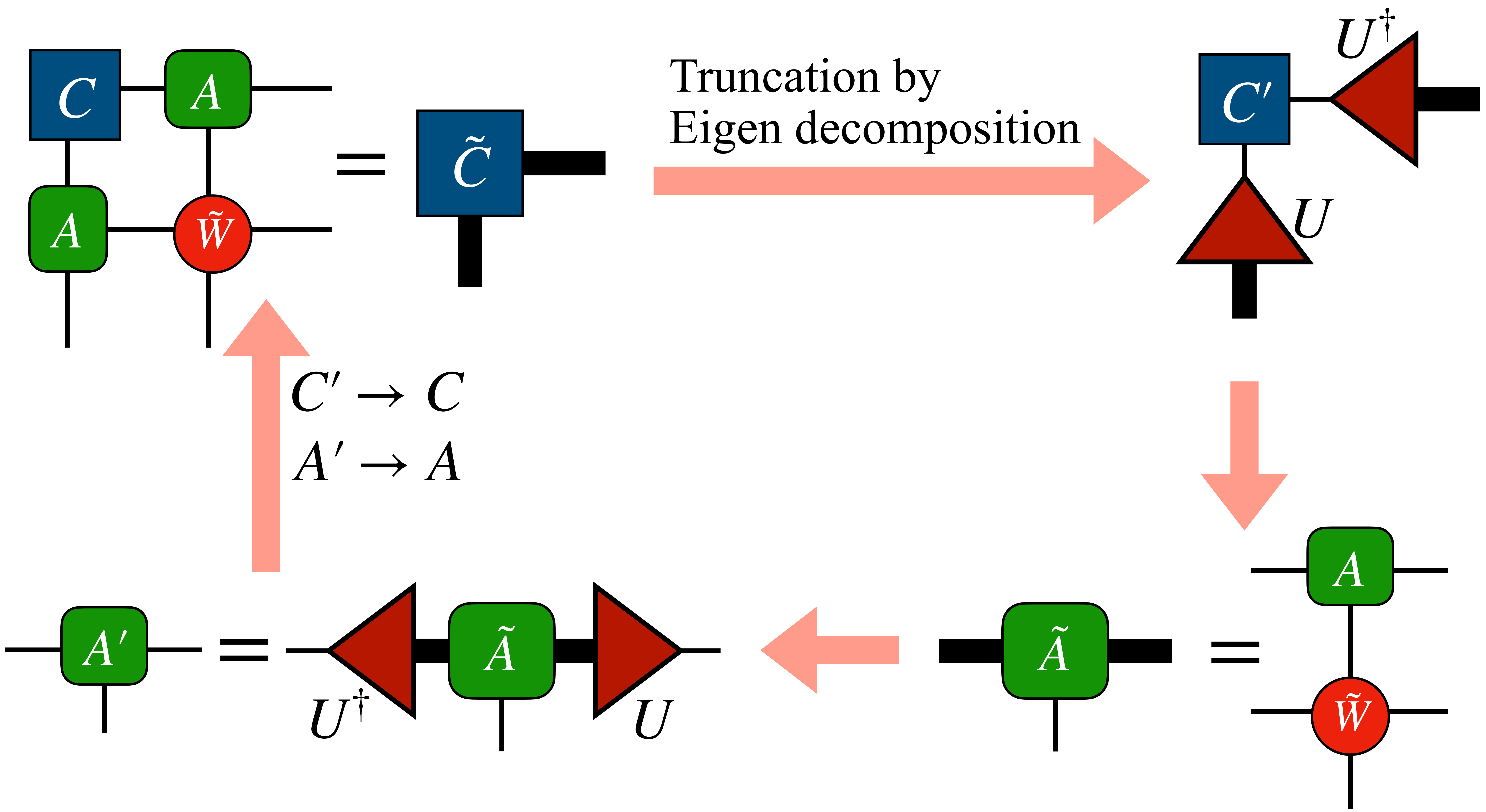
QLRO ⊕ $\mathbb{Z}_2$ NM
PM
$\mathbb{Z}_6^+$ AFM ⊕ $\mathbb{Z}_2$ NM
$\mathbb{Z}_6^-$ AFM ⊕ $\mathbb{Z}_2$ NM
$\mathbb{Z}_2$ NM
κ, CMC, $L = 64$
$|\langle \tilde{m} \rangle|$, CMC, $L = 64$
κ, CTMRG
$|\langle \tilde{m} \rangle|$, CTMRG
$\cos(q\phi)$, CMC, $L = 64$
T/J_1



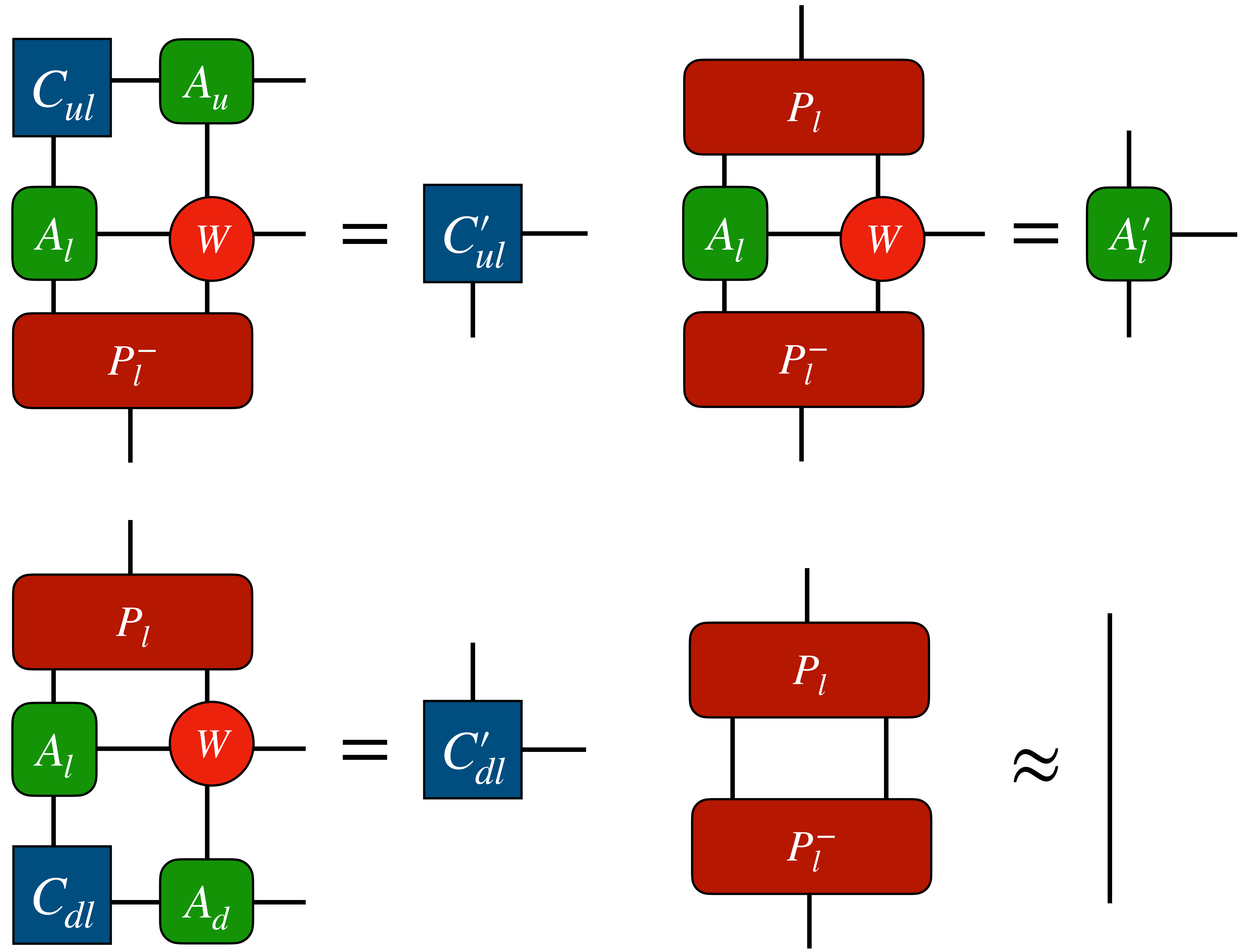
δ
W
C_{ul}
A_u^1
A_u^2
A_u^3
A_u^4
C_{ur}
A_l^1
A_l^2
A_l^3
A_l^4
A_r^1
A_r^2
A_r^3
A_r^4
C_{dl}
A_d^1
A_d^2
A_d^3
A_d^4
C_{dr}



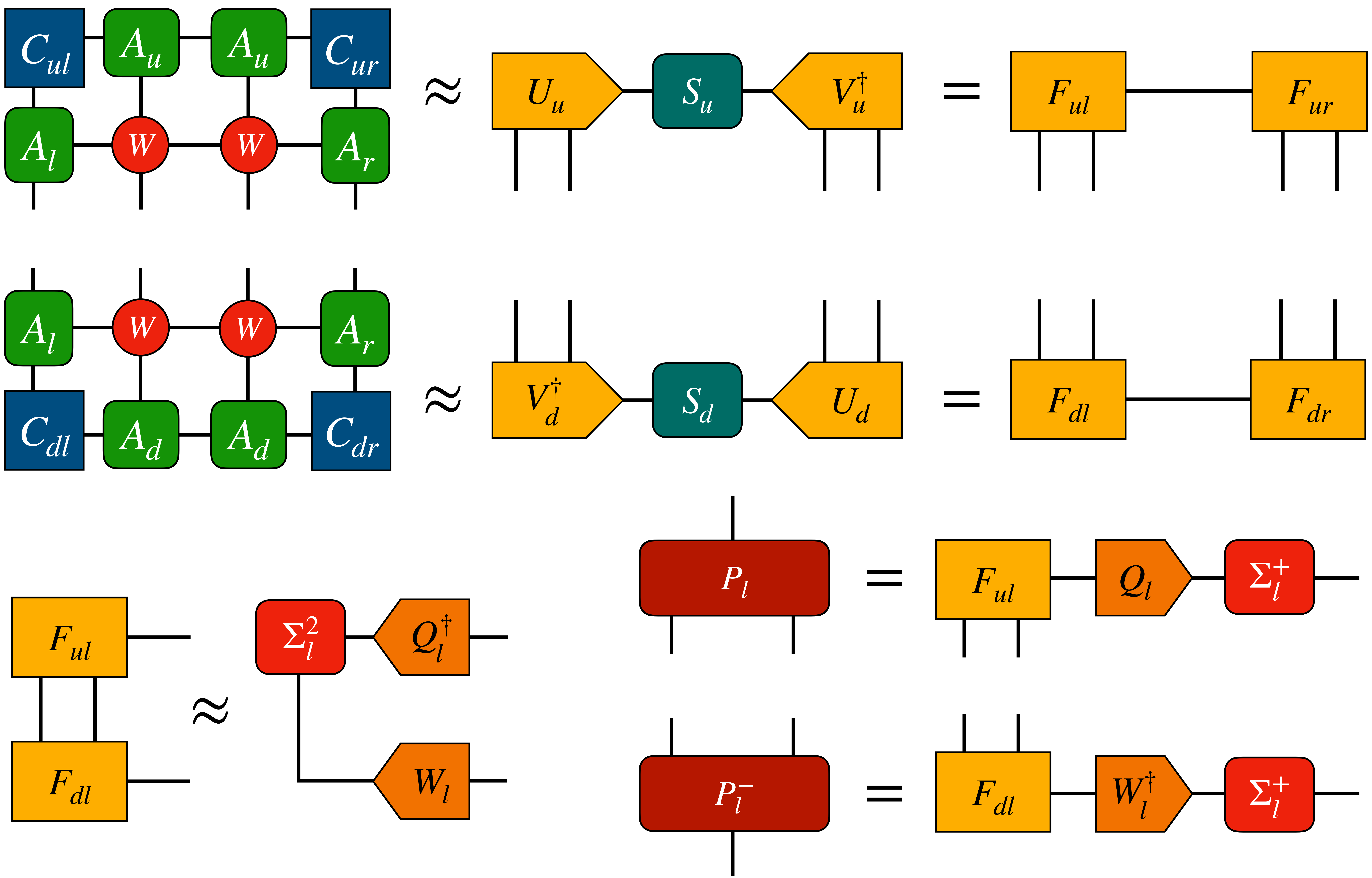
$\tilde{W}$
C
A



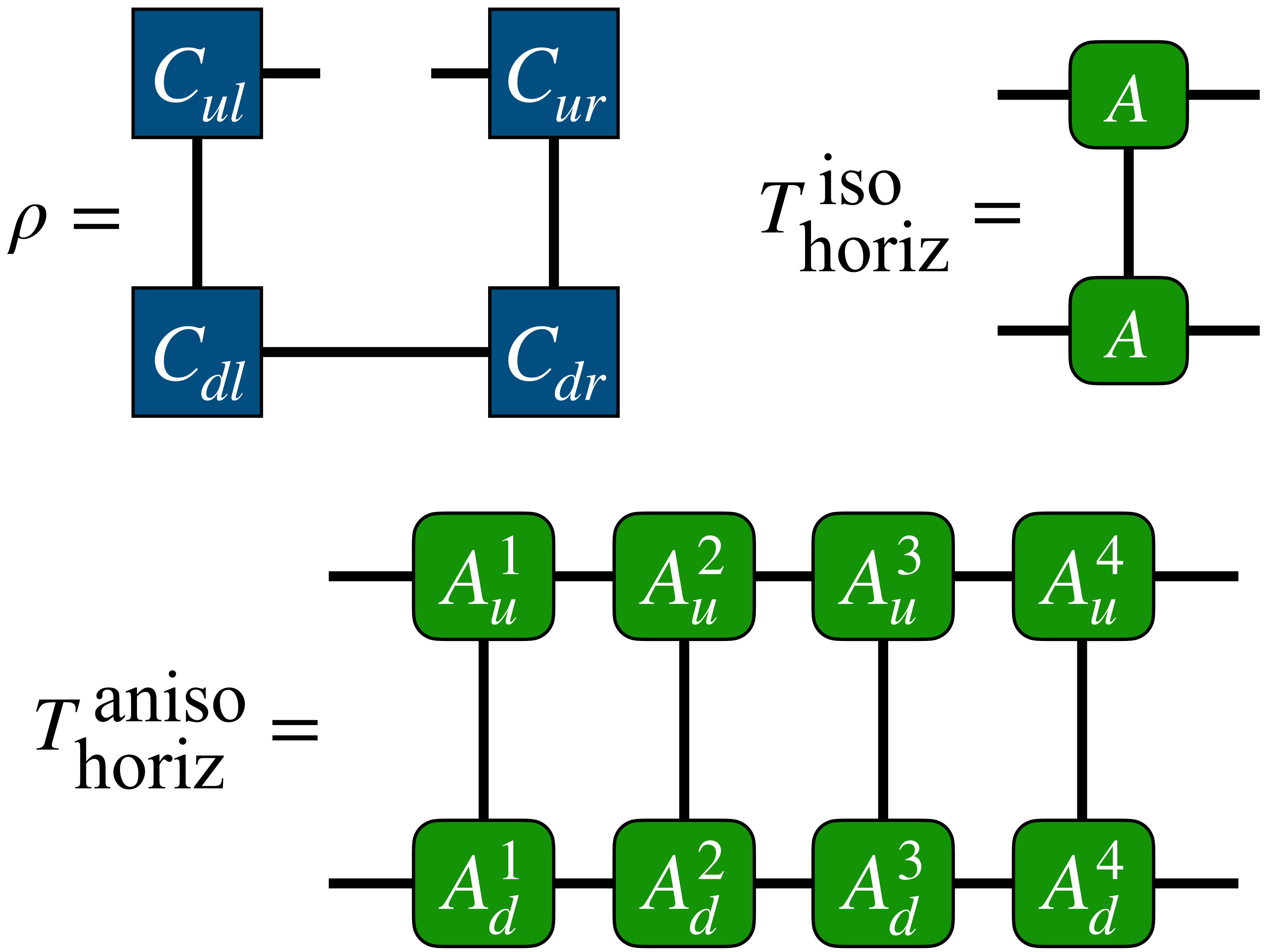
C
A
A
$\tilde{W}$
$=$
$\tilde{C}$
Truncation by
Eigen decomposition
$U^\dagger$
C'
U
$C' \to C$
$A' \to A$
A
$\tilde{W}$
$\tilde{A}$
$=$
A'
$=$
$\tilde{A}$
$U^\dagger$
U



C_{ul}
A_u
A_l
W
P_l^-
$=$
C'_{ul}
P_l
A_l
W
P_l^-
$=$
A'_l
P_l
A_l
W
C_{dl}
A_d
$=$
C'_{dl}
P_l
P_l^-
$\approx$



C_{ul} A_u A_u C_{ur}
A_l W W A_r
$\approx$ U_u S_u $V_u^\dagger$ $=$ F_{ul} F_{ur}
A_l W W A_r
C_{dl} A_d A_d C_{dr}
$\approx$ $V_d^\dagger$ S_d U_d $=$ F_{dl} F_{dr}
F_{ul}
F_{dl}
$\approx$ Σ_l^2 $Q_l^\dagger$ W_l
P_l $=$ F_{ul} Q_l Σ_l^+
P_l^- $=$ F_{dl} $W_l^\dagger$ Σ_l^+



$\rho =$
C_{ul}
C_{ur}
C_{dl}
C_{dr}
$T^{\text{iso}}_{\text{horiz}} =$
A
A
$T^{\text{aniso}}_{\text{horiz}} =$
A_u^1
A_u^2
A_u^3
A_u^4
A_d^1
A_d^2
A_d^3
A_d^4