Lecture Notes for the 2026 Trieste Spring School on Superstring Theory and Related Topics

Resurgent methods in string theory

Marcos Mariño

Département de Physique Théorique et Section de Mathématiques
Université de Genève, Genève, CH-1211 Switzerland

E-mail: marcos.marino@unige.ch

Abstract: This is an introductory set of lecture notes on resurgent methods in string theory. The lectures start with a practical review of resurgence, including examples in quantum mechanics. I then review how resurgent methods can be used in non-critical string theories and two-dimensional gravity, in particular trans-series solutions to non-linear differential equations. I also discuss large N matrix models off-criticality to emphasize the geometric nature of instanton actions coming from D-branes in those string models. This picture can be extended to topological strings on Calabi–Yau manifolds and more complicated string models, which are the subject of the last part of the notes.

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1 Introduction

Most perturbative series in quantum theory are asymptotic and divergent. String theory is not an exception, and we expect on general considerations that a genus g amplitude grows factorially like $(2g)!$ [1, 2]. This is also the same type of divergence that one finds in the $1/N$ expansion of matrix theories [3], as one would expect from large N string/gauge theory dualities.

The theory of resurgence is a very general framework to study such factorially divergent series. It allows one to obtain two things:

1. *Non-perturbative effects*: Given a divergent perturbative series, extract from its factorial growth non-perturbative amplitudes. This is part of the magic of the theory of resurgence and goes back to the work of Bender–Wu in the 1970s. However, resurgent analysis might not give you access to *all* non-perturbative sectors. This is a limitation of the method.

2. *Non-perturbative definitions*: Given a divergent perturbative series as well as non-perturbative effects, obtain a well-defined observable by resumming appropriately all these contributions. In this sense, resurgence provides an upgrade of perturbative results to exact results. However, the non-perturbative definition obtained by resummation might require choices, and it is in general not unique.

In these lectures I will provide an introduction to resurgent methods in string theory. These methods have been used for a long time (perhaps without using the keyword "resurgence"), and they played a rôle in the study of non-critical strings in the 1990s, when people had access for the first time to all-genus results in toy models of string theory. A more systematic use of resurgent technology was made after 2007, starting with the works [4–6], and continuing until today. The focus of these developments, and perhaps the clearest success story, was in the context of topological string theory, where the newly available results for higher genus amplitudes made it possible to apply resurgent methods successfully.

The organization of these lecture notes is as follows. I start in section 2 with a practical review of resurgence, which we illustrate with an example in quantum mechanics. In section 3 I introduce resurgent methods in string theory by focusing on the simplest model, namely, non-critical string theories and two-dimensional gravity, and I explain in particular how to obtain trans-series solutions of non-linear differential equations. I also discuss the next simplest case, large N matrix models off-criticality, to emphasize the geometric nature of instanton actions coming from D-branes. This picture can be extended to topological strings on Calabi–Yau manifolds and more complicated string models, which are the subject of the last section of the notes.

2 Elementary resurgent technology

In this section I give a practical introduction to resurgent technology. There are by now many introductions to resurgent techniques. The book [3] contains a pedagogical introduction. For a more mathematical perspective, one can profit from [7–9]. The lecture notes [10] review some early applications of resurgence on matrix models and strings.

2.1 Asymptotic series

Let us consider a formal power series of the form,

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{2.1}$$

We stress that such a formal power series is not a function, but rather a convenient way of collecting the coefficients a_n. We say that this formal power series gives an asymptotic approximation to the function $f(z)$, in the sense of Poincaré, and we write

$$f(z) \sim \varphi(z), \tag{2.2}$$

if the following condition hold: for every N, the remainder after $N+1$ terms of the series is much smaller than the last retained term as $z \to 0$. More precisely,

$$\lim_{z\to 0} z^{-N} \left(f(z) - \sum_{n=0}^{N} a_n z^n \right) = 0, \tag{2.3}$$

for all $N > 0$. In an asymptotic series, the remainder does not necessarily go to zero as $N \to \infty$ for a fixed z, in contrast to what happens in convergent series. Note that analytic functions might have asymptotic expansions. For example, the Stirling series for the Gamma function

$$\left(\frac{z}{2\pi}\right)^{1/2} \left(\frac{z}{\mathrm{e}}\right)^{-z} \Gamma(z) = 1 + \frac{1}{12z} + \frac{1}{288z^2} + \cdots \tag{2.4}$$

is an asymptotic series for $|z| \to \infty$.

In practice, asymptotic expansions are characterized by the fact that, as we vary N, the partial sums

$$\varphi_N(z) = \sum_{n=0}^{N} a_n z^n \tag{2.5}$$

will first approach the true value $f(z)$, and then, for N sufficiently big, they will diverge. A natural question is then to find the partial sum which gives the best possible estimate of $f(z)$. To do this, one has to find the N that truncates the asymptotic expansion in an optimal way. This procedure is called *optimal truncation*. Usually, the way to find the optimal value of N is to retain terms up to the smallest term in the series, and discard all terms of higher degree. Let us assume (as it is the case in all interesting examples) that the coefficients a_n in (2.1) grow factorially at large n,

$$a_n \sim A^{-n} n!, \qquad n \gg 1. \tag{2.6}$$

The smallest term in the series, for a fixed $|z|$, is obtained by minimizing w.r.t. N in

$$\left| a_N z^N \right| = c N! \left| \frac{z}{A} \right|^N . \tag{2.7}$$

By using the Stirling approximation, we can rewrite this as

$$c \exp \left\{ N \left(\log N - 1 - \log \left| \frac{A}{z} \right| \right) \right\} . \tag{2.8}$$

The above function has a saddle at large N given by

$$N_* = \left| \frac{A}{z} \right| . \tag{2.9}$$

If $|z|$ is small, the optimal truncation can be performed at large values of N, but as $|z|$ increases, less and less terms of the series can be used. We can now estimate the error made in the optimal truncation by evaluating the next term in the asymptotics,

$$\epsilon(z) = C_{N_*+1} |z|^{N_*+1} \sim \mathrm{e}^{-|A/z|}. \tag{2.10}$$

Therefore, the maximal "resolution" we can expect when we reconstruct a function $f(z)$ from an asymptotic expansion is of order $\epsilon(z)$. This type of ambiguity is sometimes called a *non-perturbative ambiguity*, since it is not seen in perturbation theory. Indeed, the exponential term

$$\mathrm{e}^{-A/z} \tag{2.11}$$

is not analytic at $z = 0$, and therefore it doesn't contribute to the perturbative series. We conclude that, in general, an asymptotic expansion does not determine the function $f(z)$ uniquely, and some additional information is required. Note that the absolute value of A gives the "strenght" of the non-perturbative ambiguity.

Example 2.1. *Optimal truncation and the quartic integral.* It is instructive to see optimal truncation at work in a simple example. Let us consider the quartic integral $Z(g)$ defined by

$$Z(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}z \, \mathrm{e}^{-S(z)}, \qquad S(z) = \frac{z^2}{2} + \frac{g z^4}{4}. \tag{2.12}$$

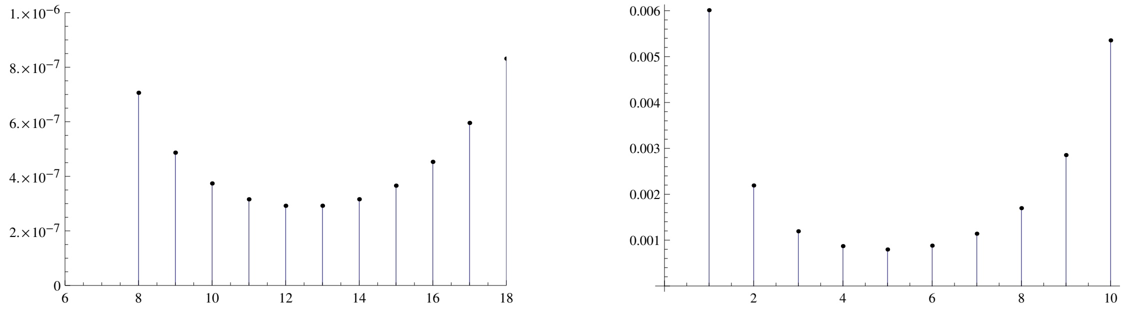

Figure 1: We illustrate the method of optimal truncation for the quartic integral (2.12) by plotting the difference (2.17) between the integral and the partial sum of order N of its asymptotic expansion, as a function of N, for $g = 0.02$ (left) and $g = 0.05$ (right).

This integral can be regarded as a zero-dimensional reduction of the $\lambda\phi^4$ theory, and we will use it for other illustrative purposes later on. As a function of g, $Z(g)$ is well-defined as long as $\mathrm{Re}(g) > 0$. At the same time, $Z(g)$ has an asymptotic expansion at small g which is obtained, as in field theory, by expanding the exponent of the quartic monomial, and integrating term by term. One finds in this way an all-orders asymptotic expansion

$$Z(g) \sim \varphi(g) \tag{2.13}$$

where

$$\varphi(g) = \sum_{k=0}^{\infty} a_k g^k, \tag{2.14}$$

and

$$a_k = \frac{(-4)^{-k}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}z \frac{z^{4k}}{k!} \mathrm{e}^{-z^2/2} = (-4)^{-k} \frac{(4k-1)!!}{k!}. \tag{2.15}$$

The formal power series (2.14) has *zero radius of convergence*. Its asymptotic behavior at large k is obtained immediately from Stirling's formula

$$a_k \sim (-4)^k k!. \tag{2.16}$$

In view of (2.16) we have that $|A| = 1/4$. In Fig. 1 we plot the difference

$$|Z(g) - \varphi_N(g)| \tag{2.17}$$

as a function of N, for two values of g. The optimal values are seen to be $N_* = 12$ and $N_* = 5$, in agreement with the estimate (2.9).

Optimal truncation gives a reasonable approximation to the original function for some values of the coupling constant, but it typically becomes a bad one for other values. In addition, in optimal truncation only a finite number of terms in the asymptotic expansion are actually used, and the remaining terms cannot be exploited to improve the approximation, as one does with convergent series. In fact, we can do better than optimal truncation and take into account the information contained in *all* the terms of the series. The way to do that is Borel resummation, which we now explain.

2.2 Borel transform and Borel resummation

Let

$$\varphi(z) = \sum_{n\geq 0} a_n z^n \tag{2.18}$$

be a formal power series in z. The *Borel transform* of $\varphi(z)$ is given by

$$\mathcal{B}(\varphi)(\zeta) = \widehat{\varphi}(\zeta) = \sum_{n\geq 0} a_n \frac{\zeta^n}{n!}. \tag{2.19}$$

Remark 2.2. Sometimes it is more convenient to use the definition

$$\widetilde{\varphi}(\zeta) = \sum_{n\geq 1} a_n \frac{\zeta^{n-1}}{(n-1)!}. \tag{2.20}$$

It is related to the previous definition by

$$\widetilde{\varphi}(\zeta) = \frac{\mathrm{d}\widehat{\varphi}(\zeta)}{\mathrm{d}\zeta}. \tag{2.21}$$

△

If the coefficients of $\varphi(z)$ grow as (2.6), then the Borel transform $\widehat{\varphi}(z)$ is analytic in a neighbourhood of $\zeta = 0$ with radius $|A|$. A *resurgent function* is a factorially divergent series $\varphi(z)$ whose Borel transform has the following property: on any line issuing from the origin, there are only a finite number of singularities of the Borel transform, and $\widetilde{\varphi}(\zeta)$ may be continued analytically along any path that follows the line, while circumventing (from above or from below) those singular points. This is Ecalle's principle of *endless analytic continuation*. In many situations in physics, the singularities are in a discrete lattice (two dimensional or even one dimensional), so they can be easily avoided and the corresponding series are resurgent functions. A resurgent function is *simple* if the singularities of its Borel transform are poles or logarithmic branch cuts.

Example 2.3. Consider the series

$$\varphi(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+b)}{\Gamma(b)} A^{-k} z^k. \tag{2.22}$$

The Borel transform is given by

$$\widehat{\varphi}(\zeta) = \sum_{k=0}^{\infty} \frac{\Gamma(k+b)}{k!\Gamma(b)} A^{-k} \zeta^k = (1-\zeta/A)^{-b}, \tag{2.23}$$

which has a singularity at $\zeta = A$. If $b = 1$, this singularity is a pole. If $0 < b < 1$, it is a branch point. The case $b = 0$ corresponds to a logarithmic singularity. More precisely, if

$$\varphi(z) = \sum_{k=0}^{\infty} \Gamma(k) A^{-k} z^k, \tag{2.24}$$

we have

$$\widehat{\varphi}(\zeta) = \sum_{k\geq 0} \frac{1}{k} A^{-k} \zeta^k = -\log\left(1 - \frac{\zeta}{A}\right). \tag{2.25}$$

♢

For simplicity of exposition, we will focus on simple resurgent functions with logarithmic branch cuts. We will also state the results for pole and power branch cut singularities.

If $\widehat{\varphi}(\zeta)$ has a log singularity at $\zeta = \zeta_\omega$, its local expansion there is of the form

$$\widehat{\varphi}(\zeta_\omega + \xi) = -\frac{\mathsf{S}_\omega}{2\pi} \log(\xi) \sum_{n\geq 0} \widehat{c}_n \xi^n + \text{regular}, \tag{2.26}$$

where the series

$$\widehat{\varphi}_\omega(\xi) = \sum_{n\geq 0} \widehat{c}_n \xi^n \tag{2.27}$$

has a finite radius of convergence. We note that we might want to make specific choices of normalization for $\widehat{\varphi}_\omega(\xi)$, and that's why we have introduced an additional (in general complex) number S_ω in (2.26), which is called a *Stokes constant*. We will regard $\widehat{\varphi}_\omega(\xi)$ as the Borel transform of the formal power series

$$\varphi_\omega(z) = \sum_{n\geq 0} c_n z^n, \qquad c_n = n! \widehat{c}_n. \tag{2.28}$$

Remark 2.4. If we have a simple resurgent function involving as well a pole singularity, with

$$\widehat{\varphi}(\zeta_\omega + \xi) = -\frac{\mathsf{S}_\omega a}{2\pi\xi} - \frac{\mathsf{S}_\omega}{2\pi} \log(\xi) \sum_{n\geq 0} \widehat{c}_n \xi^n + \text{regular}, \tag{2.29}$$

the corresponding factorially divergent series is of the form,

$$\varphi_\omega(z) = \frac{a}{z} + \sum_{n\geq 0} c_n z^n, \qquad c_n = n! \widehat{c}_n. \tag{2.30}$$

An even more general case involves functions with branch cuts of the form

$$(\zeta_\omega - \zeta)^{-b}, \qquad 0 < b < 1. \tag{2.31}$$

In this case, we have the local expansion

$$\widehat{\varphi}(\zeta_\omega + \xi) = (-\xi)^{-b} \sum_{n\geq 0} \widehat{c}_n \xi^n + \text{regular}, \tag{2.32}$$

and we will regard $\widehat{\varphi}_\omega(\xi)$ as the Borel transform of

$$\varphi_\omega(z) = z^{-b} \sum_{n\geq 0} c_n z^n, \qquad c_n = \Gamma(n+1-b) \widehat{c}_n. \tag{2.33}$$

The reason for transforming these convergent series into formal power series will be understood shortly. △

The key result so far is that, given a formal power series $\varphi(z)$, the expansion of its Borel transform around its singularities generates additional formal power series:

$$\varphi(z) \to \{\varphi_\omega(z)\}_{\omega\in\Omega}, \tag{2.34}$$

where Ω labels the set of singular points.

Example 2.5. *Formal series for the Airy function.* Let us consider the formal power series

$$\varphi_1(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \left(\mp\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} z^n \tag{2.35}$$

appearing in the asymptotics of the Airy function. In this case, the Borel transform can be computed explicitly,

$$\widehat{\varphi}(\zeta) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{3\zeta}{4}\right). \tag{2.36}$$

This has a singularity at

$$\zeta_\omega = -\frac{4}{3}. \tag{2.37}$$

It is easy to see from the general theory of hypergeometric functions that this is a logarithmic singularity, and one finds

$$\widehat{\varphi}_1(\zeta) = -\frac{1}{2\pi} \log\left(\zeta + \frac{4}{3}\right) {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; \frac{3\zeta}{4}\right) + \text{regular} \tag{2.38}$$

therefore

$$\varphi_{-4/3}(z) = \varphi_2(z) = \sum_{n\geq 0} \frac{1}{2\pi} \left(\frac{3}{4}\right)^n \frac{\Gamma(n+5/6)\Gamma(n+1/6)}{n!} z^n. \tag{2.39}$$

This is the other formal power series that appears when solving the Airy equation. ♢

Example 2.6. *Formal series from critical points.* Let us consider the formal power series (2.14) obtained as an asymptotic expansion of the quartic integral (2.12). This is one of the rare series for which the Borel transform can be computed in closed form, and one finds

$$\widehat{\varphi}(\zeta) = \frac{2K(k)}{\pi(1+4\zeta)^{1/4}}, \qquad k^2 = \frac{1}{2} - \frac{1}{2\sqrt{1+4\zeta}}, \tag{2.40}$$

where $K(k)$ is the elliptic integral of the first kind. This function has a branch point at $\zeta = A = -1/4$, of the logarithmic type, and one can extract the formal power series $\widehat{\varphi}_{-1/4}(\xi)$:

$$\widehat{\varphi}_{-1/4}(\xi) = \frac{\mathrm{i}}{\sqrt{2}} \sum_{k\geq 0} (-1)^k \frac{a_k}{k!} \xi^k \tag{2.41}$$

and $\mathsf{S} = -2\mathrm{i}$ (the choice of normalization of $\widehat{\varphi}_{-1/4}(\xi)$ will be justified in a moment). Therefore, up to an overall factor, $\varphi_{-1/4}(z)$ is the original power series up to an alternating sign. This is the same phenomenon that we found in the Airy function. It turns out that one can also find this second power series by expanding the integral $Z(g)$ around its non-trivial saddle-points. These are obtained from

$$z + gz^3 = 0 \Rightarrow z^2 = -\frac{1}{g}. \tag{2.42}$$

Therefore we have two nontrivial saddle-points $z_{1,2}$

$$z_{1,2} = \pm\frac{\mathrm{i}}{\sqrt{g}} \tag{2.43}$$

where we assumed $g > 0$. Expanding around z_1, we find

$$S(z) = -\frac{1}{4g} - y^2 + \mathrm{i}g^{1/2}y^3 + \frac{g}{4}y^4, \qquad y = z - z_1. \tag{2.44}$$

This gives a Gaussian with the wrong sign, so we have to perform a rotation $y \to \mathrm{i}y$. The resulting expansion is

$$Z(g) \sim \mathrm{e}^{\frac{1}{4g}}\varphi_{-1/4}(g), \tag{2.45}$$

and gives the trans-series associated to the Borel singularity at $\zeta = -1/4$. This is a general phenomenon. Let us consider an integral of the form

$$Z(g) = \int \mathrm{e}^{-\frac{1}{g}V(x)}\mathrm{d}x, \tag{2.46}$$

where $V(x)$ has a non-degenerate critical point at $x = 0$. Then, by expanding around this point one finds a "perturbative" series $\varphi(g)$. The additional series $\varphi_\omega(g)$ can be obtained in general by considering the expansion of $Z(g)$ around non-trivial critical points, see e.g. [11]. $\diamondsuit$

The above procedure can be repeated with the new formal power series $\varphi_\omega(z)$, to generate yet more power series. This might lead to a finite set of functions (this is the case of the Airy functions). Generically, however, one finds an infinite set of formal power series. We will label such series by the index ω, and we will write then as $\varphi_\omega(z)$. Let us assume that they are all simple resurgent functions with logarithmic singularities. Then, we have the general relation

$$\widehat{\varphi}_\omega(\zeta_{\omega'} + \xi) = -\mathsf{S}_{\omega\omega'}\frac{\log(\xi)}{2\pi}\widehat{\varphi}_{\omega'}(\xi) + \text{regular}. \tag{2.47}$$

The constants $\mathsf{S}_{\omega\omega'}$ are Stokes constants.

Example 2.7. In the Airy example, the Stokes constants are simply

$$\mathsf{S}_{12} = \mathsf{S}_{21} = 1. \tag{2.48}$$

The above structure can be extended to a more general type of singularity, with e.g. a branch cut structure.

A very convenient way to encode the information in the singularities is through the *Stokes automorphism*. To do this we need to consider the *Borel resummation* of the factorially divergent series. Let ζ_ω be a singularity of $\widetilde{\varphi}(\zeta)$. A ray in the Borel plane which starts at the origin and passes through ζ_ω is called a *Stokes ray*. It is of the form $\mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+$, where $\theta = \arg(\zeta_\omega)$. Note that a Stokes ray might pass through many singularities. A typical situation is that we have an infinite sequence of singularities on the ray, of the form $\ell\mathcal{A}$ with $\ell \in \mathbb{Z}_{>0}$.

Let $\varphi(z)$ a factorially divergent power series. If $\widehat{\varphi}(\zeta)$ analytically continues to an L^1-analytic function along the ray $\mathcal{C}^\theta := \mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+$, we define its *Borel resummation* along the direction θ by

$$s_\theta(\varphi)(z) = \frac{1}{z}\int_{\mathcal{C}^\theta}\widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta. \tag{2.49}$$

Remark 2.8. If one uses the definition (2.20) of Borel transform, the Borel resummation is given instead by

$$s_\theta(\varphi)(z) = a_0 + \int_{\mathcal{C}^\theta}\widetilde{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta. \tag{2.50}$$

$\triangle$

Let us first note that, if $s_\theta(\varphi)(z)$ exists, its asymptotic behavior for small z can be obtained by expanding the integrand and integrating term by term:

$$s_\theta(\varphi)(z) \sim \sum_{n\geq 0} a_n z^n. \tag{2.51}$$

This is the formal power series that we started with. Therefore, if we are lucky, Borel resummation produces an actual function which reproduces the original series. It is then a way to "make sense" of our original formal power series.

If we vary θ and we do not encounter singularities of $\widehat{\varphi}$, the function $s_\theta(\varphi)(z)$ is locally analytic. However, when θ is the direction of a Stokes ray, the Borel resummation is not well defined. In fact, as θ crosses a Stokes ray, it has a discontinuity. To define this discontinuity more precisely, we introduce *lateral Borel resummations*.

Let $\varphi(z)$ be a resurgent function, and let $\mathcal{C}^\theta_\pm$ be contours starting at the origin and going slightly above (respectively, below) the Stokes ray, in such a way that $\mathcal{C}^\theta_+ - \mathcal{C}^\theta_-$ is a clockwise contour. Then, the lateral Borel resummations of $\varphi(z)$ are defined as

$$s_{\theta\pm}(\varphi)(z) = \frac{1}{z}\int_{\mathcal{C}^\theta_\pm} \widetilde{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta. \tag{2.52}$$

The discontinuity is then defined by

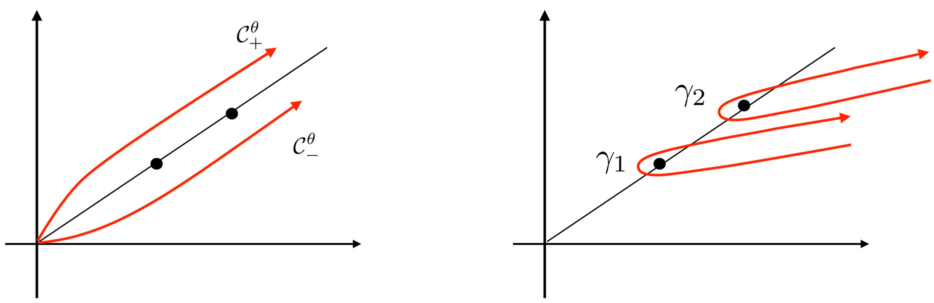

Figure 2: Contour deformation in the calculation of the discontinuity.

$$\mathrm{disc}_\theta(\varphi)(z) = s_{\theta+}(\varphi)(z) - s_{-\theta}(\varphi)(z). \tag{2.53}$$

Note that, since $s_{\theta\pm}(\varphi)(z)$ have the same asymptotics for small z, given in (2.51), the discontinuity must be invisible in a conventional asymptotic expansion. As we will now see, this difference is *exponentially small* and closely related to the local structure of the Borel transform. Indeed, let us assume e.g. that $\varphi(z)$ is a simple resurgent function, with logarithmic singularities at ζ_ω in the Stokes ray, where $\omega \in \Omega$. The difference between the two contours $\mathcal{C}^\theta_+ - \mathcal{C}^\theta_-$ can be deformed into a sum of Hankel-like contours γ_ω around the logarithmic branch cuts. We then have, for each ω,

$$\oint_{\gamma_\omega} \widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta = -\frac{\mathrm{e}^{-\zeta_\omega/z}}{2\pi}\mathsf{S}_\omega \int_{\mathcal{C}^\theta_-} \left(\log(\xi) - \log(\xi) - 2\pi\mathrm{i}\right)\widehat{\varphi}_\omega(\xi)\mathrm{e}^{-\xi/z}\mathrm{d}\xi, \tag{2.54}$$

where in the first line we have written $\zeta = \zeta_\omega + \xi$. Therefore

$$\begin{aligned} s_{\theta+}(\varphi)(z) - s_{\theta-}(\varphi)(z) &= \frac{1}{z}\sum_{\omega\in\Omega}\oint_{\gamma_\omega}\widehat{\varphi}(\zeta)\mathrm{e}^{-\zeta/z}\mathrm{d}\zeta = \mathrm{i}\sum_{\omega\in\Omega}\mathsf{S}_\omega\frac{r\mathrm{e}^{-\zeta_\omega/z}}{z}\int_{\mathcal{C}^\theta_-}\widehat{\varphi}_\omega(\xi)\mathrm{e}^{-\xi/z}\mathrm{d}\xi \\ &= \mathrm{i}\sum_{\omega\in\Omega}\mathsf{S}_\omega\mathrm{e}^{-\zeta_\omega/z}s_-(\varphi_\omega)(z). \end{aligned} \tag{2.55}$$

This formula holds for more general types of singularities, with appropriate definitions of the trans-series $\varphi_\omega(z)$. An example is given in (3.51).

The result (2.55) involves Borel resummed trans-series, but it is useful to rewrite it as a relation between formal trans-series themselves. If we regard lateral Borel resummations as operators, we introduce the *Stokes automorphism* along the ray $\mathcal{C}^\theta$, $\mathfrak{S}_\theta$, as

$$s_{\theta+} = s_{\theta-}\mathfrak{S}_\theta. \tag{2.56}$$

Then, we can write (2.55) as

$$\mathfrak{S}_\theta(\varphi) = \varphi + \mathrm{i}\sum_{\omega\in\Omega}\mathsf{S}_\omega\mathrm{e}^{-\zeta_\omega/z}\varphi_\omega(z). \tag{2.57}$$

We would like to emphasize that, although we have introduced the Stokes automorphism by using lateral Borel resummations, the expression (2.57) just collects the local information of the Borel transform near the Borel singularities on a ray. In fact, in Écalle's theory, the Stokes automorphism can be defined in terms of the so-called alien derivatives [8], which encode this local information and do not involve resummations.

Remark 2.9. If the simple resurgent function has in addition pole singularities, it is easy to see, by using Cauchy's theorem, that the pole in (2.29) gives a contribution

$$\mathrm{i}\,\mathsf{S}_\omega\frac{a}{z} \tag{2.58}$$

to the discontinuity, and the formula (2.55) still holds with the definition (2.30). If a singularity is not simple, but of the branch cut type (2.32), the discontinuity can be easily calculated and is given by

$$2\mathrm{i}\sin(\pi b)\mathsf{S}_\omega\mathrm{e}^{-\zeta_\omega/z}s_-(\varphi_\omega)(z), \tag{2.59}$$

where $\varphi_\omega(z)$ is given by (2.33). Therefore, after absorbing the prefactor $2\sin(\pi b)$ in the Stokes constant, the formula (2.55) is still valid, provided one uses the more general Borel transform (2.33). △

The expression (2.55) involves (possible infinite) sums of the formal series $\varphi_\omega(z)$ with an exponentially small prefactor $\mathrm{e}^{-\zeta_\omega/z}$. These objects are called *trans-series*. More formally, let $\varphi_\omega(z)$ be resurgent functions. A *trans-series* is a (possibly infinite) formal linear combination of formal power series

$$\Phi(z;\boldsymbol{C}) = \sum_\omega C_\omega\mathrm{e}^{-\zeta_\omega/z}\varphi_\omega(z), \tag{2.60}$$

where $\boldsymbol{C} = (C_{\omega_1}, \cdots)$ is a (possibly infinite) vector of complex numbers.

Let us make some conceptual comments on trans-series:

1. Trans-series involve at least two "small parameters": the first small parameter is the one appearing in the original, "perturbative" series. There is also an *exponentially small parameter* $e^{-\zeta_\omega/z}$. In this sense, trans-series go beyond classical asymptotics by including exponentially small corrections

2. All series in the first small parameter are factorially divergent.

3. The different series appearing in the trans-series are not independent. For example, the large order behavior of the terms in the leading perturbative series are controlled by the first series in trans-series (the one corresponding to the smallest singularity, in absolute value).

2.3 Resurgence and large order behaviour

In practice, determining the structure of singularities in the Borel transform is not easy, since we often don't even know the perturbative series in closed form. The most direct manifestation of these singularities is through the large order behavior of the original series. This also explain the name "resurgence": the new series associated to the singularities "resurge" in the original series through the behavior of the coefficients a_k when k is large. In terms of the Borel transform (which is analytic at the origin), this is essentially an old theorem of Darboux, which relates the large order behavior of the coefficients of an analytic function at the origin, to the behavior near the closest singularity (see e.g. [12]).

Let $\varphi(z)$ be a resurgent function, and let A be the singularity of the Borel transform which is closest to the origin in the complex ζ plane (we will assume for simplicity that there is only one, although the generalization is straightforward). Let us assume that the behavior near this singularity is as in (2.29), with $\zeta_\omega = A$. Then, the coefficients a_k have the following asymptotic behavior,

$$a_k \sim \frac{\mathsf{S}}{2\pi} \sum_{n\geq 1} A^{-k+n} c_n \Gamma(k-n), \qquad k \gg 1. \tag{2.61}$$

Remark 2.10. The contribution of the residue in (2.29) to the asymptotics is simply

$$\mathsf{S}\frac{a}{2\pi} A^{-k-1}\Gamma(k+1). \tag{2.62}$$

When the Borel transform has a branch cut as in (2.32), a similar argument leads to

$$a_k \sim \mathsf{S}\frac{\sin(\pi b)}{\pi} \sum_{n\geq 0} A^{-k-b+n} c_n \Gamma(k+b-n). \tag{2.63}$$

□

To understand the formula (2.61) better, it is convenient to write explicitly the very first terms:

$$a_k \sim \frac{\mathsf{S}}{2\pi} A^{-k}\Gamma(k) \left\{ c_0 + \frac{c_1 A}{k-1} + \frac{c_2 A^2}{(k-1)(k-2)} + \cdots \right\}, \qquad k \gg 1. \tag{2.64}$$

The first factor in the r.h.s. gives the leading factorial asymptotics, while the second factor gives a series of corrections in $1/k$ to the leading factorial behavior. These corrections involve the coefficients c_n of the power series obtained in (2.28). One can use this asymptotic formula in two ways: as a procedure to extract the numbers A, c_n from the knowledge of the series a_k, of conversely, as a way to obtain the large order asymptotics of these coefficients once A, c_n are known.

The asymptotics (2.61) implies that the coefficients c_n determining the series $\varphi_A(z)$ are encoded in the large order behavior of a_k. This has multiple applications. In particular, when there is no easy method to determine $\varphi_A(z)$, one can extract it from the knowledge of the large order behavior. This has become one of the most powerful heuristic methods in resurgence.

Example 2.11. In the case of the series $\varphi_1(z)$ in (2.35), the above result leads to the asymptotics

$$\begin{aligned} a_k &\sim \frac{1}{2\pi}\left(-\frac{4}{3}\right)^{-k}(k-1)!\sum_{n\geq 0} c_n \left(-\frac{4}{3}\right)^{n}\frac{\Gamma(k-n)}{\Gamma(k)} \\ &\sim \frac{1}{2\pi}\left(-\frac{4}{3}\right)^{-k}(k-1)!\left(1-\frac{4}{3}\cdot\frac{5}{48}\frac{1}{k-1}+\frac{16}{9}\cdot\frac{385}{4608}\frac{1}{(k-1)(k-2)}+\cdots\right). \end{aligned} \tag{2.65}$$

Since we have a closed form for a_k, we can calculate the asymptotics directly:

$$a_k = \frac{1}{2\pi}\left(-\frac{3}{4}\right)^k \frac{\Gamma(k+\frac{5}{6})\Gamma(k+\frac{1}{6})}{k!}, \tag{2.66}$$

and one indeed verifies that

$$\frac{\Gamma(k+\frac{5}{6})\Gamma(k+\frac{1}{6})}{k!(k-1)!} \sim 1-\frac{4}{3}\cdot\frac{5}{48}\frac{1}{k-1}+\frac{16}{9}\cdot\frac{385}{4608}\frac{1}{(k-1)(k-2)}+\cdots, \qquad k\gg 1. \tag{2.67}$$

◊

Example 2.12. A similar analysis can be done for the coefficients of the quartic integral (2.15). One finds,

$$\begin{aligned} a_k &\sim \frac{1}{\sqrt{2\pi}}\left(-\frac{1}{4}\right)^{-k}(k-1)!\sum_{n\geq 0} a_n(-1)^n \left(-\frac{1}{4}\right)^{n}\frac{\Gamma(k-n)}{\Gamma(k)} \\ &\sim \frac{1}{\sqrt{2\pi}}\left(-\frac{1}{4}\right)^{-k}(k-1)!\left(1-\frac{1}{4}\cdot\frac{3}{4}\frac{1}{k-1}+\frac{1}{16}\cdot\frac{105}{32}\frac{1}{(k-1)(k-2)}+\cdots\right). \end{aligned} \tag{2.68}$$

◊

2.4 An example in quantum mechanics

To apply the previous technology to a "realistic" example, let us consider the following problem in one-dimensional quantum mechanics, which is at the origin of the whole subject. We consider the Hamiltonian

$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + V_{\rm QO}(\mathsf{x}), \qquad V_{\rm QO}(x) = \frac{x^2}{2} + g x^4. \tag{2.69}$$

We will consider the formal perturbative series for the ground state energy in powers of g, which we will denote by $\phi_0(g)$. This is perhaps the simplest factorially divergent series appearing in quantum theory, and it provides an asymptotic approximations to the actual ground state energy $E_0(g)$. One can calculate $\phi_0(g)$ very efficiently with the *BenderWu* program of Sulejmanpasic and Unsal [13]. At the very first orders one has

$$\phi_0(g) = \sum_{n\geq 0} a_n g^n = \frac{1}{2} + \frac{3g}{4} - \frac{21g^2}{8} + \frac{333g^3}{16} + \cdots \tag{2.70}$$

The behavior of this series at large orders was studied in two pioneering papers by Bender and Wu [14, 15]. It is interesting to retrace their path, since this is a role model for other studies of resurgence and a beautiful example of experimental mathematics. First, Bender and Wu generated around 70 terms of the perturbative series. To do this, they invented a method which is much more efficient for large orders than the conventional one (and which is at the base of the *BenderWu* software). Then, they studied numerically the growth of the coefficients. They found a factorial growth of the form

$$a_n \sim \frac{\sqrt{6}}{\pi^{3/2}} 3^n (-1)^{n+1} \Gamma\left(n + \frac{1}{2}\right), \qquad n \gg 1. \tag{2.71}$$

The numbers appearing here were first guessed from numerical approximations.

Exercise 1 ✎. Use the numerical techniques of Appendix A to obtain the Bender–Wu asymptotics (2.71). This is done in the first Mathematica package provided with the lectures.

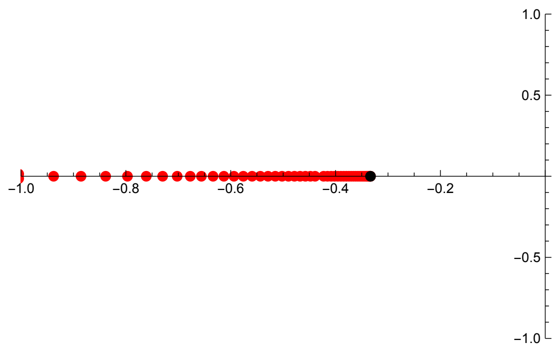

Figure 3: The singularities of the Borel transform for the series $\phi_0(g)$ of the quartic oscillator. This is obtained by a Padé approximant which uses 185 terms of the perturbative series. The black dot is the point $(-1/3, 0)$

To understand the physics behind this large order behavior, we recall that according to the general theory, it gives information about the closest singularity in the Borel plane. The expression (2.71) indicates that there is a singularity of $\widehat{\phi}_0(\zeta)$ in the Borel plane at $\zeta = -1/3$, as shown in Fig. 3. The trans-series associated to this singularity is of the form,

$$\Phi_{-1/3}(g) \approx 2\sqrt{\frac{2}{\pi g}} \mathrm{e}^{\frac{1}{3g}}, \tag{2.72}$$

up to an overall normalization. The exponent involves the location of the singularity in the Borel plane, and the overall power of g in this formula is due to the value of $b = 1/2$ in (2.71), see (2.33) and (2.63).

We note that, as $g \to 0$, the contribution (2.72) diverges. Therefore, this type of trans-series should not contribute to the actual value of the observable. In fact, in this problem it is known that the *exact* value of the ground state energy can be obtained by Borel resummation of (2.70) (see e.g. [16] for a review of these results), i.e.

$$E_0(g) = s(\phi_0)(g). \tag{2.73}$$

We note that (2.71) is experimental mathematics. So far, this result has not been derived in the context of perturbation theory alone, although there is an argument also due to Bender and Wu along these lines in [17]. The result (2.71) can be derived rigorously (see again [16]), but for this we need first its physical interpretation. Bender and Wu noted that for *negative* values of the coupling, the potential becomes unstable and the ground state decays by a tunneling effect which is exponentially small. In the theory with negative coupling $\lambda = -g$, the Borel singularity occurs in the positive real axis, and the trans-series is given by

$$\Phi_{1/3}(\lambda) \approx 2\sqrt{\frac{2}{\pi\lambda}}\mathrm{e}^{-\frac{1}{3\lambda}}. \tag{2.74}$$

The tunneling effect can be calculated by various methods. One can use e.g. the WKB approximation, or instanton methods. More precisely, in the quartic oscillator at negative coupling, the Hamiltonian has no longer bound states, but it has a spectrum of *resonances* or complex values for the energy. The imaginary part of the energies can be interpreted as the inverse lifetime, and for low-lying states is exponentially small. For the ground state energy, the WKB approximation gives

$$\mathrm{Im}\, E_0 \approx \mathrm{e}^{-\mathcal{S}_{\mathrm{WKB}}}, \qquad \lambda \ll 1. \tag{2.75}$$

Here,

$$\mathcal{S}_{\mathrm{WKB}} = 2\int_{x_-}^{x_+} p(x)\mathrm{d}x = 2\int_0^{1/\sqrt{2\lambda}} x\sqrt{1-2\lambda x^2}\,\mathrm{d}x = \frac{1}{3\lambda}, \tag{2.76}$$

where x_- is the minimum of the inverpotential, and x_+ is the turning point. This can be also regarded as the action of an instanton moving in the inverted potential. We can see that the trans-series for negative coupling gives precisely the small λ asymptotics of the imaginary part of the energy. More precisely, one has the following equality

$$2\mathrm{i}\,\mathrm{Im}\, E_0(\lambda) = s_+(\phi_0)(\lambda) - s_-(\phi_0)(\lambda), \tag{2.77}$$

and remember that this discontinuity was given precisely by the contributions of the Borel singularities along the axis. The first such contribution is $\Phi_{1/3}(\lambda)$, so we have the more precise estimate

$$\mathrm{Im}\, E_0 \approx -\sqrt{\frac{2}{\pi\lambda}}\mathrm{e}^{-\frac{1}{3\lambda}}, \qquad \lambda \ll 1. \tag{2.78}$$

3 Resurgence, non-critical strings and matrix models

3.1 The perturbative genus expansion in non-critical strings

The resurgent structure of perturbative series in QFT is a very interesting and challenging topic, but in these lectures we will focus on simple examples in string theory.

It seems clear that resurgence is most effective where we have a deep control of the perturbative series. In string theory, the genus expansion is in general very hard to compute directly unless one uses indirect routes. One of the main achievements of string theory in the early 90s was to obtain the full genus expansion in "toy" models of string theory, namely, non-critical strings. These are CFTs with $c \leq 1$ coupled to two-dimensional gravity, and the way to solve them at all genera was to construct them, at least perturbatively, in terms of the so-called *double-scaling limit* of an appropriate matrix model. Developing this in some detail would require a set of lectures in its own. I will then give a lightning summary of the simplest non-critical string, namely the $(2,3)$ minimal model coupled to gravity, also called *minimal* or *pure gravity*, see [18, 19] for reviews of the subject.

Minimal gravity can be obtained as the double-scaling limit of a large N matrix model with a quartic or cubic potential (more on matrix models in the next section). In this model, the only physical parameter is the *cosmological constant* κ. The free energy of the theory at fixed genus g has a fixed scaling with κ, given by

$$F_g(\kappa) = c_g \kappa^{(2-\gamma)(1-g)}, \qquad g \neq 1. \tag{3.1}$$

When $g = 1$ the dependence on κ is logarithmic. Here, c_g is a constant, and γ is a critical exponent (the so-called *string susceptibility*), which can be determined directly in Liouville theory coupled to matter [20]. It is given, in terms of the central charge c_M of the matter CFT, by

$$\gamma = \frac{1}{12}\left(c_M - 1 - \sqrt{(c_M-1)(c_M-25)}\right). \tag{3.2}$$

The $(2,3)$ model has $c = 0$, therefore $\gamma = -1/2$. Calculating c_g directly in string theory involves performing a complicated integral over the moduli space of Riemann surfaces. Fortunately, the matrix model approach leads to the result for all the c_g in a single strike. Let us define the *all-genus free energy*

$$F(\kappa) = \sum_{g \geq 0} F_g(\kappa). \tag{3.3}$$

Usually one would introduce a string coupling constant g_s to keep track of the genera, but it is easy to see that due to the scaling (3.1), it can be absorbed in the cosmological constant. One of the main results of the "non-critical string theory revolution" is that $F(\kappa)$ is given by

$$F''(\kappa) = -u(\kappa), \tag{3.4}$$

where $u(\kappa)$ or specific heat satisfies the famous Painlevé I equation

$$\kappa = u^2 - \frac{1}{6}u'' \tag{3.5}$$

with the asymptotic condition

$$u(\kappa) \approx \sqrt{\kappa}, \qquad \kappa \to \infty. \tag{3.6}$$

This result was obtained almost simultaneously by Douglas–Shenker [21], Brézin–Kazakov [22] and Gross–Migdal [23]. With the above information, one can obtain all the coefficients c_g. To do that we have to find the all-orders asymptotics (3.6), but this is fixed by the Painlevé I equation. One finds that

$$u(\kappa) \sim \kappa^{\frac{1}{2}} \sum_{g=0}^{\infty} u_g \kappa^{-5g/2}, \tag{3.7}$$

explicitly

$$\kappa^{-\frac{1}{2}} u(\kappa) \sim 1 - \frac{1}{484}\kappa^{-5/2} - \frac{49}{4608}\kappa^{-5} - \frac{1225}{55296}\kappa^{-\frac{15}{2}} - \frac{4412401}{42467328}\kappa^{-10} - \cdots, \tag{3.8}$$

and then one simply plugs this into the definition of the specific heat to obtain

$$F(\kappa) = -\frac{4}{15}\kappa^{\frac{5}{2}} - \frac{1}{48}\log\kappa + \frac{7}{5760}\kappa^{-\frac{5}{2}} + \frac{245}{331776}\kappa^{-5} + \cdots \tag{3.9}$$

This result creates an opportunity to understand the resurgent structure of string theory in this simple example. How would one go about this?

A direct approach would be to study the coefficients of $u(\kappa)$ in (3.7) and see how they grow. Since we have a differential equation, it is easy to find a pattern. In fact, by plugging (3.7) in the Painlevé I equation, we obtain a recursion relation

$$u_{k+1} = \frac{25k^2-1}{48} u_k - \frac{1}{2} \sum_{\ell=1}^{k} u_\ell u_{k+1-\ell}, \qquad u_0 = 1. \tag{3.10}$$

Let us be naif and suppose that the first term, of order $k^2 u_k$, dominates the second term for large k. Then, one has

$$u_{k+1} \sim \frac{25}{192}(2k)^2 u_k \tag{3.11}$$

and this implies the growth for $k \gg 1$

$$u_k \sim A^{-2k}(2k)!, \qquad A = \frac{8\sqrt{3}}{5}. \tag{3.12}$$

This can be verified numerically. In fact, one has the refined asymptotic behavior

$$u_g \sim \frac{\mathsf{S}}{\pi} A^{-2g+1/2} \Gamma\left(2g - \frac{1}{2}\right), \qquad g \gg 1, \tag{3.13}$$

where

$$\mathsf{S} = -\frac{3^{1/4}}{2\sqrt{\pi}} \tag{3.14}$$

can be regarded as a Stokes constant.

Exercise 2 ✎. Verify the above asymptotics with a numerical analysis, by using the methods explained in Appendix A. This is done in the second Mathematica package accompanying the lectures.

This is direct evidence for the famous $(2g)!$ growth of string perturbation theory. From this analysis we conclude that the Borel transform of the series for the specific heat has singularities at $\pm A$ in the Borel plane, where the relevant variable is $x = \kappa^{-5/4}$. The reason the singularities come in pairs is simply a consequence of the fact that the series is in *even* powers of the coupling constant, therefore the Borel transform is also an even function. The corresponding trans-series can be obtained by reminding ourselves the form (2.33), and the fact that to obtain $u(\kappa)$ we need an overall factor $\kappa^{1/2}$. One finds that the leading non-perturbative effects are,

$$\Phi_{\pm A}(\kappa) \approx \kappa^{-1/8} \exp\left(\pm \frac{8\sqrt{3}}{5} \kappa^{5/4}\right). \tag{3.15}$$

We will come back to the physical interpretation of these effects in a moment. We would like to know if it is possible to obtain this result in a more rigorous way, without relying on numerics. The idea is simply to consider a *trans-series ansatz* to solve the equation in the first place, namely, we set

$$u(\kappa; \mathcal{C}) = \sum_{\ell=0}^{\infty} \mathcal{C}^\ell u_{(\ell)}(\kappa), \tag{3.16}$$

where $\mathcal{C}$ is a parameter, and $u_{(0)}$ is the "perturbative" solution obtained above[1]. We will think about the $u_{(\ell)}(z)$ as "non-perturbative corrections," as it will be clear in a moment. When plugging this ansatz in the original Painlevé equation we find

$$-\frac{u''_{(0)}}{6}+u^2_{(0)}-z+\mathcal{C}\left(-\frac{u''_{(1)}}{6}+2u_{(0)}u_{(1)}\right)+\mathcal{O}(\mathcal{C}^2)=0 \tag{3.17}$$

so the equation for $u_{(1)}$ is a *linear inhomogeneous equation*

$$u''_{(1)}-12u_{(0)}u_{(1)}=0. \tag{3.18}$$

Let us use a non-perturbative ansatz for $u_{(1)}$, i.e. let us assume that

$$u_{(1)}(\kappa)\approx\kappa^{\rho}\mathrm{e}^{-A\kappa^{\sigma}}. \tag{3.19}$$

We have

$$u''_{(1)}(\kappa)\approx\left(\sigma^2A^2\kappa^{2\sigma}+\sigma A\left(1-\sigma-2\rho\right)\kappa^{\sigma}+\cdots\right)\kappa^{\rho-2}\mathrm{e}^{-A\kappa^{\sigma}}, \tag{3.20}$$

and on the other hand

$$12u_{(0)}u_{(1)}\approx12\kappa^{\rho+\frac{1}{2}}\mathrm{e}^{-A\kappa^{\sigma}}. \tag{3.21}$$

The only non-trivial solution (with $A\neq0$) occurs if we match (3.21) with the first term in the r.h.s. of (3.20), while the second term vanishes. We find in this way

$$2\sigma-2=\frac{1}{2},\qquad\sigma^2A^2=12,\qquad\rho=\frac{1-\sigma}{2}, \tag{3.22}$$

i.e.

$$\sigma=\frac{5}{4},\qquad A=\pm\frac{8\sqrt{3}}{5},\qquad\rho=-\frac{1}{8}. \tag{3.23}$$

Therefore, $u_{(1)}$ has exactly the same form than we found by a resurgent, large order analysis in (3.15)! Before further commenting on this, let us note that we can push this calculation in two directions. First of all, we can now write the correct ansatz for the first non-perturbative correction

$$u_{(1)}(\kappa)=\kappa^{1/2}\mathrm{e}^{-A\kappa^{5/4}}\phi_1(\kappa),\qquad\phi_1(\kappa)=\kappa^{-5/8}\sum_{n\geq0}u_{n,1}\kappa^{-5n/4}. \tag{3.24}$$

Explicitly,

$$\kappa^{5/8}\phi_1(\kappa)=1-\frac{5}{64\sqrt{3}}\kappa^{-\frac{5}{4}}+\frac{75}{8192}\kappa^{-\frac{5}{2}}-\frac{341329}{23592960\sqrt{3}}\kappa^{-\frac{15}{4}}+\cdots \tag{3.25}$$

The normalization of the first term is arbitrary. Note that the powers in $\phi_1(z)$ are half of the powers in $u_0(z)$. In addition, we can obtain equations for all the $u_{(\ell)}(\kappa)$,

$$-\frac{1}{6}u''_{(\ell)}+\sum_{k=0}^{\ell}u_{(k)}u_{(\ell-k)}=0, \tag{3.26}$$

which for $\ell\geq2$ are linear inhomogeneous. They have the structure

$$u_{(\ell)}(\kappa)=\kappa^{1/2}\mathrm{e}^{-\ell A\kappa^{5/4}}\phi_k(\kappa),\qquad\phi_k(\kappa)=\kappa^{-5k/8}\sum_{n\geq0}u_{n,k}\kappa^{-5n/4}. \tag{3.27}$$

[1]Restricted versions of this strategy were used in [24, 25] to obtain non-perturbative information from the Painlevé I equation and other equations describing non-critical strings.

Evidently, the coefficients $u_{n,k}$ can be obtained systematically from the above equations.

This is all very nice, but are these exponentially small solutions related to the resurgent structure of the original perturbative series of $u(\kappa)$? Do they correspond to the singularities of the Borel transform? The answer is yes, they are! This was shown by Écalle in very beautiful work, reviewed e.g. in [8]. Note that in principle the information about trans-series at the singularities involves analyzing discontinuities, while the trans-series $u_{(\ell)}$ appear as extended solutions of the ODE. What Écalle showed is that there is a bridge between these two types of phenomena. In particular, the series $u_{(1)}(z)$ is the trans-series associated to the first Borel singularity of the Borel transform of $u_{(0)}$, as we verified above. It is a nice exercise in resurgence to check that the additional coefficients in (3.25) arise from a numerical analysis of the large order behavior of the coefficients u_g.

We can now come back to the interpretation of these non-perturbative effects. As anticipated by Polchinski, they are due to D-branes, which then explain in this case the doubly-factorial divergence of the perturbative series. One way to check this is to verify that the action of the instanton A is given by the disk partition function in the minimal CFT coupled to Liouville theory, with D-brane boundary conditions. This was done in [25, 26] for a variety of minimal CFTs.

An elementary corollary of the results above is the calculation of the first trans-series for the free energy [5],

$$F^{(1)}(\kappa) = \frac{1}{8 \cdot 3^{\frac{3}{4}} \sqrt{\pi}} \kappa^{-\frac{5}{8}} \exp\left(-\frac{8\sqrt{3}}{5} \kappa^{\frac{5}{4}} \right) \left\{ 1 - \frac{37}{64\sqrt{3}} \kappa^{-\frac{5}{4}} + \cdots \right\}, \tag{3.28}$$

which is obtained by thinking about (3.4) as relation between trans-series, and not only asymptotic series. The trans-series above leads to the large order behavior

$$F_g \sim \frac{1}{8 \cdot 3^{3/4} \pi^{3/2}} \left(\frac{8\sqrt{3}}{5} \right)^{-2g+5/2} \Gamma\left(2g - \frac{5}{2}\right) \left(1 - \frac{37}{80g} + \cdots \right), \qquad g \gg 1. \tag{3.29}$$

Another way of understanding these effects is by looking at the double-scaled matrix model which is used to construct the theory. We will do that in the next section.

So far we have used resurgence to obtain non-perturbative effects or sectors in the theory, by using a direct analysis of the Borel transform or Écalle's idea of considering trans-series solutions. Is it possible to obtain actual non-perturbative solutions? Indeed, one can do lateral Borel resummations of the trans-series (3.16) to obtain a one-parameter family of solutions to Painlevé I with the asymptotic behaviour (3.6) (see e.g. [27]). The information in this resummed trans-series is however not different from the one we would obtain by solving the Painlevé I equation directly. The trans-series gives us rather a *decoding* of this solution as the contribution of the perturbative sector and the non-perturbative sectors. In this family of solutions, the trans-series parameter $\mathcal{C}$ is not fixed, and it can be regarded as a *non-perturbative ambiguity*: all the solutions in this family have the same perturbative content, but differ in their non-perturbative corrections.

We conclude then that the Painlevé I equation does not lead by itself to a *unique* non-perturbative free energy for pure two-dimensional gravity. This seems to be a feature of all unitary CFTs coupled to 2d gravity: the double-scaling limit does not define the theory uniquely, at the non-perturbative level. This is also the case for more recent attemps, like JT gravity, for example. There have been various proposals to fix the non-perturbative ambiguity of 2d gravity (see e.g. [28, 29]). The theory of resurgence does not give to new radical insights on this problem, but the representation of solutions as resummed trans-series makes manifest the explicit non-perturbative content of the different completions.

Remark 3.1. Since the Painlevé I equation is of second order, one should be able to construct more general solutions with *two* trans-series parameters. This construction is also naturally motivated if one wants to understand the asymptotics of the coefficients in the instanton sectors, like e.g. (3.25). These two-parameter solutions were constructed in detail in [30], and their implications have been studied in detail in various works [31–34]. △

Example 3.2. *Painlevé II.* Similar considerations can be applied to the Painlevé II equation,

$$u''(\kappa) - 2u^3(\kappa) + 2\kappa u(\kappa) = 0, \tag{3.30}$$

which describes minimal supergravity [35]. There is a formal solution to PII with the asymptotics (3.6):

$$u^{(0)}(\kappa) = \sqrt{\kappa} - \frac{1}{16\,\kappa^{\frac{5}{2}}} - \frac{73}{512\,\kappa^{\frac{11}{2}}} - \frac{10657}{8192\,\kappa^{\frac{17}{2}}} - \frac{13912277}{542888\,\kappa^{\frac{23}{2}}} + \cdots, \qquad \kappa \to \infty. \tag{3.31}$$

One considers again a formal solution with the structure (3.16), and solves for the subleading $u^{(\ell)}$. For $\ell = 1$ one finds

$$u''_{(1)} + 2\kappa u_{(1)} - 6u_{(0)}^2 u_{(1)} = 0. \tag{3.32}$$

The subleading corrections have the form

$$u_{(\ell)}(\kappa) = \kappa^{-\frac{3\ell}{4}} \mathrm{e}^{-\ell A \kappa^{3/2}} \epsilon^{(\ell)}(\kappa), \quad \kappa \to \infty, \tag{3.33}$$

where

$$A = \frac{4}{3}, \qquad \epsilon^{(\ell)}(\kappa) = \sum_{n=0}^{\infty} u_{\ell,n} \kappa^{-3n/2} \tag{3.34}$$

In this case it has been proposed that the non-perturbative free energy of minimal supergravity is given by a particular solution of Painlevé II with asymptotics (3.6), namely the Hastings–McLeod solution. A resurgent analysis makes it possible to decode this solution in terms of a trans-series [6]. ♢

3.2 The $1/N$ expansion in matrix models

We will consider matrix models for an $N \times N$ Hermitian matrix M, with a potential $V(M)$. In the simplest case, this is a polynomial potential,

$$V(\lambda) = \frac{1}{2}\lambda^2 + \sum_{p\geq 3} \frac{g_p}{p} \lambda^p, \tag{3.35}$$

where the g_p are coupling constants of the model. The partition function is defined by

$$Z(N, g_s) = \frac{1}{\mathrm{vol}(U(N))} \int \mathrm{d}M\, \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr}V(M)}, \tag{3.36}$$

where g_s is an additional coupling constant, sometimes referred to as the string coupling constant. Matrix models have a $U(N)$ "gauge" symmetry

$$M \to UMU^\dagger, \tag{3.37}$$

therefore one can go to the "diagonal gauge" and write this partition function in terms of the eigenvalues of M, denoted by λ_i. The resulting N-dimensional integral is given by

$$Z_\gamma(N, g_s) = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{2\pi} \, \Delta^2(\lambda) \, \mathrm{e}^{-\frac{1}{g_s} \sum_{i=1}^{N} V(\lambda_i)}, \tag{3.38}$$

where

$$\Delta(\lambda) = \prod_{a<b} (\lambda_a - \lambda_b) \tag{3.39}$$

is the Vandermonde determinant. We have not specified an integration contour in (3.38), although this can be done in such a way that the integral is convergent. We will focus here on how to extract formal perturbative series from (3.38), in the $1/N$ expansion, and we will for the moment being consider only so-called *one-cut matrix models*.

To define one-cut matrix models in the $1/N$ expansion, we first consider the *'t Hooft limit* as the limit

$$g_s \to 0, \quad N \to \infty, \qquad N g_s = t \text{ fixed}. \tag{3.40}$$

t is called the 't Hooft parameter. In this regime, it turns out that the free energy of the model admits a $1/N$ expansion that we will write in the form

$$F(t, g_s) = \log Z(N, g_s) \sim \sum_{g \geq 0} F_g(t) g_s^{2g-2}. \tag{3.41}$$

The quantities $F_g(t)$ are called genus g free energies. We note that, since t is fixed, an expansion in g_s is the same thing as an expansion in $1/N$.

It is always good to test ideas and procedures in simple models. The simpler one-cut matrix model is the *Gaussian matrix model*. It is defined by the matrix integral (3.38) with the potential

$$V(\lambda) = \frac{\lambda^2}{2}. \tag{3.42}$$

In this case, the matrix integral can be computed exactly at finite N, by using e.g. the Selberg integral or orthogonal polynomials. The result reads,

$$Z^{\mathrm{G}}(N, g_s) = \frac{g_s^{N^2/2}}{(2\pi)^{N/2}} G_2(N+1), \tag{3.43}$$

where $G_2(N+1)$ is the Barnes function

$$G_2(N+1) = \prod_{i=0}^{N-1} i!. \tag{3.44}$$

In this case, the large N expansion (3.41) follows from the asymptotics of this function. One finds,

$$\begin{aligned} F_0^{\mathrm{G}}(t) &= \frac{1}{2} t^2 \Big(\log t - \frac{3}{2} \Big), \\ F_1^{\mathrm{G}}(t) &= -\frac{1}{12} \log t + \frac{1}{12} \log g_s + \zeta'(-1), \\ F_g^{\mathrm{G}}(t) &= \frac{B_{2g}}{2g(2g-2)} t^{2-2g}, \quad g > 1, \end{aligned} \tag{3.45}$$

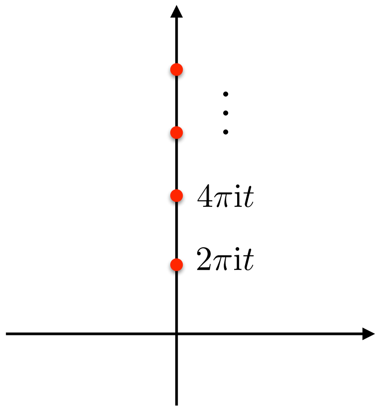

Figure 4: The singularities of the Borel transform (3.49) are located at non-zero integer multiples of $2\pi i t$. In the figure we show the Stokes ray through the singularities $2\pi i \ell t$ with $\ell \in \mathbb{Z}_{>0}$.

where B_{2g} are Bernoulli numbers. Notice that $F_1^{\rm G}(t)$ depends also logarithmically on g_s.

What is the resurgent structure of the free energies in the Gaussian matrix model? It turns out that, at fixed t, we have

$$F_g^{\rm G}(t) \sim (2g)!, \tag{3.46}$$

This follows for example from the formula for the Bernoulli numbers

$$B_{2g} = (-1)^{g-1} \frac{2(2g)!}{(2\pi)^{2g}} \sum_{\ell \geq 1} \ell^{-2g}. \tag{3.47}$$

One can do much better and analyze in detail the resurgent structure, as done in [36]. Let us consider the formal power series

$$\varphi(g_s) = \sum_{g \geq 2} b_{2g} g_s^{2g-2}, \qquad b_{2g} = \frac{B_{2g}}{2g(2g-2)} t^{2-2g}. \tag{3.48}$$

In this case, it turns out to be more convenient to use the Borel transform (2.20). One finds,

$$\begin{aligned} \widetilde{\varphi}(\zeta) &= \sum_{g \geq 2} \frac{b_{2g}}{(2g-3)!} \zeta^{2g-3} = \sum_{g \geq 2} \frac{B_{2g}}{2g(2g-2)!} t^{2-2g} \zeta^{2g-3} \\ &= \frac{1}{\zeta} \left\{ -\frac{1}{12} + \frac{t^2}{\zeta^2} - \frac{1}{4 \sinh^2\left(\frac{\zeta}{2t}\right)} \right\}. \end{aligned} \tag{3.49}$$

(Note that the version of the Borel transfom defined in (2.19) is given by a primitive of this function, and that is why it is difficult to obtain an explicit expression for it.)

The singularities of the Borel transform (3.49) are located at

$$\zeta = 2\pi i \ell t, \qquad \ell \in \mathbb{Z} \backslash \{0\}, \tag{3.50}$$

see Fig. 4. They are double poles. Let us consider the Stokes ray going through the singularities with $\ell > 0$, at the angle $\theta = \pi/2$. The discontinuity of the lateral Borel resummations for that angle is simply computed by the sum of residues at those poles:

$$\begin{aligned} s_+(\varphi)(g_s) - s_-(\varphi)(g_s) &= -2\pi \mathrm{i} \sum_{\ell=1}^{\infty} \mathrm{Res}_{\zeta=2\pi \mathrm{i}\ell\lambda} \left(\widetilde{\varphi}(\zeta) \mathrm{e}^{-\zeta/g_s} \right) \\ &= \frac{\mathrm{i}}{2\pi} \sum_{\ell \geq 1} \left\{ \frac{1}{\ell} \left(\frac{\mathcal{A}}{g_s} \right) + \frac{1}{\ell^2} \right\} \mathrm{e}^{-\ell \mathcal{A}/g_s}, \end{aligned} \tag{3.51}$$

where

$$\mathcal{A} = 2\pi i t. \tag{3.52}$$

If we consider the singularities in the negative imaginary axis, we find the same result, but with negative ℓ. By comparing the second line in (3.51) to (2.55), we can read the trans-series:

$$\varphi_{\ell\mathcal{A}}(g_s) = \frac{1}{2\pi} \left\{ \frac{1}{\ell} \left(\frac{\mathcal{A}}{g_s} \right) + \frac{1}{\ell^2} \right\} \mathrm{e}^{-\ell\mathcal{A}/g_s}, \qquad \ell \in \mathbb{Z}\backslash\{0\}. \tag{3.53}$$

The second line of (3.51) looks like a sum over multi-instantons with action $\mathcal{A}$. We will refer to (3.53) as a *Pasquetti–Schiappa ℓ-instanton amplitude*. The expansion around each instanton is truncated at next-to-leading order in the coupling constant g_s. The instanton action in this case does not have a clear interpretation in terms of a saddle configuration. However, it can be interpreted geometrically, as we will see in a moment.

What should we do in a more general matrix model? Let us write the partition function as

$$Z(N, g_s) = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{2\pi} \exp\left\{ -\frac{1}{g_s} \left(\sum_{i=1}^{N} V(\lambda_i) - \frac{t}{N} \sum_{i \neq j} \log\left(\lambda_i - \lambda_j\right)^2 \right) \right\}. \tag{3.54}$$

We want to take a limit where g_s is small and t is fixed, so that N is large. The two terms inside the parenthesis are then of the same order (i.e. $\mathcal{O}(N)$) and we see that the 't Hooft parameter controls the strength of the Vandermonde interaction. Let us suppose that t is very small, so that we can neglect this interaction. In this limit (3.54) factorizes into a product of standard saddle-point integrals,

$$Z(N, g_s) \to \left(f(g_s)\right)^N, \qquad t \to 0, \tag{3.55}$$

where

$$f(g_s) = \int \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{-\frac{1}{g_s} V(\lambda)} \tag{3.56}$$

which could be computed by saddle–point methods. When t is no longer small, we have to take into account the Vandermonde determinant. Since this induces a repulsion between eigenvalues, they will no longer sit at the saddle points of $V(\lambda)$, but rather along an interval or arc $\mathcal{C} = [a, b]$ around the saddle-point. In the one-cut matrix models we will consider, they will spread along a single critical point of the potential, $x_\star$. The distribution of the eigenvalues on the interval $\mathcal{C}$ around $x_\star$ is described by a *density function* $\rho(\lambda)$, which is normalized to one, i.e.

$$\int_{\mathcal{C}} \rho(\lambda) \mathrm{d}\lambda = 1. \tag{3.57}$$

It turns out that, for polynomial potentials, this function can be computed in closed form, and determines the $1/N$ expansion (3.41), see Appendix B for a quick summary.

For our purposes, the most important effect is that the potential $V(\lambda)$ gets corrected, at finite t, to an *effective potential* of the eigenvalues,

$$V_{\text{eff}}(\lambda) = V(\lambda) - 2t \int \mathrm{d}\lambda' \, \rho(\lambda') \log |\lambda - \lambda'|, \tag{3.58}$$

The saddle-point condition for obtaining the $1/N$ expansion in this one-cut configuration is that, on the interval $\mathcal{C}$, the effective potential must be constant, i.e. independent of λ:

$$V_{\text{eff}}(\lambda) = t\xi(t), \qquad \lambda \in \mathcal{C}. \tag{3.59}$$

By taking a further derivative w.r.t. λ one finds the equation

$$\frac{1}{2t} V'(\lambda) = \mathrm{P} \int_{\mathcal{C}} \frac{\rho(\lambda')\mathrm{d}\lambda'}{\lambda - \lambda'}, \qquad \lambda \in \mathcal{C} \tag{3.60}$$

where P denotes the principal value of the integral. This is an integral equation that allows one in principle to compute $\rho(\lambda)$, given the potential $V(\lambda)$, as a function of the 't Hooft parameter t. For example, for the Gaussian matrix model above, it is given by

$$\rho(\lambda) = \frac{1}{2\pi t} \sqrt{4t - \lambda^2}, \qquad b = -a = 2\sqrt{t}, \tag{3.61}$$

which is the famous Wigner–Dyson semicircle law. In general, the density function of a polynomial potential in the one-cut matrix model has the form

$$\rho(\lambda) = \frac{M(\lambda)}{2\pi t} \sqrt{(b - \lambda)(\lambda - a)}, \tag{3.62}$$

where $M(\lambda)$ is a polynomial in λ. It is sometimes useful to introduce the *spectral curve*

$$y(x) = M(x)\sqrt{(x - b)(x - a)}, \tag{3.63}$$

which gives another way to parametrize the density of eigenvalues. Geometrically, $y(x)$ describes a (in general, degenerate) Riemann surface with a cut along the interval $[a, b]$. The fact that the integral of $\rho(\lambda)$ is normalized to 1 on the interval $\mathcal{C}$ means that

$$\int_a^b y(x)\mathrm{d}x = 2\pi \mathrm{i} t. \tag{3.64}$$

The quantity in the left was identified as an instanton action in the Gaussian matrix model. We can now see that it has a geometric interpretation, as promised above: it is a *period* of the differential $y(x)\mathrm{d}x$ on the Riemann surface described by $y(x)$ itself, along the cycle encircling the cut $[a, b]$.

One of the most important applications of spectral curves is that they allow to compute in principle the full $1/N$ expansion by using the technology of *topological recursion* [37]. This means that to define the theory at all orders in perturbation theory it is sufficient to give the spectral curve (and a differential on it, which in the case above is simply $y(x)\mathrm{d}x$.) However, topological recursion by itself does not allow to define the theory beyond perturbation theory[2].

[2]The "non-perturbative topological recursion" of [38] is a method of computing non-perturbative effects in matrix models largely based on [5], but it does not provide by itself a non-perturbative definition.

Example 3.3. In the Gaussian matrix model, we have

$$y(x) = \sqrt{x^2 - 4t} \tag{3.65}$$

and one verifies (3.64) immediately. ◊

The effective potential is constant in $[a, b]$, but outside of this interval is a function which might have additional critical points. In fact, by using the properties of the density of eigenvalues, it is possible to show that

$$V'_{\text{eff}}(x) = y(x), \qquad x \notin \mathcal{C}, \tag{3.66}$$

therefore points outside $\mathcal{C}$ where $M(x) = 0$ are critical points of the effective potential. Let us assume that there is such a critical point, which we will denote by x_0 (when $t \to 0$, this point becomes a critical point of $V(x)$). Let us also assume that, instead of integrating the N eigenvalues of the matrix model around the original critical point at $\lambda = x_\star$, we integrate one of the eigenvalues around the critical point x_0, while the other $N-1$ are still integrated around $x_\star$. This is the analogue of calculating an exponential integral around a different saddle point. This new sector will have a weight

$$\exp\left(-\frac{1}{g_s}(V_{\text{eff}}(x_0) - V_{\text{eff}}(b)\right), \tag{3.67}$$

where in the exponent we have the difference between the effective potential at x_0, and the effective potential in the interval $[a, b]$ around $x_\star$ (where it is constant). Therefore, we can think of

$$A = V_{\text{eff}}(x_0) - V_{\text{eff}}(b), \tag{3.68}$$

which is a non-trivial function of t, as the action of an instanton. The integral where one eigenvalue has "tunneled" to x_0 gives then an exponentially suppressed trans-series. These are examples of *large N instantons*, which in the case of matrix models can then be obtained by this mechanism of *eigenvalue tunneling* [2, 39]. We should note that the instanton action has also a geometric interpretation (this was pointed out in [40] in the context of non-critical strings). In view of (3.66), we have

$$A = \int_b^{x_0} y(x)\mathrm{d}x. \tag{3.69}$$

where we have assumed that $x_0 > b$. This is a period integral along the other cycle of the (degenerate) Riemann surface described by $y(x)\mathrm{d}x$, from the endpoint of the cut $\mathcal{C}$ to the singular point x_0, as shown in Fig. 5.

By using matrix model technology, it is possible to calculate very explicitly the corresponding trans-series. It has the structure

$$Z^{(1)} = \mathrm{i}\, g_s^{1/2}\, F_{1,1}\, \exp\left(-\frac{A}{g_s}\right)\left\{1 + \sum_{n=1}^{\infty} F_{1,n+1} g_s^n\right\}. \tag{3.70}$$

where the coefficients $F_{1,n+1}$ are explicit functions of the 't Hooft coupling [5, 6, 41]. There is an alternative procedure to calculate these trans-series, based on orthogonal polynomials, which makes direct contact with the simple approach based on Painlevé I in non-critical strings [6]. It turns out that one can encode both the perturbative genus expansion and the non-perturbative corrections, at least in some simple matrix models, by studying *finite difference equations*.

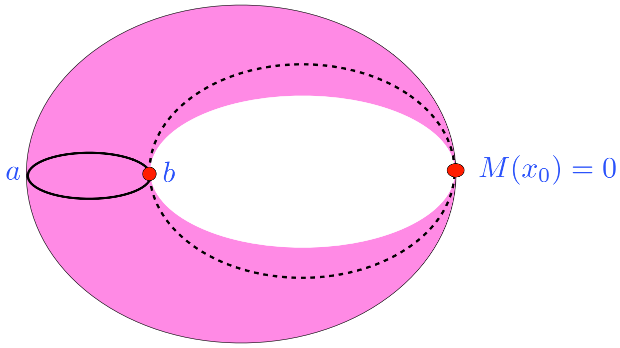

Figure 5: The spectral curve $y(x)$ has a singular point at the nontrivial saddle x_0.

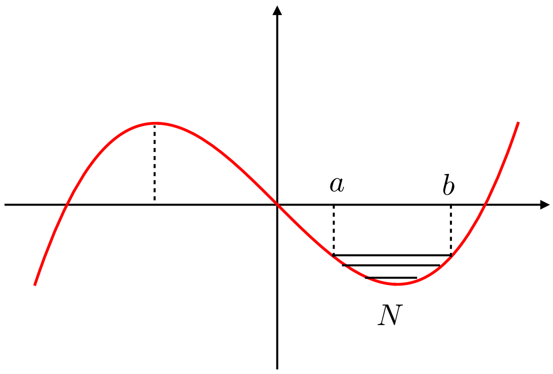

Figure 6: The cubic potential (3.71) has two critical points at $x = \pm 1$. We consider the one-cut configuration where N eigenvalues sit around the stable critical point at $x = 1$.

Let us give an interesting example which also makes contact with the pure gravity theory considered in the previous section. Consider the matrix model with potential

$$V(x) = -x + \frac{x^3}{3}. \tag{3.71}$$

This potential has a stable critical points at $x_\star = 1$, and an unstable critical point at $x = -1$, see Fig. 6. We will consider the one-cut configuration around $x_\star$. We will parametrize the endpoints of the cut by

$$a = \upsilon - \delta, \qquad b = \upsilon + \delta. \tag{3.72}$$

One can use matrix model technology to show that υ and δ are determined by

$$\upsilon(1 - \upsilon^2) = t, \qquad \delta^2 = 2(1 - \upsilon^2), \tag{3.73}$$

We note that $\upsilon = 1 + \mathcal{O}(t)$. The spectral curve can be also obtained in an easy way and it is given by

$$y(x) = (x + \upsilon)\sqrt{x^2 + 2x\upsilon + 3\upsilon^2 - 2}. \tag{3.74}$$

Exercise 3 ✎. Derive (3.73) and (3.74) from (B.24) and (B.26), respectively.

It follows from (3.74) that there is a critical point at $x_0 = -\upsilon$, which as $t \to 0$ becomes the unstable critical point of the potential at $x = -1$. One can easily calculate the effective potential by integrating $y(x)\mathrm{d}x$, and the instanton action for tunneling one eigenvalue to $x_0 = -\upsilon$ turns out to be given by

$$A(t) = \int_a^{x_0} y(x)\mathrm{d}x = \frac{2}{3}\sqrt{6\upsilon^2 - 2} + 2t\log\left(2\upsilon - \sqrt{6\upsilon^2 - 1}\right) - t\log(2 - 2\upsilon^2). \tag{3.75}$$

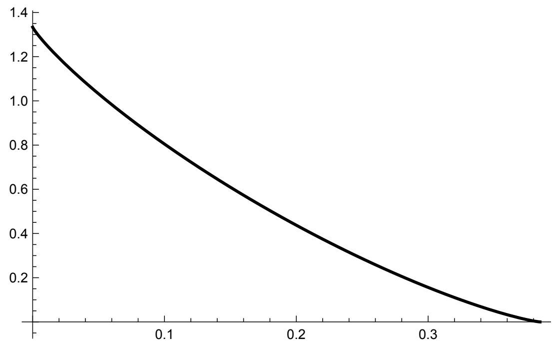

Figure 7: The instanton action $A(t)$ in (3.75) as a function of $0 \leq t \leq t_c$, where t_c is defined in (3.77). Near $t = 0$ is just the "classical" difference of actions between the critical points of the potential, while near $t = t_c$ has critical behavior (3.80) characterized by the exponent $1 - \gamma/2$.

This function has many interesting properties, see its plot in Fig. 7. First of all, as $t \to 0$ we find

$$A(t) = \frac{4}{3} + t\left(\log\left(\frac{t}{8}\right) - 1\right) + \mathcal{O}(t^2), \tag{3.76}$$

and as expected the leading term is the difference between the potentials at the two different critical points $V(-1) - V(1)$. This is a manifestation of the fact that the matrix model is a deformation of the usual one-dimensional integral. The second property of this instanton action is that it controls the factorial divergence of the $1/N$ expansion of the cubic matrix model in the one-cut phase. Finally, the action *vanishes* for a special value of t, namely

$$t_c = \frac{2}{3\sqrt{3}}, \qquad \upsilon_c = \frac{1}{\sqrt{3}}. \tag{3.77}$$

It is easy to see that, at this point, a collides with x_0, and the $1/N$ expansion essentially breaks down (in fact, $t = t_c$ signals the first singularity of the functions $F_g(t)$). This point indicates a *phase transition* and we can now "zoom" around that point. Let us denote

$$\tau = t_c - t, \qquad \zeta = \upsilon - \upsilon_c. \tag{3.78}$$

It is easy to see that

$$\zeta \approx 3^{-\frac{1}{4}}\tau^{1/2}, \qquad \tau \to 0, \tag{3.79}$$

and in particular, near the critical point, the instanton action scales as

$$A \approx \frac{8}{5} 3^{5/8} \tau^{5/4}, \qquad \tau \to 0. \tag{3.80}$$

We note the non-trivial scaling exponent, equal to $1 - \gamma/2$, where γ is the string susceptibility. We know introduce the scaling variable

$$\kappa^{5/4} = \mathfrak{c} \tau^{5/4} g_s^{-1}, \tag{3.81}$$

where $\mathfrak{c}$ is a normalization constant. The *double-scaling limit* is defined as the limit in which $\tau, g_s \to 0$ but κ is fixed. In particular, the instanton action becomes in this limit

$$\frac{A}{g_s} = \frac{8}{5} 3^{5/8} \mathfrak{c}^{-1} \kappa^{5/8}, \tag{3.82}$$

which is precisely the instanton action of pure gravity with the choice $\mathfrak{c} = 3^{1/8}$. With this very same choice, one can check that the genus expansion (3.41) of the cubic matrix model becomes (3.9), in the double scaling limit.

It is worth pointing out that the spectral curve has itself a double-scaled form near criticality. We zoom near the point where a and x_0 coalesce, so we set

$$x = -v_c + 2\zeta\xi, \tag{3.83}$$

where ξ is the variable parametrizing the double-scaled curve. It can be easily seen that the power of ζ in the scaling equation (3.83) is the appropriate one for a non-trivial limit, and the coefficient 2 is chosen for convenience. One finds in this way

$$\frac{1}{g_s} y(\xi) \mathrm{d}\xi = \frac{4}{3}(1 + 2\xi)\sqrt{2 - 2\xi}. \tag{3.84}$$

The points x_0 and a are mapped to $-1/2$ and 1, respectively, and the instanton action in the double-scaled theory can be directly computed as

$$A = \int_{-1/2}^{1} y(\xi) \mathrm{d}\xi. \tag{3.85}$$

Note that the double-scaled curve is substantially simpler. All the perturbative information about correlation functions and free energies of pure gravity can be obtained from the curve above by using topological recursion.

The main conclusion of this discussion is the following: matrix models at finite 't Hooft coupling, away from criticality, provide a rich generalization of non-critical strings. Their non-perturbative effects can still be computed in some detail. Instanton actions turn out to be *periods* of the natural differential over the spectral curve describing the planar limit. Full instanton trans-series can be computed in many ways, either by doing a direct saddle–point calculation [5] or by using finite difference equations (which rely on orthogonal polynomial techniques for matrix models).

The techniques we have described above can be used to study more recent incarnations of non-critical strings, like JT gravity [42] or the Virasoro minimal string (VMS) [43]. We will briefly consider the case of JT gravity (for VMS, see the recent paper [44]).

Example 3.4. *Instantons in JT gravity.* JT gravity can be defined by specifying a spectral curve, and defining open and closed string invariants associated to it by topological recursion. The relevant curve was first proposed by Eynard and Orantin in [45] as a description of Mirzakhani's recursion for Weil–Petersson volumes in terms of topological recursion. Later, it was argued in [42] to describe Jackiw–Teitelboim gravity in two dimensions. It is given by

$$y(x) = \frac{1}{2\pi} \sin\left(2\pi\sqrt{x}\right), \tag{3.86}$$

and corresponds to a density of eigenvalues of the form

$$\rho(E) = \frac{1}{4\pi^2} \sinh\left(2\pi\sqrt{E}\right), \tag{3.87}$$

in which we take $x = -E$ and pick the imaginary part of $y(x)$, as in the relation between (3.62) and (3.63). We note that, as in the double-scaled curve (3.84), there is a single square root branch point, in this case at $x = 0$. We can write $y(x)$ as in (3.63),

$$y(x) = M(x)\sqrt{x}, \qquad M(x) = \frac{\sin\left(2\pi\sqrt{x}\right)}{2\pi\sqrt{x}}. \tag{3.88}$$

We note that $M(x)$ is in fact an entire function of x, with non-trivial zeroes at

$$x_\ell = \frac{\ell^2}{4}, \qquad \ell \in \mathbb{Z}_{>0}. \tag{3.89}$$

We can then use the approach above to calculate the instanton actions describing non-perturbative effects in the closed string sector [38, 42, 46]. The instanton actions are given by

$$A_\ell = \int_0^{x_\ell} y(x)\mathrm{d}x = \frac{(-1)^{\ell-1}}{4\pi^2}\ell. \tag{3.90}$$

This can be used to predict the asymptotic behaviour. In the case of JT gravity, the free energies F_g compute Weil–Petersson volumes $V_{g,0}$ of moduli spaces of Riemann surfaces with no punctures $\mathcal{M}_{g,0}$. The leading instanton corresponds to $\ell = 1$ and from the above one sees immediately that

$$V_{g,0} \sim (4\pi^2)^{2g}(2g)!, \qquad g \gg 1. \tag{3.91}$$

By using the techniques of [5] one can compute the one-instanton to one loop and obtain [38, 42]

$$Z^{(1)} \approx \frac{\mathrm{i} g_s^{5/2}}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{4\pi^2 g_s}}, \tag{3.92}$$

which leads to the refined asymptotics [42]

$$V_{g,0} \sim \frac{1}{\sqrt{2\pi^3}} \left(4\pi^2\right)^{2g-5/2} \Gamma\left(2g - \frac{5}{2}\right). \tag{3.93}$$

Note the structural similarity with the result in (3.29). ◊

3.3 More general matrix models?

In polynomial matrix models, and in the one-cut case, one can determine exponentially small corrections with the techniques above. However, many matrix models relevant for string theory are not polynomial. The most important example is perhaps the ABJM matrix model. It is given by [47]

$$Z(N, g_s) = \frac{1}{(N!)^2} \int \prod_{i=1}^{N} \frac{\mathrm{d}\mu_i \mathrm{d}\nu_i}{(2\pi)^2} \frac{\prod_{i<j} \left(2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right)^2 \left(2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right)^2}{\prod_{i,j} \left(2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right)^2} \mathrm{e}^{-\frac{1}{2g_s}\sum_i(\mu_i^2-\nu_j^2)}. \tag{3.94}$$

It calculates the partition function of ABJM theory [48] on the three-sphere, and it is dual to type IIA theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$, or to M-theory on $\mathbb{S}^7/\mathbb{Z}_k$. The string coupling constant g_s is related to k as

$$g_s = \frac{2\pi \mathrm{i}}{k}, \tag{3.95}$$

and the 't Hooft coupling is usually taken to be

$$\lambda = \frac{N}{k}. \tag{3.96}$$

The matrix model (3.94) can be solved at all orders in the $1/N$ expansion [49, 50], but the calculation of trans-series, exponentially small corrections to this expansion (or, in the string dual, of D-brane corrections to the genus expansion) is far from obvious. One way to address this is to exploit the relationship between the above matrix model and topological strings, and therefore one has to understand first how to find trans-series in topological string theory. This is also a natural step since topological strings are probably the next stage in complexity in string theory, after non-critical strings. They are in a sense similar to matrix models off-criticality. I will not have time to develop this subject in detail here, but I will give some indications about how this goes.

4 Resurgence and topological strings

4.1 Topological strings and the resolved conifold

It is impossible to introduce topological strings in some detail here, and even presenting a coherent summary is challenging. Let me however try the latter.

Non-critical minimal strings are obtained by coupling minimal 2d CFTs to 2d gravity. Similarly, topological string theory is obtained by coupling a 2d topological CFT to 2d topological gravity. The topological CFT is based on a non-linear sigma model whose target is a Calabi–Yau threefold M. The marginal deformations of the topological CFT are the "moduli" of the CY, which we will denote collectively by $\mathbf{t} = (t_1, \cdots, t_s)$. The simplest observable in topological string theory is the free energy $F(\mathbf{t}; g_s)$. As usual, we have a genus expansion of the form

$$F(\mathbf{t}; g_s) \sim \sum_{g \geq 0} F_g(\mathbf{t}) g_s^{2g-2}. \tag{4.1}$$

Since topological strings were introduced in the early 1990s, many ideas and methods have been developed to understand them and to calculate the free energies $F_g(\mathbf{t})$. As is generically the case, these free energies grow doubly-factorially with the genus, so one can ask what is their resurgent structure. In the last four years or so, a very precise picture of this structure has been proposed. In the rest of these lectures I will consider some very simple examples of this problem, and then give you a summary of the complete proposal, since it ties very nicely with the case of non-critical strings and with the crucial rôle of D-branes in string theory in general.

The simplest example of a topological string theory happens when the target is a non-compact CY manifold known as the *resolved conifold*. This manifold is a plane bundle over the two-sphere:

$$X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1. \tag{4.2}$$

There is a single modulus t, which can be regarded as the (complexified) area of the $\mathbb{P}^1$. The results for the free energies at genus g are

$$\begin{aligned} F_0(t) &= \mathrm{Li}_3(\mathrm{e}^{-t}), \\ F_1(t) &= \frac{1}{12}\mathrm{Li}_1(\mathrm{e}^{-t}), \\ F_g(t) &= \frac{(-1)^{g-1}B_{2g}}{2g(2g-2)!}\mathrm{Li}_{3-2g}(\mathrm{e}^{-t}), \qquad g \geq 2. \end{aligned} \tag{4.3}$$

In these expressions,

$$\mathrm{Li}_n(z) = \sum_{k\geq 1} \frac{z^k}{k^n} \tag{4.4}$$

is the polylogarithm function of order n. Let us note that each of these free energies is a (convergent) series in e^{-t}, i.e.

$$F_g(t) = \sum_{d\geq 1} N_{g,d}\mathrm{e}^{-dt}. \tag{4.5}$$

This is in fact a sum over so-called *worldsheet instantons*, i.e. maps from a Riemann surface Σ_g to the CY M (see Fig. 8 for a pictorial representation.) The coefficients $N_{g,d}$ are obtained by doing a complicated integral over the moduli space of such instantons, and are examples of Gromov–Witten invariants. The results (4.3) can be derived by a direct calculation of these invariants [51].

Remark 4.1. The free energies in (4.3) correspond to worldhsheet instantons of finite area. There are also perturbative contributions, given by a polynomial in t of degree 3 in genus zero, and of degree 1 in genus one. For the higher genus amplitudes $g \geq 2$ there is also a contribution of constant maps, or worldsheet instantons of zero area, which we will analyze later on in Exercise 4. △

It is easy to see that, at fixed t, the sequence $F_g(t)$ grows doubly-factorially with the genus, by using the formula

$$\mathrm{Li}_{3-2g}(\mathrm{e}^{-t}) = \Gamma(2g-2) \sum_{k\in\mathbb{Z}} \frac{1}{(2\pi k\mathrm{i} + t)^{2g-2}}, \tag{4.6}$$

which is valid for $g \geq 2$ and $\mathrm{e}^{-t} \neq 1$. What is the resurgent structure of the sequence (4.3)? Thanks to (4.6), we can write the resolved conifold free energies as an infinite sum of Gaussian matrix model free energies:

$$F_g(t) = \sum_{m\in\mathbb{Z}} \frac{B_{2g}}{2g(2g-2)} (\mathrm{i}t + 2\pi m)^{2-2g}, \qquad g \geq 2. \tag{4.7}$$

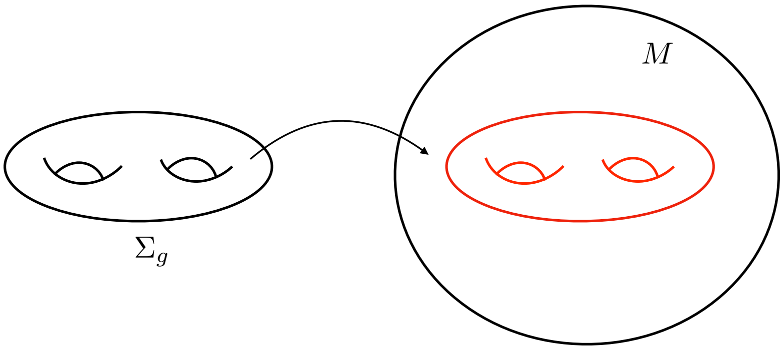

Figure 8: A pictorial representation of a holomorphic map from a Riemann surface Σ_g into a CY M.

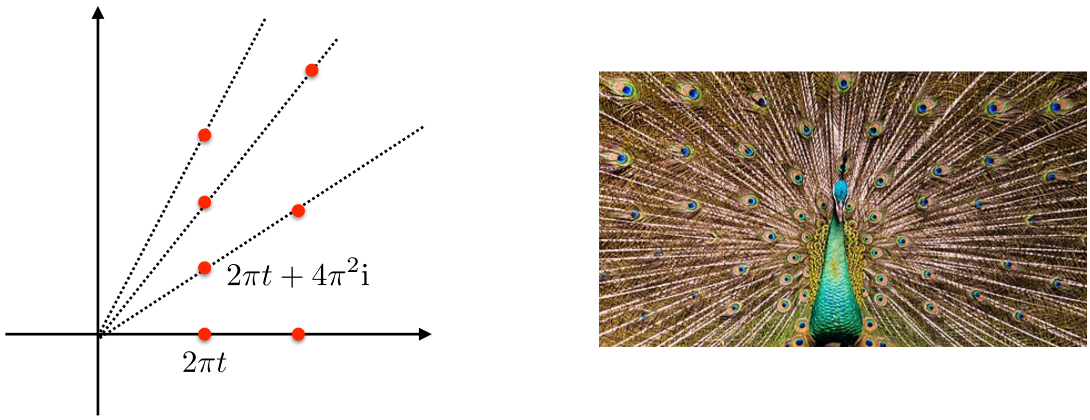

Figure 9: The singularities of the Borel transform in the case of the resolved conifold are located at non-zero integer multiples of $2\pi t + 4\pi^2 \mathrm{i} m$, where $m \in \mathbb{Z}$. As m goes through $\mathbb{Z}$, the singularities form towers. The Stokes rays are somewhat similar to the feathers in a peacock's tail, hence the name "peacock patterns" for this structure of singularities (the picture on the right was made by Mallory Cessair and is courtesy of *Wikimedia Commons*).

Therefore, the singularities of the Borel transform are of the form $\ell \mathcal{A}_m$, where $\ell \in \mathbb{Z}\backslash\{0\}$ and the "action" $\mathcal{A}_m$ is labelled by an additional integer m:

$$\mathcal{A}_m = 2\pi t + 4\pi^2 \mathrm{i} m, \qquad m \in \mathbb{Z}. \tag{4.8}$$

The corresponding trans-series are also of the Pasquetti–Schiappa form (this resurgent structure was also found in [36]). A plot of the very first singularities is shown in Fig. 9. Note that they are organized in infinite towers, and the singularities in each tower are obtained by changing the value of m in (4.8). The Stokes rays going through the singularities $\ell \mathcal{A}_m$ for a fixed m and $\ell \in \mathbb{Z}_{>0}$ accumulate along the imaginary axis. Such a pattern of Borel singularities is common in topological string theory on local CY manifolds, but also in complex Chern–Simons theory. Graphically, the set of Stokes rays going through Borel singularities is similar to the tail of a peacock, and for this reason these patterns were called "peacock patterns" in [52].

What is the interpretation of the new ingredient appearing in (4.8), namely, the towers with $m \in \mathbb{Z}$? It turns out that they correspond to m *D0 branes* bound to a D2 brane wrapping the $\mathbb{P}^1$ in the resolved conifold (if m is negative, we should view them ast $|m|$ anti-D0 branes). If we think about the CY manifold as a compactification manifold for type IIA strings or M-theory, a D-brane configuration of this type leads to BPS particles in four (or five) dimensions. Therefore, the resurgent structure of the free energies is governed by this D-brane configuration. We will make this clearer in a moment.

The above description of the resurgent structure looks very different from the one we made in the context of matrix models, but it is not so. The reason is *mirror symmetry*. In the above description, we have assumed the so-called *A-model* of topological string theory, where the free energies are obtained by looking at maps from Riemann surfaces to the CY manifold X. There is another type of topological string, called the B-model. This is also a non-linear sigma model but the target is a different manifold, the *mirror manifold*. In the case of the resolved conifold this manifold is also a spectral curve, described by the equation

$$
\mathrm{e}^x + \mathrm{e}^y + \mathrm{e}^{-y+x} + \mathrm{e}^{-t} = 0. \tag{4.9}
$$

We note that we have written the equation in exponential coordinates. We can now consider periods of the differential $y(x)\mathrm{d}x$ on cycles of this curve, and we note that $y(x)$ has a logarithmic form,

$$
y(x) = \log\left[-\frac{\mathrm{e}^x + \mathrm{e}^{-t} \pm \sqrt{(\mathrm{e}^x + \mathrm{e}^{-t})^2 - 4\mathrm{e}^x}}{2}\right]. \tag{4.10}
$$

It turns out that the periods of this curve are precisely of the form (4.8), and the appearance of D0 branes is related to the multivaluedness of the logarithm. In fact, one can extract a density function from (4.10) given by the discontinuity across the branch cut of the square root, which reads

$$
\rho(x) = \frac{1}{\pi}\tan^{-1}\left[\frac{\sqrt{4\mathrm{e}^x - (\mathrm{e}^x + \mathrm{e}^{-t})^2}}{\mathrm{e}^x + \mathrm{e}^{-t}}\right]. \tag{4.11}
$$

One can check that

$$
\int_{a_-}^{a_+} \rho(x)\mathrm{d}x = t, \qquad a_\pm = \log\left(2 - \mathrm{e}^{-t} \pm 2\sqrt{1 - \mathrm{e}^{-t}}\right). \tag{4.12}
$$

We note that $a_\pm = \pm 2\sqrt{t} + \mathcal{O}(t)$ for small t, so this density becomes Wigner semi-circle law by setting $x = \hbar u$, $t = \hbar^2 \tau$, and taking the limit $\hbar \to 0$.

Exercise 4 ✎. The free energies (4.3) represent the contributions of worldsheet instantons of finite area, associated to a non-trivial homology class in the target. However, one can consider *constant* maps from Σ_g to a CY M. The contribution of such maps at fixed genus $g \geq 2$ is given by $c_g \chi$ [53], where χ is the Euler characteristic of M and [51, 54]

$$
c_g = \frac{(-1)^{g-1} B_{2g} B_{2g-2}}{4g(2g-2)(2g-2)!}. \tag{4.13}
$$

Show that it can be written as

$$
c_g = -\frac{B_{2g}}{2g(2g-2)} \sum_{m=1}^{\infty} (2\pi m)^{2-2g} \tag{4.14}
$$

Deduce that, if

$$\varphi_{\rm cm}(g_s) = \sum_{g\geq 2} c_g g_s^{2g-2}, \tag{4.15}$$

one has the discontinuity formula

$$s_+(\varphi_{\rm cm})(g_s) - s_-(\varphi_{\rm cm})(g_s) = -\frac{\rm i}{2\pi}\sum_{\ell\geq 1}\sigma(\ell)\left\{\frac{1}{\ell}\left(\frac{\mathcal{A}}{g_s}\right)+\frac{1}{\ell^2}\right\}{\rm e}^{-\ell\mathcal{A}/g_s}, \tag{4.16}$$

where

$$\mathcal{A} = 4\pi^2{\rm i}, \qquad \sigma(\ell) = \sum_{m|\ell}\left(\frac{\ell}{m}\right)^2. \tag{4.17}$$

This result was derived in [55] with a different technique, This can be interpreted as due to a tower of BPS D0 branes attached to the CY manifold (see e.g. [56]).

4.2 Resurgence and D-branes on general Calabi–Yau manifolds

In an *arbitrary* CY M we have $s = b_2(M)$ homology two-classes, and corresponding moduli $\mathbf{t} = (t_1, \cdots, t_s)$, which support both worldsheet instantons and D2-D0 BPS states. What is the resurgent structure in this case? A good starting point to address this issue is to appeal to a famous result of Gopakumar and Vafa [57] which says that the free energies of the topological string can be computed by counting BPS states obtained from configurations of D2-D0 branes wrapping the s homology classes. These BPS states have mass

$$\mathcal{A}_{\mathbf{d},m} = 2\pi\mathbf{d}\cdot\mathbf{t} + 4\pi^2{\rm i}m, \qquad m\in\mathbb{Z}, \tag{4.18}$$

where d_i, $i = 1, \cdots, s$, refers to the number of times that the D2 branes "wraps" the i-th homology cycle, and m is the number of D0 branes. The precise formula is

$$F^{\rm GV}(\mathbf{t}; g_s) = \sum_{g\geq 0}\sum_{\mathbf{d}}\sum_{w=1}^{\infty}\frac{1}{w}n_g^{\mathbf{d}}\left(2\sin\frac{wg_s}{2}\right)^{2g-2}{\rm e}^{-w\mathbf{d}\cdot\mathbf{t}}, \tag{4.19}$$

where $n_g^{\mathbf{d}}$ are the so-called *Gopakumar–Vafa (GV) invariants*. Physically, they are indices counting the BPS states with mass (4.18) (with signs). The integer g (which should not be confused with the genus of the wordlsheet instantons) is related to the spin of the BPS particles. Mathematically, these invariants can be interpreted, roughly, as Euler characteristics of moduli spaces of D2 branes in the CY manifold (see e.g. [58] for a beautiful discussion with examples).

Exercise 5 ✎. Show that the expression (4.19) leads to the following formula for $F_g(\mathbf{t})$ [58]. For $g = 0$, $g = 1$, one has

$$\begin{aligned} F_0(\mathbf{t}) &= \sum_{\mathbf{d}} n_0^{\mathbf{d}}\,{\rm Li}_3\left({\rm e}^{-\mathbf{d}\cdot\mathbf{t}}\right), \\ F_1(\mathbf{t}) &= \sum_{\mathbf{d}}\left(\frac{n_0^{\mathbf{d}}}{12}+n_1^{\mathbf{d}}\right){\rm Li}_1\left({\rm e}^{-\mathbf{d}\cdot\mathbf{t}}\right), \end{aligned} \tag{4.20}$$

while for $g \geq 2$ one has

$$F_g(\mathbf{t}) = \sum_{\mathbf{d}}\left(\frac{(-1)^{g-1}B_{2g}n_0^{\mathbf{d}}}{2g(2g-2)!}+\frac{2(-1)^g n_2^{\mathbf{d}}}{(2g-2)!}+\cdots-\frac{g-2}{12}n_{g-1}^{\mathbf{d}}+n_g^{\mathbf{d}}\right){\rm Li}_{3-2g}\left({\rm e}^{-\mathbf{d}\cdot\mathbf{t}}\right). \tag{4.21}$$

We note that the results for the resolved conifold (4.3) are a particular case of this, and in this case $n_0^d = 1$ for $d = 1$ and zero otherwise. □

The GV form of the free energies gives us some information on the resurgent structure of the topological string, which was already discussed in [31, 36, 59] and further developed in [55, 56]. Let us consider the expression (4.21) for $g \geq 2$, which is valid for the free energies in the large radius frame, and near the so-called large radius point $\mathrm{Re}(t_i) \gg 1$. The first term is a sum of free energies for the resolved conifold, and it is easy to see that it is the only term growing factorially with g. Therefore, we expect that, close enough to the large radius point, we will have a sequence of Borel singularities at $\ell \mathcal{A}_{\mathbf{d},m}$, where $\mathcal{A}_{\mathbf{d},m}$ is given by (4.18) and $\mathbf{d}$ are the values of the degrees which lead to a non-zero GV invariant $n_0^{\mathbf{d}}$. The trans-series associated to these singularities are of the Pasquetti–Schiappa form

$$n_0^{\mathbf{d}}\, \varphi_{\ell \mathcal{A}_{\mathbf{d},m}}(g_s). \tag{4.22}$$

Therefore, $n_0^{\mathbf{d}}$ (which is an integer) is to be interpreted as the Stokes constant associated to the sequence of singularities (4.18). Explicit numerical calculations show that the Borel singularities (4.18) indeed do occur, and their Stokes constants are given by the genus zero GV invariants [55, 59].

Therefore, the same D0-D2 BPS states that lead to the GV representation and invariants also determine a part of the resurgent structure. Note however that, from all the GV invariants, only the ones with $g = 0$ appear as Stokes constants. However, in a general CY manifold there are many other BPS states associated to D-branes wrapping more general cycles. For definiteness we will first use the A-model picture of BPS states, in which D-branes wrap even-dimensional cycles of the CY M.

BPS states are characterized by a charge $\boldsymbol{\gamma} \in \Gamma$, where

$$\Gamma = H^{\mathrm{ev}}(M, \mathbb{Z}), \tag{4.23}$$

is a lattice of rank is $2(s+1)$, where we recall that $s = b_2(M)$. If we choose a basis for this lattice, we can write $\boldsymbol{\gamma}$ in terms of two pairs of vectors of rank $s+1$, with entries, $\boldsymbol{\gamma} = (c^I, d_I)$, where $I = 0, 1, \cdots, s$. We can think of c^0, d_0 as D6 and D0 brane charges, respectively, and of c^a, d_a, $a = 1, \cdots, s$, as D4 and D2 brane charges, respectively. The central charge corresponding to such an element of Γ can be calculated from the moduli and the genus zero free energy of the topological string on M. In order to obtain formulae valid for compact CYs, we have to use projective coordinates. This means that, in addition to the Kähler moduli t_i, $i = 1, \cdots, s$, one needs an additional period X^0. We then define

$$X^I = X^0 t_I, \qquad I = 0, \cdots, s, \tag{4.24}$$

with the understanding that $t_0 = 1$. One also defines,

$$\mathfrak{F}_0(X^I) = (X^0)^2 F_0(\mathbf{t}). \tag{4.25}$$

as well as

$$\mathcal{F}_I = \frac{\partial \mathfrak{F}_0}{\partial X^I}. \tag{4.26}$$

In terms of these quantities, the central charge of a D-brane with charge $\boldsymbol{\gamma}$ is given by

$$Z_{\boldsymbol{\gamma}} = c^I \mathcal{F}_I + d_I X^I, \tag{4.27}$$

where summation over the repeated indices is understood. Let us note that, in the case of toric CY manifolds, D6 branes decouple, and the charge $\boldsymbol{\gamma}$ is specified by $2s+1$ integers which we will denote by c^a, d_a and m, with $a = 1, \cdots, s$. The central charge reads then

$$Z_{\boldsymbol{\gamma}} = c^a \frac{\partial F_0}{\partial t_a} + d_a t_a + 4\pi^2 \mathrm{i} m. \tag{4.28}$$

To each of these D-brane configurations one can associate a generalized BPS index called, in the mathematics literature, a *Donaldson–Thomas invariant* $\Omega_{\boldsymbol{\gamma}}$. In the particular case of a D0-D2 BPS state, with $\boldsymbol{\gamma} = (0, \cdots, 0, d_I)$, one has

$$\Omega_{\boldsymbol{\gamma}=(0,\cdots,0,d_I)} = n_0^{\mathbf{d}}, \qquad \mathbf{d} = (d_1, \cdots, d_s), \tag{4.29}$$

The general conjecture is that the Borel singularities of the topological string free energies occur at non-zero integer multiples of the central charge $\ell Z_{\boldsymbol{\gamma}}$ of BPS states with a non-zero index $\Omega_{\boldsymbol{\gamma}}$. These indices or invariants are Stokes constants of the resurgent structure. This conjecture imples that there are many other singularities in the Borel transform, other than the ones (4.18) obtained from the GV formula. For example, D-branes whose mass vanish at conifold points (these are typically D4 branes or D6 branes) appear in the resurgent structure. This follows from the universal behavior of the topological string at conifold points [60] and it can be verified in some CYs by using the numerical techniques reviewed in Appendix A. In Fig. 10, we show the numerical Borel plane for the free energies in the case of the so-called local $\mathbb{P}^2$ geometry (the total space of the canonical bundle over $\mathbb{P}^2$). In addition to the tower of singularities (4.18), one can see a pair of singularities in the imaginary axis which correspond to a D4 brane.

We should also mention that in the so-called B-model picture, obtained after mirror symmetry, the location of the Borel singularities can be interpreted as periods, as in the simplest cases of non-critical strings and matrix models. In the mirror CY manifold $M^\star$ there is a holomorphic 3-form and one has

$$Z_{\boldsymbol{\gamma}} = \int_{\mathcal{G}} \Omega, \qquad \mathcal{G} = c^I \beta_I + d_I \alpha^I, \tag{4.30}$$

where α^I, β_I is a symplectic basis of 3-cycles in $M^\star$.

The trans-series associated to these singularities can be calculated in closed form. For general charge $\boldsymbol{\gamma}$, they are no longer of the Pasquetti–Schiappa form. Let us give an example in the case of toric CY manifolds with one single modulus. Let us consider a singularity of the form

$$\mathcal{A} = c \frac{\partial F_0}{\partial t} + dt + 4\pi^2 \mathrm{i} m, \qquad m \in \mathbb{Z}. \tag{4.31}$$

We define the modified genus zero free energy as

$$\widehat{F}_0(t) = F_0(t) + \frac{d}{2c} t^2 + \frac{4\pi^2 \mathrm{i} m}{c} t, \tag{4.32}$$

which is such that

$$\mathcal{A} = c \frac{\partial \widehat{F}_0(t)}{\partial t}. \tag{4.33}$$

Let us consider the formal total free energy

$$\widehat{F}(t; g_s) = \widehat{F}_0(t) g_s^{-2} + \sum_{g \geq 1} F_g(t) g_s^{2g-2}, \tag{4.34}$$

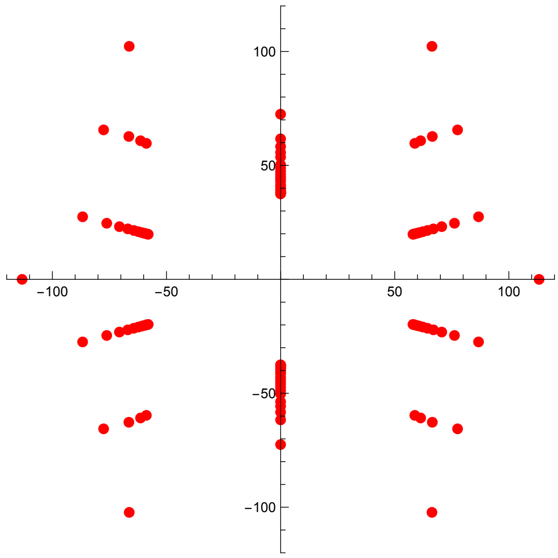

Figure 10: A numerical approximation to the Borel singularities of the free energies $F_g(t)$ in the large radius frame, for local $\mathbb{P}^2$, and for a value of the modulus $z = -10^{-4}$ (see e.g. [61] for details). This is based on a sequence of free energies up to $g = 120$. The singularity in the positive imaginary axis corresponds to (3.52) and is due to the D4 state that becomes massless at the conifold point. It occurs at $2\pi\mathrm{i}\lambda \approx 37.5\mathrm{i}$, where $2\pi\mathrm{i}\lambda$ is a period of the mirror CY. Since $2\pi t \approx 57.9 + 2\pi^2\mathrm{i}$, the other singularities in the plot correspond to the tower (4.18) with $d = 1$, $m = 0, 1, -2$. As expected, singularities appear in pairs $\mathcal{A}, -\mathcal{A}$.

which differs from $F(t; g_s)$ only in the genus zero part. Then, the tran-series associated to the first instanton sector is given by

$$F^{(1)}(t; g_s) = \frac{\mathsf{S}}{2\pi} \left(1 + c g_s \partial_t \widehat{F}(t - c g_s; g_s)\right) \mathrm{e}^{\widehat{F}(t - c g_s; g_s) - \widehat{F}(t; g_s)}, \tag{4.35}$$

where S is an appropriate Stokes constant. Note that, by expanding the exponent in series in g_s, we find immediately the behaviour $\exp(-\mathcal{A}/g_s)$, where $\mathcal{A}$ is (4.33). This formula was first found in [61], building on [62, 63]. A simplified derivation can be found in [64]. It has many interesting aspects. First of all, the trans-series is completely determined by the perturbative sector (this is also the case for higher order instanton sectors). Second, it is closely related to eigenvalue tunneling. If we think about t as $N g_s$, the exponent appearing here can be understood as the tunneling of c eigenvalues from the perturbative background. Third, this formula gives the correct large N instanton trans-series for more general matrix models, including the ABJM matrix model (3.94) [65].

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A Resurgence in practice: numerical methods

The main goals of resurgence are to determine the structure of singularities of Borel transforms, and to perform lateral Borel resummations. In practice, we have a finite number of terms in the power series, and we have to extract as much information as possible from these terms. We will now explain how to do this in practice.

A.1 Large order behavior

The "leading" singularity of the Borel transform is the one which is closest to the origin, and we know from (2.61) and (2.63) that it governs the large order behavior of the perturbative series. Let us first sketch a proof of this result.

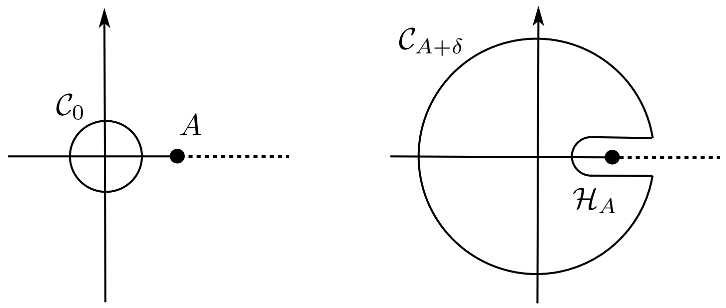

Figure 11: Contour deformation in (A.2).

The coefficients of the Borel transform are given by the Cauchy formula

$$\frac{a_k}{k!} = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}_0} \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta, \tag{A.1}$$

where $\mathcal{C}_0$ is a small circle around $\zeta = 0$. Let us choose a $\delta > 0$. We now enlarge the contour $\mathcal{C}_0$ to a contour $\mathcal{C}_{A+\delta} \cup \mathcal{H}_A$, where $\mathcal{C}_{A+\delta}$ is a circle of radius $A+\delta$, minus an arc, and $\mathcal{H}_A$ is a Hankel contour centered around A, see Fig. 11. By deforming the contour we find

$$\frac{a_k}{k!} = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}_{A+\delta}} \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{H}_A} \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta. \tag{A.2}$$

The first integral can be estimated to be of order $\mathcal{O}\left((A+\delta)^{-k}\right)$. Since, as we will now show, the leading large k asymptotics goes like A^{-k}, and $A+\delta > A$, this is a subleading, exponentially small correction as k grows large, and it does not contribute to the leading $1/k$ asymptotics. The integral around the contour $\mathcal{H}_A$ can be evaluated by using (2.26). and we obtain

$$\frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{H}_A} \frac{\widehat{\varphi}(\zeta)}{\zeta^{k+1}} \mathrm{d}\zeta = \frac{1}{2\pi} \sum_{n \geq 0} \hat{c}_n \int_0^\delta \frac{\xi^n}{(A+\xi)^{k+1}} \mathrm{d}\xi, \tag{A.3}$$

where we have set $\zeta = A + \xi$. An easy estimate shows that, at fixed n,

$$\begin{aligned} \int_0^\delta \frac{\xi^n}{(A+\xi)^{k+1}} \mathrm{d}\xi &= \int_0^\infty \frac{\xi^n}{(A+\xi)^{k+1}} \mathrm{d}\xi + \mathcal{O}\left((A+\delta)^{-k}\right) \\ &= A^{n-k} \frac{\Gamma(k-n)\Gamma(n+1)}{\Gamma(k+1)} + \mathcal{O}\left((A+\delta)^{-k}\right). \end{aligned} \tag{A.4}$$

We conclude that the asymptotics of a_k is given by (2.61). We can then try to extract from the behavior of the a_k at large k the information of the trans-series (2.33) associated to this singularity, namely its location (A), the exponent of the prefactor (b), and the coefficients c_n.

The procedure is the following. First, we construct an *auxiliary sequence* which asymptotes the number $\mathcal{N}$ we want to extract. Second, we perform *accelerations* of the sequence which make convergence faster. This gives a *numerical estimate* of the number $\mathcal{N}$. Finally, one can try to *guess* an exact form for this number from the numerical estimate, by writing it e.g. as a suitable combination of rational and transcendental numbers.

We will illustrate this procedure with the perturbative series for the ground state energy of the quartic oscillator.

The first number we want to extract is the location of the singularity, A. There are three situations to consider. If $A < 0$, we obtain an alternating sequence for a_n, for sufficiently large n. If $A > 0$, the sequence is non-alternating, and the coefficients a_n have all the same sign as n becomes large enough. If A is complex, the situation is more delicate. If the numbers a_n are real, as it is often the case, this means that the asymptotic behaviour is controlled by the trans-series associated to A and its complex conjugate $A^\star$. This leads generically to an *oscillatory* asymptotics, and one needs more sophisticated methods (discussed e.g. in the Appendix of [66]). A somewhat simpler example of this behavior appears in the pure quartic oscillator, as discussed in section 2.4. Here, we will only consider the case in which A is real.

The auxiliary sequence

$$\frac{n a_n}{a_{n+1}} \sim A + \sum_{k \geq 1} \frac{\sigma_k}{n^k}, \tag{A.5}$$

asymptotes the number A that we want to obtain. Therefore, if we know the coefficients a_n up to the number $n = N^*$, the above sequence gives as its best approximation to A the number

$$A_{N^*} = \frac{(N^* - 1) a_{N^*-1}}{a_{N^*}}. \tag{A.6}$$

This approximation is not necessarily very good, even if N^* is large. For example, in the case of the quartic oscillator, by using 150 terms in $\phi_0(g)$ we obtain

$$A_{150} \approx -0.33435... \tag{A.7}$$

which has only two correct digits. This is clearly seen in the plot of the sequence (A.5), see Fig. 12.

There are however ways to improve substantially the estimate of A without the need to generate further terms in the perturbative series. The idea is to use acceleration techniques, which are indeed crucial in numerical analysis. One of these techniques is the Richardson transform, or Richardson acceleration. Given a sequence with the asymptotic behavior

$$s_n \sim \sum_{k=0}^{\infty} \frac{\sigma_k}{n^k}, \tag{A.8}$$

its M-th Richardson transform is defined by

$$s_n^{(M)} = \sum_{\ell=0}^{M} \frac{s_{n+\ell} (n+\ell)^M (-1)^{\ell+M}}{\ell! (M-\ell)!}. \tag{A.9}$$

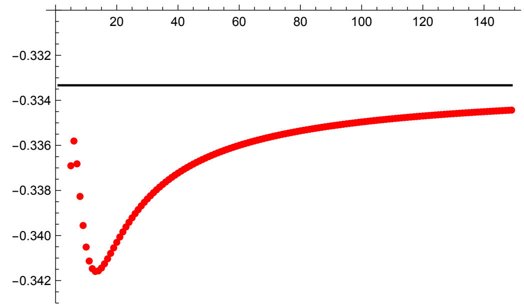

Figure 12: The red points correspond to the sequence (A.5). The horizontal line is the asymptotic limit $-1/3$.

The effect of this transformation is to remove the M subleading tails in (A.8), and therefore it leads to a sequence which convergences to σ_0 much faster than the original one. Note that, if we have a sequence of N terms, then the M-th Richardson transform produces a sequence of $N-M$ terms. If we have N^* terms in the original sequence, the M-th Richardson transform provides an estimate of the asymptotic limit given by $s^{(M)}_{N^*-M}$. As M increases, this approaches the correct value, although there is an accuracy limit set by the original (finite) sequence. In the example above, after a single Richardson transform we obtain a much better estimate, given by

$$A^{(1)}_{149} = -0.3333584391... \tag{A.10}$$

This is also clear in the plot of the original sequence and its first Richardson transform, see Fig. 13. Further Richardson transforms improve the estimate, and we can get the right value $-1/3$ with 22 significant digits! This is a huge improvement as compared with the original estimate (A.7).

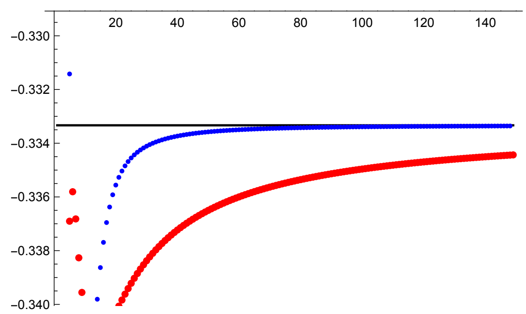

Figure 13: In addition to the the sequence (A.5), in red points, we show its first Richardson transform, in blue points. The convergence to the asymptotic value is much faster.

Once we have determined the action A, we can determine the value of b. To do this, one possibility is to note that the first correction in (A.5) is given by

$$\sigma_1 = -Ab. \tag{A.11}$$

Therefore, we can consider the sequence

$$-n\left(\frac{1}{A}\frac{na_n}{a_{n+1}}-1\right)\sim b+\mathcal{O}\left(\frac{1}{n}\right) \tag{A.12}$$

and apply again Richardson acceleration. By using the same method, it is easy to determine that

$$b=\frac{1}{2} \tag{A.13}$$

with 20-21 significant digits. Once b is determined, the prefactor can be easily obtained by considering the sequence

$$(-1)^{n+1}\frac{3^{n+1/2}}{\Gamma\left(n+\frac{1}{2}\right)}a_n. \tag{A.14}$$

By using again Richardson transforms, we obtain the coefficient

$$\frac{\sqrt{2}}{\pi^{3/2}} \tag{A.15}$$

with a precision of 22-23 digits.

A.2 Borel resummation

We recall that the Borel transform has in principle a finite radius of convergence, and we need an analytic continuation to a neigbourhood of the positive real axis in order to be able to calculate the Laplace transform. When one has a finite number of terms in a series, the more efficient way to produce an approximate analytic continuation is the method of Padé approximants. Given a series

$$\varphi(z)=\sum_{k=0}^{\infty}a_k z^k, \tag{A.16}$$

its Padé approximant $[l/m]_\varphi$, where l, m are positive integers, is the rational function

$$[l/m]_\varphi(z)=\frac{p_0+p_1 z+\cdots+p_l z^l}{q_0+q_1 z+\cdots+q_m z^m}, \tag{A.17}$$

where q_0 is fixed to 1, and one requires that

$$\varphi(z)-[l/m]_\varphi(z)=\mathcal{O}(z^{l+m+1}). \tag{A.18}$$

This fixes the coefficients involved in (A.17).

Given a series $\varphi(z)$, we can use Padé approximants to reconstruct the analytic continuation of its Borel transform. There are various methods to do this, but one simple approach is to use the following Padé approximant,

$$\mathcal{P}_n^\varphi(\zeta)=[[n/2]/[(n+1)/2]]_{\widehat{\varphi}}(\zeta) \tag{A.19}$$

which requires knowledge of the first $n+1$ coefficients of the original series. The integral

$$s(\varphi)_n(z)=z^{-1}\int_0^\infty \mathrm{d}\zeta\, \mathrm{e}^{-\zeta/z}\mathcal{P}_n^\varphi(\zeta) \tag{A.20}$$

gives an approximation to the Borel resummation of the series (A.16), which can be systematically improved by increasing n.

B A cheat sheet on matrix models at large N

In order to make the lectures a little bit more self-contained, I have included in this Appendix a summary of how to solve matrix models at large N, largely based on [3].

Let us first show how to compute $F_0(t)$ once $\rho(\lambda)$ is known. In the saddle-point approximation, the free energy is given by

$$\frac{1}{N^2}F = S_{\rm eff}(\rho) + \mathcal{O}(N^{-2}), \tag{B.1}$$

where the effective action in the large N limit is a functional of ρ which can be read from (3.54):

$$S_{\rm eff}(\rho) = -\frac{1}{t}\int_{\mathcal{C}} \rho(\lambda)V(\lambda){\rm d}\lambda + \int_{\mathcal{C}\times\mathcal{C}} \rho(\lambda)\rho(\lambda')\log|\lambda-\lambda'|{\rm d}\lambda\,{\rm d}\lambda'. \tag{B.2}$$

Therefore, the planar free energy is given by

$$F_0(t) = t^2 S_{\rm eff}(\rho). \tag{B.3}$$

We can obtain the equilibrium condition (3.60) by computing the extremum of the functional

$$S(\rho,\xi) = S_{\rm eff}(\rho) + \xi\left(\int_{\mathcal{C}} \rho(\lambda){\rm d}\lambda - 1\right) \tag{B.4}$$

with respect to ρ. Here, ξ is a Lagrange multiplier that imposes the normalization condition of the density. It is in general a function of the 't Hooft parameter. This leads to

$$\frac{1}{t}V(\lambda) = 2\int \rho(\lambda')\log|\lambda-\lambda'|{\rm d}\lambda' + \xi(t), \tag{B.5}$$

which is nothing but (3.59). After integrating w.r.t. λ we obtain (3.60). The Lagrange multiplier ξ appears in this way as an integration constant that only depends on the coupling constants. It can be computed by evaluating (B.5) at a convenient value of λ (say, $\lambda=0$ if $V(\lambda)$ is a polynomial). Since the effective action is evaluated on the distribution of eigenvalues that solves (3.60), one can simplify the expression to

$$F_0(t) = -\frac{t}{2}\int_{\mathcal{C}} \rho(\lambda)V(\lambda){\rm d}\lambda - \frac{1}{2}t^2\xi(t). \tag{B.6}$$

The saddle–point equation (B.5) says that the effective potential is *constant* on the interval $\mathcal{C}$, which is the condition (3.59).

Once $\rho(\lambda)$ is known, we can compute some useful correlation functions in the planar limit. For example,

$$\frac{1}{N}\left\langle {\rm Tr}\, M^\ell \right\rangle = \int_{\mathcal{C}} \lambda^\ell \rho(\lambda){\rm d}\lambda. \tag{B.7}$$

We then see that the planar limit is characterized by a *classical* density of states $\rho(\lambda)$, and the planar limit of the above quantum averages can be computed as a moment of this density.

We want now to solve the equation (3.60) to find $\rho(\lambda)$. This is a singular integral equation which has been studied in detail in other contexts of physics. A very elegant way to solve it is to introduce an auxiliary function called the *resolvent*. The resolvent is defined as a correlator in the matrix model:

$$\omega(p) = \frac{1}{N}\left\langle {\rm Tr}\frac{1}{p-M}\right\rangle, \tag{B.8}$$

which is in fact a generating functional of the correlation functions (B.7):

$$\omega(p) = \frac{1}{N} \sum_{k=0}^{\infty} \left\langle \mathrm{Tr} M^k \right\rangle p^{-k-1}. \tag{B.9}$$

Since it is a generating functional of connected correlators, general large N counting shows that it has the genus expansion

$$\omega(p) = \sum_{g=0}^{\infty} g_s^{2g} \omega_g(p), \tag{B.10}$$

and the genus zero piece can be written in terms of the eigenvalue density as

$$\omega_0(p) = \int \frac{\rho(\lambda)}{p-\lambda} \mathrm{d}\lambda. \tag{B.11}$$

The genus zero resolvent (B.11) has three important properties. First of all, due to the normalization property of the eigenvalue distribution (3.57), it has the asymptotic behavior

$$\omega_0(p) \sim \frac{1}{p}, \qquad p \to \infty. \tag{B.12}$$

Second, as a function of p, it is analytic on the whole complex plane, but it has a branch cut at the interval $\mathcal{C}$. The discontinuity in crossing the branch cut can be computed by standard contour deformation arguments. We have

$$\omega_0(p + \mathrm{i}\epsilon) = \int_{\mathbb{R}} \frac{\rho(\lambda)}{p + \mathrm{i}\epsilon - \lambda} \mathrm{d}\lambda = \int_{\mathbb{R} - \mathrm{i}\epsilon} \frac{\rho(\lambda)}{p-\lambda} \mathrm{d}\lambda = \mathrm{P} \int \frac{\rho(\lambda)}{p-\lambda} \mathrm{d}\lambda + \int_{C_\epsilon} \frac{\rho(\lambda)}{p-\lambda} \mathrm{d}\lambda, \tag{B.13}$$

where $0 < \epsilon \ll 1$, and C_ϵ is a semi-circle contour around $\lambda = p$ in the lower half plane, oriented counterclockwise. This can be evaluated as a residue, and we finally obtain,

$$\omega_0(p + \mathrm{i}\epsilon) = \mathrm{P} \int \frac{\rho(\lambda)}{p-\lambda} \mathrm{d}\lambda - \pi \mathrm{i} \rho(p). \tag{B.14}$$

Similarly,

$$\omega_0(p - \mathrm{i}\epsilon) = \int_{\mathbb{R} + \mathrm{i}\epsilon} \frac{\rho(\lambda)}{p-\lambda} \mathrm{d}\lambda = \mathrm{P} \int \frac{\rho(\lambda)}{p-\lambda} \mathrm{d}\lambda + \pi \mathrm{i} \rho(p), \tag{B.15}$$

One then finds the key equation

$$\rho(\lambda) = -\frac{1}{2\pi \mathrm{i}} \left(\omega_0(\lambda + \mathrm{i}\epsilon) - \omega_0(\lambda - \mathrm{i}\epsilon) \right). \tag{B.16}$$

Another way to derive this equation is to start from the definition of $\rho(\lambda)$ and use the basic identity

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} = \lim_{\epsilon \to 0} \mathrm{Im} \frac{1}{\pi} \frac{1}{x - \mathrm{i}\epsilon}. \tag{B.17}$$

This leads to

$$\rho(\lambda) = \frac{1}{\pi} \mathrm{Im} \frac{1}{N} \left\langle \mathrm{Tr} \frac{1}{\lambda - M - \mathrm{i}\epsilon} \right\rangle, \tag{B.18}$$

and from the definition of the resolvent we just find,

$$\rho(\lambda) = -\frac{1}{2\pi \mathrm{i}} \left(\omega(\lambda + \mathrm{i}\epsilon) - \omega(\lambda - \mathrm{i}\epsilon) \right) = -\frac{1}{\pi} \mathrm{Im}\, \omega(\lambda), \tag{B.19}$$

which is an equation at all orders in N.

From these equations we deduce that, if the resolvent at genus zero is known, the planar eigenvalue distribution follows from (B.16), and one can compute the planar free energy. On the other hand, by using again (B.13) and (B.15) we can compute

$$\omega_0(p+\mathrm{i}\epsilon)+\omega_0(p-\mathrm{i}\epsilon)=2\mathrm{P}\int\frac{\rho(\lambda)}{p-\lambda}\mathrm{d}\lambda, \tag{B.20}$$

and we then find the equation

$$\omega_0(p+\mathrm{i}\epsilon)+\omega_0(p-\mathrm{i}\epsilon)=\frac{1}{t}V'(p), \tag{B.21}$$

which determines the resolvent in terms of the potential. In this way we have reduced the original problem of computing $F_0(t)$ to the problem of computing $\omega_0(\lambda)$. There is in fact a closed form expression for the planar resolvent which is very useful and valid for a very general class of potentials, not only polynomial:

$$\omega_0(p)=\frac{1}{2t}\oint_{\mathcal{C}}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{V'(z)}{p-z}\left(\frac{(p-a)(p-b)}{(z-a)(z-b)}\right)^{1/2}, \tag{B.22}$$

where $\mathcal{C}$ denotes a closed contour around the interval $[b,a]$. This equation is easily proved by converting (B.21) into a discontinuity equation:

$$\widehat{\omega}_0(p+\mathrm{i}\epsilon)-\widehat{\omega}_0(p-\mathrm{i}\epsilon)=\frac{1}{t}\frac{V'(p)}{\sqrt{(p-a)(p-b)}}, \tag{B.23}$$

where $\widehat{\omega}_0(p)=\omega_0(p)/\sqrt{(p-a)(p-b)}$. The discontinuity equation determines $\omega_0(p)$ to be given by (B.22) up to an analytic function of p, but because of the asymptotics (B.12), this function must vanish. The asymptotics of $\omega_0(p)$ also gives two more conditions. By taking $p\to\infty$, one finds that the r.h.s. of (B.22) behaves like $c+d/p+\mathcal{O}(1/p^2)$. Requiring the asymptotic behavior (B.12) imposes $c=0$ and $d=1$, and this leads to

$$\begin{aligned}\oint_{\mathcal{C}}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{V'(z)}{\sqrt{(z-a)(z-b)}}&=0,\\ \oint_{\mathcal{C}}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{zV'(z)}{\sqrt{(z-a)(z-b)}}&=2t.\end{aligned} \tag{B.24}$$

These equations are enough to determine the endpoints of the cuts, a and b, as functions of the 't Hooft coupling t and the coupling constants of the model.

When $V(z)$ is a polynomial, one can find a very convenient expression for the resolvent: if we deform the contour in (B.22) to infinity, we pick up a pole at $z=p$, and another one at infinity, and we get

$$\omega_0(p)=\frac{1}{2t}\left(V'(p)-M(p)\sqrt{(p-a)(p-b)}\right), \tag{B.25}$$

where the *moment function*

$$M(p)=\oint_{\infty}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{V'(z)}{z-p}\frac{1}{\sqrt{(z-a)(z-b)}}, \tag{B.26}$$

can be written as a contour integral around $z = 0$

$$M(p) = \oint_0 \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{V'(1/z)}{1 - pz} \frac{1}{\sqrt{(1-az)(1-bz)}}. \tag{B.27}$$

These formulae, together with the expressions (B.24) for the endpoints of the cut, completely solve the one-matrix model with one cut in the planar limit, for polynomial potentials.

Example B.1. *The Gaussian matrix model.* Let us now apply this technology to the simplest case, the Gaussian model with $V(M) = M^2/2$. Let us first look for the position of the endpoints from (B.24). Deforming the contour to infinity and changing $z \to 1/z$, we find that the first equation in (B.24) becomes

$$\oint_0 \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{1}{z^2} \frac{1}{\sqrt{(1-az)(1-bz)}} = 0, \tag{B.28}$$

where the contour is now around $z = 0$. Therefore $a + b = 0$, in accord with the symmetry of the potential. Taking this into account, the second equation becomes:

$$\oint_0 \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{1}{z^3} \frac{1}{\sqrt{1 - a^2 z^2}} = 2t, \tag{B.29}$$

and gives

$$a = 2\sqrt{t}. \tag{B.30}$$

We see that the support of the density of eigenvalues $[-a, a] = [-2\sqrt{t}, 2\sqrt{t}]$ opens as the 't Hooft parameter grows up, and as $t \to 0$ it collapses to the minimum of the potential at the origin, as expected. We immediately find from (B.25)

$$\omega_0(p) = \frac{1}{2t} \left(p - \sqrt{p^2 - 4t} \right), \tag{B.31}$$

and from the discontinuity equation we derive the density of eigenvalues (3.61). Once we know $\rho(\lambda)$, we can compute $\xi(t)$. Evaluating (B.5) at $\lambda = 0$, we find

$$\xi(t) = -2 \int \mathrm{d}\lambda \rho(\lambda) \log|\lambda| = \frac{1 - \log t}{2} \tag{B.32}$$

and

$$-\frac{t}{4} \int \mathrm{d}\lambda \rho(\lambda) \lambda^2 = -\frac{t^2}{4}. \tag{B.33}$$

Therefore,

$$F_0(t) = \frac{1}{2} t^2 \log t - \frac{3}{4} t^2. \tag{B.34}$$

This agrees with the first line of (3.45). ◊

Example B.2. We now consider the so-called quartic matrix model, with potential

$$V(z) = \frac{1}{2} z^2 + g z^4. \tag{B.35}$$

In this case, it is customary to write the support of the density of eigenvalues as $[-2a, 2a]$. The resolvent can be easily computed from (B.22) and reads,

$$\omega_0(z) = \frac{1}{2t} \left(z + 4gz^3 - \left(1 + 8ga^2 + 4gz^2\right) \sqrt{z^2 - 4a^2} \right). \tag{B.36}$$

The density of eigenvalues is given by the discontinuity of this function,

$$\rho(\lambda) = \frac{1}{2\pi t}\left(1 + 8ga^2 + 4g\lambda^2\right)\sqrt{4a^2 - \lambda^2}. \tag{B.37}$$

In order to determine the position of the endpoints as a function of g, we notice that

$$\oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{zV'(z)}{\sqrt{z^2 - 4a^2}} = 2(a^2 + 12a^4 g), \tag{B.38}$$

and equating this to $2t$ we find,

$$a^2 = \frac{1}{24g}\left(-1 + \sqrt{1 + 48gt}\right). \tag{B.39}$$

As expected, $a \to 0$ as $t \to 0$. The spectral curve is given by

$$y(z) = \left(1 + 8ga^2 + 4gz^2\right)\sqrt{z^2 - 4a^2}. \tag{B.40}$$

The free energy at genus zero can be computed by using (B.6). First of all, we have

$$-\frac{t}{2}\int_{\mathcal{C}} \mathrm{d}\lambda\, \rho(\lambda) V(\lambda) = -\frac{1}{4}a^4(1 + 20a^2 g + 72a^4 g^2). \tag{B.41}$$

In order to determine the integration constant ξ, we evaluate (B.5) at $\lambda = 0$ to obtain

$$-t^2\frac{\xi}{2} = t^2\int_{\mathcal{C}} \mathrm{d}\lambda\, \rho(\lambda) \log|\lambda| = \frac{ta^2}{2}\left(-1 - 6a^2 g + \left(2 + 24a^2 g\right)\log(a)\right). \tag{B.42}$$

Adding both contributions and using (B.39) one finds,

$$F_0(g) = -\frac{1}{24}(9t^2 + 10ta^2 - a^4) + \frac{t^2}{2}\log a^2. \tag{B.43}$$

It is useful however to subtract the part of the free energy that corresponds to the Gaussian model. This is obtained by evaluating the above quantity at $g = 0$ i.e. at $a^2 = t$, which agrees exactly with the Gaussian limit (3.45), and one finds:

$$\mathcal{F}_0(g) = F_0(g) - F_0(g = 0) = -\frac{1}{24}(a^2 - t)(9t - a^2) + \frac{t^2}{2}\log\frac{a^2}{t}. \tag{B.44}$$

This function has the following expansion around $t = 0$,

$$\mathcal{F}_0(g) = -t^2 f_0(gt), \tag{B.45}$$

where

$$f_0(z) = \sum_{k=1}^{\infty} a_k z^k, \qquad a_k = (-12)^k \frac{(2k-1)!}{k!(k+2)!}, \tag{B.46}$$

and the first few terms in the expansion of $\mathcal{F}_0(g)$ are

$$\mathcal{F}_0(g) = -t^2\left(2\,(gt) - 18\,(gt)^2 + 288\,(gt)^3 + \mathcal{O}(g^4)\right). \tag{B.47}$$

◊

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