Intro to Gyrokinetics

D. R. Hatch

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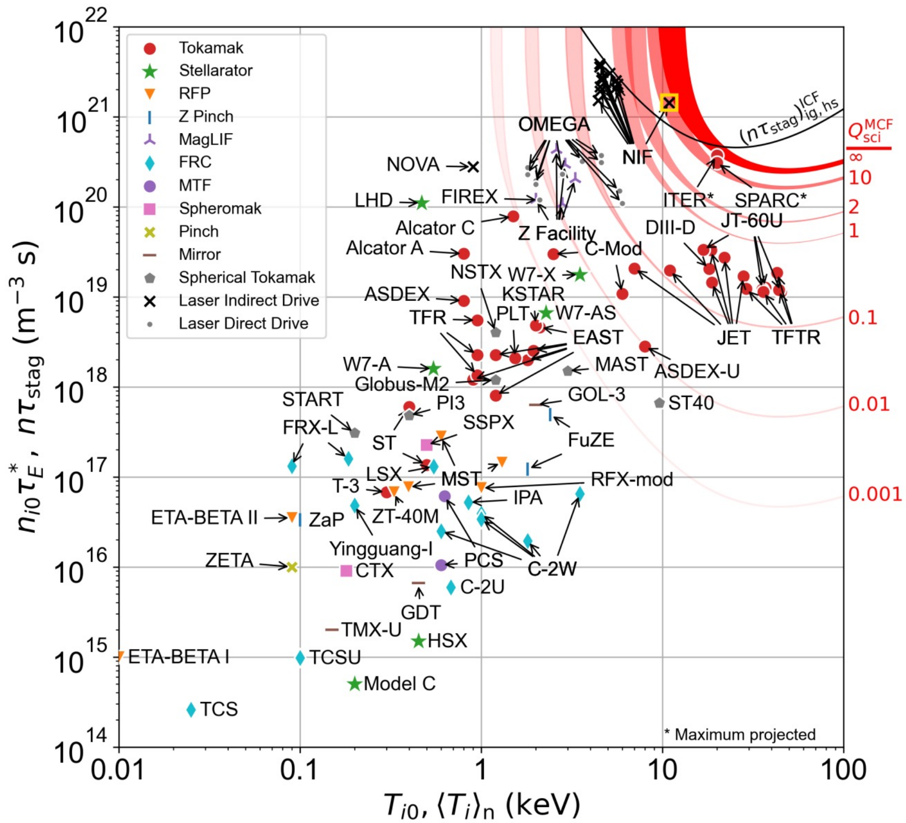
Proximity to High Fusion Gain

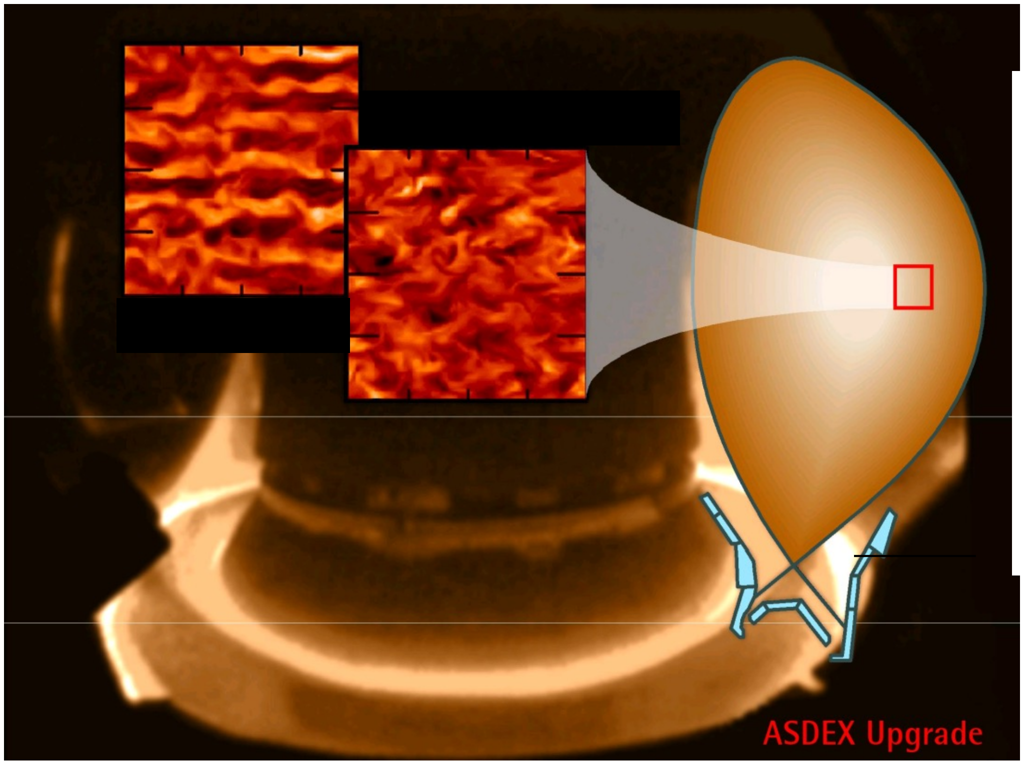
$$nT\tau > 3 \times 10^{21} m^{-3} keV s$$

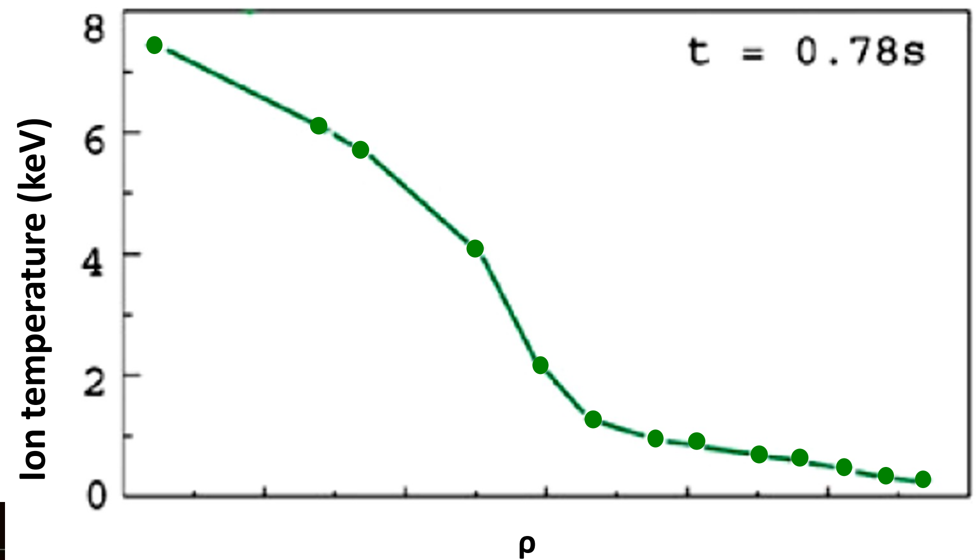
Confinement time

τ = Stored Energy / Heating Power



‘Micro’-Turbulence Sets Confinement Time





“Nature abhors a gradient”

Turbulence produced by small-scale (gyro-radius) instabilities driven by density and temperature gradients.

Ion Temperature Gradient (**ITG**) instability
Trapped Electron Mode (**TEM**) instability

Gyrokinetics: Plasma Micro-Turbulence in Fusion Plasmas

Global Gyrokinetic Simulation of

Turbulence in

ASDEX Upgrade

GENE

gene.rzg.mpg.de

Projected Cost of a Tokamak Fusion Pilot Plant is Extremely Sensitive to **Confinement time**

$\$ \sim R^3$

Simple estimate from scaling laws*:

$R \sim 1/H$ (H is normalized confinement time)

** Hartmut Zohm "On the Minimum Size of DEMO", Fusion Sci. Technol., 58:2, 613-624 (2010)*

Confinement is Key

- Confinement is key to achieving high fusion gain
- Confinement is key to reducing cost of fusion devices
- The **gyrokinetic model** is the main tool for understanding and predicting turbulence and confinement

Can Describe All the Plasma Dynamics with the Distribution Function

$$f_\sigma(\mathbf{x}, \mathbf{v}, t)$$

Moments of Distribution Function

$$f_\sigma(\mathbf{x}, \mathbf{v}, t)$$

$$n_\sigma(\mathbf{x}, t) = \int f_\sigma(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}$$

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$$\mathbf{u}_\sigma(\mathbf{x}, t) = \frac{1}{n_\sigma} \int \mathbf{v} f_\sigma(\mathbf{x}, \mathbf{v}, t) d\mathbf{v} \quad \longrightarrow \quad J = \sum_\sigma q_\sigma n_\sigma u_\sigma$$

Moments of Distribution Function

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$$\mathbf{u}_\sigma(\mathbf{x}, t) = \frac{1}{n_\sigma} \int \mathbf{v} f_\sigma(\mathbf{x}, \mathbf{v}, t) d\mathbf{v} \longrightarrow J = \sum_\sigma q_\sigma n_\sigma u_\sigma$$

Couple to **Maxwell's equations** and describe the entire system

How To Solve for Distribution Function?

$$\frac{Df_\sigma}{Dt} = 0$$

How To Solve for Distribution Function?

$$\dot{\mathbf{x}} \qquad \dot{\mathbf{v}}$$

$$\frac{\partial f_\sigma}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q_\sigma}{m_\sigma}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{v}} f_\sigma = 0$$

+

Maxwell's Equations

Fokker-Planck: Theory of (almost) Everything for Fusion Plasma

$$\frac{\partial f_\sigma}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q_\sigma}{m_\sigma}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_\sigma = C_{\sigma\alpha}(f_\sigma)$$

- Also need collision operator

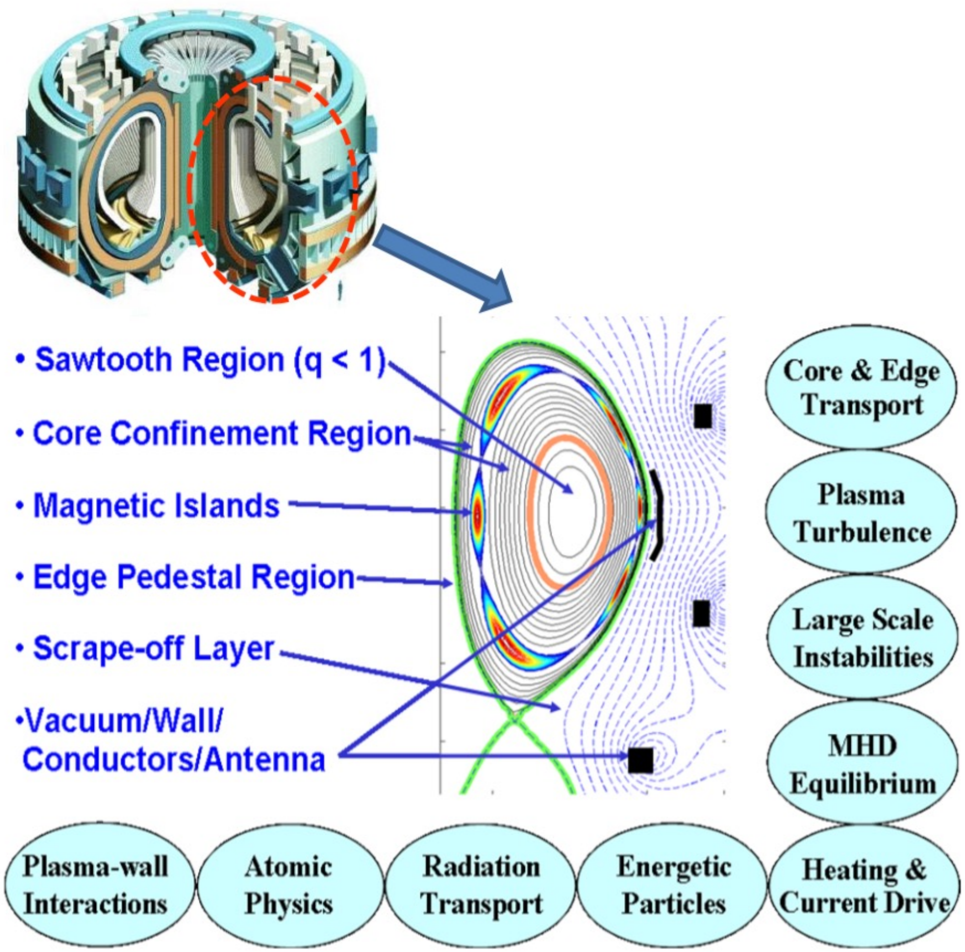
\+

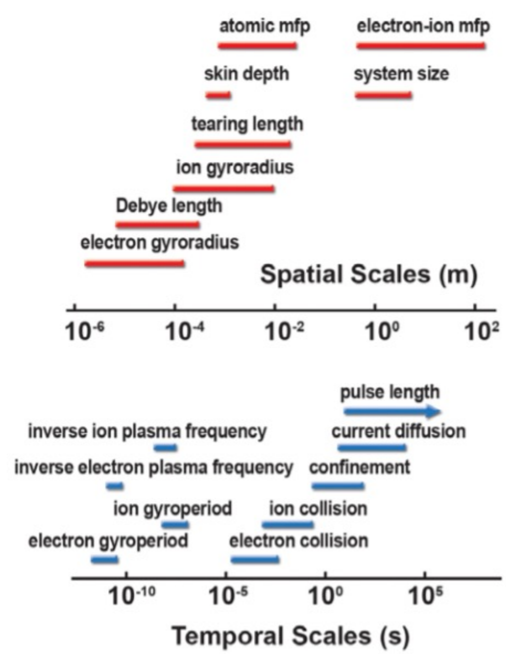
Maxwell's Equations

- LHS: interaction of particles with fields produced collectively by particles
  - ➔ conservation of particles in phase space
- RHS: collision operator representing short-scale particle interaction
- This equation is capable of describing all relevant dynamics over all space and time scales
  - (Exceptions—plasma material interaction, atomic physics at the boundary)
- Consequently it is too complex to be of much practical use
- But it's the best starting point for formulating other models that optimize rigor and tractability

Fokker-Planck: Theory of (almost) Everything for Fusion Plasma

$$\frac{\partial f_\sigma}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q_\sigma}{m_\sigma}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_\sigma = C_{\sigma\alpha}(f_\sigma)$$





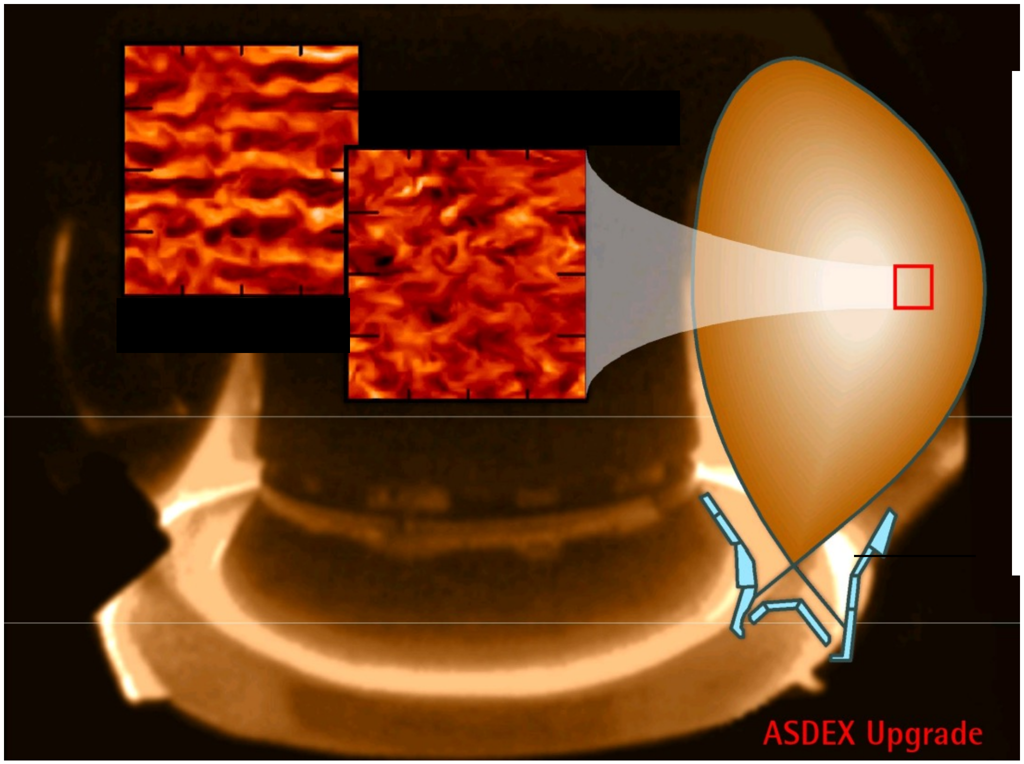
How to Model a Fusion Reactor?

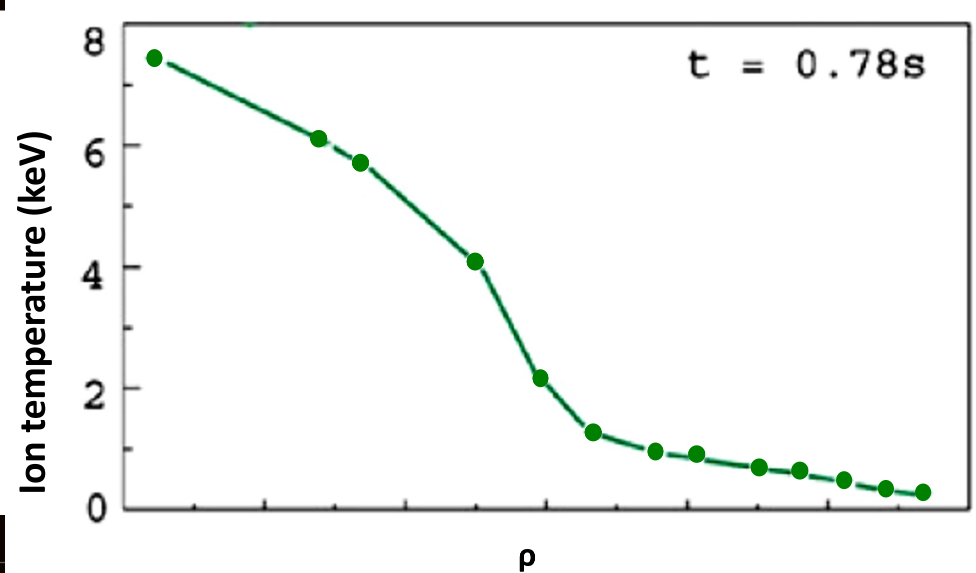
- Seek to find optimal balance of rigor and tractability
- Today's lecture: gyrokinetics
  - How the main models describing a fusion plasma can be derived from first principles (Fokker-Planck) based on a well-defined, rigorously justified ordering scheme.
  - The orderings and assumptions may seem arbitrary to you
  - But they are actually very well justified based on experimental observations of the systems we are trying to describe
  - **The main thing that makes this possible: high magnetic field B**
  - *This remains very close to first principles*
  - References:
    - Multiscale Gyrokinetics: Abel et al Reports on Progress in Physics 2013
    - Simple derivation avoiding tokamak geometry: Astrophysical Gyrokinetics, Howes et al. The Astrophysical Journal, 2006
    - GENE dissertations (Merz, Told, Goerler)

Sneak Peak: What We Will Do

- Establish a rigorous ordering system—i.e. define a small parameter in terms of the relevant space and time scales, etc
- Transform into a natural coordinate system for a magnetized plasma—drift coordinates
- Split distribution function into
  - Background, slow time scale, large spatial scale part
  - Fluctuating, ‘fast’ time scale, small spatial scale part
- Expand kinetic equation with these orderings and solve order by order

Sneak Peak: What We Will Achieve





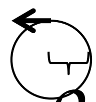
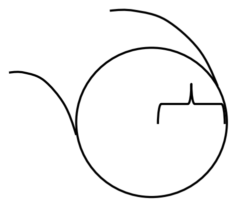
Equations for:

1. Macroscopic equilibrium (**Grad-Shafranov**):

   $\nabla P = J \times B$

   Without this there would be no confinement
2. Small amplitude, small scale, fast time scale fluctuations (**Gyrokinetics**)
3. Large scale, slow time scale transport and flows (**Drift-Kinetic→Neoclassical**)
4. Slowly evolving background temperature and density (**Transport Equations**)

Examples: Scales of a Fusion Plasma

Quantity	Typical Value	ρ_*
Gyroradius: $\rho_j = \frac{m_j v_{Tj}}{q_j B}$  ρ_j	ions ~a few cm electrons <~mm	$\frac{\rho_i}{a} \equiv \rho_{*i} \approx 10^{-3}$
Minor radius: a  a=minor radius:	~1 m	$\frac{\rho_e}{a} \equiv \rho_{*e} \approx 10^{-5}$
Gyrofrequency: $\Omega_j = \frac{q_j B}{m_j}$	ion~10^9 Hz electron~10^{12} Hz	$\frac{\omega_{*i}}{\Omega_i} \equiv \rho_{*i}$
Drift frequency: $\omega_* = \frac{v_{Tj}}{a}$	ion~10^6 Hz electron~10^8 Hz	$\frac{\omega_{*e}}{\Omega_e} \equiv \rho_{*e}$
Collision frequency $\nu \propto nT^{-3/2}$	~$5\text{x}10^4$ Hz	

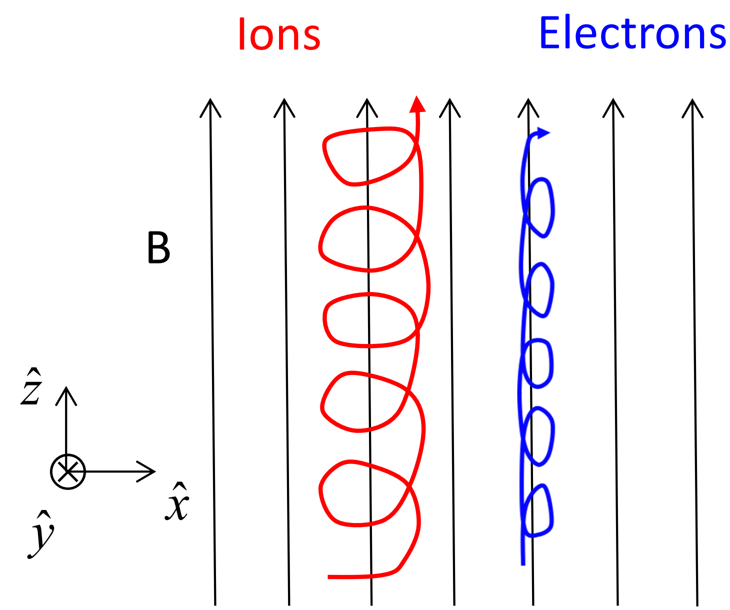
Plasma physics basics – Particle motion in a Magnetic Field

$$m_s \frac{dw}{dt} = q_s w \times B \qquad \mathbf{B} = \mathbf{B_0}\hat{\mathbf{z}}$$

$$w_{||} = w_{||0}$$

$$w_x = w_\perp cos\Omega t \qquad x = \frac{w_\perp}{\Omega} sin(\Omega t) + x_0$$

$$w_y = -w_\perp sin\Omega t \qquad y = \frac{w_\perp}{\Omega} cos(\Omega t) + y_0$$



gyro-frequency:

$$\Omega_s = q_s B / m_s$$

gyro-radius:

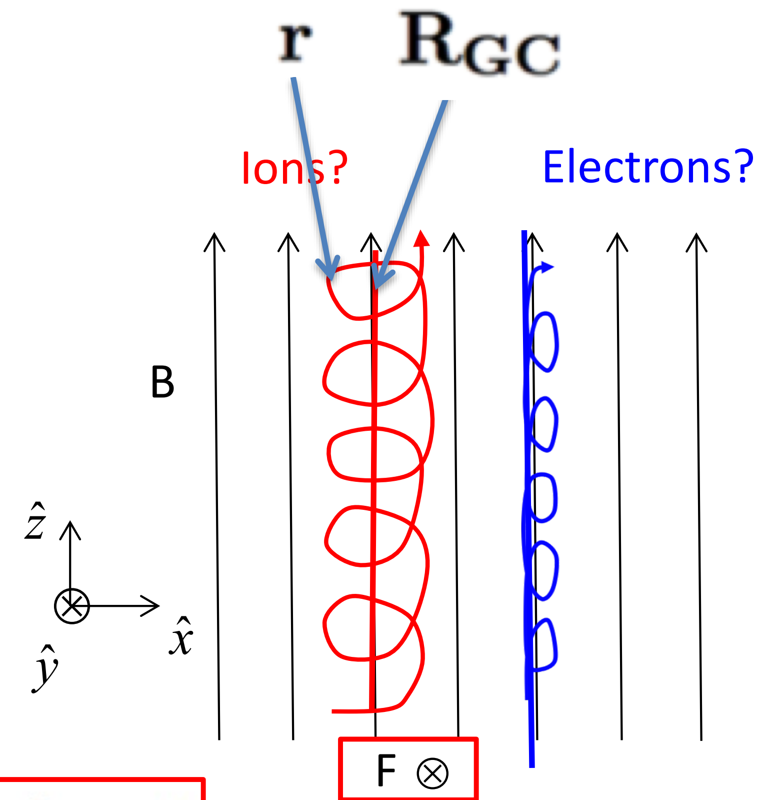
$$\rho_s = w_\perp / \Omega_s$$

What Happens if We Add an Additional Force?

$$m\frac{dv}{dt} = F + qv \times B$$

Assume small gyroradius
and slowly evolving fields
--i.e. a magnetized plasma

$$\mathbf{r} = \mathbf{R_{GC}} + \rho_{\mathbf{L}}$$



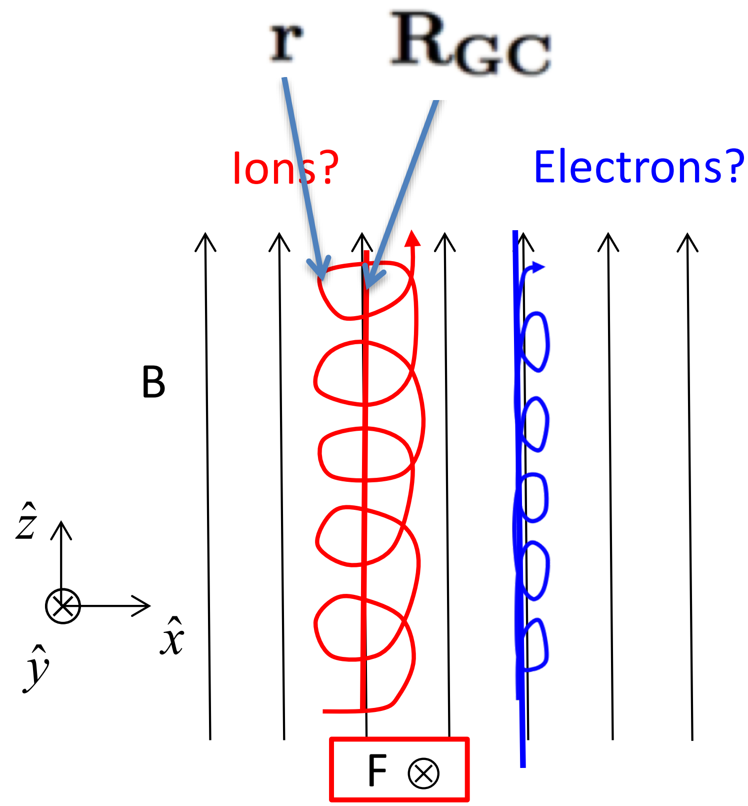
$$m\frac{d}{dt}[v_{GC} + w_L] = F + q[v_{GC} + w_L] \times B$$

This is small

We know this solution already

$$w_x = w_\perp cos\Omega t \qquad w_y = -w_\perp sin\Omega t$$

What Happens if We Add an Additional Force?



$$m\frac{dv}{dt} = F + qv \times B$$

Assume small gyroradius
and slowly evolving fields
--i.e. a magnetized plasma

$$\mathbf{r} = \mathbf{R_{GC}} + \rho_{\mathbf{L}}$$

What's left?

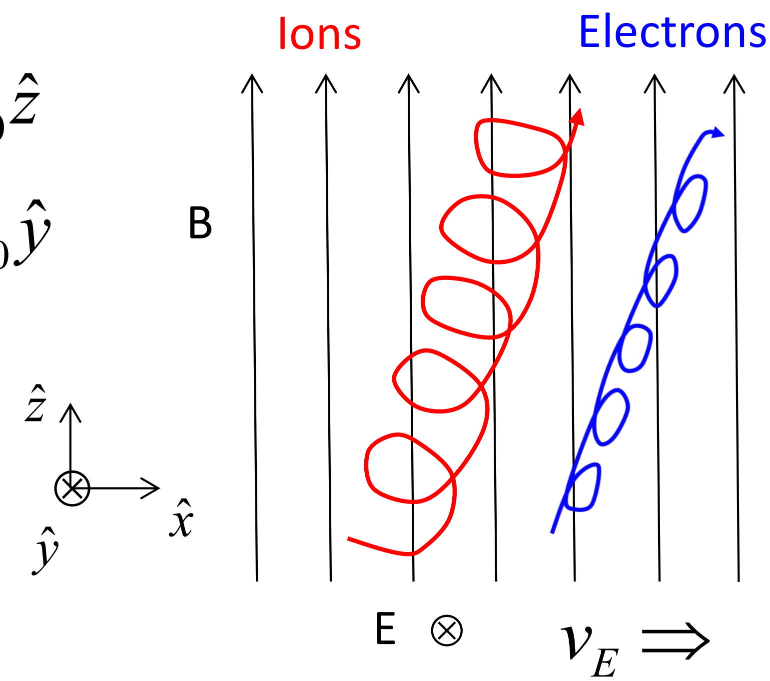
$$F + qv_{GC} \times B = 0$$

$$\Rightarrow \quad v_{\perp,GC} = \frac{F_{\perp} \times B}{qB^2}$$

Plasma physics basics –
Magnetic Field Plus Electric Field

$$\vec{B} = B_0 \hat{z}$$

$$\vec{E} = E_0 \hat{y}$$



ExB drift:

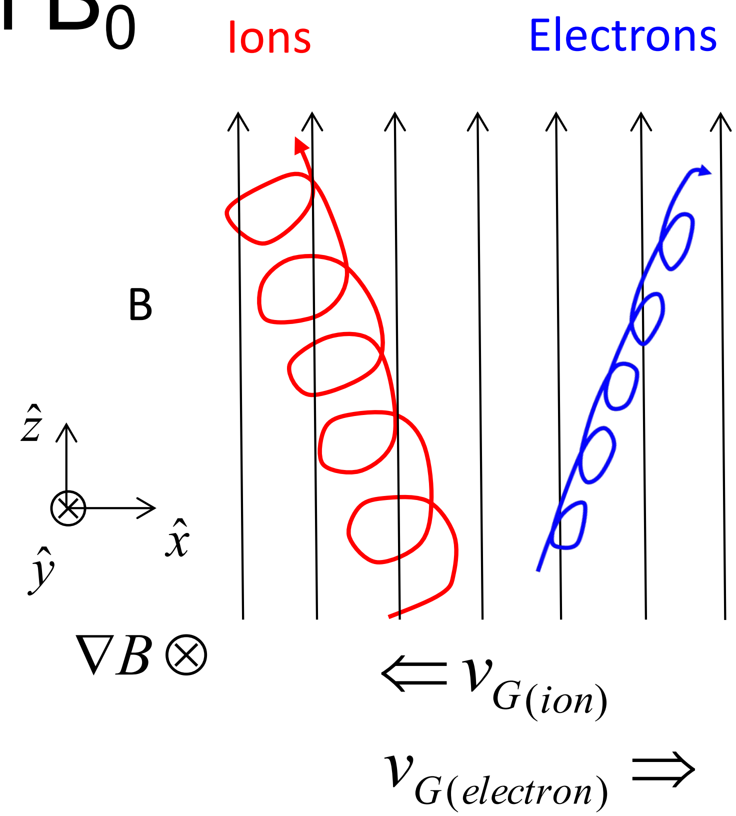
$$v_{\perp,GC} = v_E = \frac{E \times B}{B^2}$$

(Note: same direction for ions and electrons)

Plasma physics basics – Gradient in B_0

$$\vec{B} = B(y)\hat{z}$$

(Only small variations on gyroradius scale)



Grad B drift:

$$v_{\perp,GC} = v_{\nabla B} = m_j w_\perp^2 \frac{B \times \nabla B}{2 q_j B^2}$$

(Note: this drift depends on the charge, so it is in opposite direction for ions and electrons)

Starting Point: Fokker-Planck + Maxwell

- Fokker-Planck

$$\frac{\mathrm{d} f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \boldsymbol{v} \cdot \nabla f_s + \frac{Z_s e}{m_s}\left(\widetilde{\boldsymbol{E}} + \frac{1}{c}\boldsymbol{v} \times \widetilde{\boldsymbol{B}}\right) \cdot \frac{\partial f_s}{\partial \boldsymbol{v}} = C[f_s] + S_s,$$

- Maxwell's Equations

$$\nabla \cdot \widetilde{\boldsymbol{E}} = 4\pi\widetilde{\varrho},$$

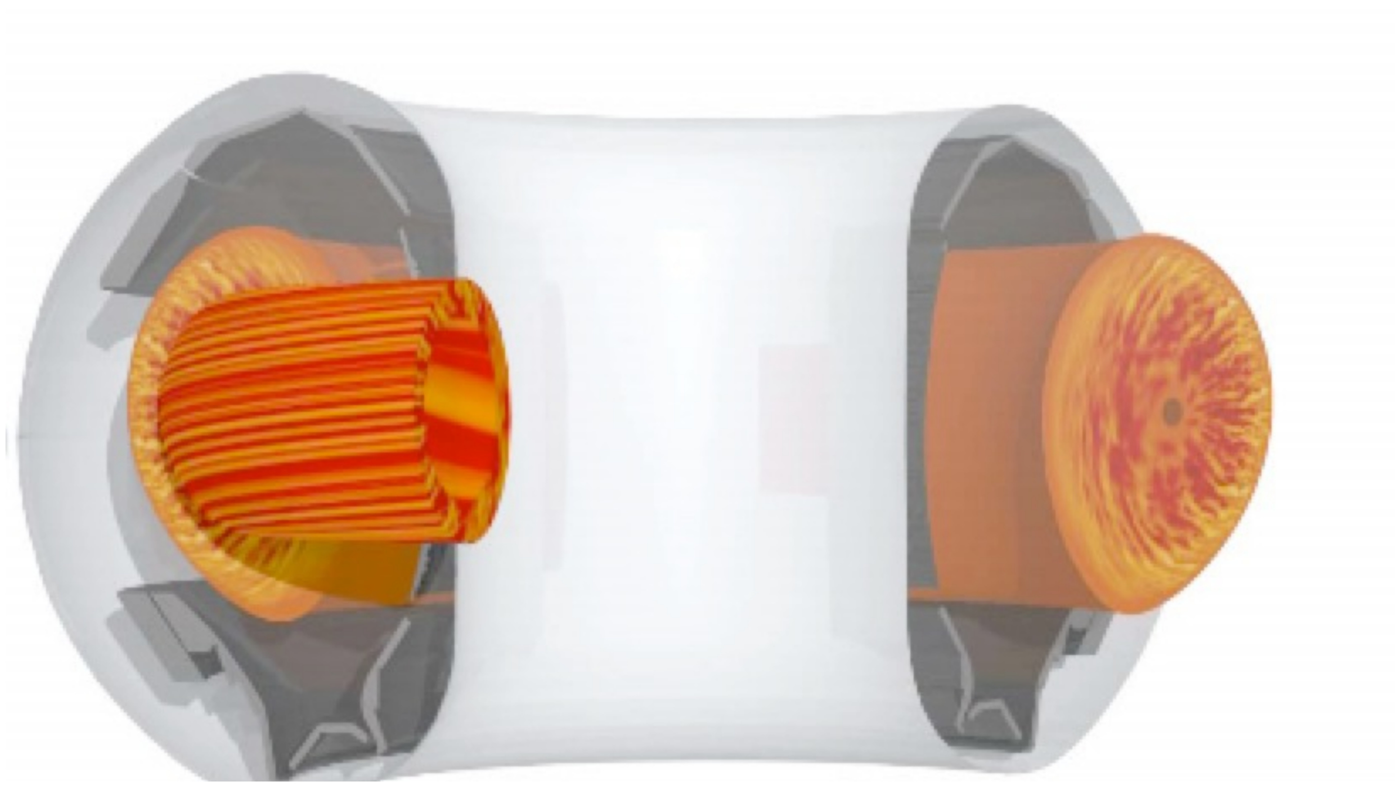
$$\nabla \cdot \widetilde{\boldsymbol{B}} = 0,$$

$$\frac{\partial \widetilde{\boldsymbol{B}}}{\partial t} = -c\nabla \times \widetilde{\boldsymbol{E}},$$

$$\nabla \times \widetilde{\boldsymbol{B}} = \frac{4\pi}{c}\widetilde{\boldsymbol{j}} + \frac{1}{c}\frac{\partial \widetilde{\boldsymbol{E}}}{\partial t},$$

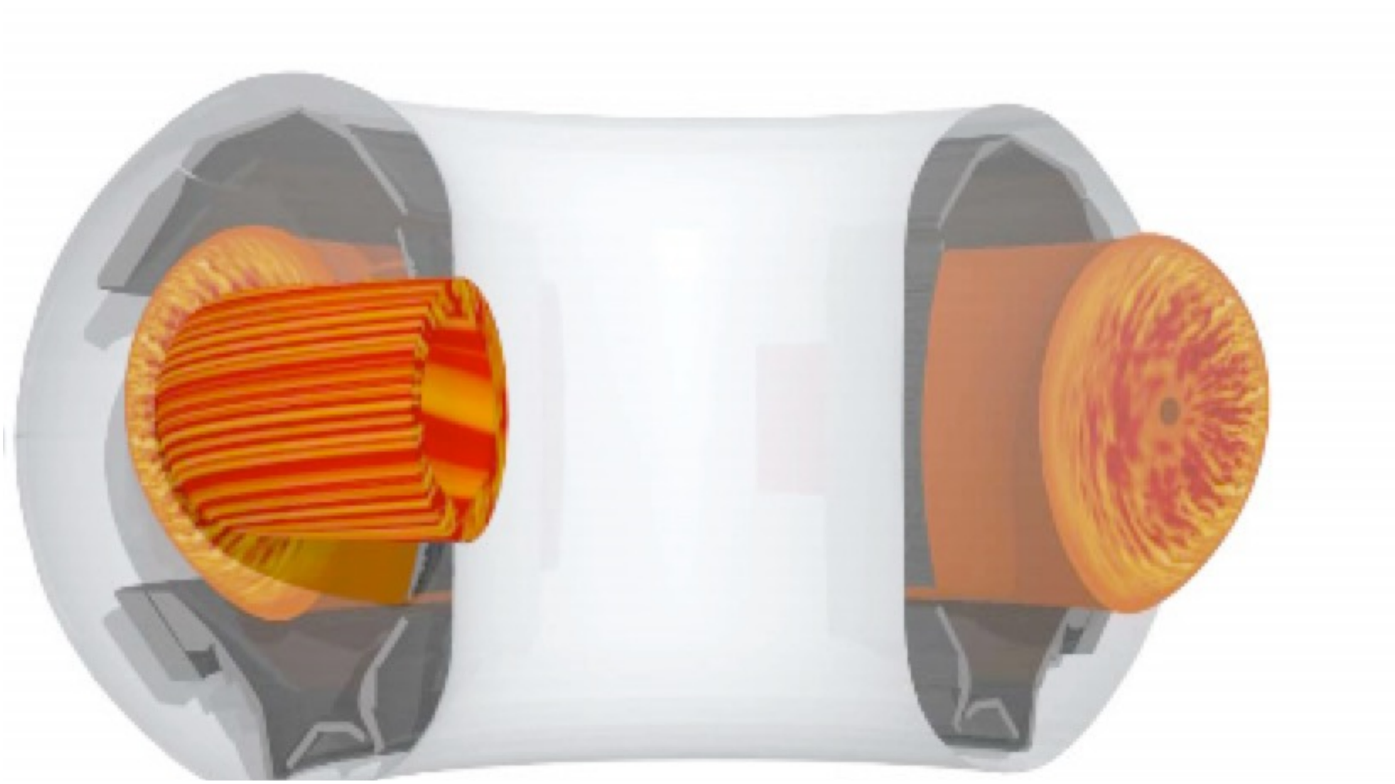
Note: using equations largely from Abel et al 2013 for convenience (some changes in notation from earlier slides—e.g. now Gaussian units)

Gyrokinetic Ordering: Exploit Known Time and Length Scales of Turbulence



- Well-established scale separation between turbulence time and length scales and those of background
- Multi-scale processes: challenge and opportunity
  - Challenge if you try brute force
  - Opportunity if you exploit it (which is what we do in this talk)

Gyrokinetic Ordering: Small Spatial Scales

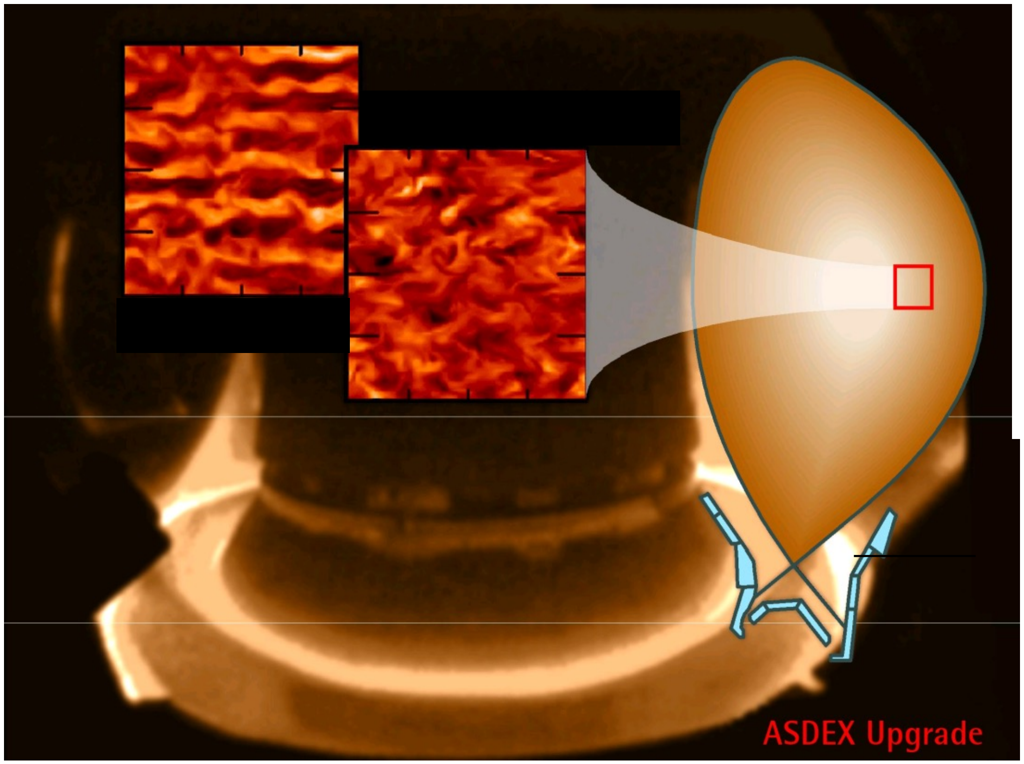


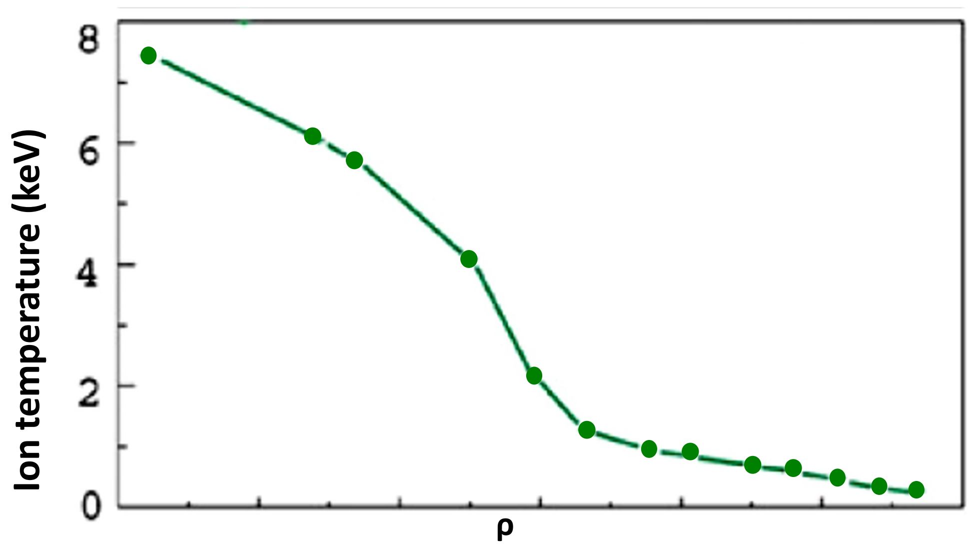
- Fluctuations have scales comparable to gyroradius
- Gyroradius is small compared to, e.g., machine size
- Use this as small parameter to due multi-scale expansion
- Note: this is a condition for a 'strongly magnetized' plasma

$$k_\perp \sim 1/\rho_s$$

$$\frac{\rho_s}{a} = \epsilon.$$

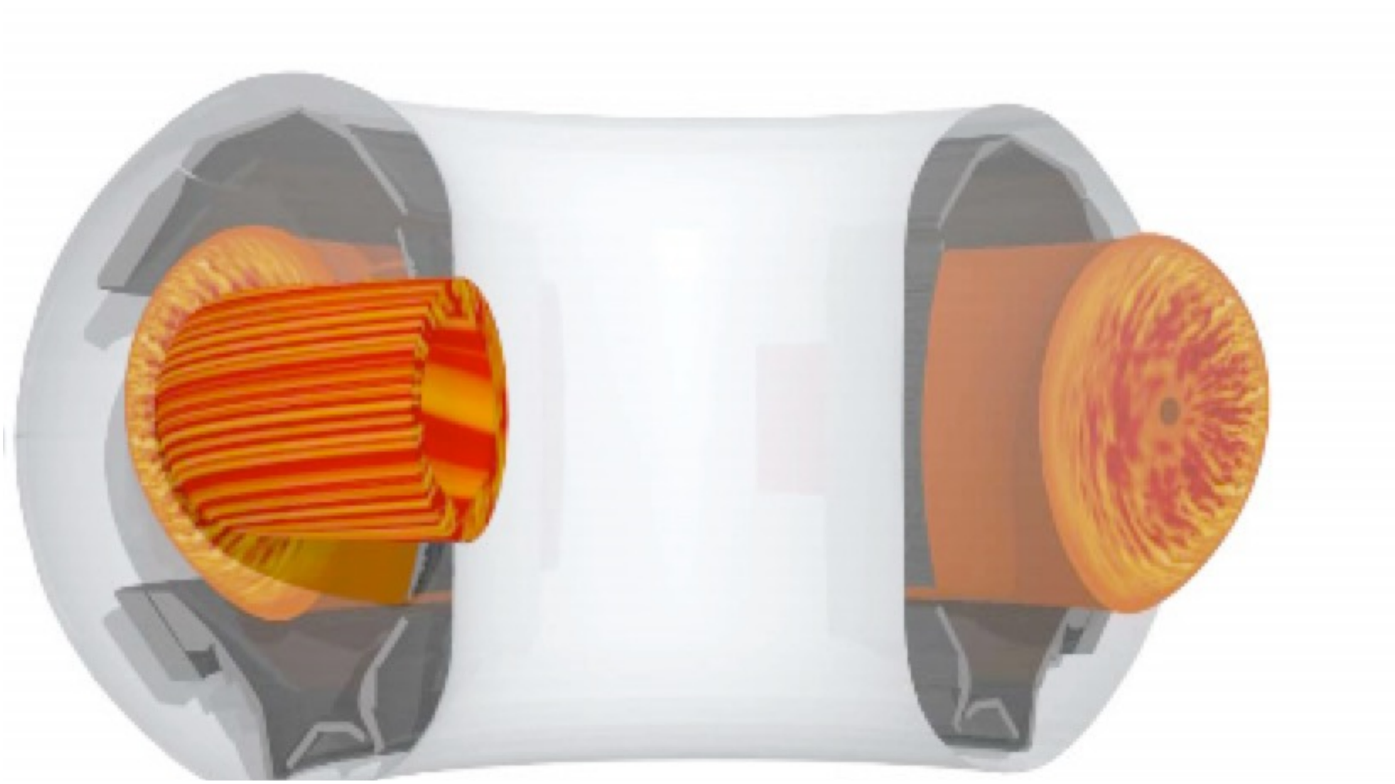
Gyrokinetic Ordering: Small Amplitude Fluctuations





- Fluctuations are small compared to background $\frac{|\delta \boldsymbol{B}|}{|\boldsymbol{B}|} \sim \frac{|\delta \boldsymbol{E}|}{|\boldsymbol{E}|} \sim \frac{\delta f_s}{f_s} \sim \epsilon.$

Gyrokinetic Ordering: Time Scales

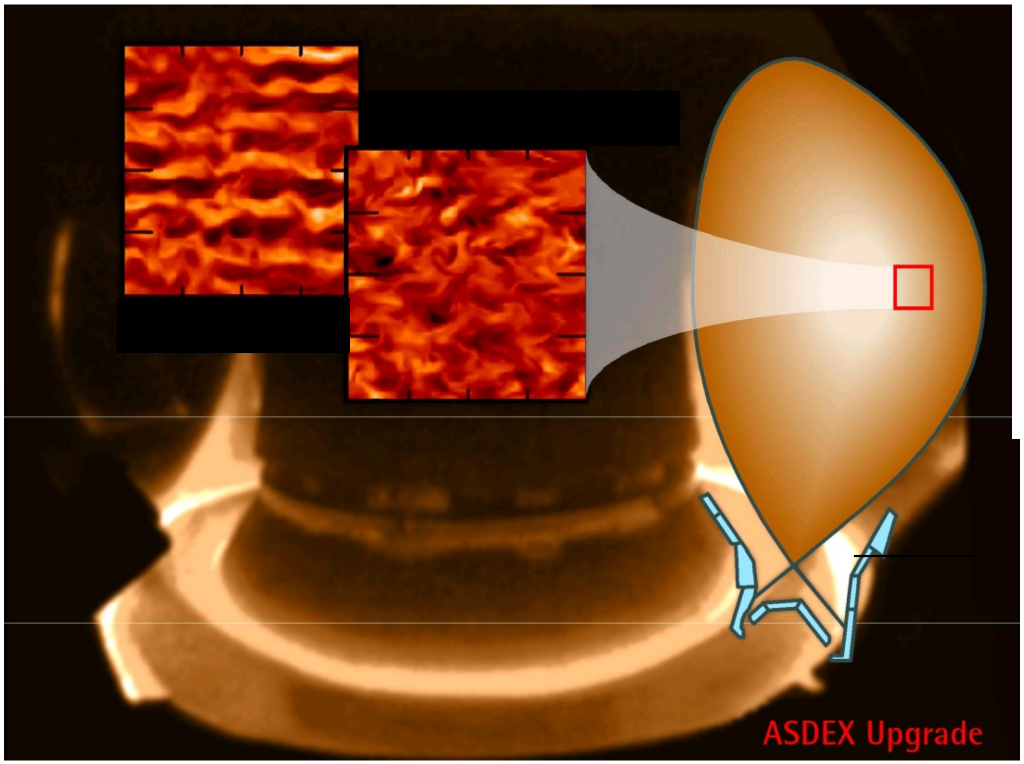


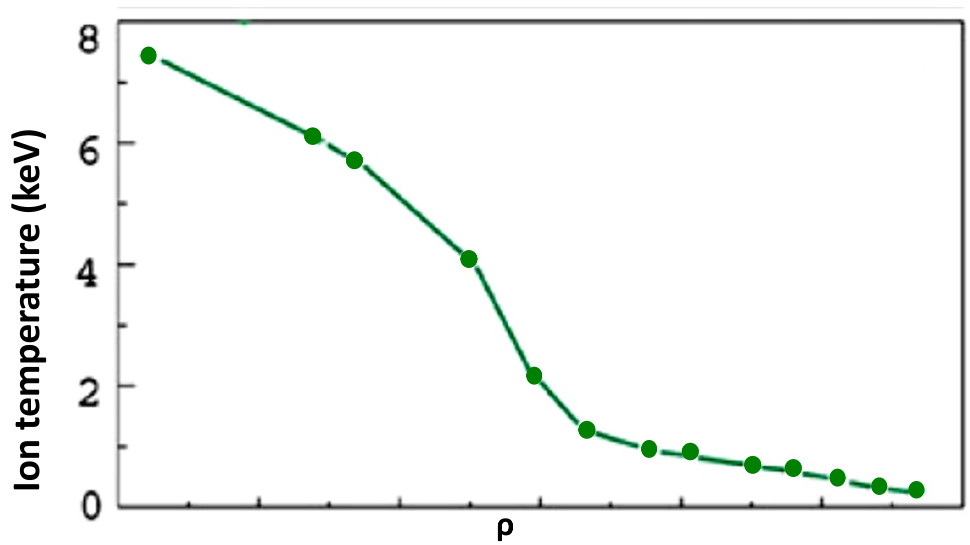
- Fluctuation time scales are large compared to gyrofrequency
- Fluctuation time scales are small compared to confinement time
  - i.e. time scale of background evolution
- Note: this is also a condition for a 'strongly magnetized' plasma

$$\omega \sim c_s/a$$

$$\frac{\omega}{\Omega} \sim \epsilon$$

Gyrokinetic Ordering: Small Amplitude Fluctuations

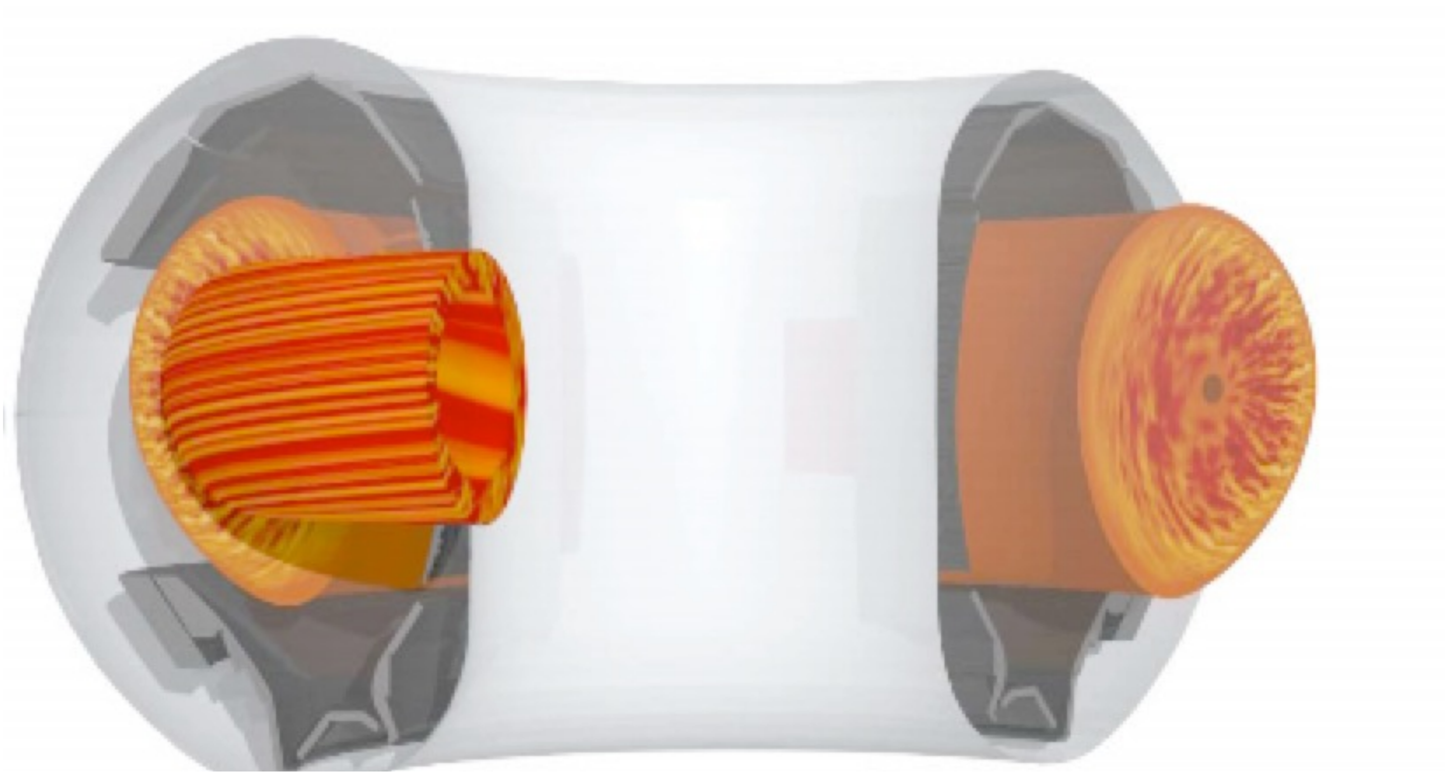




- Background evolves much slower than fluctuations (gyroBohm scaling)

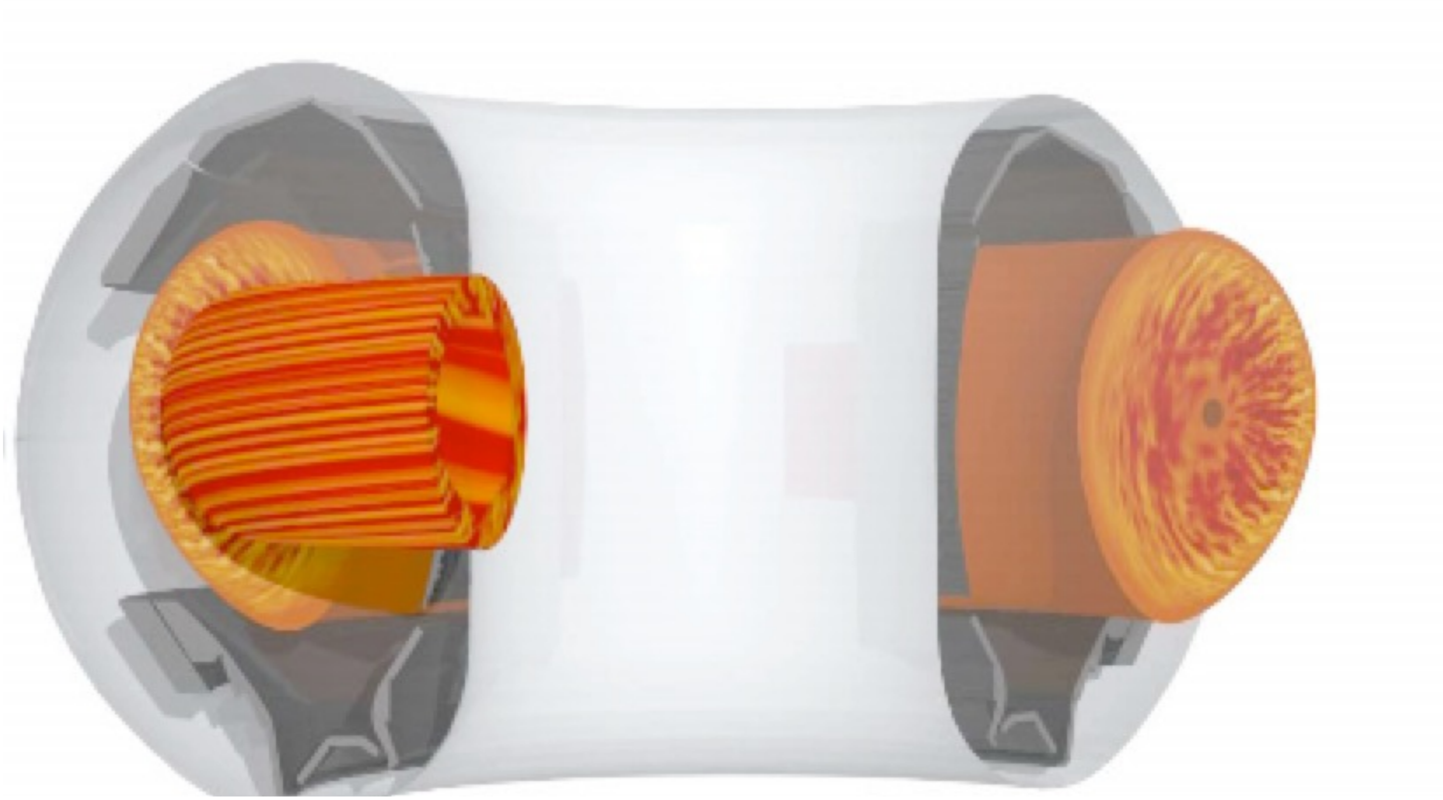
$$\frac{1}{\tau_E} \sim \frac{\chi_T}{a^2} \sim \frac{\omega}{\Omega_s}\left(\frac{\rho_s}{a}\right)^2 \Omega_s \sim \epsilon^3 \Omega_s$$

Gyrokinetic Ordering: Parallel vs Perpendicular Scales



- Perpendicular scales are much smaller than parallel $\frac{k_\parallel}{k_\perp} \sim \epsilon.$

Gyrokinetic Ordering: Summary



$$\frac{|\delta \boldsymbol{B}|}{|\boldsymbol{B}|} \sim \frac{|\delta \boldsymbol{E}|}{|\boldsymbol{E}|} \sim \frac{\delta f_s}{f_s} \sim \frac{k_\parallel}{k_\perp} \sim \frac{\omega}{\Omega_s} \sim \frac{\rho_s}{a} = \epsilon.$$

$$\frac{1}{\tau_E} \sim \frac{\chi_T}{a^2} \sim \frac{\omega}{\Omega_s}\left(\frac{\rho_s}{a}\right)^2 \Omega_s \sim \epsilon^3 \Omega_s$$

Simplified Maxwell's Equations

- Small Debye length and non-relativistic $k_\perp^2 \lambda_{\mathrm{De}}^2 \ll 1$ $\dfrac{v_{\mathrm{th}_s}{}^2}{c^2} \ll 1$

$$\nabla \cdot \widetilde{\boldsymbol{E}} = 4\pi \widetilde{\varrho},$$

$$\nabla \cdot \widetilde{\boldsymbol{B}} = 0,$$

$$\frac{\partial \widetilde{\boldsymbol{B}}}{\partial t} = -c \nabla \times \widetilde{\boldsymbol{E}},$$

$$\nabla \times \widetilde{\boldsymbol{B}} = \frac{4\pi}{c} \widetilde{\boldsymbol{j}} + \frac{1}{c} \frac{\partial \widetilde{\boldsymbol{E}}}{\partial t},$$

$\Rightarrow$

$$\widetilde{\varrho} = 0,$$

$$\widetilde{\boldsymbol{E}} = -\nabla \widetilde{\varphi} - \frac{1}{c} \frac{\partial \widetilde{\boldsymbol{A}}}{\partial t},$$

$$\widetilde{\boldsymbol{B}} = \nabla \times \widetilde{\boldsymbol{A}}.$$

$$\nabla \times \widetilde{\boldsymbol{B}} = \frac{4\pi}{c} \widetilde{\boldsymbol{j}}.$$

$$\widetilde{\varrho} = \sum_s Z_s e \int \mathrm{d}^3 \boldsymbol{v} \, f_s,$$

$$\widetilde{\boldsymbol{j}} = \sum_s Z_s e \int \mathrm{d}^3 \boldsymbol{v} \, \boldsymbol{v} f_s.$$

Average Over 'Intermediate' Scales to Separate Fluctuations from Background

Space

$$a \gg \lambda \gg \rho_s$$

$$\langle g(\boldsymbol{r}, \boldsymbol{v}, t)\rangle_\perp = \int_{\lambda_\perp^2} \mathrm{d}^2 \boldsymbol{r}'_\perp g(\boldsymbol{r}'_\perp, l, \boldsymbol{v}, t) \Big/ \int_{\lambda_\perp^2} \mathrm{d}^2 \boldsymbol{r}_\perp,$$

Time

$$\tau_E \gg T \gg \omega^{-1}$$

$$\langle g(\boldsymbol{r}, \boldsymbol{v}, t)\rangle_T = \frac{1}{T}\int_{t-T/2}^{t+T/2} \mathrm{d}t' g(\boldsymbol{r}, \boldsymbol{v}, t'),$$

$$f_s = F_s + \delta f_s, \qquad F_s = \langle f_s \rangle_{\mathrm{turb}},$$

$$\widetilde{\boldsymbol{E}} = \boldsymbol{E} + \delta \boldsymbol{E}, \qquad \boldsymbol{E} = \langle \widetilde{\boldsymbol{E}} \rangle_{\mathrm{turb}},$$

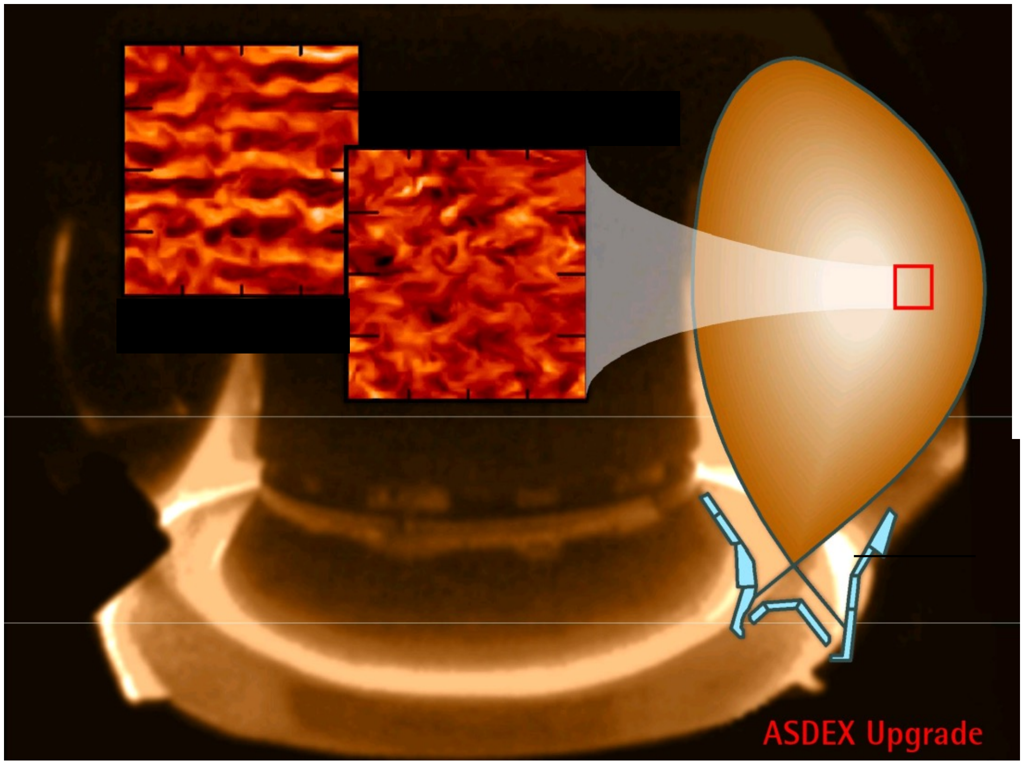
$$\widetilde{\boldsymbol{B}} = \boldsymbol{B} + \delta \boldsymbol{B}, \qquad \boldsymbol{B} = \langle \widetilde{\boldsymbol{B}} \rangle_{\mathrm{turb}},$$

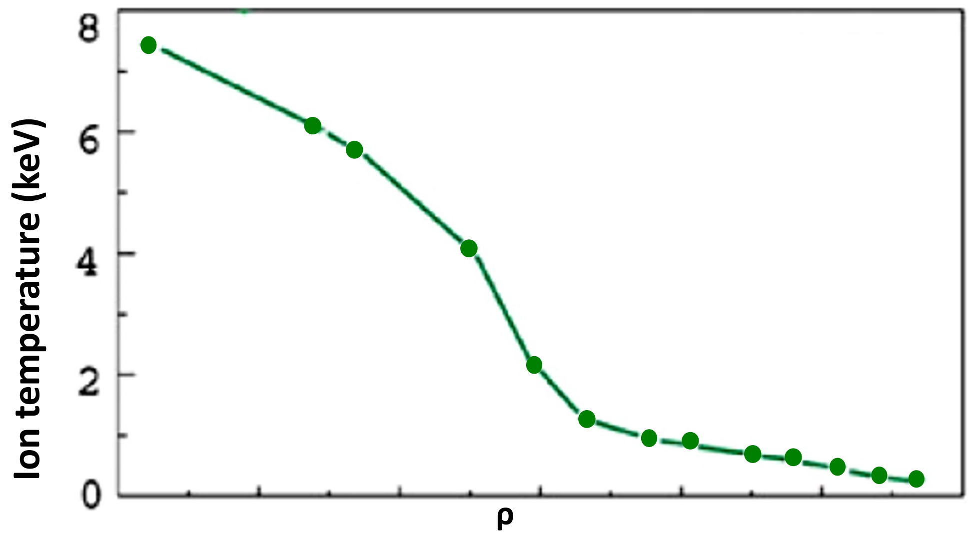
$$\widetilde{\boldsymbol{A}} = \boldsymbol{A} + \delta \boldsymbol{A}, \qquad \boldsymbol{A} = \langle \widetilde{\boldsymbol{A}} \rangle_{\mathrm{turb}},$$

$$\widetilde{\varphi} = \varphi + \delta \varphi, \qquad \varphi = \langle \widetilde{\varphi} \rangle_{\mathrm{turb}}.$$

$$\langle g(\boldsymbol{r}, \boldsymbol{v}, t)\rangle_{\mathrm{turb}} = \big\langle \langle g \rangle_\perp \big\rangle_T.$$

First Step: Separating Background, Macroscopic, Slowly Evolving Quantities from Fluctuating Quantities





$$f_s = F_s + \delta f_s, \qquad F_s = \langle f_s \rangle_{\mathrm{turb}},$$

$$\widetilde{\boldsymbol{E}} = \boldsymbol{E} + \delta \boldsymbol{E}, \qquad \boldsymbol{E} = \langle \widetilde{\boldsymbol{E}} \rangle_{\mathrm{turb}},$$

$$\widetilde{\boldsymbol{B}} = \boldsymbol{B} + \delta \boldsymbol{B}, \qquad \boldsymbol{B} = \langle \widetilde{\boldsymbol{B}} \rangle_{\mathrm{turb}},$$

$$\widetilde{\boldsymbol{A}} = \boldsymbol{A} + \delta \boldsymbol{A}, \qquad \boldsymbol{A} = \langle \widetilde{\boldsymbol{A}} \rangle_{\mathrm{turb}},$$

$$\widetilde{\varphi} = \varphi + \delta \varphi, \qquad \varphi = \langle \widetilde{\varphi} \rangle_{\mathrm{turb}}.$$

Next Step: Convert into ‘Drift Coordinates’

Gyrokinetic Variables

(Ωt) gyrophase angle

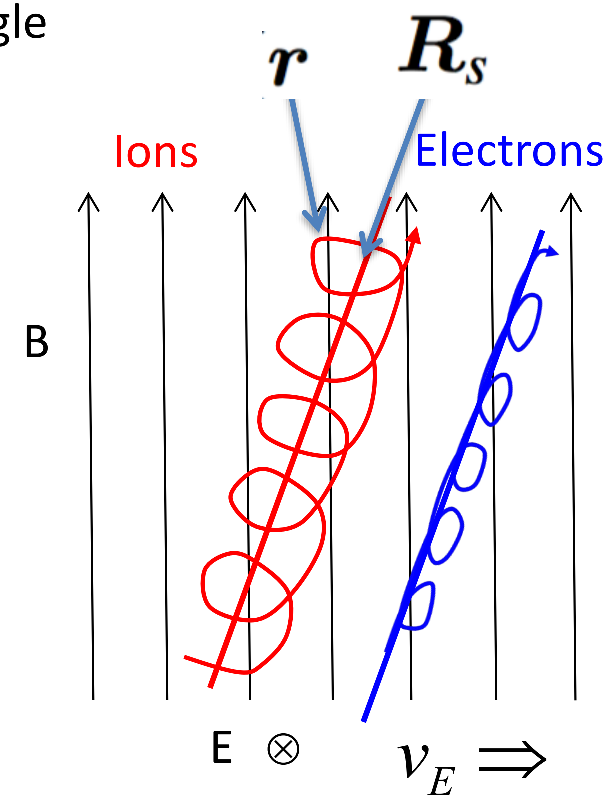
$$\boldsymbol{w} = w_\parallel \boldsymbol{b} + w_\perp \left(\cos\vartheta\, \boldsymbol{e}_2 - \sin\vartheta\, \boldsymbol{e}_1\right),$$

Particle location

Location of guiding center

$$(r, w) \to (\boldsymbol{R}_s, \varepsilon_s, \mu_s, \vartheta, \sigma)$$

$$\boldsymbol{R}_s = \boldsymbol{r} - \frac{\boldsymbol{b} \times \boldsymbol{w}}{\Omega_s},$$



Useful Velocity Space Variables

Gyrophase angle

$$(r, w) \rightarrow (\boldsymbol{R}_s, \varepsilon_s, \mu_s, \vartheta, \sigma)$$

$$\varepsilon_s = \frac{1}{2} m_s w^2 \qquad \mu_s = \frac{m_s w_\perp^2}{2B}, \qquad \sigma = \frac{w_\parallel}{|w_\parallel|},$$

Alternatively (GENE uses these):

$w_\parallel$ and $\mu_s = \frac{m_s w_\perp^2}{2B},$

Convert Entire Kinetic Equation into Gyrokinetic Variables

$$\frac{\mathrm{d} f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + w \cdot \nabla f_s + \frac{Z_s e}{m_s}\left(\widetilde{\boldsymbol{E}} + \frac{1}{c} w \times \widetilde{\boldsymbol{B}}\right) \cdot \frac{\partial f_s}{\partial \boldsymbol{v}} = C[f_s] + S_s,$$

$$(r, w) \rightarrow (\boldsymbol{R}_s, \varepsilon_s, \mu_s, \vartheta, \sigma)$$

$$\frac{\mathrm{d} f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \dot{\boldsymbol{R}}_s \cdot \frac{\partial f_s}{\partial \boldsymbol{R}_s} + \dot{\mu}_s \frac{\partial f_s}{\partial \mu_s} + \dot{\varepsilon}_s \frac{\partial f_s}{\partial \varepsilon_s} + \dot{\vartheta} \frac{\partial f_s}{\partial \vartheta} = C[f_s] + S_s,$$

We now have a kinetic equation in its ‘natural’ coordinates for a strongly magnetized plasma.
Now we have, instead of a distribution of particles, a distribution of guiding centers

Convert Entire Kinetic Equation into Gyrokinetic Variables

$$\frac{\mathrm{d}f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + w \cdot \nabla f_s + \frac{Z_s e}{m_s}\left(\widetilde{\boldsymbol{E}} + \frac{1}{c} w \times \widetilde{\boldsymbol{B}}\right) \cdot \frac{\partial f_s}{\partial \boldsymbol{v}} = C[f_s] + S_s,$$

$$(r, w) \to (\boldsymbol{R}_s, \varepsilon_s, \mu_s, \vartheta, \sigma)$$

$$\frac{\mathrm{d}f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \dot{\boldsymbol{R}}_s \cdot \frac{\partial f_s}{\partial \boldsymbol{R}_s} + \dot{\mu}_s \frac{\partial f_s}{\partial \mu_s} + \dot{\varepsilon}_s \frac{\partial f_s}{\partial \varepsilon_s} + \dot{\vartheta} \frac{\partial f_s}{\partial \vartheta} = C[f_s] + S_s,$$

What is this?

Convert Entire Kinetic Equation into Gyrokinetic Variables

$$\frac{\mathrm{d} f_s}{\mathrm{d} t} = \frac{\partial f_s}{\partial t} + w \cdot \nabla f_s + \frac{Z_s e}{m_s}\left(\widetilde{\boldsymbol{E}} + \frac{1}{c} w \times \widetilde{\boldsymbol{B}}\right) \cdot \frac{\partial f_s}{\partial \boldsymbol{v}} = C[f_s] + S_s,$$

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$$\frac{\mathrm{d} f_s}{\mathrm{d} t} = \frac{\partial f_s}{\partial t} + \dot{\boldsymbol{R}}_s \cdot \frac{\partial f_s}{\partial \boldsymbol{R}_s} + \dot{\mu}_s \frac{\partial f_s}{\partial \mu_s} + \dot{\varepsilon}_s \frac{\partial f_s}{\partial \varepsilon_s} + \dot{\vartheta} \frac{\partial f_s}{\partial \vartheta} = C[f_s] + S_s,$$

What is this?
This encompasses the drift velocities etc

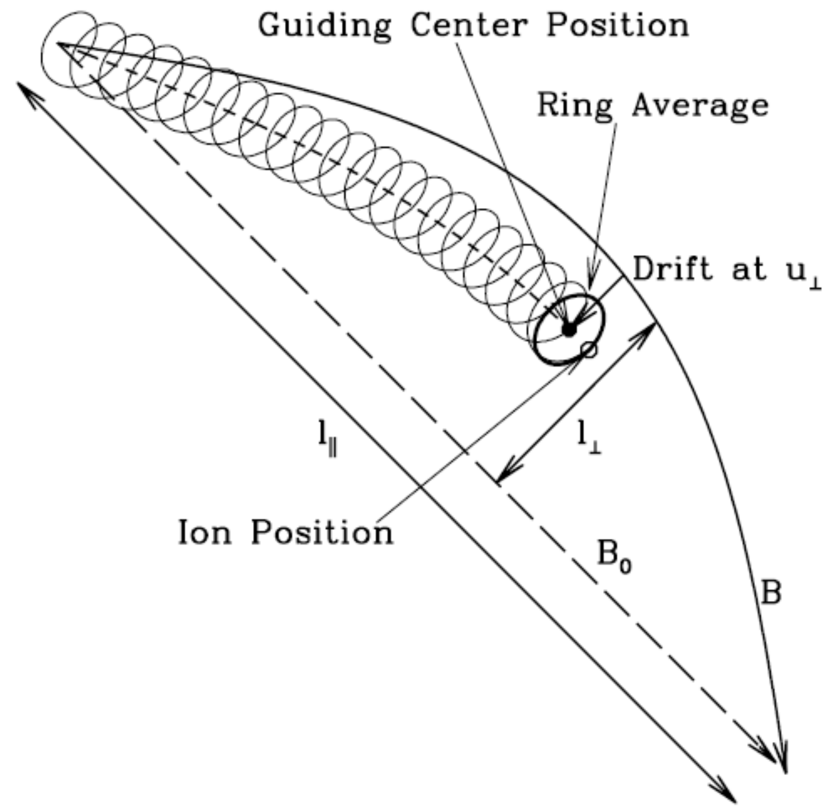
$$v_E = \frac{E \times B}{B^2}$$

$$v_{\nabla B} = m_j w_\perp^2 \frac{B \times \nabla B}{2 q_j B^2}$$

Gyro-average

$$\langle g \rangle_{\boldsymbol{R}} = \frac{1}{2\pi} \oint \mathrm{d}\vartheta \, g(\boldsymbol{R}_s, \varepsilon_s, \mu_s, \vartheta, \sigma),$$

Averaging out the gyrophase angle eliminates this extremely fast time scale --very useful!



Split the Distribution Function

Background Maxwellian

Neoclassical Distribution Function

For bookkeeping

Background,
Slow time scale
Large space scale

$$F_s = F_{0s} + F_{1s} + F_{2s} + \cdots ,$$

$$\delta f_s = \delta f_{1s} + \delta f_{2s} + \cdots ,$$

Fluctuating,
'Fast' time scale
Small space scale

For bookkeeping

Turbulence (gyrokinetics)

$$F_{0s} \sim f_s,\ F_{1s} \sim \delta f_{1s} \sim \epsilon f_s,\ F_{2s} \sim \delta f_{2s} \sim \epsilon^2 f_s, \text{etc.}$$

Now Expand in Terms of Our Ordering Scheme

$$\frac{\mathrm{d}f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \dot{\boldsymbol{R}}_s \cdot \frac{\partial f_s}{\partial \boldsymbol{R}_s} + \dot{\mu}_s \frac{\partial f_s}{\partial \mu_s} + \dot{\varepsilon}_s \frac{\partial f_s}{\partial \varepsilon_s} + \dot{\vartheta} \frac{\partial f_s}{\partial \vartheta} = C[f_s] + S_s,$$

$$\frac{|\delta \boldsymbol{B}|}{|\boldsymbol{B}|} \sim \frac{|\delta \boldsymbol{E}|}{|\boldsymbol{E}|} \sim \frac{\delta f_s}{f_s} \sim \frac{k_\parallel}{k_\perp} \sim \frac{\omega}{\Omega_s} \sim \frac{\rho_s}{a} = \epsilon.$$

$$\frac{1}{\tau_E} \sim \frac{\chi_T}{a^2} \sim \frac{\omega}{\Omega_s}\left(\frac{\rho_s}{a}\right)^2 \Omega_s \sim \epsilon^3 \Omega_s$$

Solve order by order

Some examples

$$\frac{\mathrm{d} f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \dot{\boldsymbol{R}}_s \cdot \frac{\partial f_s}{\partial \boldsymbol{R}_s} + \dot{\mu}_s \frac{\partial f_s}{\partial \mu_s} + \dot{\varepsilon}_s \frac{\partial f_s}{\partial \varepsilon_s} + \dot{\vartheta} \frac{\partial f_s}{\partial \vartheta} = C[f_s] + S_s,$$

$$\frac{\partial f}{\partial t} = \frac{\partial F_0}{\partial t} + \frac{\partial F_1}{\partial t} + \frac{\partial \delta f_1}{\partial t} + \ldots$$

Ordering in terms of ΩF_0 ?

Some examples

$$\frac{\mathrm{d} f_s}{\mathrm{d} t} = \frac{\partial f_s}{\partial t} + \dot{\boldsymbol{R}}_s \cdot \frac{\partial f_s}{\partial \boldsymbol{R}_s} + \dot{\mu}_s \frac{\partial f_s}{\partial \mu_s} + \dot{\varepsilon}_s \frac{\partial f_s}{\partial \varepsilon_s} + \dot{\vartheta} \frac{\partial f_s}{\partial \vartheta} = C[f_s] + S_s,$$

$$\frac{\partial f}{\partial t} = \frac{\partial F_0}{\partial t} + \frac{\partial F_1}{\partial t} + \frac{\partial \delta f_1}{\partial t} + \ldots$$

$$\epsilon^3 \Omega F_0 \qquad \epsilon^4 \Omega F_0 \qquad \epsilon^2 \Omega F_0$$

Some examples

$$\frac{\mathrm{d} f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \dot{\boldsymbol{R}}_s \cdot \frac{\partial f_s}{\partial \boldsymbol{R}_s} + \dot{\mu}_s \frac{\partial f_s}{\partial \mu_s} + \dot{\varepsilon}_s \frac{\partial f_s}{\partial \varepsilon_s} + \dot{\vartheta} \frac{\partial f_s}{\partial \vartheta} = C[f_s] + S_s,$$

$$v_E \cdot \nabla f = v_E \cdot \nabla(F_0 + F_1 + \delta f_1 + ...)$$

$$v_E \cdot \nabla F_0 \to \epsilon v_{th}(1/a)F_0 \sim \epsilon^2 \Omega F_0$$

$$v_E \cdot \nabla F_1 \to \epsilon v_{th}(1/a)F_1 \sim \epsilon^3 \Omega F_0$$

$$v_E \cdot \nabla \delta f_1 \to \epsilon v_{th}(1/\rho)\delta f_1 \sim \epsilon^2 \Omega F_0$$

Note:
v_E is perp to B
$v_E \sim \epsilon v_{th}$

Summary of Equations Order by Order

- In the following: lots of averaging, use of several identities, Boltzmann H-theorem, etc.
- 0th order: Background distribution function is independent of gyro-phase

$$\Omega_s \left. \frac{\partial F_{0s}}{\partial \vartheta} \right|_{\boldsymbol{R}_s, \mu_s, \varepsilon_s} = 0.$$

- First order: Background is a Maxwellian with density and temperature 'flux functions'

$$F_{0s} = N_s(\psi(\boldsymbol{R}_s)) \left[\frac{m_s}{2\pi T_s(\psi(\boldsymbol{R}_s))} \right]^{3/2} \mathrm{e}^{-\varepsilon_s / T_s(\psi(\boldsymbol{R}_s))}.$$

- First order: fluctuating distribution function is made of two parts—Boltzmann response and a part that is gyro-phase independent (➔ no fast time dependence in h!)

$$\delta f_{1s} = -\frac{Z_s e \delta\varphi'(\boldsymbol{r})}{T_s} F_{0s} + h_s\left(\boldsymbol{R}_s, \mu_s, \varepsilon_s, \sigma, t\right),$$

Summary of Equations Order by Order

- Second order:

$$\frac{\partial h_s}{\partial t} + \left(\dot{\boldsymbol{R}}_s \cdot \frac{\partial}{\partial \boldsymbol{R}_s} + \dot{\mu}_s \frac{\partial}{\partial \mu_s} + \dot{\varepsilon}_s \frac{\partial}{\partial \varepsilon_s}\right)(F_{0s} + F_{1s} + h_s) = -\Omega_s \frac{\partial}{\partial \vartheta}(F_{2s} + \delta f_{2s}) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Z_s e \delta\varphi'}{T_s} F_{0s}\right) + C[F_{0s} + F_{1s} + h_s],$$

- Gyro-averaging eliminates higher order distribution functions
- Average over fluctuations ➔ drift-kinetic equation (neoclassical)
- Ampere's law with F_0, F_1 (e.g., bootstrap current): Grad-Shafranov

$$\boldsymbol{j} = \frac{c}{4\pi} \nabla \times \boldsymbol{B}, \qquad \boldsymbol{j} = \sum_s Z_s e \int \mathrm{d}^3 \boldsymbol{w}\, \boldsymbol{w} F_s,$$

$$\Rightarrow \frac{1}{c} j \times B = \sum_s \nabla p_s \Rightarrow$$

Macroscopic Equilibrium

Gyrokinetic Equation:
Describes the Time Evolution of **Guiding Centers**
5D instead of 6D
No fast gyro-frequency time scales

- Second order:

$$\frac{\partial h_s}{\partial t} + \left(\dot{\boldsymbol{R}}_s \cdot \frac{\partial}{\partial \boldsymbol{R}_s} + \dot{\mu}_s \frac{\partial}{\partial \mu_s} + \dot{\varepsilon}_s \frac{\partial}{\partial \varepsilon_s}\right)(F_{0s} + F_{1s} + h_s) = -\Omega_s \frac{\partial}{\partial \vartheta}(F_{2s} + \delta f_{2s}) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Z_s e \delta\varphi'}{T_s} F_{0s}\right) + C[F_{0s} + F_{1s} + h_s],$$

- Keep fluctuating part: gyrokinetic equation and again gyroaverage to eliminate f2 terms (note, I have simplified the following equation w.r.t. Abel 2013)

$$\frac{\partial}{\partial t}(h_s - \frac{Z_s e F_{0s}}{T_s} < \delta\phi >_R) + (w_{||}\hat{b} + v_{Ds} + < v_E >_R) \cdot \frac{\partial h_s}{\partial R_s} + < v_E >_R \cdot \nabla F_0 = < C_l[h_s] >_R$$

Gyrokinetic Equation:
Describes the Time Evolution of **Guiding Centers**
5D instead of 6D
No fast gyro-frequency time scales

$$\frac{\partial}{\partial t}\left(h_s - \frac{Z_s e F_{0s}}{T_s} < \delta\phi >_R\right) + (w_{||}\hat{b} + v_{Ds} + < v_E >_R) \cdot \frac{\partial h_s}{\partial R_s} + < v_E >_R \cdot \nabla F_0 = < C_l[h_s] >_R$$

(electrostatic, no background flow)

$$< v_E >_R = \frac{c}{B}\hat{b} \times \frac{\partial \delta\phi}{\partial R}$$ (gyro-averaged ExB drift)

(Grad B drift)

$$\boldsymbol{V}_{Ds} = \frac{\boldsymbol{b}}{\Omega_s} \times \left[w_{\|}^2 \boldsymbol{b} \cdot \nabla \boldsymbol{b} + \frac{1}{2} w_{\perp}^2 \nabla \ln B \right]$$

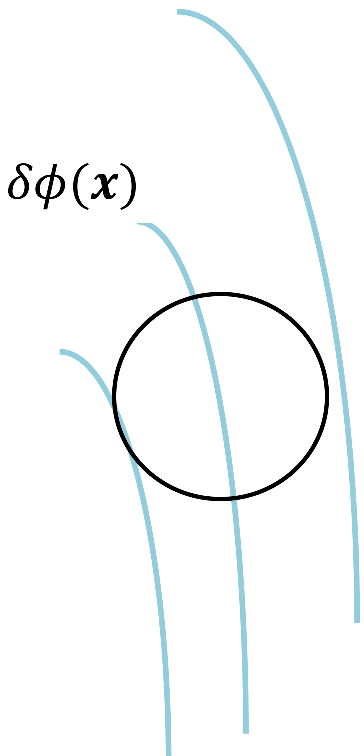
Curvature drift

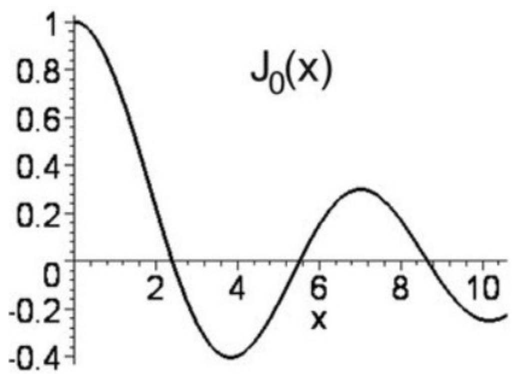
What's the meaning of the gyroaverages?

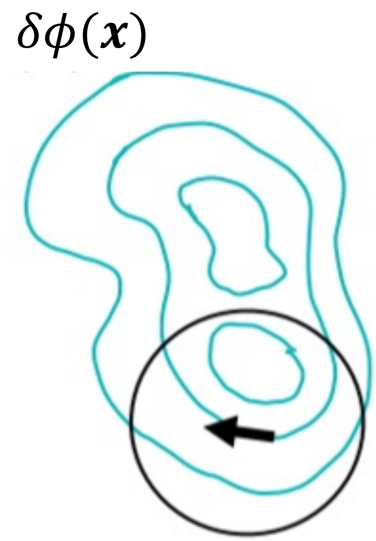
Consequences of Gyroaveraging

In Fourier space (k), gyroaverage operator
Can be expressed as $J_0(k\,\rho)$ (Bessel function)

$$< \delta\phi(R) >_R \rightarrow J_0(k_\perp \rho)\delta\phi_k$$







Large scales: $J_0(k\,\rho) \sim 1$

Small scales ($k\,\rho \sim 1$): $J_0(k\,\rho) \rightarrow 0$

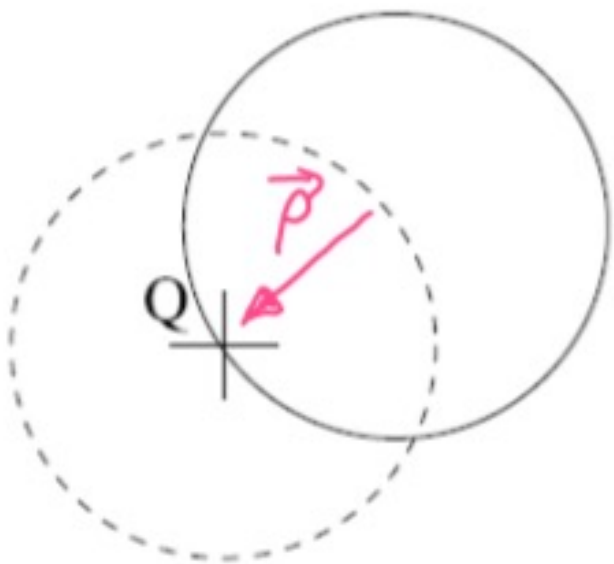
➔ Instabilities suppressed at scales much smaller than gyroradius

GK Poisson Equation: Need Distribution Function for Particles (not gyro-centers) for Poisson Eqn.

For fields: need particle (not gyro center) distribution function

➔ Get particle distribution function

$$\sum_s \frac{Z_s^2 e^2 n_s \delta\varphi}{T_s} = \sum_s Z_s e \int \mathrm{d}^3 w \langle h_s \rangle_r,$$



Gyroaverage at constant particle position r

Get contribution of each gyrocenter with particles at location ‘Q’

What We Have Achieved with Gyrokinetics

- Extracted a rigorous equation for the fluctuations + turbulence
- This equation describes the guiding center distribution function
    - Gyro-average: removes fast gyro-frequency time scale
    - Exploits the anisotropy of the fluctuations (parallel vs perpendicular)
    - Big savings!
- Captures *all* the important micro-instabilities for a fusion plasma (ITG, TEM, ETG, MTM, KBM, RBM, drift Alfven…)
    - (But this means some transparency is lost—may require some more work to understand physics)
- What's left out
    - Some MHD behavior (current-driven MHD instabilities, low n modes?)
    - Some fast particle instabilities
    - Anything with time scales faster than the gyro-frequency
        - Not much in a fusion plasma
        - But some space / astro waves (whistlers, fast Alfven wave)
- Later discussion: in the edge 'transport barrier' some of these orderings are not as robust as they are in the main plasma (we'll talk about this later)

Major Theoretical Speedups

G.Hammett

relative to original Vlasov/Maxwell system on a naive grid, for ITER $\rho_* = \rho/a \sim 1/1000$

- Nonlinear gyrokinetic equations
  - eliminate plasma frequency: $\omega_{pe}/\Omega_i \sim m_i/m_e$ — $\times 10^3$
  - eliminate Debye length scale: $(\rho_i/\lambda_{De})^3 \sim (m_i/m_e)^{3/2}$ — $\times 10^5$
  - average over fast ion gyration: $\Omega_i/\omega \sim 1/\rho_*$ — $\times 10^3$
- Field-aligned coordinates
  - adapt to elongated structure of turbulent eddies: $\Delta_{||}/\Delta_{\perp} \sim 1/\rho_*$ — $\times 10^3$
- Reduced simulation volume
  - reduce toroidal mode numbers (i.e., 1/15 of toroidal direction) — x15
  - $L_r \sim a/6 \sim 160\ r \sim 10$ correlation lengths — x6
- Total speedup — $\times 10^{16}$
- For comparison: Massively parallel computers (1984-2009) — $\times 10^7$

You'll Find Varying Notation for GK Equation; This is a Standard for GENE (in k space)

$$\frac{\partial g}{\partial t} = Z + \mathcal{L}[g] + \mathcal{N}[g]\,,$$

This notation puts all time derivatives on g

$$\mathcal{L}[g] = -\left(\omega_n + (v_\parallel^2 + \mu B_0 - \frac{3}{2})\omega_{Tj}\right) F_{0j} i k_y \chi + \frac{\beta T_{0j}}{q_j B_0^2} v_\parallel^2 \omega_p \Gamma_{jy} - \frac{v_{Tj}}{J B_0} v_\parallel \Gamma_{jz}$$
$$- \frac{T_{0j}(2v_\parallel^2 + \mu B_0)}{q_j B_0}(K_y \Gamma_{jy} + K_x \Gamma_{jx}) + \frac{v_{Tj}}{2 J B_0} \mu \partial_z B_0 \frac{\partial f_j}{\partial v_\parallel} + \langle C_j(f) \rangle\,,$$

$$N[g] = \sum_{\vec{k}'_\perp} \left(k'_x k_y - k_x k'_y\right) \chi(\vec{k}'_\perp) g_j(\vec{k}_\perp - \vec{k}'_\perp),$$

$$\chi_j = \bar{\phi}_j - v_{Tj} v_\parallel \bar{A}_{1\parallel j} \qquad f_j = g_j - \frac{2 q_j}{m_j v_{Tj}} v_\parallel \bar{A}_{1\parallel} F_{0j}$$

$$\Gamma_{x,y} = i k_{x,y} g + \frac{q_j}{T_{0j}} F_0 i k_{x,y} \chi$$

↑

Gamma is h from earlier slides

Third Order: Transport Equations

- Third order: transport equation describing slow evolution of background

$$\frac{\partial F_s}{\partial t} + (\boldsymbol{u}+\boldsymbol{w})\cdot\nabla F_s + \left[\boldsymbol{a}_s - \frac{\partial \boldsymbol{u}}{\partial t} - (\boldsymbol{u}+\boldsymbol{w})\cdot\nabla\boldsymbol{u}\right]\cdot\frac{\partial F_s}{\partial \boldsymbol{w}} + \left\langle \delta\boldsymbol{a}_s\cdot\frac{\partial \delta f_s}{\partial \boldsymbol{w}}\right\rangle_{\text{turb}} = C[F_s] + S_s,$$

$$\frac{\partial n_s}{\partial t} + \frac{1}{V'}\frac{\partial}{\partial \psi}(V'\overline{\langle\Gamma_s\rangle}) = \overline{\langle S_n\rangle},$$

$$\frac{\partial \overline{L}}{\partial t} + \sum_s \frac{1}{V'}\frac{\partial}{\partial \psi}(V'\overline{\langle\pi_s\rangle}) = \frac{1}{4\pi}\overline{\nabla\cdot\langle\delta\mathbf{B}\,\delta\mathbf{B}\cdot\nabla\phi R^2\rangle} + \sum_s \overline{\langle S_{L_s}\rangle},$$

$$\frac{3}{2}\frac{\partial p_s}{\partial t} + \frac{1}{V'}\frac{\partial}{\partial \psi}(V'\overline{\langle Q_s\rangle}) = -\overline{\langle H_s\rangle} + \frac{3}{2}n_s\sum_u \nu^{\varepsilon}_{su}(T_u - T_s) + \overline{\langle S_p\rangle},$$

$$\Gamma \equiv \nabla\psi\cdot\int d^3\mathbf{v}(\mathbf{v}_\chi\delta f_1 + \mathbf{v}_B\langle F_1\rangle + \boldsymbol{\rho}C[\boldsymbol{\rho}\cdot\nabla f_0]),$$

$$Q \equiv \nabla\psi\cdot\int d^3\mathbf{v}\frac{mv^2}{2}(\mathbf{v}_\chi\delta f_1 + \mathbf{v}_B\langle F_1\rangle + \boldsymbol{\rho}C[\boldsymbol{\rho}\cdot\nabla f_0]),$$

$$\pi \equiv \nabla\psi\cdot\int d^3\mathbf{v}(mR^2\mathbf{v}\cdot\nabla\phi)\mathbf{v}_\chi\delta f_1,$$

Example: Heat Flux

$$Q \equiv \nabla\psi \cdot \int d^3\mathbf{v} \frac{mv^2}{2} (\mathbf{v}_\chi \delta f_1 + \mathbf{v}_B \langle F_1 \rangle + \boldsymbol{\rho} C[\boldsymbol{\rho} \cdot \nabla f_0]),$$

- Classical collisional heat flux
- Most obvious / basic transport mechanism
- Step size = gyroradius
- Step time = inverse collision frequency
- Very high confinement time

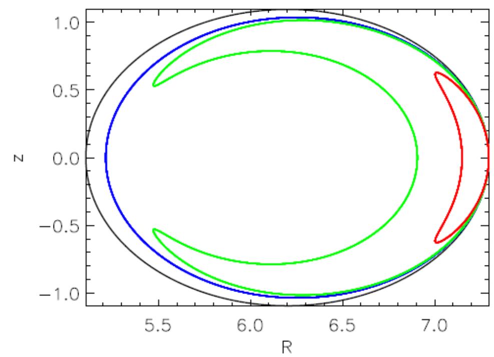
$$D = \rho_e^{\ 2} \nu_c \approx 10^{-3} m^2 s^{-1}$$

Classical → $\tau_E \approx 1000 s$

Example: Heat Flux

$$Q \equiv \nabla\psi \cdot \int d^3\mathbf{v}\frac{mv^2}{2}(\mathbf{v}_\chi \delta f_1 + \boxed{\mathbf{v}_B\langle F_1\rangle} + \boldsymbol{\rho} C[\boldsymbol{\rho}\cdot\nabla f_0]),$$

- **Neo**classical collisional heat flux
- Classical (linear, not turbulent)
- Accounts for broad particle orbits, etc.
- Relevant in some parameter regimes
- Still very high confinement time



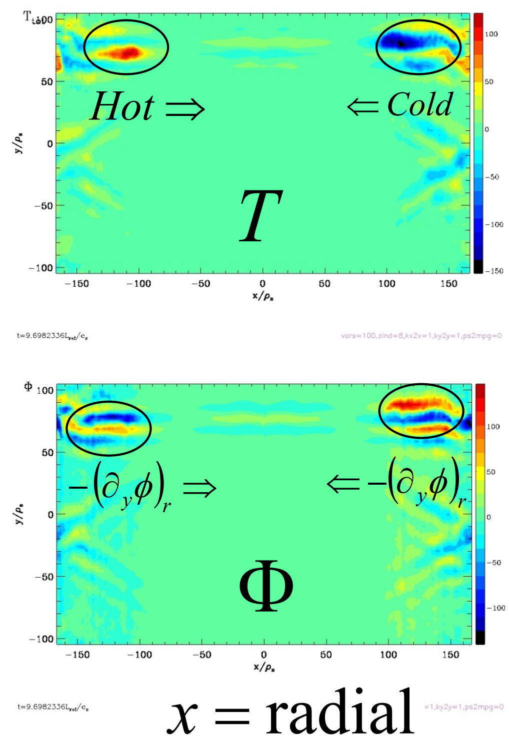
$$\boxed{\tau_{E(neo)} \approx \frac{\tau_{E(classical)}}{100} \approx 10s}$$

Example: Heat Flux

$$Q \equiv \nabla\psi \cdot \int d^3\mathbf{v}\frac{mv^2}{2}\left(\mathbf{v}_\chi \delta f_1 + \mathbf{v}_B\langle f_1\rangle + \boldsymbol{\rho}C[\boldsymbol{\rho}\cdot\nabla f_0]\right)$$

Hot $\Rightarrow$

- **Turbulent** transport
- Advection of temperature fluctuations by velocity fluctuations
- Dominant transport mechanism in fusion devices
- Lower confinement time
  - ➔Understanding and controlling plasma turbulence is a major part of fusion research



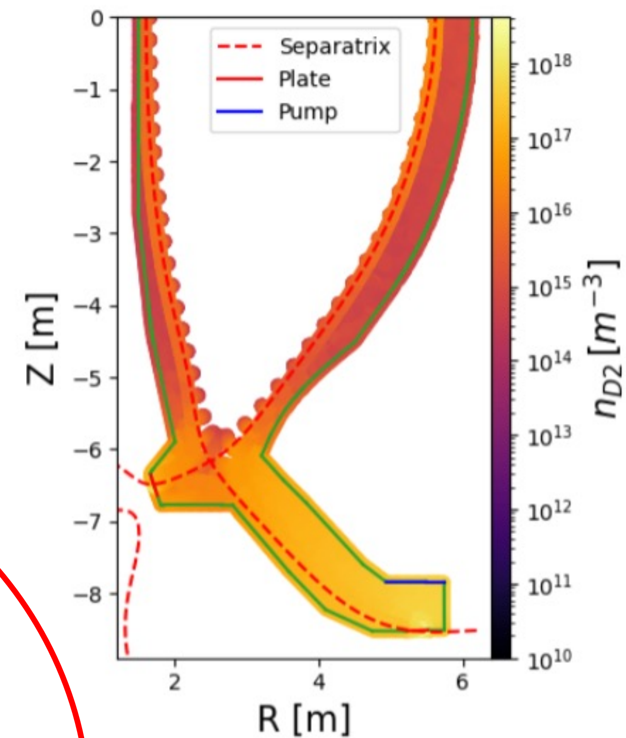
Typical Fusion Parameters➔ $\tau_E \approx 0.1 - 1.0s$

End Result

- Starting from first principles kinetic equation
- Exploiting scale separation
- Arriving at a set of four equations that is still extremely close to first principles (in the core):
    - Grad-Shafranov for background equilibrium
    - Drift-kinetic for neoclassical (second order macroscopic distribution)
    - Gyro-kinetic for fluctuations (turbulence)
    - Transport equation for slow evolution of background profiles

New Generation of Gyrokinetic Codes: Full-f

XGC
GENE-X
Gkeyll
Etc.



$$f_s = F_s + \delta f_s, \qquad F_s = \langle f_s \rangle_{\mathrm{turb}},$$
$$\widetilde{\boldsymbol{E}} = \boldsymbol{E} + \delta \boldsymbol{E}, \qquad \boldsymbol{E} = \langle \widetilde{\boldsymbol{E}} \rangle_{\mathrm{turb}},$$
$$\widetilde{\boldsymbol{B}} = \boldsymbol{B} + \delta \boldsymbol{B}, \qquad \boldsymbol{B} = \langle \widetilde{\boldsymbol{B}} \rangle_{\mathrm{turb}},$$
$$\widetilde{\boldsymbol{A}} = \boldsymbol{A} + \delta \boldsymbol{A}, \qquad \boldsymbol{A} = \langle \widetilde{\boldsymbol{A}} \rangle_{\mathrm{turb}},$$
$$\widetilde{\varphi} = \varphi + \delta \varphi, \qquad \varphi = \langle \widetilde{\varphi} \rangle_{\mathrm{turb}}.$$