Curvature-homogeneous Riemannian Manifolds

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However, the converse is not true for $n \geq 3$, as will be seen.

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The codimension of $\mathcal{O}_r$ in $\mathcal{K}_n$ is

$$c(\mathcal{O}_r) = \tfrac{1}{12} n^2(n^2{-}1) - \left(\tfrac{1}{2} n(n{-}1) - d_r\right) = \tfrac{1}{12} n(n{-}1)(n{-}2)(n{+}3) + d_r$$

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Since g, modulo diffeomorphisms, depends on only $\frac{1}{2}n(n-1)$ functions of n variables, this second-order system of PDE for g is overdetermined as soon as $n > 3$, no matter which $r \in K_n$ is considered.

Cartan-Spencer analysis

Let (M^n, g) is curvature-homogeneous of type $r \in K_n$, and let $F \to M$ be the g-orthogonal frame bundle over M. Cartan's first and second structure equations are

$$\mathrm{d}\omega = -\theta \wedge \omega \qquad\qquad \mathrm{d}\theta = -\theta \wedge \theta + \langle R, \omega \wedge \omega \rangle.$$

- $\omega : TF \to \mathbb{R}^n$ is the tautological 1-form
- $\theta : TF \to \mathfrak{so}(n)$ is the Levi-Civita connection 1-form, and
- $R : F \to K_n$ is the curvature function, satisfying $R(u \cdot h) = \rho_n(h)^{-1}\big(R(u)\big)$ for $h \in \mathrm{O}(n)$.

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By hypothesis, $F \to M$ contains an H_r-subbundle

$$B_r = \big\{ u \in F \mid R(u) = r \ \big\} \subset F$$

Let $\mathfrak{so}(n) = \mathfrak{h}_r \oplus \mathfrak{h}_r^{\perp}$ and write $\theta = \psi + \tau$ where ψ has values in $\mathfrak{h}$ and τ has values in $\mathfrak{h}_r^{\perp}$.

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So $\tau = p\omega$ for some $p : B_r \to \mathfrak{a}_r \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathfrak{h}_r^\perp) = \mathfrak{h}_r^\perp \otimes (\mathbb{R}^n)^*$, where

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$$\begin{aligned} \mathrm{d}\psi &= -\psi \wedge \psi + \Pi_{\mathfrak{h}_r}\big(< r, \omega \wedge \omega > -(p\omega) \wedge (p\omega)\big) \\ \mathrm{d}(p\omega) &= -\psi \wedge (p\omega) - (p\omega) \wedge \psi + \Pi_{\mathfrak{h}_r^\perp}\big(< r, \omega \wedge \omega > -(p\omega) \wedge (p\omega)\big) \end{aligned}$$

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This last equation takes the form $\mathrm{d}p = -\psi.p + q\omega$ for some $q : B_r \to \mathfrak{a}_r \otimes (\mathbb{R}^n)^*$ that satisfies

$$\delta(q) = Q_r(p) + Lr$$

where $Q_r(p){+}Lr : B_r \to \mathfrak{h}_r^{\perp} \otimes \Lambda^2(\mathbb{R}^n)^*$ and $\delta : \mathfrak{a}_r \otimes (\mathbb{R}^n)^* \to \mathfrak{h}_r^{\perp} \otimes \Lambda^2\big((\mathbb{R}^n)^*\big)$ is the Spencer skewsymmetrization.

The upshot is that the structure equations for the coframing $\eta = \omega \oplus \psi : TB_r \to \mathbb{R}^n \oplus \mathfrak{h}_r$ and the structure function $p : B_r \to \mathfrak{a}_r$ on B_r take the Cartan-Spencer form

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- Usually: Neither condition holds, and prolongation is required before the Cartan-Kähler machinery can be applied. Effectively, this involves expressing $\mathrm{d}q$ in terms of ψ and ω.

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In dimensions $n \geq 5$, it's easy to give examples of r for which the second Bianchi identity alone rules out existence of curvature-homogeneous metrics with curvature type r.

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B.–Florit–Ziller (2024) completed the classification of curvature-homogeneous hypersurfaces in space-forms, finding new explicit examples of dimension 3.

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1. If $c_1 c_2 c_3 > 0$, there is a 3-dimensional Lie group G with left-invariant 1-forms ω_i satisfying

$$\mathrm{d}\omega_i = \frac{c_i(c_j+c_k)}{\sqrt{2c_1 c_2 c_3}}\,\omega_j \wedge \omega_k, \quad \text{where } (i,j,k) \text{ is an even permutation of } (1,2,3)$$

with $\mathrm{Ric}\big({\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2\big) = c_1\,{\omega_1}^2 + c_2\,{\omega_2}^2 + c_3\,{\omega_3}^2$.

Curvature-Homogeneous Riemannian 3-Manifolds

A (M^3, g) is curvature-homogeneous if and only if the Ricci curvature of g has constant eigenvalues (c_1, c_2, c_3).

Remark: Not every triple of eigenvalues (c_1, c_2, c_3) occurs for some homogeneous Riemannian 3-manifold. By Bianchi's classification, for $c_1 < c_2 < c_3$, there are two types of possibilities:

1. If $c_1 c_2 c_3 > 0$, there is a 3-dimensional Lie group G with left-invariant 1-forms ω_i satisfying

$$\mathrm{d}\omega_i = \frac{c_i(c_j+c_k)}{\sqrt{2c_1c_2c_3}}\,\omega_j \wedge \omega_k, \quad \text{where } (i,j,k) \text{ is an even permutation of } (1,2,3)$$

with $\mathrm{Ric}\big({\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2\big) = c_1\,{\omega_1}^2 + c_2\,{\omega_2}^2 + c_3\,{\omega_3}^2$.

2. If $(c_j^2+c_k^2)/(c_j+c_k) \le c_i < \frac{1}{2}(c_j+c_k) < 0$ for some (i,j,k) distinct, there are constants p_1, p_2, p_3, p_4 and a 3-dimensional Lie group G with left-invariant 1-forms satisfying

$$\mathrm{d}\omega_i = 0, \quad \mathrm{d}\omega_j = (p_1\,\omega_j + p_2\,\omega_k) \wedge \omega_i, \quad \mathrm{d}\omega_k = (p_3\,\omega_j + p_4\,\omega_k) \wedge \omega_i$$

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Theorem (K. Yamato; 1992): Let $r = (c_1, c_2, c_3)$ be real, distinct constants satisfying

$$c_2 > 0 > c_1 + c_2 > c_3 \geq \frac{{c_1}^2 + {c_2}^2}{c_1 + c_2}$$

Then there exists a complete, inhomogeneous metric g_r on $\mathbb{R}^3$ whose Ricci eigenvalues are distinct and equal to the r_i.

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Yamato's method is a direct Ansatz construction: Assume a metric on $\mathbb{R}^3$ of the form

$$g = \mathrm{d}z^2 + E(z)\,\mathrm{d}x^2 + 2F(z)\,\mathrm{d}x\,\mathrm{d}y + G(z)\,\mathrm{d}y^2$$

and write down the set of ODE for the functions E, F, and G that will make the resulting metric g have Ricci eigenvalues (c_1, c_2, c_3). Such a metric will be complete iff the solutions $(E(z), F(z), G(z))$ are defined on the entire real line.

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Note that these metrics generally have cohomogeneity 1, since they are clearly invariant under x- and y-translations.

Theorem (Kowalski,Prüfer; 1994): For any distinct constants c_1, c_2, c_3, there exist local non-homogeneous Riemannian 3-manifolds (M, g) with the eigenvalues of the Ricci tensor equal to (c_1, c_2, c_3).

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More precisely: When written in geodesic normal coordinates that diagonalize the Ricci tensor at the origin, the space of k-jets of curvature-homogeneous metrics in dimension 3 with distinct Ricci eigenvalues has dimension $3\binom{k}{2} = \frac{3}{2}k(k-1)$.

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The characteristic variety: The underlying PDE, which is diffeomorphism invariant, is not 'transversely elliptic', In fact, for the linearization of the PDE at a metric $g = {\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2$ that satisfies $\mathrm{Ric}(g) = c_1\,{\omega_1}^2 + c_2\,{\omega_2}^2 + c_3\,{\omega_3}^2$ with the c_i distinct, one finds that a covector

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The PDE for the metric is 'transversely symmetric hyperbolic', so that a curvature-homogeneous metric is determined up to diffeomorphism by data specified along a 'non-characteristic' surface $\Sigma \subset M^3$, i.e., a surface on which $\omega_i{\wedge}\omega_j$ is non-vanishing for $i \neq j$.

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Henceforth, I will set $(c_1, c_2, c_3) = (2c, 2c, 2r)$ for constants $c \neq r \neq 0$.

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Conversely, every (M^3, g) with Ricci eigenvalues $(2c, 2c, -2)$ is locally of the form $\bigl(F, \beta(\mathrm{d}\sigma^2)\bigr)$ for some $(L, \mathrm{d}\sigma^2)$ satisfying $|K| < 1$ and $\Delta\bigl(\cos^{-1}(K)\bigr) = -2c\sqrt{1-K^2}$.

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where $(x, y, z) \in \mathbb{R}^3$ lie in the open 'slab' $S \subset \mathbb{R}^3$ defined by $|x-y| < \pi/2$, and this representation is unique up to replacement by $(x + 2k\pi, y + 2k\pi, z)$ for some integer k.

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Then the Schmidt-Wolfson metrics are the β-metrics of surface metrics of the form

$$\mathrm{d}\sigma^2 = \frac{4\,\sec 2v(x)\,\mathrm{d}x\circ\mathrm{d}y}{\cos^2(x-y)}.$$

on the strip $|x-y| < \pi/2$ in the xy-plane, where v is a 2π-periodic function of x that satisfies $0 < v < \pi/4$.

However, one does not need v to be periodic in order to define the β-metric of this $\mathrm{d}\sigma^2$, one merely needs that $0 < v < \pi/4$. Moreover, the β-metric will be complete whenever v is constrained to take values in $[\epsilon, \pi/4 - \epsilon]$ for any positive ϵ.

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The analysis of the almost-complex manifold X_c is an ongoing project.

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Let $B \subset \mathrm{SU}(3)$ be the special unitary bases frames $f = (f_0\ f_1\ f_2)$ such that $p = [f_0] \in \mathbb{CP}^2$ lies in C and f_0 and f_1 span a 2-plane that represents the tangent line to C at $p = [f_0]$. Then

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Conversely, every connected (M^3, g) with Ricci eigenvalues $(6, 6, 2)$ is of this form for some holomorphic curve $C \subset \mathbb{CP}^2$.

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Hence there is induced a Riemannian metric (M^3, g) with Ricci eigenvalues $(-6, -6, -2)$ on the frame bundle $F \to L$, and every (M^3, g) with Ricci eigenvalues $(-6, -6, -2)$ is locally of this form.

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$$g = \frac{1}{8}\left(\mathrm{d}y^2 - \mathrm{d}x\,\mathrm{d}z + x\,\mathrm{d}u^2 - 2y\,\mathrm{d}u\,\mathrm{d}v + z\,\mathrm{d}v^2\right)$$

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$$\begin{aligned}
\omega_1 &= \cosh(2\sqrt{3}\,x_2)\,\mathrm{d}x_1\,,\\
\omega_2 &= \mathrm{d}x_2 - \big(h(x_1) - f(x_1)\,x_3 - g(x_1)\,x_4\big)\,\mathrm{d}x_1\,,\\
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and f, g, and h are smooth functions on $\mathbb{R}$.

The case $a = 1$, $b = s = -1$, and $c = 3$

On $\mathbb{R}^5$ with coordinates x, y, z, u, v, consider the quadratic form

$$g = \frac{1}{8}\left(\mathrm{d}y^2 - \mathrm{d}x\,\mathrm{d}z + x\,\mathrm{d}u^2 - 2y\,\mathrm{d}u\,\mathrm{d}v + z\,\mathrm{d}v^2\right)$$

pulled back to the $M^4 \subset \mathbb{R}^5$ defined by the equation $xz - y^2 = 1$ and the inequalities $x, z > 0$. This defines a positive definite metric on M^4, with a symmetry group that is a direct product of $\mathrm{SL}(2, \mathbb{R})$ with $\mathbb{R}^2$ (translations in u and v).

The case $a = b = s = -1$ and $c = 3$

On $\mathbb{R}^4$ with coordinates x_1, x_2, x_3, x_4, let $g = {\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2 + {\omega_4}^2$ where

$$\begin{aligned}
\omega_1 &= \cosh(2\sqrt{3}\,x_2)\,\mathrm{d}x_1\,, \\
\omega_2 &= \mathrm{d}x_2 - \big(h(x_1) - f(x_1)\,x_3 - g(x_1)\,x_4\big)\,\mathrm{d}x_1\,, \\
\omega_3 &= \mathrm{d}x_3 - f(x_1)\,x_2\,\mathrm{d}x_1\,, \\
\omega_4 &= \mathrm{d}x_4 - g(x_1)\,x_2\,\mathrm{d}x_1\,.
\end{aligned}$$

and f, g, and h are smooth functions on $\mathbb{R}$. For generic choice of f, g, and h, the metric g has trivial symmetry group and is complete.

The first (exceptional) case $a = b = 1$, $s = -1/2$, and $c = 3$

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$$a^{-1}\,\mathrm{d}a = \begin{pmatrix} 0 & \sigma_3 & -\sigma_2 \\ -\sigma_3 & 0 & \sigma_1 \\ \sigma_2 & -\sigma_1 & 0 \end{pmatrix}.$$

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Consider the metric on $\hat{M}$ defined by

$$g = \frac{1}{3}\left(\mathrm{d}t^2 + (2\sinh t\ \sigma_1)^2 + \mathrm{e}^{-2t}\,{\sigma_2}^2 + \mathrm{e}^{2t}\,{\sigma_3}^2\right)$$

This is the unique example (originally found by Tsukada in 1988). It has cohomogeneity 1.

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This is the unique example (originally found by Tsukada in 1988). It has cohomogeneity 1. To get the actual manifold M, one must divide by the isometry of order 4

$$f(t,a) = (-t, aC)\,, \quad \text{where} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{satisfies} \quad C^4 = I_3,$$

and 'blow down' the locus $t = 0$ to a 2-sphere in M.

Congratulations, Claude!

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Thank you for your attention!

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