Non-Relativistic General Relativity: Not for public distribution: under construction

Ira Z. Rothstein[a1]

[1] *Carnegie Mellon University*

Dept. of Physics,

Pittsburgh PA 15213, USA

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I. INTRODUCTION

In this chapter we will apply the ideas presented in the previous chapter to general relativity (GR)[1]. Some rudimentary understanding of GR will be assumed, although perhaps surprisingly, very little knowledge of the details of GR are required to generate interesting results relevant to gravitational wave astronomy. We will not go into any details regarding the phenomenology of this subject, but instead refer the reader to the literature [2]. The reader should be familiar with geometric quantities such as the Riemann tensor and its contractions, the scalar curvature, and the Ricci tensor. Some of the calculations, in particular those involving the calculation of multipole moments, are standard and can be found in text books such as [2], but they have been included in order to keep the chapter more self contained.

We will be interested in calculating the gravity wave signal generated during the inspiral of two gravitationally bound compact objects. We will assume that initially the orbit is sufficiently large that the constituents are moving at velocities small compared to the speed of light and, as such, the space is approximately flat (see the discussion in the next section). Towards the end of the inspiral, when the velocity grows, the curvature effects become so large that, at present, calculating analytically is not feasible. So we will be restricting our analysis to the early stages of the inspiral where we can do perturbation theory around flat space. Our goal will be to calculate the gravitational power loss as well as the wave forms. To accomplish this we follow the line of reasoning of the previous chapter, proceeding using the two stage approach. We start with two (or perhaps more) finite size objects with typical size of order R, separated by a distance of order r, where $r \gg R$. Next we integrate out all of the physics at distances shorter than R. This leads to an effective theory of point particles. This matching is done in isolation, and fully covariantly, as each particle could have different properties leading to differing matching coefficients for the operators which reproduce the finite size effects. At this stage the only relevant power counting parameter is the ratio of the size of the objects to some long distance scale, typically the wavelength of the radiation. Next, we consider the interaction of the bodies, at which point a new power counting parameter will be determined once one specifies the nature of the two body system. In this chapter that parameter will be the relative velocity of the two constituents, assumed to be small compared to c. Another case of interest is the so-called extreme mass ratio

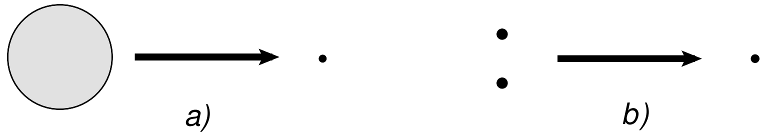

FIG. 1. The two stages of our effective field theory. In the first stage a) we consider each body in isolation and coarse grain to generate an effective field theory of point particles. In the second stage b) we consider the theory of two such point particles we generate an EFT of a single particle, representing the bound state, with time dependent multipole moments.

scenario, where a small black hole orbits a large black hole. In this case one expands around the Schwarzchild solution and, as such, the results are analytically challenging. We will not be discussing this case here and refer the reader to [4] as an example of the application of the EFT technique to this case.

Next we consider what the system looks like at distances much longer than r since we will be observing the binaries asymptotically far away. At these very long distances the binary looks like single particle with time dependent multipole moments. The power loss and wave-form then follow from calculations in this final effective theory. The steps of the coarse graining procedure are illustrated in figure 1.

II. THE SINGLE PARTICLE EFFECTIVE THEORY

Consider an isolated compact body, such as a black hole or neutron star. We wish to write down an effective theory which accounts for finite size effects that is consistent with all of the symmetries, which now include, general coordinate invariance and world line reparameterization invariance (RPI). The former implies that our action must be built out of scalar invariants. At the level of a point particle we parameterize the particles world line via $x^\mu(\lambda)$. The action for the point particle is then given by[1]

$$S^0 = -M \int d\tau - 2M_{pl}^2 \int d^4x \sqrt{g} R, \tag{1}$$

where

$$d\tau = d\lambda \sqrt{\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} g^{\mu\nu}(x(\lambda))}. \tag{2}$$

[1] Note we are sticking with our choice of units where $\hbar = c = 1$. Thus even though we will be only interested in classical results, M_{pl} sits around during intermediate stages of the calculation. This is purely historical and a consequence of the author's educational upbringing. Changing back to more conventional units is hopefully not a great burden on the reader.

Varying this action with respect to the path leads to the geodesic equation for a point like particle.

Now we would like to include terms which account for the finite size of the object and generate non-geodesic motion. There are an infinite number of such terms in an expansion in the size of the object. The lowest dimensional terms that we can write down that are generally coordinate and RPI are [5]

$$S_2 = C_R \int d\tau R(x(\tau)) + C_{vvR} \int d\tau v^\mu v^\nu R_{\mu\nu}(x(\tau)). \tag{3}$$

Where to avoid cumbersome factors of v^2 we choose to work with the proper time parameterization to ensure RPI.

However these terms can be removed by a field redefinition (see the previous chapter), that is, they vanish by use of the vacuum field equations

$$R_{\mu\nu} = 0. \tag{4}$$

Note that we are allowed to use the vacuum equations because the sources all live on the world line. If we included these source terms the field redefinition will introduce singular delta functions ($\delta(0)$) on the world line which should be dropped (see exercise below). If we have multiple particles the redefinition will introduce counter-terms for terms where the particles intersect. Such local interactions should be dropped as they are not part of the effective theory given that they violate our primary assumption of large separations.

Exercise n.1

In the text it was claimed that we could eliminate terms which vanished by Einsteins equations in the bulk. Determine what field re-definitions are necessary to eliminate the following possible two terms which are allowed by the symmetries

$$L = c_1 \int d\tau R(x(\tau)) + c_2 \int d\tau R_{\mu\nu}(x(\tau) R^{\mu\nu}(x(\tau). \tag{5}$$

For the first operator it is suggest that one consider a conformal transformation of the metric. In both cases one should consider shifts in the metric proportional to the diffeomorphism invariant quantity,

$$\int d\lambda \delta^{(4)}(x - x(\lambda))/\sqrt{g}. \tag{6}$$

Choose the coefficient of this shift to cancel the operator of interest. Notice that one must also transform the world line operator as well. This leads to the product of delta functions on the world line proportional to C^2 and correspond to counter-terms for higher dimensional operators (see section).

Given that these operators can be eliminated by a field redefinition, their coefficients must be unphysical. This is a very useful piece of information because it informs us that any divergences which can be absorbed into counter-terms for these operators are irrelevant. It is indeed true that UV divergences will arise in the calculation of the potential at 3PN. Determine which set of diagrams can contribute these (logarithmic) divergences. Hint: Determine the number of mass insertions needed by fixing the scaling of the Wilson coefficient using the same reasoning as discussed near eq. (111). Once you have determined the scaling of the Wilson coefficient, power count and show that naively this operator looks like it should contribute at 2PN. The leading contribution to the potential actually starts at 3PN, because the (leading order) spatial derivatives lead to contact interactions. So one must consider the derivatives in the operator to be temporal to get a non-vanishing potential.

The only possible terms that can not be eliminated must involve the Riemann tensor. As we will see this also implies that there are no terms linear in the metric tensor. This is consistent with Birkhoff's theorem, which states that the spherically symmetric solution to Einstein's equation is fixed by only one parameter, the mass[2]. Furthermore, on shell, we can write down all of the operators in terms of the trace free Weyl tensor defined by (square brackets indicating anti-symmetrization)

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - (g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{1}{3}g_{\rho[\mu}g_{\nu]\sigma}R. \tag{7}$$

We can further decompose the Weyl tensor into pieces which transform covariantly under parity. Given a time like vector, which we will take to be the world line tangent vector (v^β), we define the electric and magnetic part of the Weyl tensor as

$$E_{\rho\mu} = C_{\rho\sigma\mu\nu}v^\sigma v^\nu \equiv R_{\rho\sigma\mu\nu}v^\sigma v^\nu \tag{8}$$

$$B_{\rho\mu} = \frac{1}{2}\star C_{\rho\sigma\mu\nu}v^\sigma v^\nu \equiv \frac{1}{2}\star R_{\rho\sigma\mu\nu}v^\sigma v^\nu \tag{9}$$

where the Hodge dual is defined as

$$\star C_{\rho\sigma\mu\nu} = \epsilon_{\rho\sigma\beta\alpha}C^{\beta\alpha}{}_{\mu\nu}. \tag{10}$$

and the equivalences (8,9) hold only on shell. E and B are traceless and symmetric. Physically we can see that in the static/Newtonian limit $E_{ij} \sim \partial_i\partial_i h_{00}$ is a measure of tidal acceleration. The relative acceleration between two point particles separated by $\vec{x}$ in a static field is $\vec{a}_{tidal} \equiv \vec{a}_1(\vec{x}+\vec{R}) - \vec{a}_2(\vec{x}) \approx R_i E_{ij}$. In analogy with the electro-magnetic field B_{ij} mixes with E_{ij} under a boost.

The Weyl tensor can be decomposed as follows

$$C^{\mu\nu}{}_{\rho\sigma} = -4E^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]} + 8E^{[\mu}{}_{[\rho}v^{\nu]}v_{\sigma]} + 4\epsilon^{\mu\nu}{}_{\beta\delta}v^\delta B^\beta{}_{[\rho}v_{\sigma]}. \tag{11}$$

Symmetry allows us to include two terms that are bi-linear in the Weyl tensor

$$S_{E+B} = \int d\tau(C_E E_{\mu\nu}E^{\mu\nu} + C_B B_{\mu\nu}B^{\mu\nu}). \tag{12}$$

[2] Any term in the action which has a linear piece in the metric will effect the one-point function, i.e. field value.

Note that a term a proportional to $E \cdot B$ violate parity and will have a vanishing coefficient for the cases of interest. The effects of such a term is discussed in [6]

The action which includes (1) and (12) represents the first step in coarse graining as shown in the figure (1a). The corrections to this action are suppressed by powers of the radius (R). The scaling of the coefficients $C_{E,B}$ in R (not to be confused with the scalar curvature) is complicated for most compact objects and will depend upon the equation of state. However, for black holes we can take the size as the Schwarzchild radius which is related to the mass via [3]

$$r_s = \frac{M}{16\pi M_{pl}^2}. \tag{13}$$

Because of the dimensionful coupling in gravity however, we can only conclude that

$$C_{E,B} \sim (r_s^3)(r_s M_{pl})^n. \tag{14}$$

Later one we will fix n by making sure the scaling makes sense classically.

This world-line theory is valid for observables for which there are no relevant scales shorter than r_s (or whatever the radius is of the object(s) under consideration). The coefficients $C_{E,B}$ are analogous to the polarizabilities discussed in the case of electrodynamics in the previous chapter. In the present case, they represent how easily the object is tidally distorted and are known as the leading order (electric, magnetic) "Love numbers". Terms with more derivatives would generate higher order multipole responses, while terms beyond quadratic in the Weyl tensor would generate non-linear responses.

To fix the Wilson coefficeints (C_E, C_B) one needs a theory which describe the inner dynamics of the object of interest. The Love numbers are then calculated via a matching calculation just as the polarizability was in the last section. We are free to match in various ways, either off-shell or on-shell. An example of the former is putting the system in a background field and calculating its response as done in [39, 40]. While the ladder case is exemplified by calculating a scattering rate in the full and effective theory [47]. We will not go into the details here of the matching procedure and only report on the remarkable result that the tidal Love number for Schwarzchild black hole is zero [38, 39]. In fact all of the static tidal Love numbers vanish. This is also true for a spinning black hole as well [46],

[3] The use of M_{pl} instead of G_N is a consequence of the authors early education. Given that $\hbar$ is irrelevant for this problem it is probably more suiting to use G_N. However in some case this would lead to a clunky $\sqrt{G_N}$ at intermediate stages of the calculation which is displeasing.

however this vanishing seems to be unique to four dimensions. Furthermore a calculation in [43] concludes that the non-linear response to the $l = 2$ also vanishes corresponding to the Wilson coefficient of the operator tri-linear in E.

At face value, this vanishing implies a fine tuning of both Dirac and t'Hooft nature. At first glance there are many possible diagrams (including power divergent ones) that should renormalize these Wilson coefficients. However, if you try to sit down and draw all possible diagrams with two external graviton lines and five mass insertions that is proportional to G^4, you will see that, unless you allow for diagrams with closed (quantum) graviton loops, there simply are no diagrams. This means that there is no *classical* radiative instability (t'Hooft naturalness), though quantum mechanically there will be. It is natural to ask whether or not there is a symmetry which protects the static Love numbers. The ultimate reason, not surprisingly, for this vanishing is the existence of the event horizon. In particular, when solving for perturbations around Schwarschild one must impose regularity at the horizon, which forces the static Love numbers to zero. In fact, the near-horizon geometry has more killing vectors than the full Schwarzchild solution that can prohibit Love numbers [41, 42]. A complete argument for the vanishing of the Love numbers would entail finding a symmetry of the world line action which would forbid the operators. In the case of scalar Love numbers (i.e. coupling the world line to a bulk scalar field and looking for its response in the presence of a scalar background), such as symmetry has been demonstrated to be a conformal transformation [42].

A. Interacting World-Lines: The Post-Newtonian Expansion

The power of the world-line EFT manifests itself when considering interacting bodies. By matching for the Wilson coefficients in isolation we extract the relevant data needed for calculating the finite size corrections to the interactions in a much simpler setting. That is to say, we have "divided and conquered". By treating one scale at a time we avoid the complication involved in solving the whole problem at once. Furthermore, once we have these coefficients we can use them in many different setting/approximations. As discussed in the introduction, there are several approximations that can be considered. In the "extreme mass ratio" approximation, one considers a small black hole with Schwarzchild radius r_{s1} orbiting a large black hole with radius ($r_{s2} \gg r_{s1}$). One then approximates the small black hole dynamics to evolve as a point particle with finite size effects included using (12). When the light hole become relativistic, we may still use the point particle approximation, but we must work in the full Schwarzchild background with self-force corrections which scale as powers of m_1/m_2. This case is calculationally nettlesome due to the fact that graviton propagator in a curved background is quite complicated, and one usually has to resort numerical methods (for a review see [44]).

Here will be concerned with the case where the space-time in which the black holes propagate is approximately flat. This criteria will be satisfied when the black holes [4] are sufficiently far apart, i.e. when the distance (r) between them obeys $r \gg r_1, r_2$. Expanding around flat space is called the post-Minkowskian approximation. This approximation need not correspond to a non-relativistic one, as it is an expansion in r_s/r. However, typically when two holes are interacting at long distances they will be bound, in which case the virial theorem implies

$$v^2 \sim \frac{M}{M_{pl}^2 r} \sim r_s/r, \tag{15}$$

and we see that approximate flatness is automatic if $v \ll 1$. This regime is called "Post-Newtonian" (PN)[3]. There exists a family of preferred frames for such system. Typically one chooses to work in the center of mass (COM) frame where the COM position is taken to be at the origin[5]. If we consider the evolution of a binary inspiral then we expect the PN approximation to be valid during the early stages only. At later stages of the inspiral, the holes enter the strong coupling regime and merge.

We will now utilize our world-line EFT to calculate gravity wave signals in the PN approximation. To do so we would need to generate a power counting scheme in v. The scheme we will use is derivative of NRQCD. Indeed, QCD and gravity (expanded around flat space) are both non-linear theories with quadratic propagator kernels. As in NRQCD we expect to have potential modes and radiation modes. Unlike NRQCD however, the soft modes are not relevant for classical gravity as they generate quantum corrections. In principle, we could include soft modes but they would generate corrections suppressed by $1/L$ where L is the angular momentum of the binary. This is easily uncovered once we repristinate the $\hbar$'s in our calculation and use dimensional analysis. Furthermore, we will be

[4] More generally we are also interested in neutron stars or other compact objects, but we will use black holes for all the examples we are considering, just for simplicity.

[5] Note that the center of mass position receives corrections from non-linearities. See the associated problem at the end of this chapter.

interested in classical sources, and as such, the origin of the relevant modes does not follow from the Landau criteria [6] but from the time dependence of the sources.

To formally develop the power counting we first consider all of the relevant scales in the problem,

$$R, r, M, M_{pl}. \tag{16}$$

where M is the reduced mass, R is the radius of the compact body, and r is the radius of the orbit. The scale R will not play a role in the dynamics and can be ignored for the moment. With the remaining parameters we can form two independent dimensionless quantities:

$$\lambda = \frac{M}{r}\frac{1}{M_{pl}^2} \sim v^2 \tag{17}$$

where we have used (15). As the other dimensionless quantity we choose the angular momentum of the orbit

$$L = \frac{M^2}{vM_{pl}^2}. \tag{18}$$

Next we wish to write down an action with each term scaling homogeneously in each of these expansion parameters. We will work at leading order in $1/L$ since quantum corrections are more than negligible.

In the next step we take the action (1) for i bodies, ignoring finite size effects for the moment,

$$S = -2M_{pl}^2 \int d^4x\sqrt{g}R + \sum_i M_i \int d\tau_i \tag{19}$$

and expand it out so that each term in the action scales homogeneously in L and v. In the previous chapter we saw that the photon field itself does not scale homogeneously in v since it has support in both potential and radiation modes (see the discussion in the previous chapter), and the same reasoning applies here. It should not be surprising that the same modes will contribute despite the fact that we are now interested in classical sources, since the classical limit can be reached by ignoring recoil effects.

In analogy to what was done in electrodynamics in the previous chapter, we split the

[6] This comment is relevant only to the readers of the earlier chapters on QCD.

graviton into potential (H) and radiation ($\bar{h}$) modes such that

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{pl}} \equiv \eta_{\mu\nu} + \frac{H_{\mu\nu}}{M_{pl}} + \frac{\bar{h}_{\mu\nu}}{M_{pl}}. \tag{20}$$

At first this decomposition might lead one to think there may be issues of double counting as well as problems with diffeomorphism invariance. However, this decomposition is indeed nothing more then a manifestation of the background field method originally developed by DeWitt [35]. The radiation mode should be thought of as a long wavelength background field which is frozen on the time scales of the potential modes.

Exercise n.3 Consider a theory of scalar-tensor theory gravity. In such a theory there will be a new set of worldline couplings that one can write down that can not be eliminated by field redefinitions. Use PN power counting to determine the order of the first finite size operator on the worldline.

Let us now expand the action beginning with the world-line term

$$S_M = -M \int d\tau, \tag{21}$$

we expand around Minkowski space and choose to parameterize the worldline by the global time coordinate

$$d\tau = dt \sqrt{\frac{dx^\mu}{dt}\frac{dx_\mu}{dt} + \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{h_{\mu\nu}(x(t))}{M_{pl}}}. \tag{22}$$

Then the action is given by

$$S_m = -M \int dt \left((1 - \frac{1}{2}\vec{v}^2)(1 + (\frac{1+\vec{v}^2}{2})\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{h_{\mu\nu}(x(t))}{M_{pl}}) - \frac{1}{8}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{h_{\mu\nu}(x(t))}{M_{pl}}\frac{dx^\rho}{dt}\frac{dx^\beta}{dt}\frac{h_{\rho\beta}(x(t))}{M_{pl}} + ... \right) \tag{23}$$

For the moment let us concentrate on calculating the potentials, so we will set aside the radiation graviton. Recall that the potential field is the instantaneous part of the graviton. If our effective field theory is to have homogenous scaling then we must enforce this instantaneity at the level of the action. To accomplish this we follow the strategy in the last chapter and pull out the large part of the potential's momentum by doing a partial Fourier transform [7] via

$$\mathbf{H}_{\mu\nu}(t,\vec{x}) = \int [d^3k] e^{i\vec{k}\cdot\vec{x}} H_{\vec{k}\ \mu\nu}(t) \equiv \int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} H_{\vec{k}\ \mu\nu}(t). \tag{24}$$

The potential modes have a typical wavelength of order r, which is to say that $\mathbf{k}$ scales as $1/r$. Now to calculate the propagator for the potential mode we need to gauge fix the action. It is convenient to choose the (harmonic) gauge fixing term as

$$S_{GF} = M_{Pl}^2 \int d^4x \sqrt{g} \Gamma_\mu \Gamma^\mu, \tag{25}$$

with $\Gamma_\mu = \partial_\alpha H^\alpha_\mu - \frac{1}{2}\partial_\mu H^\alpha_\alpha$. When we come back to the issue of radiation we will have to modify this gauge fixing term. In this gauge, the $\mathcal{O}(H^2)$ terms in the action are (after performing a rescaling of $H_{\mu\nu}$ to obtain a canonically normalized kinetic term)

$$\mathcal{L}_{H^2} = -\frac{1}{2}\int_{\vec{k}} \left[\vec{k}^2 H_{\vec{k}\mu\nu} H^{\mu\nu}_{-\vec{k}} - \frac{\vec{k}^2}{2} H_{\vec{k}} H_{-\vec{k}} - \partial_0 H_{\vec{k}\mu\nu} \partial_0 H^{\mu\nu}_{-\vec{k}} + \frac{1}{2}\partial_0 H_{\vec{k}} \partial_0 H_{-\vec{k}} \right], \tag{26}$$

where $H_{\vec{k}} = H^\mu_{\mu\vec{k}}$. As we will see the terms in the second line of this equation are suppressed relative to the first line by a power of v^2 and must be treated as perturbations which correspond to corrections to instantaneity. The propagator for $H_{\vec{k}\mu\nu}$ can be read off from the first term

$$\langle H_{\vec{k}\mu\nu}(x^0) H_{q\alpha\beta}(0) \rangle = -(2\pi)^3 \delta^3(\vec{k}+\vec{q}) \frac{i}{\vec{k}^2} \delta(x_0) P_{\mu\nu;\alpha\beta}, \tag{27}$$

where $P_{\mu\nu;\alpha\beta} = \frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\alpha\beta}\right]$ with d the spacetime dimension. Note the propagator is manifestly instantaneous (independent of energy). Assigning the usual non-relativistic scalings, $x^0 \sim r/v$, $\vec{k} \sim 1/r$, we learn from this that a potential graviton scales as $H_{\vec{k}\mu\nu} \sim v^{1/2} r^2$, while in coordinate space $H/M_{pl} \sim v^2/\sqrt{L}$.

We now expand (23) to get a set of vertices (interactions) each of which has definite scaling in v (and L).

$$\begin{aligned} S_{v^0} &= -\int \frac{M}{2M_{pl}} H_{00} dt \\ S_{v^1} &= -\int \frac{M}{M_{pl}} v^i H_{0i} dt \\ S_{v^2} &= -\int (\frac{M}{4M_{pl}} v^2 H_{00} - \frac{M}{2M_{pl}} v^i v^j H_{ij} + \frac{M}{8}\frac{H_{00}H_{00}}{M_{pl}^2}) dt \end{aligned} \tag{28}$$

[7] In NRQCD we called this a label formalism.

where there is an implied integral over momentum which has been suppressed for notational simplicity. The terms linear in H scale as $\sqrt{L}$ while the quadratic term scales as L^0. As we shall see these the are proper scalings to generate classical potentials.

With the results (27) and (28) in hand we are ready to begin integrating out the potential mode in order to generate potentials. The leading order (Newtonian) potential follows by considering the coupling of the potential graviton between two world lines, which yields [8]

$$
\begin{aligned}
-i\int V dt &= (-i)^2 \int dt_1 dt_2 M_1 M_2 \frac{1}{4M_{pl}^2}\int [d^3k][d^3q] e^{i\vec{k}\cdot\vec{x}_1(t_1)} e^{i\vec{q}\cdot\vec{x}_2(t_2)} \langle 0 \mid T\left[H_{\vec{k}00}(t_1) H_{\vec{q}00}(t_2)\right] \mid 0\rangle, \\
&= \int dt M_1 M_2 \frac{1}{8M_{pl}^2}\int [d^3k] e^{-i\vec{k}\cdot(\vec{x}_1(t)-\vec{x}_2(t))}\frac{i}{\vec{k}^2}. \qquad (29)
\end{aligned}
$$

To perform this integral we will first evaluate the more general integral

$$
\int \frac{d^dk}{(2\pi)^d}\frac{1}{(\vec{k}^2)^\alpha} e^{-i\vec{k}\cdot\vec{x}} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)}\left(\frac{\vec{x}^2}{4}\right)^{\alpha-d/2}. \qquad (30)
$$

which will be useful when we consider higher order corrections. Using this result we find

$$
V_0(r) = -\frac{M_1 M_2}{32\pi M_{pl}^2 r}. \qquad (31)
$$

Notice that for the graviton correlator we wrote down a time ordered product, which in the end is extraneous since the correlator is instantaneous, as can be seen in (27), and we will drop the time ordering in potential correlators. As a. matter of notation, there is always an overall time integral left over, which we will drop in any final expression for the potential.

After re-deriving this monumental result let us pause to discuss the power counting in L. Since the action has the units of angular momentum all the classical potential contribution to the action must scale like L, $\int dtV \sim L$. We can see that this is true using the fact that $dt \sim r/v$ and $M^2/M_{pl}^2 \sim vL$. However, we always wish to power count at the level of the action, including the graviton field, that way we can determine which diagrams (a combination of vertices) contribute at leading order in L, i.e. classically (as expected quantum contributions will involve closed graviton loops). As promised below eq. (28) we see that by power counting the vertices in L we can read off the scaling of the potential.

[8] The feynman diagram generates $iM = -iV$.

Let us now calculate the 1PN (order v^2) potential[9]. We begin by enumerating all of the possible contributions at order v^2 (not including diagrams where the masses are interchanged, which are trivially added when we are done)[10]. First we have all the possible single graviton diagrams which connect vertices whose net momentum scaling is v^2.

1. The simplest such example stems from the terms $v^2 h_{00}$ and $v^i v^j h_{ij}$. By inserting this term we generate exactly the same expression as in (29) so that this contribution is

$$V^{(1)}_{v^2} = \frac{3}{2}(v_1^2 + v_2^2)V_0 \tag{32}$$

2. We can also have two insertions of the term

$$S_m = -M \int dt v^i e^{ik\cdot x}\frac{H_{ki0}}{M_{pl}} \tag{33}$$

one on each line. Expanding the exponential of L_{int} to second order in this perturbation leaves

$$\begin{aligned} V^{(2)}_{v^2} &= -i\int dt_1 dt_2 M_1 M_2 \frac{1}{M_{pl}^2} v_1^i(t_1) v_2^j(t_2)(\langle H_{\vec{p}i0}(x_1(t_1)) H_{\vec{q}j0}(x_2(t_2))\rangle. \\ &= M_1 M_2 \frac{1}{M_{pl}^2}\vec{v}_1(t)\cdot\vec{v}_2(t)\frac{1}{2}\int [d^3p] e^{-i\vec{p}\cdot(\vec{x}_1(t)-\vec{x}_2(t))}\frac{1}{\vec{p}^2} \\ &= -4\vec{v}_1(t)\cdot\vec{v}_2(t)V_0. \end{aligned} \tag{34}$$

3. There is a first order correction to the propagator coming from the last line in (26)

$$V^{(3)}_{v^2} = -\int dt_1 dt_2 M_1 M_2 \frac{1}{8M_{pl}^2}\int d^4p e^{ip\cdot(x_1(t_1)-x_2(t_2))}\frac{p_0^2}{\vec{p}^4}. \tag{35}$$

Had we not expanded in v at the level of the action, so that it scaled homogeneously,

[9] The nomenclature is such that that the N-th PN correction to the potential is order v^{2N}. The reason for this is that the conservative potential pieces must scale with even powers of the velocity by time reversal invariance.

[10] Using Feynman diagrams to calculate classical potentials goes back to the work [7], although in this reference the potential is extracted from S-matrix elements. The calculations in terms of classical sources can be found in [9].

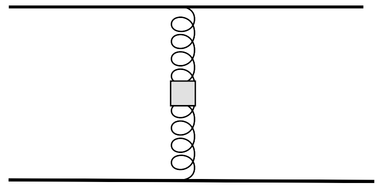

FIG. 2. Propagator correction which accounts for the leading deviation from instantaneity which is order v^2.

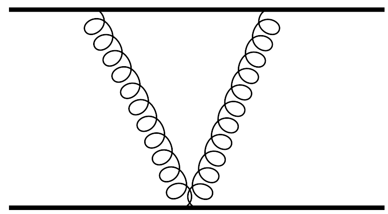

FIG. 3. Seagull contribution to the potential at order v^2.

then we would have seen this correction by expanding the graviton propagator.

$$\frac{1}{p_0^2 - \vec{p}^2} \approx \frac{1}{\vec{p}^2} + \frac{p_0^2}{\vec{p}^4}. \tag{36}$$

In terms of Feynman diagrams such propagator corrections are drawn as in figure (2).

To perform this integral, it is simplest to integrate by parts, yielding

$$\begin{aligned} V_{v^2}^{(3)} &= -\int dt_1 dt_2 M_1 M_2 \frac{[d^4p]}{8M_{pl}^2} \left(\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2} e^{-ip_0(t_1-t_2)}\right) e^{i\vec{p}\cdot(\vec{x}_1(t_1)-\vec{x}_2(t_2))}\frac{1}{\vec{p}^4} \\ V_{v^2}^{(3)} &= \vec{v}_1\cdot\vec{v}_2\frac{V_0}{2} - (\vec{v}_1\cdot\vec{X})(\vec{v}_2\cdot\vec{X})\frac{V_0}{2X^2}, \end{aligned} \tag{37}$$

where $\vec{X} \equiv \vec{x}_1 - \vec{x}_2$. Notice that when we integrated by parts there exist an ambiguity. In particular, we could have chosen to write $\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_1}$. However, this can not change the results for any physical observable since a total derivative can not affect the equations of motion.

4. The first non-linear contribution comes from the term quadratic in H in the expansion of the world-line action (23). This interaction generates so-called “sea-gull” diagrams as shown in figure (3). We consider a correlator of this operator with two insertions of the leading order interaction. We have

$$
\begin{aligned}
V^{(4)}_{(v^2)} &= \int dt_1 dt_2 dt_2' M_1 M_2^2 \frac{1}{8\times 2\times 2}\langle \frac{H_{\vec{k}00}(x_1(t_1))}{M_{pl}} \frac{H_{\vec{q}00}(x_1(t_1))}{M_{pl}} \frac{H_{\vec{l}00}(x_2(t_2))}{M_{pl}} \frac{H_{\vec{p}00}(x_2(t_2'))}{M_{pl}} \rangle \\
&+ \ (1 \leftrightarrow 2). \qquad (38)
\end{aligned}
$$

There are two possible contractions which give identical contributions, but we also pick up a factor of 1/2 from each propagator

$$
\begin{aligned}
V^{(4)}_{(v^2)} &= \int dt_1 dt_2 dt_2' \frac{M_1 M_2^2}{M_{pl}^4} \frac{1}{8\times 2^4} \left(\int [d^3p] e^{-i\vec{p}\cdot(\vec{x}_2(t_2)-\vec{x}_1(t_1))} \frac{i}{\vec{p}^2} \times \int [d^3p'] e^{-i\vec{p}'\cdot(\vec{x}_2(t_2)-\vec{x}_1(t_1))} \frac{i}{\vec{p}'^2} \right) \\
&+ \ (1 \leftrightarrow 2). \qquad (39)
\end{aligned}
$$

Let us pause to explain the symmetry factor. There is a 1/8 from the operator, and two factors of 1/2 from the leading order operator. There is a factor of 2 since there are two possible Wick contractions, but this is cancelled by a factor of 1/2 coming from the expansion of the exponential. Finally we have two additional factors of 1/2 coming from the propagators. Thus the net contribution from this "sea-gull" diagram is

$$
V^{(4)}_{(v^2)} = -\int dt \frac{M_1 M_2^2}{128 M_{pl}^4} \frac{1}{(4\pi)^2} \frac{1}{r^2} \quad + (1 \leftrightarrow 2). \qquad (40)
$$

Notice that to generate a term liner in L it must be that the world line term quadratic in H scale as L^0 as we found in eq.(28).

5. Finally we have the contribution from the three graviton vertex as shown in the left hand side of figure (II A). To see why this type of diagram shows up at 1PN we need to understand the scaling of the bulk non-linear vertices. Since the vertices arise as an expansion of the bulk curvature term. The generic form of the vertex involves two derivatives giving a term in the action which scales as

$$
S_{3g} \sim \frac{1}{M_{pl}} \int d^4x h \partial h \partial h \sim \frac{1}{M_{pl}} \frac{r^4}{v} (\frac{\sqrt{v}}{r})^3 \frac{1}{r^2} \sim \frac{M}{M_{pl}} \frac{v}{rMv} \sqrt{v} = \frac{v^2}{\sqrt{L}}. \qquad (41)
$$

Note that we have taken the measure to scale as r^4/v since we are dealing with potential gravitons (see previous chapter). This scaling will change when we discuss radiation gravitons. Given that each leading order world-line vertex scales as $\sqrt{L}$, we find that the diagram on the left hand side of figure (4) scales as Lv^2 as a 1PN potential should.

The Feynman rule for the three graviton vertex can be found in the appendix of reference [1]. We will not go through its derivation simply because as we shall see in the next section, one can make a simpler choice for the metric ansatz which will allow us to avoid the three graviton vertex at 1PN. But it is instructive to use the three graviton vertex since it will need to be included when working at higher orders. The calculation simplifies since the propagator which hooks up h_{00} gravitons obeys certain simple identities

$$P_{00:\alpha\beta}P^{00:\alpha\beta} = 1. \tag{42}$$

$$P_{00:\alpha\beta}\eta^{\alpha\beta} = -1. \tag{43}$$

$$P_{00:\alpha\beta}P^{00:\alpha\delta} = \frac{1}{4}\delta^{\delta}_{\alpha}. \tag{44}$$

$$k^{\alpha}P_{00:\alpha\beta} = -\frac{1}{2}k_{\beta}. \tag{45}$$

Performing all of the requisite contractions the net result is

$$\begin{aligned} V_{3g} &= i\frac{M_1^2M_2}{2!}(-i/(2M_{pl}))^3(-2i/M_{pl})(i)^3\int dt\frac{[d^{d-1}k]}{-\vec{k}^2}\frac{[d^{d-1}p]}{-\vec{p}^2}\frac{[d^{d-1}r]}{-\vec{r}^2}\frac{-1}{8}(\vec{p}^2+\vec{k}^2+\vec{r}^2)\times \\ & e^{i\vec{k}\cdot\vec{x}_1(t)}e^{i\vec{p}\cdot\vec{x}_1(t)}e^{i\vec{r}\cdot\vec{x}_2(t)}(2\pi)^3\delta^{(3)}(\vec{k}+\vec{p}+\vec{r}). \end{aligned} \tag{46}$$

Note that first two of the three terms in this integral vanish by definition in dimensional regularization. To see this, consider the first term and perform the r integral leaving

$$V_{3g}^{div} \sim \iint dt\frac{[d^{d-1}k]}{-\vec{k}^2}[d^{d-1}p]\frac{1}{-(\vec{p}+\vec{k})^2}e^{i\vec{k}\cdot(\vec{x}_1(t)-\vec{x}_2(t)}e^{i\vec{p}\cdot(\vec{x}_1(t)-\vec{x}_2(t))}. \tag{47}$$

then shifting $p \to p-k$ leaves the k integral scaleless (see section **??** for details) It is important to understand that there is nothing magical about dimensional regularization. Had we used a different regulator, one in which this integral did not vanish, it still can not have any physical effect. To see this let us study the origin of this divergence. The $\vec{p}^2$ in the numerator cancels the propagator and forces the bulk vertex to live on the world line. The k integral becomes a loop which closes onto itself (this is sometimes called a tadpole) and corresponds to a mass renormalization as shown in second diagram in figure (4).

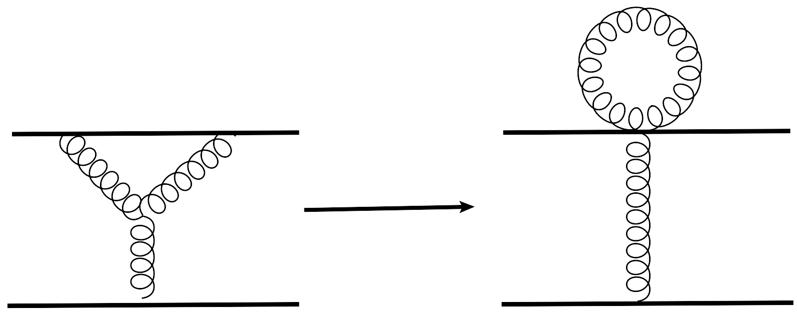

FIG. 4. The three graviton contribution to the EIH potential. Terms with momenta in the numerator eliminate one propagator and leads to a vanishing "tadpole" contribution as shown in the right hand side of the diagram.

The result for the three graviton diagram is then given by

$$V_{3g} = \frac{i}{8}\frac{M_1^2 M_2}{2!}(-i/(2M_{pl}))^3(-2i/M_{pl})(i)^3 \int d\tau \frac{[d^{d-1}k]}{-\vec{k}^2}\frac{[d^{d-1}p]}{-\vec{p}^2} e^{i\vec{k}\cdot(\vec{x}_1-x_2)} e^{i\vec{p}\cdot(\vec{x}_1-x_2)}. \quad (48)$$

$$V^{(3g)}_{(v^2)} = \int dt \frac{M_1^2 M_2}{64 M_{pl}^4}\frac{1}{(4\pi)^2}\frac{1}{r^2} + (1 \leftrightarrow 2). \quad (49)$$

After symmetrizing in the masses, and adding all the contributions $(1-6)$ we find the (Einstein-Infeld-Hoffmann) potential, written in terms of $G_N = \frac{1}{32\pi M_{pl}^2}$ is

$$V_{EIH} = \frac{G_N M_1 M_2}{2r}\left[-3(v_1^2+v_2^2) + 7 v_1\cdot v_2 + \frac{(v_1\cdot r)(v_1\cdot r)}{r^2}\right] + \frac{G_N^2 M_1 M_2 (M_1+M_2)}{2r^2}. \quad (50)$$

It is again important to emphasize that this potential depends upon the choice of gauge fixing as well as the coordinate system.

B. The Quantum Calculation versus the Classical Calculation of the Potential [11].

When considering diagrams such as those in figure (4) it is important not to interpret the source line as a fluctuating field. In NRGR we are considering classical sources. We draw the (non-graviton) lines connecting the vertices to represent time propagation, though this is slightly misleading. Instead one could draw the diagram corresponding to figure (4) by using classical source insertions as shown in figure (5). Drawing diagrams in this way has the advantage that it also makes clear what diagrams should *not* be included. For instance, including the matter line propagators would lead to diagrams with the box topology, which

[11] The section assumes the reader has some familiarity with NRQCD as discussed in chapter

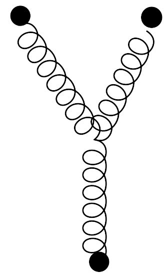

FIG. 5. An alternative way to draw first the diagram in figure (4) which makes it clear that the masses act as classical sources which do not fluctuate. The down side of this way of drawing things is that it does not make clear that one still integrates over the times of the source insertions.

one might naively include. However, if we instead draw the diagrams with source insertions it becomes clear that these are disconnected diagrams which should not be included when calculating the energy. Such diagrams would serve the purpose of exponentiating the energy (see section ()).

It is natural compare and contrast the classical QCD and GR potentials. One thing we can discern immediately is that all corrections to the QCD potential are at most logarithmic if the quarks are static. This is obvious once we use the fact that in the static limit the quark mass plays no role and a classical conformal symmetry arises. There is simply no potential that can be written down aside from $V \sim \log(r)/r$ that has the right units. These logarithmic corrections are necessarily quantum mechanical [12] and may be accounted for by allowing the coupling to run. However, once we allow for non-static sources (i.e. terms suppressed by the mass) there will be velocity dependent classical corrections to the Coulomb potential. In gravity even the static limit leads to corrections to Newtons law as can be seen in the last term of the EIH result (50).

Let us further explore the differences between QCD and GR by considering the relative sizes of non-linear contributions. Note that g determines the strength of both the "bulk" coupling, i.e. the gluon self interactions, as well as the coupling to matter, at least up to a representation dependent group theory factor. As such, in Yang-Mills theory both the quantum corrections as well as the non-linearities are controlled by the dimensionless coupling g. In GR, on the other hand, the bulk couplings (k/M_{pl}) are independent of the mass. Moreover, the mass is an independent parameter, and as such, the coupling to matter can be arbitrarily large. Thus, in GR all quantum corrections (which are necessarily non-

[12] To see this note that if there are no finite size world line corrections divergences can only renormalize the coupling. That is, the divergence is associated with a bulk (quantum) loop.

linearities) are small as long as the momentum is small compared to the Planck scale. While classical non-linearities can be order one, if $kM/M_{pl}^2 \sim kr_s \sim O(1)$. Thus for a classical black hole $r_s \gg 1/M_{pl}$, non-linearities will always be order one near the horizon, but quantum corrections will be negligible. Had this not been the case, classical solutions to GR (such as the Schwarzchild or FRW solution) would be poor approximations to the true vacua.

In YM theory when the coupling gets large both the quantum and classical corrections grow. If we consider a matter representation with a very large Casimir, such that $Q_{eff} = C_r g \gg g$ then the classical contribution will dominate the quantum corrections assuming the mass is large enough that quark vacuum polarizations are suppressed.

Exercise n.5 Let us consider the contribution of higher dimensional bulk operators to the potential. Such operators are necessary counter-terms which arise upon calculating bulk gravitons loops. As such, we had better find their contribution to the potentials are suppressed by powers of $1/L$. Consider the operator proportional square of the Riemann tensor, R^2, as an example. Show that each term in the expansion of this operator scales as

$$S_{R^2} \sim \int d^4x \sum_{m\geq 2} \frac{1}{M_{pl}^m}(\partial^4) h^m \sim v^{2m-1} L^{-m/2}. \tag{51}$$

Then show that the contribution of this operator to the potential will always be suppressed by $1/L$, as expected.

C. A Simpler Way to Calculate

As mentioned in the previous section, the highly non-linear nature of GR makes higher order diagrams very cumbersome. At 1PN we run into the three graviton vertex which has six indices that leads to a plethora of contractions. In general, using the standard representation of the metric we will need the $n+2$ graviton vertex at nPN order. However, it was pointed out in [11] that there is a better way to parameterize the metric which pushes back the need for this vertex by one order. To accomplish this one utilizes the so-called

Kaluza-Klein[13] variables where the metric is parameterized as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \equiv e^{2\phi}(dt - A_i dx^i)^2 - e^{-2\phi}\gamma_{ij}dx^i dx^j. \quad (52)$$

In terms of these variables the Einstein-Hilbert action is given by

$$S = -2M_{pl}^2 \int d^4x \sqrt{\gamma}\left(-R(\gamma) - \frac{1}{4}e^{4\phi}F_{ij}F^{ij} + 2\partial^i\phi\partial_i\phi + ...\right) \quad (53)$$

where the dots represent terms with time derivatives that are higher order in the PN expansion and $F_{ij} = \partial_i A_j - \partial_j A_i$. Now consider the 1-PN three-graviton contribution. At 1PN the world line couplings all involve $h_{00} \equiv \phi$ since any open vector indices must be contracted with velocities. Notice that there are no try-linear scalar (ϕ) couplings in this choice of variables, and thus the three-graviton coupling will not show up until 2PN[14]. Even with this simplification, as one goes to higher orders in the PN expansion one will need to calculate diagrams with even more vertices, and and some point the calculation will become very computationally intensive. The 3PN calculation using the KK variables was done on a computer [16] and involved 20 differing topologies for diagrams.

Exercise n.4

Use the KK variables to derive the 1PN potential. To accomplish this begin by showing that the propagators for the various pieces of the metric are given by

$$\begin{aligned}
\langle\phi_{\vec{p}}(t_a)\phi_{\vec{q}}(t_b)\rangle &= -\frac{1}{8}(2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\frac{i}{\vec{p}^2}\delta(t_a - t_b)\\
\langle A^i_{\vec{p}}(t_a)A^j_{\vec{q}}(t_b)\rangle &= \frac{1}{2}(2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\frac{i\delta_{ij}}{\vec{p}^2}\delta(t_a - t_b)\\
\langle\gamma^{ij}_{\vec{p}}(t_a)\gamma^{kl}_{\vec{q}}(t_b)\rangle &= -(2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\frac{iP^{ij,kl}}{\vec{p}^2}\delta(t_a - t_b)
\end{aligned} \quad (54)$$

where $P^{ij,kl} = \frac{1}{2}(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - 2\delta^{ij}\delta^{kl})$.

[13] Kaluza-Klein theories are extra dimensional attempts at unification. For instance, a five dimensional metric is split up into a 4-D metric and a vector fields which play the role of the electromagnetic gauge potential.

[14] The 2PN potential is calculated using these techniques in [15] and the 3PN was done in [16].

III. THE POST-MINKOWSKIAN (PM) APPROXIMATION

Our calculations in the PN approximation have been performed in a pedantic and tedious way. In particular we have made every vertex scale homogenously in v, which is indeed the proper formal way to do things, but there are short cuts. In particular, our field theory methods are tailor made to do relativistic calculations, so we might as well calculate without first expanding in v. In doing so we will capture many higher order corrections at once, but only those at a fixed order in G. This procedure is called that "Post-Minkowskian" expansion and will reduce the number of diagrams considerably. For instance, at leading order in G we can capture all of the diagrams at 1PN, except for the sea-gull and the three graviton contribution, with the simple one graviton exchange. Though, as we will see, we can also eliminate the need for the sea-gull. The PM expansion is thus not a systematic expansion for the case of the binary inspiral, since calculating at n'th order in G, does not capture all of the nPN corrections. Of course the $n+1PM$ result will capture all of the $n-PN$ result. The PM expansion is, however, systematic in the case of a relativistic scattering process where $G\mu/b \ll 1$ (see for instance [5]), where b is the impact parameter and μ is the reduced mass. This limit corresponds to having the impact parameter much larger than the Schwartzchild radius.

The PM expansion is most easily implemented by using the so-called Polyakov form of the action where one introduces a world-line metric, or "einbein". For completeness I will review here how this action is generated. We start with the first order form of the action with a Lagrange multiplier (e which will play the role of the einbein) used to impose the on-shell contraint [15]

$$S = \int d\lambda (p \cdot \dot{x} + e(p^2 - m^2)). \tag{55}$$

Then using the vanishing of the variation of the action with respect to p to solve for p and plugging the result back into the action gives

$$S = \int d\lambda (-\frac{1}{4e} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - e m^2). \tag{56}$$

e acts as a 1-d metric in the sense that it transforms under world-line diffeomorphisms

[15] The need for a constraint follows from looking at the second order form of the action and noting that the Hessian, $\frac{\partial^2 L}{\partial \dot{x}^\mu \dot{x}^\nu}$ has a vanishing determinant, implying that the relation between p and $\dot{q}$ is not invertible. More prosaically, $p^\mu = m\frac{\dot{x}^\mu}{\sqrt{\cdot x^2}}$ so that $p^2 = m^2$.

($\lambda \to \lambda'(\lambda)$ (RPI)) as $e \to \frac{d\lambda'}{d\lambda}e$, so that the action is still invariant. We next choose a gauge (parameterization) such that $e = \frac{1}{2m}$ and

$$S = -\frac{m}{2}\int g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu d\tau \tag{57}$$

where the constant term has been dropped and the norm is chosen so as to fix the proper kinetic energy in the non-relativistic limit. This form of the action is simpler to work with since we no longer have the square root and thus all of the non-linear world line terms are gone.

Now we could calculate the potential as we did previously by calculating the effective action as a function of the world-line trajectories. However, since we are working to all orders in the velocity, the graviton propagator is no longer instantaneous and therefore, the time integrals are non-trivial and will depend upon the world-line trajectory. As such, calculating the potentials directly is not the most convenient avenue. Instead it's prudent to calculate the scattering angle (χ) by differentiating the oh-shell action which will be calculated by iteratively solving the equation of motion and plugging back in. The Hamiltonian can be determined from χ directly. Calculating in this way carries certain advantages: χ is gauge invariant since it only depends upon measurements made at infinity. The time integrals can be performed since the world line trajectories are small deviations from straight lines. The integrands which show up in the scattering angle case simplify greatly when plugging in the trajectories into the action. Finally, quite remarkably, the scattering data can be used directly to calculate many physical (gauge invariant) bound state observables directly, such as the periastron advance and the orbital binding energy, without the need of determining an intermedite, gauge dependent, Hamiltonian through analytic continuation [12].

As an example, let us work out the 1PM scattering angle. We first parameterize the incoming wordlines

$$x^\mu_{i0}(\tau_i) = \bar{v}^\mu_i \tau + b^\mu_i, \tag{58}$$

$\bar{v}^\mu_i$ is the asymptotic four velocity at particle i at past infinity and $b^\mu_i = x^\mu_{i0} - (\bar{v}_i \cdot x_{i0})\bar{v}_i$, such that $b_i \cdot \bar{v}_i = 0$. $b^\mu \equiv b^\mu_1 - b^\mu_2$ is the impact parameter in the center of mass frame. The zero subscript marks the fact that one expands the path around this configuration. That is, the idea is to write

$$x_i^\mu(\tau_i) = \bar{v}_i^\mu \tau + b_i^\mu + \sum_n \delta^n x_i^\mu \tag{59}$$

where $\delta^n x_i^\mu$ is the deflection induced by the order n correction in G.

Given a trajectory we can calculate the impulse for the i'th particle via the equation of motion

[16]

$$\Delta p_i^\mu = -\eta^{\mu\nu} \int_{-\infty}^{\infty} (\frac{\partial L}{\partial x_i^\nu}) d\tau_i. \tag{60}$$

In order to perform this calculation systematically at $n'th$ order in G we need to calculate $L^{(m)}$, the mth order Lagrangian determined by integrating out the graviton iteratively, and plug in $x^{(k)}$ such that $m + k = n$, where $x^{(k)}$ is the trajectory at $k'th$ order.

Let us calculate the leading order result, for which we only need $L^{(1)}$ and $x^{(0)}$. The former is given by the single graviton exchange diagram which is

$$\begin{aligned} S^{(1)} &= -im_1m_2 \int d\tau_1 d\tau_2 (\frac{-i}{2M_{pl}})^2 \int [d^4k] \frac{ie^{ik\cdot(x_1-x_2)}}{k^2+i\epsilon} v_1^\mu(\tau_1) v_1^\nu(\tau_1) v_2^\rho(\tau_2) v_2^\sigma(\tau_2) P_{\mu\nu;\rho\sigma} \\ &= -\frac{m_1m_2}{8M_{pl}^2} \int d\tau_1 d\tau_2 \int [d^4k] \frac{ie^{ik\cdot(x_1-x_2)}}{k^2+i\epsilon} [2(v_1\cdot v_2)^2 - v_1^2 v_2^2]. \end{aligned} \tag{61}$$

To calculate the 1PM impulse (Δp^μ) we use (60) plugging in the leading order trajectory, i.e. setting $\delta^n x_i^\mu = 0$. Now we can see why it is convenient to study scattering, since we can do the proper time integrals. Using (60) we find

$$\Delta p_i^\mu = -\eta^{\mu\nu} \frac{m_1m_2}{8M_{pl}^2} \frac{\partial}{\partial b_i^\nu} \int \frac{[d^4k]}{k^2+i\epsilon} e^{ik\cdot b} (2\pi)^2 \delta(k\cdot\bar{v}_1)\delta(k\cdot\bar{v}_2)(2\gamma^2-1). \tag{62}$$

The remaining integral is easily evaluated by going to the rest from of one of the particles. The energy integral then becomes trivial leading to a $\delta(k_0)$ and the propagator becomes instantaneous. This is because there can be no radiation at this order given the lack of any kick to the world-lines. At higher orders the $i\epsilon$ prescription becomes relevant and care must be taken to distinguish between the radiation and potential pieces. To perform the integral

[16] This is a nice example of Hamilton-Jacobi theory where the ("principle") action ($S(q,t)$) generates a canonical transformation between the initial and final states. Other generating functionals are the "reduced" action $W(q,E)$, the "radial action", at fixed angular momentum, $J_R(E,J)$, and the Eikonal $\chi(b,E)$, where b is the impact parameter. Each has its own place dependending upon the type of problem. e.g. the radial action is suited for circular orbits, while the eikonal is suited for the high energy scattering limit.

we must regulate it and the final result, after covariantizing, is given by

$$\Delta p_1^\mu = -2m_1 m_2 G \frac{(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} \frac{b^\mu}{\mid b^2 \mid}. \tag{63}$$

Using this result we can extract the scattering angle by going to the center of mass frame where trigonometircally

$$\sin(\frac{\chi}{2}) \equiv \frac{\mid \vec{\Delta} p \mid}{2 \mid \vec{p}_\infty \mid}. \tag{64}$$

One can extract a, gauge dependent, Hamiltonian by first writing the center of mass momentum as an expansion around its asymptotic value

$$\vec{p}^{\,2} = p_\infty (1 + \sum_a f_a (\frac{GM}{r})^a) \tag{65}$$

and f_a are written in terms of the scattering angle at a given order in G [**?**]. The Hamiltonian is then parameterized as

$$H = \sum_{a=0} \frac{c_a(\vec{p}^{\,2})}{a!} (G/r)^a \,, \tag{66}$$

and the coefficient are read off by comparing the total energy expressions

$$\sqrt{\vec{p}^{\,2} + m_1^2} + \sqrt{\vec{p}^{\,2} + m_2^2} = \sum_{a=0} \frac{c_a(\vec{p}^{\,2})}{a!} (G/r)^a \,, \tag{67}$$

then using (65) one can iteratively solve for the c_a.

1. *Going to higher order in the PM expansion*

Let us now consider going to next order in the PM expansion. The leading order action arose from a simple graviton exchange, so it is easily understood in terms of Feynman diagrams/rules. But at higher orders we will need the first order correction to the trajectories $\delta^1 x_i^\mu$ as well as the one loop correction to the action evaluated on the leading order trajectories. So we have to do some intermediate work to reach the integrands of interest. The corrections to the path are determined by perturbatively solving the equations of motion which are given by

$$m_1 a_1^\mu(\tau_1) = \frac{m_1 m_2}{8M_{pl}^2}\int_{-\infty}^{\infty} d\tau_2 \int \frac{[d^4k]}{k^2+i\epsilon} e^{ik\cdot(x_1-x_2)}[k^\mu(2(v_1\cdot v_2)^2 - v_1^2 v_2^2) - 4(a_1\cdot v_2 v_2^\mu - 2a_1^\mu v_2^2)$$
$$-ik\cdot v_1(4(v_1\cdot v_2)v_2^\mu - 2v_1^\mu v_2^2)] \tag{68}$$

where $a^\mu = \dot{v}^\mu$ and the dependence on the proper time has been suppressed. We next integrate this equation in τ_2 and τ_1 to determine $\delta^1 v_1^\mu$ by plugging in the leading order trajectories such that

$$\delta^1 v_1^\mu(\tau_1) \;=\; -i\frac{m_2}{4M_{pl}^2}\int \frac{[d^4k]}{k^2+i\epsilon}(2\pi)\delta(k\cdot\bar{v}_2)\int_{-\infty}^{\tau_1} d\bar{\tau}_1 e^{ik\cdot v_1\bar{\tau}_1 + ik\cdot(b_1-b_2)}[k^\mu(\frac{1}{2}-\gamma^2) - k\cdot\bar{v}_1(2\gamma\bar{v}_2^\mu - \bar{v}_1^\mu)] \tag{69}$$

Re-writing the integral of $\bar{\tau}_1$ by integrating all the way to infinity and inserting the integral representation of the step function $\theta(\tau_1 - \bar{\tau}_1)$ yields

$$\delta^1 v_1^\mu(\tau_1) \;=\; -\frac{m_2}{4M_{pl}^2}\int \frac{[d^4k]}{k^2+i\epsilon}\frac{e^{ik\cdot b + i\bar{v}_1\cdot k\tau_1}}{\bar{v}_1\cdot k - i\epsilon}(2\pi)\delta(k\cdot\bar{v}_2)[k^\mu(\frac{1}{2}-\gamma^2) - k\cdot\bar{v}_1(2\gamma\bar{v}_2^\mu - \bar{v}_1^\mu)] \tag{70}$$

Integrating again gives

$$\delta^1 x_1^\mu \;=\; i\frac{m_2}{4M_{pl}^2}\int \frac{[d^4k]}{k^2+i\epsilon}\delta(k\cdot\bar{v}_2)\frac{e^{ik\cdot\bar{v}_1\tau_1 + ik\cdot(b_1-b_2)}}{(k\cdot\bar{v}_1 - i\epsilon)^2}[k^\mu(\frac{1}{2}-\gamma^2) - k\cdot\bar{v}_1(2\gamma\bar{v}_2^\mu - \bar{v}_1^\mu)]. \tag{71}$$

The calculation for $\delta^1 x_2^\mu$ and $\delta^1 v_2^\mu$ follows almost identically interchanging 1 and 2 except for a few sign changes, the calculation of which is left as an exercise for the reader. The resulting correction to the action that arises from inserting these first order changes of the path into the leading order action eq.(61) is given by

$$\delta S^{(0,1)} = \frac{m_1 m_2}{4M_{pl}^2}\int d\tau_1 d\tau_2((2\gamma^2-1)(il\cdot(\delta x_1 - \delta x_2) - 2\gamma(\delta v_1\cdot\bar{v}_2 + \delta v_2\cdot\bar{v}_1) - \bar{v}_1\cdot\delta v_1 - \bar{v}_2\cdot\delta v_2))$$
$$\times\frac{[d^4l]e^{il\cdot b + il\cdot(\bar{v}_1\tau_1 - \bar{v}_2\tau_2)}}{l^2+i\epsilon}. \tag{72}$$

Exercise n.5

Show that the impulse due to using the first order shift in the trajectory with the leading order action. is given by

$$\Delta p_1^\mu = \frac{im_1 m_2^2}{128 M_{pl}^4} \int [d^4k][d^4q]((2\gamma^2-1)^2 q^2 - 16\gamma^2 (k\cdot\bar{v}_1)^2) \frac{(q^\mu - k^\mu)\delta(q\cdot\bar{v}_2)\delta(k\cdot\bar{v}_2)\delta(q\cdot\bar{v}_1)}{k^2(q-k)^2(k\cdot\bar{v}_1 - i\epsilon)^2} e^{iq\cdot b}. \tag{73}$$

To complete the 2PM result we need to plug the leading order trajectory into the 1PM correction to the action, resulting from the calculation of the diagrams with an insertion of the three point graviton vertex. The results, along with the relevant Feynman rules, can be found in [17]. What is interesting about this result is the existence of the HQET propagators (see chapter), an avatar of a scattering amplitude for fields with large masses. This should not be surprising given that the classical action is the saddle point of the path integral, and we should be able to extract the classical action by taking a limit of the quantum amplitude.

IV. USING QUANTUM SCATTERING AMPLITUDE AND ON-SHELL TECHNIQUES

As we go to higher order in the PN or PM expansion, the number of diagrams we must draw begins to grow as in gravity there is no bound to the number of gravitons that can emitted from a single vertex (at least when we work with the canonical field variables. For alternative, actions see [19]). This is a standard complication in quantum field theory, but there is a well known methodology which simplifies the problem by using (on-shell) S-matrix techniques whereby one calculates the full integrand by sowing together on-shell tree amplitudes (for a review see [20]). While the techniques for this go beyond the scope of this book, a few sentences can capture the basic idea of the methodology. The salient point is that in gauge theories, such as gravity, the action contains a large amount of redundancy. There are unphysical degrees of freedom that play no role in physical observables and lead to very complicated intermediate steps of calculations. Feynman diagrams include these unphysical states and if one looks at unphysical objects, such as individual diagrams, which are not gauge invariant, one often finds hopelessly complicated expressions. However, the sum of the diagrams usually is quite compact. It turns out that these higher order vertices are all part of the gauge redundancy in gravity and that one can calculate all S-matrix elements without reference to anything beyond the three point vertex. Since the potentials can be related to S-matrix elements through a matching procedure [21] (see chapter on NRQCD) one can then calculate the potentials in a greatly simplified manner without summing Feynman diagrams with all their intrinsic redundancies. This also implies that the gravitational S-matrix contains within it solutions to Einsteins equation, i.e. classical space times. [9]. One down-side of this method is that the result includes quantum corrections, which are of no interest to us here, and must be discarded, though usually this is only a minor annoyance.

Given that the amplitudes are calculated relativistically, it is natural to work in the PM expansion [22]. However, to extract potentials we need to make sure what we are not double counting. That is, the amplitude will include what we would have considered disconnected diagrams when calculating within the EFT. One way to make sure we are counting correctly is to match onto a theory of "potentials". This last term is quoted because potentials are naturally a non-relativistic concept and here we are working to all order in v/c. However, as long as we are careful not to keep radiation poles this is a well defined concept. The potentials are fixed in PM expansion by matching onto a theory of relativistic point particles [22], such that in the COM frame can be written as as expansion in G

$$V(\vec{p},\vec{r}) = G\frac{c_1(\vec{p}^2)}{r} + G^2\frac{c_2(\vec{p}^2)}{r^2} + ... \tag{74}$$

where the exporessions for c_i up to 2PM can be found in [22] and up to 4PM in [23]. As one goes to higher orders one runs into the complication of separating radiation which can be both conservative (tail effect) and dissipative (direct power loss). The conservative radiative effects are not directly extractable from a hyperbolic orbit.

We see that there are two distinct approaches to the classical scattering problem: The direct solution of the equations of motion for two interacting worldlines and the field theoretic[17] calculation of scattering amplitudes. In the former methodology we caculate the impulse (Δp) or the scattering angle χ, via a calculation of the on-shell action, whereas for the amplitude calculation we can extract the Hamiltonian and then solve the dynamics).

[17] One can also calculate the amplitude by quantizing the world-lines [24] .

A. HQET, Wilson Lines and Classical Scattering[18]

In solving the classical scattering problem we have run into some familiar objects. In particular we notice in eq.(73) the existence of HQET like propagator factors. This should not be terribly surprising, as we have stated that in HQET the quarks are, non-fluctuating (classical sources) with nearly fixed four velocity. Small deviations, due to changes in the residual momentum k, are the best we can do in terms of effecting the heavy quark trajectory. When the quark gets hit by a soft gluon it goes slightly off-shell, meaning that for some intermediate time it does not obey its free dispersion relation $v \cdot k = 0$. This "off-shellness" should not be considered a quantum effect, it is simply the statement that the particle is now moving in a potential. If quarks were not confined, then the particle would be off-shell until it reaches asymptotia where we assume that the potential turns off and we have free scattering states. We could say the same for a quark obeying the NRQCD dispersion related $E = p^2/2m$, but we would not call such a particle a classical source. Indeed, even in NRQCD the particle does not form closed loops, so perhaps we should say that it too does not flcutuate? However, the distinction between an HQET and NRQCD is that, in HQET the wave function (which can have arbitrary spatial dependence) does not spread as function of time as we will discuss below.

Let us return to understanding why we see, the square, of HQET propagators ($\frac{1}{(v\cdot k)^2}$) in our classical calculation. Mechanically this distinction arises because eq.(57) involves two time derivatives for the particle as opposed to one in the case of HQET. How will HQET reproduce the classical limit? Let us consider two to two scattering in HQET beyond the leading order one gluon/graviton exchange. At next order we have two topologies (ignoring non-linearities for the moment), the box and the cross box. Focussing on one of the two lines we find the amplitude is proportional to

$$iM \sim \frac{1}{v \cdot k + i\epsilon} + \frac{1}{-v \cdot k + i\epsilon} = 2\pi\delta(v \cdot k). \tag{75}$$

when we ignore group theory factors. If we included factors of T_a the the non-commuting pieces proportional to C_A, can be considered a quantum mechanical effect[19]. The resulting delta function on the RHS is picking out the zero mode of the gluon/graviton field, and thus

[18] This section can be skipped by readers who are not itnerested in the connections to HQET.
[19] The classical limit of QCD corresponds to taking the quark charges to be large ($C_A/C_F \to 0$).

the box plus cross boxs diagrams sum to the iteration of one particle exchange which will exponentiate to form the usual divergent (see eq. (62)) phase. How then do we generate the factor of $1/(v \cdot k)^2$ in e.g. (**??**)? This factor can only be generated in HQET once we allow for a small recoil, i.e. a correction to the straight line trajectory. In HQET this comes from an insertion of the higher dimensoional operator $k^2/(2M)$.

It is worthwhile to see how this works in position space. We would like to calculate the analogue of the Wilson loop [20]and to do so we will follow the leading order (RPI) gauge fixed action

$$\overline{W} = TrPe^{i\frac{m_1}{2M_{pl}}\int_{-\infty}^{\infty} h_{\mu\nu}v_1^\mu v_1^\nu ds + i\frac{m_2}{2M_{pl}}\int_{-\infty}^{\infty} h_{\mu\nu}v_2^\mu v_2^\nu ds}, \tag{76}$$

where the two sources follow straight line trajectories and the ends of the loop, at time-like infinity, are assumed to not be irrelevant. This object is a tool to calculate soft graviton interactions with a classical source

$$T_{\mu\nu}(x) = \frac{m}{2}\int ds\ v_\mu(s)v_\nu(s)\delta^4(x^\mu - x^\mu(s)). \tag{77}$$

Let us consider the contributions with two insertions of the interaction. Concentrating on the emissions from one leg for the moment, we have two "time" (path) orderings.

$$\begin{aligned} &\int_c^b ds_1 \int_a^c ds_2 h_{\mu\nu}(s_1)h_{\rho\sigma}(s_2) + \int_c^b ds_2 \int_a^c ds_1 h_{\rho\sigma}(s_2)h_{\mu\nu}(s_1) \\ =& \int_a^b ds_1 \int_a^b ds_2(\theta(s_1 - s_2) + \theta(s_2 - s_1))h_{\rho\sigma}(s_2)h_{\mu\nu}(s_1) \end{aligned} \tag{78}$$

where we have used the fact that the fields ordering is irrelevant given that they will be contracted with the line on the other side [21]. Thus we see that we will end up with the square of the one graviton exchange which gets exponentiated (see the exercise below). This space-time picture of the heavy quark propagation further illuminates the situation. Taking $v^\mu = (1, \vec{0})$ we have

$$\langle h_\alpha(t, \vec{x})\bar{h}_\beta(0)\rangle = i\delta_{\alpha\beta}\int [d^4k]\frac{e^{-ik\cdot x}}{v \cdot k + i\epsilon} = \theta(t)\delta^3(\vec{x})\delta_{\alpha\beta}, \tag{79}$$

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[20] Unlike QCD in gravity this will not be a gauge invariant object, as now the path becomes gauge dependent and must be specified in a gauge invariant way [**?**].

[21] Contractions on the same side correspond to mass renormalization that is power law divergent and unphysical.

the spatial delta function prevents the wave function spread and maintains the classical nature of the process. The theta function generates the time ordering in eq.(78) when t is chosen as the affine parameter.

Exercise n.6

Above we have sketched the calculation of the energy between two sources using the gravitational Wilson lines. Take the sources to be static, parameterize the world line by t and take $a - b = T$, then show that if we ignore all the bulk non-linearities we can resum the series into the form

$$\overline{W} = e^{iTE(R)} \tag{80}$$

where $E(R)$ is the Newtonian energy for two statice sources. The exponentiation of the non-linear graphs in QCD implies a definative color structure in the exponent (this gives meaning to the notion of exponentiation). For a pedagogical introduction see [29]. In the context of SCET (see chapter) this exponentiation can be derived via a renormalization group argument.

V. INCLUDING DISSIPATION

[22]

Given that we have developed a systematic way to power count in the PN expansion we can ask the following question. At what order in the PN expansion do finite size effects show up? To answer this question all we have to do is to power count the finite size operators (12). For black holes this is straightforward since we know that the only relevant parameter is the Schwarzchild radius. Using the operator scaling in eq. (14) the action then scales as

$$S_{FS} \sim (r_s)^3 (r_s M_{pl})^n \int dt (\partial\partial \frac{h}{M_{pl}})^2 \sim L^{n/2-1} v^{9+n/2}. \tag{81}$$

Given that the leading contribution of this operator to the potential comes from the seagull topology (see figure 3) and that each mass insertion gives a factor of $\sqrt{L}$, we demand

[22] This section assumes knowledge of the in-in formalism, which is discussed in the previous chapter.

that $n = 2$. Thus the first finite size effect does not start until v^{10}. This diminishment of finite size effects in general relativity is often termed "effacement" [31]. Note that if we allow for quantum effects, we only have a bound $n \leq 2$. An alternative way to determine the scaling for C_E is to consider matching the graviton scattering cross section [**?**]. In the EFT the leading order scattering cross will scale as

$$\sigma \sim \mid C_E k^4 M_{pl}^{-2} \mid^2 \tag{82}$$

and the full theory cross section can be written on dimensional grounds as

$$\sigma \sim r_s^2 f(k r_s) \tag{83}$$

matching powers of k we see that $C_E \sim r_s^5 M_{pl}^2$ as expected. Note our dimensional analysis uses the fact that r_s is the only relevant length scale, i.e. that the mass is always accompanied by a af factor of G_N, which follows for a black hole. For a neutron star however, we can get an enhancement since the radius will, in general, be larger than the Schwarzchild radius. Given that it scales as the fifth power of the radius the enhancement can be siginificant.

Exercise: n.6 Calculate the leading order potential generated by a non-vanishing C_E.

There are additional finite size corrections which are not associated with the local terms (12) that arise from dissipative effects. Given that we're allowing our compact objects to deform, it is necessary to account for the fact that there is work done in this process. Some amount of energy will be absorbed by the compact object, i.e. it will heat up thus increasing its mass. How do we account for this effect? Since we are working in the point particle approximation we have already integrated out all of the degrees of freedom which would account for this heating process. Thus we will need to add back in some degrees of freedom that live on the world-line if we are to have any hope of accounting for this dissipation. This situation is reminiscent of our discussion of the Van der Waals (VdW) interaction in the previous chapter. The crucial distinction being that here we will be interested in dissipation. [23] We introduce a generic field $\phi(\tau)$ that lives on the world-line to give life

[23] Note that the VdW case also has a dissipative component, due to radiation, which we ignored.

to the aforementioned degrees of freedom. $\phi(\tau)$ transforms as some representation of the (local) Lorentz group which we will fix in a moment. We want to couple $\phi(\tau)$ to the metric in a diffeomorphism invariant fashion. Such invariants can be formed via contractions with the Riemann tensor, but, as discussed in section (II), we should not bother with invariants which vanish by of Einsteins' equations, so we should couple ϕ to the electric and magnetic pieces of the Weyl tensor. Assuming parity invariance, we introduce two distinct fields on the worldline, $Q_{\mu\nu}(\tau)$ and $M_{\mu\nu}(\tau)$ which are even and odd under parity respectively and couple to E and B such that the action is given by

$$S_{dis} = \int d\tau (Q_{ab}E^{ab} + M_{ab}B^{ab}). \tag{84}$$

Here we have written the expression in terms of small Roman letters to denote local orthogonal coordinates, as opposed to the global coordinates denoted by greek letters. Working in the local frame will allow us to simplify the correlators of the $Q's$ to be introduced below. The two coordinate systems are related by the vierbein which satisfies

$$e^{\mu}_{a} e^{\nu}_{b} \eta^{ab} = g^{\mu\nu} \qquad e^{a}_{\mu} e^{b}_{\nu} g^{\mu\nu} = \eta^{ab}. \tag{85}$$

At lowest order the distinction between global and local frames is irrelevant. Q_{ab} and M_{ab} are irreps of the local Lorentz group.

Now we would like to calculate how these couplings affect the potential in the PN expansion. To do so we need to power count the operators in (84), and thus we need to know how the Q's scale. We will return to this issue once we understand how to make sense of them. We will take the Q and M to be operators in a Hilbert space though we know in the end that this is a classical problem and they really represent the (quasi)normal mode spectrum of the black holes. But as we have learned sometimes the quantum framework organizes our calculation in a clearer fashion.

How can these interactions affect the potentials? We begin by integrating out the potential graviton that is exchanged between two world-lines with one mass insertion and one Q/M insertion. At lowest order there is no distinction between the local and global coordinate, and thus all contractions can be performed using a flat space metric. Expanding the definition of the electric Weyl tensor to linear order and keeping only the leading order piece in the PN expansion we find

$$E_{ij} = -\frac{1}{2M_{pl}}\partial_i\partial_j h_{00} \tag{86}$$

such that

$$\langle E_{ij}(t_1, x_1)h_{00}(t_2, x_2)\rangle = \frac{i}{16\pi M_{pl}}\delta(t_2 - t_1)\partial_i\partial_j\frac{1}{\mid \vec{x}_1 - \vec{x}_2 \mid}, \tag{87}$$

The resulting, instantaneous, potential is given by

$$S^E_{pot} = -G\int dt(m_1 Q_2^{ij}(t) + m_2 Q_1^{ij}(t))q_{ij}(t) \tag{88}$$

where the tidal field is $q_{ij}(t) = \partial_i\partial_j\frac{1}{|\vec{x}_1 - \vec{x}_2|}$ and we are working in the rest frame so the indices are spatial. The expectation value of this potential vanishes assuming the compact obejct is spherically symmetric, i.e. $\langle\Omega \mid Q_{ij} \mid \Omega\rangle = 0$, where $\mid \Omega\rangle$ is ground state. To get a non-vanishing result we need two insertions of the potential S_{pot}, as shown in figure (6b). The magnetic potential will be sub-leading in the PN expansion and will be ignored.

Let us try to gain some intuition for the degrees of freedom that live on the world line. If we are to have control over our calculations, it had better be that the correlators of M and Q are analytic around $\omega = 0$, since we are working in a derivative expansion. Physically, this implies that the correlations functions die off faster than any power in time. Classically we know that the system will have a set of quasi-normal modes whose propagators have poles off the real axis. In the low energy limit we will be sampling the tails of these ringing modes. Modeling the system as a set of damped harmonic oscillators the two point functions should schematically look like

$$G_R(\omega) \equiv \int dte^{i\omega t}\langle[Q(t), Q(0)]\rangle \sim \frac{1}{\omega^2 - \omega_0^2 + i\Gamma\omega}. \tag{89}$$

If we consider the case of a black hole, where the quasi-normal spectrum is known, there is only one relevant scale r_s and $\omega_0 \sim \Gamma \sim r_s^{-1}$. Note that the system is effectively ungapped even though the real part of the pole is non-zero. This is due to the dissipation (width), as any small amount of energy can be absorbed, as we would expect from a classical macroscopic object.

Given our assumption of exponential die off in time, we may expand the correlation function in a series in ω. Since the real time propagator is real valued, the imaginary part of the frequency space propagator is odd and its expansion takes the form

$$G_R(\omega) = A_1 + i\omega B_1 + A_2\omega^2 + ... \tag{90}$$

It is important to understand that once we include the couplings to the internal degrees of freedom (84) we will be double counting if we also include the local terms in (12). However, we can allow the coupling (84) to subsume the local interactions. A_2 is related to the first dynamic Love number. These coefficients can have non-trivial RG flow. That is, there are terms in the series that are proportional to $\omega^2 Log(\omega)$ [45].

We pause here for a moment to determine whether or not we should be calculating in the in-in or in-out formalisms? We learned in the previous chapter, that if we want to get sensible, causal, equations of motion, we should work with in-in. But we also know that if we are dealing with instantaneous interactions, as we are here, the $i\epsilon$ becomes irrelevant. However, as we will now discuss, dissipative systems are distinct when considering variational problems.

A. Dissipation and Variational Principles

In the previous chapter we discussed the fact that the usual in-out path integral prescription was not appropriate for describing time dependent observables and instead used the in-in formalism. A similar story arises when we want to describe dissipation. If we track all the degrees of freedom in a system, then it will be definition, conservative. So instead of trying to cook up an action that describes dissipation we will start from first principles with a full conservative system and then integrate out a subsystem, leaving behind, hopefully, the action for the remaining degrees of freedom which is dissipative. However, this process will fail as we will now describe.

The sticking points are elucidated by a simple example [50]. Consider two coupled one dimensional harmonic oscillators

$$S = \sum_{i=1,2} \int dt \; \frac{1}{2}(m_i x_i^2 - \omega_i^2 x_i^2 + \lambda x_1 x_2). \tag{91}$$

We will be interested in the dynamics of x_1 whereas x_2 will play the role of an environment and represent some large number of degrees of freedom that are responsible for dissipation which arises when x_1 interacts with x_2. The energy transferred is dispersed into a large volume of phase space and the time scale for the entire system to return to its initial state is called the "Poincare Recurrence time". Assuming the system is chaotic, we may take this time scale to be infinite. The loss of information is the tell tale sign of dissipation.

Since we are not tracking x_2 we integrate it out by replacing it with its solution given some fixed initial data, i.e. we will integrate it out. Solving the equation of motion for x_2 in terms of the retarded Greens function and plugging back into the action we have

$$S = \int dt\, \frac{1}{2}(m_1 x_1^2 - \omega_1^2 x_1^2) + \frac{\lambda^2}{2m_2}\int dtdt' x_1(t) G_F(t-t') x_1(t'). \tag{92}$$

where $G_F(t) = \frac{1}{2i\omega_2}(\theta(t)e^{i\omega_2 t} + \theta(-t)e^{-i\omega_2 t})$ is the Feynman propagtor that arises from the in-out path integral (see the previous chapter). We can see immediately that this wont suffice as it is acausal. Note this would be true even if we, by hand, force the Greens function to be retarded (i.e. only keep one of the two theta functions in G_F). This should not surprise us as the variational principle we have utilized is inherently time symmetric since it involves varying over all paths with the end points at t_i and t_f fixed and is time reversal invariant. So we are not surprised that dissipation is out of the reach of a time symmetric action using the standard variational principle. When we work in the in-in formalism however, we dont fix the final states, as discussed in the previous chapter. By doubling the variables, in the quantum language we would say we square the matrix element, the "final state" is now at the initial time. That is we are changing our target as follows

$$\langle q_i, t_i \mid q_f, t_f \rangle \to \sum_X \langle q_i, t_i \mid X, t_f \rangle \langle X, t_f \mid q_f, t_i \rangle. \tag{93}$$

So the forward and time reversed paths are distinct from each other.

B. The dissipative equations of motion for a binary inspiral

Let us now return to the effective action. We next integrate out the world-line degrees of freedom. The leading order interaction come from the insertion of two potentials (88) into

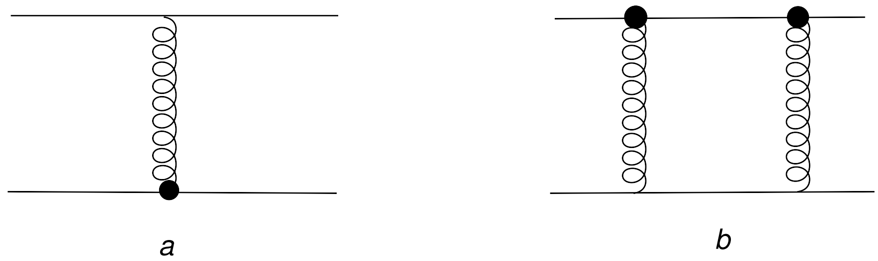

FIG. 6. Diagram (a) shows the generation of the leading order potential between the two world lines that depends upon Q_{ij}, and vanishes for a spherically symmetric object. Diagram (b) shows the first non-vanishing contribution. The heavy dot denotes an insertions of the quadrapole operator while the other vertex is the lowest order mass insertion. Not shown are the symmetric diagrams which interchange (1) and (2).

the in-in matrix element which generates the action

$$S_{6b} = -\frac{1}{2} \sum_{A \neq C = 1,2} m_C^2 G^2 \int dt dt' q_{aij}(t) G^{ab,E(A)}_{ijkl}(t-t') q_{bkl}(t') \tag{94}$$

where $q_{ij} = \partial_i \partial_j \frac{1}{r}$ and $G^{ab,E(A)}$ is the two point function in the Keldysh basis for $Q^{(A)}$. The reader should convince themselves that magnetic potential is sub-leading in the PN expansion.

The force on particle A is determined by varying the action with respect to x^A_- and setting it to zero leaving

$$F^A_l = -m_A^2 G^2 \int dt' (\partial_l q_{ij}(t)) G^{E,C(ret)}_{ijkl}(t-t') q_{kl}(t'). \tag{95}$$

We can determine $G^{E,ret}_{ijkl}$, the retarded propagator for the Q correlators describing the internal dynamics of particle C, via a matching procedure. However, note this is a very unusual matching in that we are matching for an *infra-red* quantity, whereas normally in an EFT we match to fix parameters arising from *ultra-violet* physics. So it is perhaps better to use the term "extracting" instead of matching. We need new information in order to properly integrate back in the relevant degrees of freedom.

We can extract the response function from the calculation of the scattering cross section which has both elastic as well as inelastic pieces. For dissipation will be interested in the absorptive piece which will correspond to the imaginary part of the Greens functions, as will be discussed below. In the EFT the absorptive cross section is given by

$$\sigma_{abs} = \frac{1}{2\omega}\sum_f \sum_\lambda \left(\mid \langle 0 \mid Q_{\rho\nu}(0) \mid f\rangle W^{\rho\nu} \mid^2 + \mid \langle 0 \mid M_{\rho\nu}(0) \mid f\rangle X^{\rho\nu} \mid^2\right)(2\pi)\delta(P_i^0 + \omega - P_f^0) \tag{96}$$

where P_f is the momentum of the final state f which may or may not include one or more gravitons. λ are the graviton polarizations. $W_{\rho\nu}$ and $X_{\rho\nu}$ are tensors quadratic in momenta and include the polarization tensors. Keep in mind that we are interested in the long wavelength limit, so we are expanding around flat space. At leading order in this expansion we may ignore the differences between the local and global coordinate systems.

At leading order the purely absorptive piece includes one incoming graviton and none in the final state. Furthermore, restricting ourselves to the physical polarization, the non-vanishing components of W and X are purely spatial (working in the rest frame).

Exercise: n.7 Using the linearized versions of E_{ij} and B_{ij} in momentum space

$$E_{ij} = \frac{1}{2M_{pl}}\omega^2\epsilon^\lambda_{ij} \qquad B_{ij} = \frac{\omega}{2M_{pl}}\epsilon_{ikl}k_l\epsilon^\lambda_{kj}, \tag{97}$$

$$\sigma_{abs}(\omega) \;=\; \sum_\lambda \frac{1}{8M_{pl}^2}\left(\omega^3 A^{E+}_{ijkl}(\omega)\epsilon^\lambda_{ij}\epsilon^{\lambda\star}_{kl} + \omega A^{B+}_{ijkl}(\omega)(\vec{k}\times\epsilon^\lambda)_{ij}(\vec{k}\times\epsilon^{\lambda\star})_{kl}\right) \tag{98}$$

where $(\vec{k}\times\epsilon^\lambda)_{ij} = \epsilon_{ial}k_l\epsilon^\lambda_{aj}$.

Utilizing our assumption of rotational symmetry and the tracelessness of Q_{ij}, we may parameterize the correlator in frequency space as:

$$A^{E+}_{ijkl}(\omega) \equiv \int dt e^{-i\omega t}\langle 0|\, Q_{ij}(t)Q_{kl}(0)\,|0\rangle \equiv A^{E+}(\omega)(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \frac{2}{3}\delta^{ij}\delta^{kl}) \tag{99}$$

with a similar parameterization for the magnetic Wightman function A^{B+}. The tensor structure on the RHS of this identity in the space of spin two tensors. Then using the sum over physical polarization (in transverse traceless gauge)

$$\sum_h \epsilon^h_{ij}(k)\epsilon^{*h}_{rs}(k) \equiv P_{ij;rs}(\mathbf{k}) = \frac{1}{2}\Bigg[\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr} - \delta_{ij}\delta_{rs} + \frac{1}{\mathbf{k}^2}\left(\delta_{ij}k_r k_s + \delta_{rs}k_i k_j\right) - \frac{1}{\mathbf{k}^2}\left(\delta_{ir}k_j k_s + \delta_{is}k_j k_r + \delta_{jr}k_i k_s + \delta_{js}k_i k_r\right) + \frac{1}{\mathbf{k}^4}k_i k_j k_r k_s\Bigg], \quad (100)$$

gives

$$\sigma_{abs}(\omega) = \frac{\omega^3}{2M_{pl}^2}\left(A^{E+}(\omega) + A^{B+}(\omega)\right). \quad (101)$$

Then we may use the fluctuation dissipation theorem to relate the Wightman function (A^+) to the imaginary part of the retarded propagtor.

Exercise: n.8 Prove the fluctuation dissipation theorem

$$\theta(\omega)A^+(\omega) - \theta(-\omega)A_-(\omega) = 2ImG_{ret}(\omega) \quad (102)$$

by performing a spectral decomposition of $A_\pm$. $G_{ret}(\omega)$ is defined via

$$G^{ijkl}_{ret} = G_{ret}(\omega)(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \frac{2}{3}\delta^{ij}\delta^{kl}). \quad (103)$$

Now show that the imaginary part of the retarded propagator is odd under $\omega \to -\omega$, while the real part is even.

Next we use the fact that the full theory scattering equation, often called the Regge-Wheeler equation [34], is invariant under electric magnetic duality transformations, which implies that $A^{+E} = A^{+B} \equiv A^+$ (so that $A_+ = M_{pl}^2\sigma_{abs}/\omega^3$).

Then the dissipative force is given by

$$F^A_l = -i\sum_a \frac{m_a^2 G^2}{32\pi}\int dt' \int [d\omega] e^{i\omega(t-t')}\frac{\sigma_{abs}(\omega)}{\omega^3}(\partial_l q_{ij}(t))(\frac{2}{3}\delta^{ij}\delta^{kl} - \delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})q_{kl}(t'). \quad (104)$$

Given that $Im(G_{ret}(\omega))$ is odd in ω, σ_{abs} must be an even. Moreover, since the G_{ret} should decay in time, it is analytic in ω near the origin. Thus writing

$$\sigma^i_{abs}(\omega) = A_{4i}R_i^6\omega^4 + A_{6i}R_i^8\omega^6 + ... \quad (105)$$

where R_i is the radius of object i. The series starts at order ω^4 since the action is local in time given our assumptions about the correlators and the potential nature of the interaction.

The force exerted on the world line is then

$$\vec{F}_1 = -\vec{F}_2 = \frac{9G}{8\pi}\sum_{a\neq b} A_{4b} R_b^6 m_a^2 \left(\frac{1}{\vec{x}^8}(\vec{v} + 2\frac{\vec{x}\cdot\vec{v}}{\vec{x}^2}\vec{x})\right) \tag{106}$$

Let us now specialize to the case of a black hole. At leading order in the derivative expansion the absorptive cross section for a graviton scattering on a black hole is given by [33]

$$\sigma_{abs} = \frac{4\pi r_s^6 \omega^4}{45}, \tag{107}$$

so that $R = r_s = 2Gm$ and $A_4 = \frac{4\pi}{45}$ and

$$F_1^{BHi} = -F_2^{BHi} = (m_a^2 m_b^6 + m_b^2 m_a^6)\frac{32G^7}{5}\left(\frac{1}{\vec{x}^8}(\vec{v} + 2\frac{\vec{x}\cdot\vec{v}}{\vec{x}^2}\vec{x})\right) \tag{108}$$

The power loss then follows by dotting this result into the velocity.

Now let us return to the question of the v scaling of the wordline line terms (130). We would like to determine the scaling of the absorptive potential (94) in both L and v. As previsouly emphasized we can read off the scaling by studying the scalings of the operators which make up the Feynman diagram (7b). The scaling of Q can be read off by looking at its correlator

$$Im \int dt e^{i\omega t}\langle 0 \mid T Q_{ab}(t) Q_{cd}(0) \mid 0\rangle \propto \omega (GM)^6 M_{pl}^2. \tag{109}$$

In the context of of calculating the potential in the PN expansion ω scales as v/R and dt scales as R/v such that

$$Q(\tau)Q(0) \sim (v/R)^2 M^6/M_{pl}^{10} = v^4/L^2 (M/M_{pl})^8 1/M_{pl}^2 = L^2 v^8/M_{pl}^2. \tag{110}$$

So that Q scales as Lv^4/M_{pl}. Now we may consider the scaling of action

$$S = \int dt Q_{\mu\nu} E^{\mu\nu} \sim \frac{r}{v} \times L v^4 M_{pl}^{-1} \times \frac{1}{M_{pl}} r^{-2} \times v^{1/2} r^{-1} \sim L\frac{v^{7/2}}{r^2 M_{pl}^2} \sim L\frac{v^{11/2}}{r^2 M^2 v^2} \times vL \sim v^{13/2}. \tag{111}$$

Two insertions of this operator with two mass insertions leads to an overall scaling of v^{13}.

The lack of L suppression is consistent with a classical correction. A similar calculation shows that the magnetic contribution has the identical scaling. The effective action should then scale as v^{13}. The odd power of v is constitent with a non-conservative effect since it is not time reversal invariant.

To fix these finite size coefficient one must do a matching calculation. As we have emphasized previously we are free to match in any way we wish. In the sense that we could match by putting the object in a static external field (see the last citation in [38]), or we could match via an on-shell scattering calculation [47].

VI. GRAVITATIONAL RADIATION

In the end, the relevant observables for binary inspirals are the phase and amplitude of the wave which is being detected. From a theoretical standpoint the phase is most easily calculated via the power loss. Taking for illustration a circular orbit in the extreme mass ratio limit (i.e. keeping only terms to leading order in the light mass) and using conservation of energy, $dE/dt = -P$, the gravitational wave phase seen by the detector is then given by

$$\Delta\phi(t) = \int_{t_0}^{t} dt'\omega(t') \tag{112}$$

where ω is the wave's frequency. $\omega(t')$ is determined from the power loss (P) by equating it to dE/dt for a circular orbit [24]

$$\frac{dE(\omega(t))}{dt} = -P(\omega). \tag{113}$$

Given our calculation of $E(r)$ we can trade it for $E(\omega)$ in a circular orbit and then integrate (113) to get $\omega(t)$.

Notice that the phase is sensitive to the time (unknown) t_0 at which the signal enters the detector. This is a parameter in the theory that must be fixed by an initial measurement much like a parameter in the action. In fact, we will use this parameter to eliminate infrared divergences which arise in our calculations, as the initial phase is sensitive to the long time history of the target. Calculating the amplitude at the detector itself corresponds to determining the field value far away from the source. In this section we will show how one

[24] This is only an approximation as $\dot{r}$ is being neglected. This is an adiabatic approximation in that one assumes that $|\dot{r}| \ll r\omega$.

calculates both the power and the amplitude.

To calculate the radiation we first match onto a new world-line theory that is valid only at scales larger than the orbit, as we did in the case of electrodynamics in the previous chapter, except now the non-linearities enrich the theory considerably leading to a plethora of interesting new effects. The matching onto the multipole moments is now non-trivial as there are radiative corrections in the matching procedure. as opposed to the case of electrodynamics, where we only needed to multipole expand the world-line action.

We begin by integrating out the modes whose wavelength is of order the size of the binary. These are the potential modes responsible for binding the system. Once we have removed these modes we have coarse grained to the point where the details of the bound state are no longer accessible to us. As such, to be consistent (see the previous chapter) we must multipole expand the couplings of the radiation field to the sources, which is broken down into two steps. First the direct couplings of the radiation field to the world lines must be expanded, which is a straightforward exercise. In addition, we must expand the coupling of the radiation field to the potential, since the graviton itself carries stress-energy. It is at this point that the power of the EFT becomes manifest. The net effect of these two steps is to generate an effective action for a *single* point particle with a set of multipole moments coupled to the metric in a way consistent with the symmetries. Once we have determined the multipole moments, via a matching procedure, we can calculate the power loss via the imaginary part of the the energy as in the last chapter.

When calculating the power loss, or amplitude, from the effective action, there are several other manifestations of the non-linearities of GR. For instance, when we calculate the power we must account for the fact that the radiation can scatter off the background (Schwarzchild) metric, leading to what is known as the "tail effect". Furthermore, it is also possible for the the radiation to scatter off itself. Both of these effects can be considered as manifestations of "radiative moments". We will return to these effects after we have calculated some of simple, non-radiative, moments.

When we integrate out the potential modes it desirable to maintain general coordinate invariance for the resulting action, which is a function of the radiation field $\bar{h}$. This is accomplished by working in the background field gauge, whose gauge fixing term is given by taking eq. (114) and replacing the derivatives with covariant derivatives whose connection is built from the radiation field

$$\Gamma_\mu = D_\alpha H^\alpha_\mu - \frac{1}{2} D_\mu H^\alpha_\alpha. \tag{114}$$

In this gauge after integrating out the potential modes and performing the multipole expansion, the resulting action has the form

$$S_1 = \frac{1}{2} \int dt (Q^{\mu\nu} E_{\mu\nu} - \frac{4}{3} J^{\mu\nu} B_{\mu\nu} + \frac{1}{3} O^{\mu\nu\rho} \nabla_\rho E_{\mu\nu} + ...), \tag{115}$$

where are the fields are evaluated at the origin. As in the case of finite size operators we express the gravitational field in terms of the electric and magnetic pieces of the Weyl tensor since they are irreducible under use of the equations of motion. The moments which couple to E and B are called "mass" and "current" multipole moments respectively. Notice that this action is composed of only radiation fields. But in addition there is a *new* potential field that that is generated by the static sources within the the effective one body. Which is to say that there will be a potential field generated by the conserved quantities, the mass and angular momentum. We can write down the general form of this action as follows

$$S_{pot} = -\frac{1}{2M_{pl}} \int dt (M f(h) + P_i f_i(h) + L_{ij} f_{ij}(h)), \tag{116}$$

where the conserved charges are the mass M, momentum P_i and angular momentum L_{ij} [25]. . We can fix the function f, f_i, f_{ij} by imposing invariance under small (those which keep $g_{\mu\nu} \approx \eta_{\mu\nu}$) coordinate transformations.

$$x^\mu \to x^\mu + \xi^\mu \qquad \bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \tag{117}$$

It is important to remember that the gauge parameter is also multipole expanded. In which case, by inspection, if we make the choice

$$f = a h_{00}, f_i(h) = b h_{0i}, f_{ij} = c \partial_{[i} h_{0j]}, \tag{118}$$

then this system is almost invariant. The first term is invariant since $\delta h_{00} \sim \dot{\xi}_0(t, 0)$ and the

[25] At higher orders the back reaction effects of radiation must be taken into account if the charges are to be conserved.

mass ($M = \int d^3x T_{00}$) is a conserved quantity. Notice that the gauge parameter is multipole expanded for consistency. However, let's consider now the second term.

$$\int dt P_i h_{i0} \to \int dt P_i (h_{i0}(t,0) + \dot{\xi}_i(t,0) + \partial_i \xi_0(t,0)) \tag{119}$$

The term proportional to $\dot{\xi}_i$ is zero since $\dot{P}_i = 0$. But the second term is not zero. However, we should expect action has a term of the form

$$\delta L = \frac{b}{2} \int dt M X_i \partial_i h_{00}, \tag{120}$$

whose variation will lead to an invariant action. When we do the explicit matching below we will see that the term (120) does indeed arise. Finally notice that we did not transform the Wilson coefficients. The reason is that

$$\int d^4x \delta T^{\mu\nu} h_{\mu\nu} = \int d^4x (\partial^\mu \xi^\beta T^\nu_\beta + \partial^\nu \xi^\beta T^\mu_\beta) \eta_{\mu\nu} = 0. \tag{121}$$

The coefficients (a, b, c) need to fixed by a matching calculation, as do the conserved charges $M, \vec{P}, \vec{L}$ which are functions of (x_i, v_i).

Now let us consider the higher multipole moments which couple to the radiation graviton. Each multipole moment transforms as an irreducible representation of the rotation group in the local rest frame (small Roman letters). The mass and current quadrapoles Q and J are symmetric and traceless, and form the $l = 2$ representation of $SO(3)$, while the octopole O is the completely symmetric and traceless $l = 3$ representation. Each of the multipole moments is orthogonal to each other in the sense that there will be no interference terms between them when they radiate gravitons. This is obvious from a group theory standpoint.

A. Calculating Multipole Moments

By definition, the graviton couples to the (pseudo[26]) stress energy tensor as follows

$$S = -\frac{1}{2M_{pl}} \int d^4x T_{\mu\nu}(x) h^{\mu\nu}(x), \tag{122}$$

[26] This abusive term refers to the fact that the gravitational field itself contributes to the stress energy.

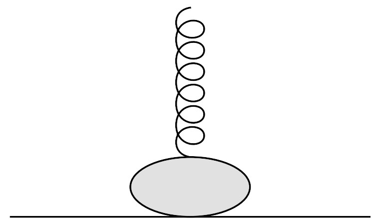

FIG. 7. Diagram showing the generic form of a tadpole diagram which generates a term of the form $h_{\mu\nu}T^{\mu\nu}$ in the effective action. The blob presents the coupling to gravitons through the non-linearities.

here h contains both the radiation field $\bar{h}$ as well as the potential generated by the *pair*.

The strategy will be to multipole expand (122) and manipulate it into the form given by (115) and (116). Recall that such an expansion is forced upon us to maintain manifest power counting at the level of the action, a tenet of EFT. Once we have done this, we will be able to read off an expression for the multipole moments in terms of derivatives of the stress energy tensor, which is generated both by the world-lines as well as the potential gravitons that bind the world-lines. The stress energy can be calculated via the expectation value of $\langle h_{\mu\nu}\rangle$, which is often called the "one-point" function and corresponds to Feynman diagrams with one open graviton line as illustrated in figure (8).

Before beginning our calculation, we must understand the v scaling of the radiation modes. Given that these modes are on shell, the energy and momentum must scale in the same way. Since the energy of the radiation scales with the frequency, $E \sim v/r$, we have $k^\mu \sim (v/r, \vec{v}/r)$. Then using, what are hopefully familiar by now, arguments, we find that the coordinate space radiation field scales as v/r. Table (I) gives a summary of the scaling rules.

TABLE I. Scaling relations

M^2/M_{pl}^2	vL
$Md\tau$	$\frac{L}{v^2}$
$\bar{h}/M_{pl}$	$\frac{v^{5/2}}{\sqrt{L}}$
H/M_{pl}	$\frac{v^2}{\sqrt{L}}$

To generate an action of the form (115) we start by multipole expanding the graviton field. Again, since we have maintained general coordinate invariance we know that the result of expanding must lead to a coupling to the curvature (or Weyl tensor). Let us consider expanding around the center of mass (taken to be the origin) of the binary to quadratic order

$$-\frac{1}{2M_{pl}}\int d^4x T^{\mu\nu}(x)\bar{h}_{\mu\nu}(x) \approx -\frac{1}{2M_{pl}}\int d^4x T^{\mu\nu}(x)\left(\bar{h}_{\mu\nu}(0) + x^i\partial_i\bar{h}_{\mu\nu}(0) + \frac{1}{2}x^ix^j\partial_i\partial_j\bar{h}_{\mu\nu}(0)\right). \tag{123}$$

Notice that the first two terms in this series are absent in (115) since the Weyl tensor is quadratic in derivatives. These terms will correspond to those in (116). We can see that $M = \int d^3x T_{00} =$ and $P_i = \int d^3x T_{0i}$ as expected. Where the ladder vanishes in the COM frame.

Exercise n.9 Show that the multipole expansion generates the coupling

$$S = -\frac{1}{2}\int dx^0 L^{ij}\omega_0^{ij}, \tag{124}$$

where the linearized spin connection is given by

$$\omega_\mu^{ij} = \frac{1}{2M_{pl}}(\partial^i\bar{h}_\mu^j - \partial^j\bar{h}_\mu^i) \tag{125}$$

and $L^{ij} = -\int d^3x(T^{0i}x^j - T^{0j}x^i)$ is the orbital angular momentum. When we allow for spin L becomes total angular momentum. Since the angular momentum is conserved, this piece will not be relevant for real radiation, but plays a roll in the tail effect, discussed below.

The coefficients a, b, c defined in (118) can then be read off. The fate of the $h_{ij}T_{ij}$ term will become clear below. Furthermore, we still need to calculate T_{00} and T_{0i} from a matching calculation.

Given that the conserved quantities dont radiate the corresponding terms in the action would appear to be irrelevant. However, as previously mentioned, the total mass generates the Schwarzchild field of the effective one body, and couples to h_{00}. Radiation scatters off this background generating the aforementioned "tail effects" which we will discuss later in this chapter. Thus it is perhaps a misnomer to call $\bar{h}_{00}$ a radiation field, as it's really a potential field with no definate momentum scaling since the orbital scale r has been integrated out at this point.

Now we would like to determine how the higher multipole depend upon the stress energy tensor. From Newtonian physics we know that the leading order result for the quadrapole should took the form

$$Q_{ij} = \int T_{00}(x)(x_i x_j - \frac{1}{3}\delta_{ij}\vec{x}^2)d^3x. \tag{126}$$

But we expect that the non-linearities of GR will amend this result.

We need to take full advantage of the information at our disposal so that we don't end up reinventing the wheel. For starters, we know that the real radiation is transverse so we might as well start by considering the coupling to h_{ij}. Diffeomorphism invariance, i.e. the fact we know that the action has the form (115), will fix the dependence upon the other components of h [27]. This implies that the multipole moments will be functions of T_{ij}. Note that given that the world-line action does not explicitly depend upon the coordiates, T_{ij} starts at $O(v^2)$ since it must be proportional to $v_i v_j$. At this order the one point function will include the diagram where radiation is emitted off of a potential mode[28]. On the other hand T_{00} starts at leading order and is more readily calculated. We may take advantage of this by by utilizing moment relations between T_{ij} and T_{00}, T_{0i}.

Exercise n.10 Using conservation of the stress energy (pseudo) tensor, prove the following moment relation

$$\int d^3x T_{ij} = \frac{1}{2}\int d^3x x_i x_j \ddot{T}_{00}. \tag{127}$$

The stress energy tensor gains time dependence through proportionality to a delta function $\delta(x - y(t))$ which forces it on the world line. Thus given the multipole expansion we have

$$S = -\frac{1}{2M_{pl}}(\int d^3x dt T_{ij}(x))h_{ij}(0) = -\frac{1}{4M_{pl}}\int dt \ddot{h}_{ij}\int d^3x T_{00}(x)x_i x_j + ... \tag{128}$$

We see that the moment relations have accomplished two things: A relationship between a $O(v^2)$ object (T_{ij}) and a $O(v^0)$ objects (T_{00}). It generated two extra derivatives which is what we need to generate couplings to the curvature tensor.

[27] By calculating in the rest frame the moments will only have spatial components.
[28] Showing this is left as an exercise for the reader.

Using the linearized version of E and B

$$\begin{aligned} E_{ij} &= \frac{1}{2M_{pl}}((h_{i0,0j} - h_{00,ij}) - (h_{ij,00} - h_{j0,i0})) \\ B_{ij} &= \frac{\epsilon_{imn}}{2M_{pl}}(h_{n0,jm} + h_{jm,0n}) \end{aligned} \tag{129}$$

and the definition (115) leads to the leading order quadrapole result

$$Q_{ij} = \int d^3 x x_i x_j T_{00}. \tag{130}$$

This representation of the quadrapole does not transform irreducibly under rotations [29] since it is not trace free. Thus the quadrapole should formally have the trace subtracted, though practically it does not matter since

$$E_{ii} = B_{ii} = 0. \tag{131}$$

When one goes to higher orders ensuring irreducibility will not be a simple formality. In any case, the result for the leading order mass quadrapole is the canonical form

$$Q_{ij}^{(0)} = \int d^3 x (x_i x_j - \frac{1}{3}\delta_{ij}) T_{00}^{(0)}, \tag{132}$$

and we have successfully derived the first term in (115). Where $T_{00}^{(0)}$ is the leading order contribution in v to the stress energy tensor. The (0) superscript in Q is there to remind the reader that this is only the leading order form of the mass quadrapole. The corrections to this result will be discussed below.

Let us next derive the current quadrapole and mass octopoles which arise from the first derivative term in (123). We decompose the tensor product $x_j T_{ik}$ into irreducible representations of $SO(3)$ via $1 \otimes 2 = 3 \oplus 2 \oplus 1$. The 1 wont couple since it corresponds to a three index object with two indices chewed up by a delta function, which leads to a zero contraction due to (131). The 3 is the totally symmetric ocotopole while the 2 is the mixed

[29] It is assumed here that the reader has a rudimentary knowledge of the rotation group.

symmetry magnetic (current) quadrapole. We can decompose the product as follows

$$x_j T_{ik} = \frac{1}{3}(x_j T_{ik} + x_i T_{jk} + x_k T_{ij}) + \frac{1}{3}(2x_j T_{ik} - x_i T_{jk} - x_k T_{ij}). \tag{133}$$

Using the moment relations

$$\begin{aligned} \frac{1}{2}\int d^3x x^i x^j x^k \ddot{T}_{00} &= \int d^3x(x_i T_{jk} + x_j T_{ik} + x_k T_{ij}) \\ \int d^3x(x_i x_j \dot{T}_{0k}) &= \int d^3x(x_i T_{kj} + x_j T_{ki}) \end{aligned} \tag{134}$$

the first term, the octopole, can be reduced to

$$\int d^3x \frac{1}{3}(x_i T_{jk} + x_j T_{ik} + x_k T_{ij}) = \frac{1}{6}\int d^3x \left[x^i x^j x^k \ddot{T}_{00}\right]^{TF}, \tag{135}$$

where TF implies the trace has been removed, while the current quadrapole can be written as

$$\int d^3\mathbf{x} \frac{1}{3}\left[2x_j T_{ki} - (x_i T_{jk} + x_k T_{ij})\right]^{TF} = \int d^3\mathbf{x} \frac{1}{3}\left[\dot{T}_{0i} x_j x_k + \dot{T}_{0k} x_j x_i - 2\dot{T}_{0j} x_i x_k\right]^{TF}. \tag{136}$$

Plugging this result into (123), integrating by parts, and picking out the piece of the Weyl tensor proportional to derivatives of h_{ij} leaves

$$-\frac{1}{2M_{pl}}\frac{1}{2}\int d^3x(\partial_j h_{ik})T_{ki}x_j = \frac{1}{6}O^{ijk}_{(0)}\partial^j E^{ik} - \frac{2}{3}B_{ij}J^{ij}_{(0)} \tag{137}$$

where

$$\begin{aligned} J^{ij}_{(0)} &= \int d^3x \left[\epsilon^{ilk} T^{0k} x^j x^l\right]_{STF} \\ O^{ijk}_{(0)} &= \int d^3x T^{00}\left[x^i x^j x^k\right]_{TF} \end{aligned} \tag{138}$$

where STF implies one must symmetrize in addition to removing the trace. The derivative acting in the octopole term may be automatically lifted to its covariant form since we have maintained diffeomorphism invariance by working in background field gauge.

If we wish to calculate the subleading contribution to the mass quadrapole, then we must consider two sources of corrections to the result (132). Higher order terms in the Taylor series expansion (123), that will change the form of the moment, and subleading corrections to T_{00}. The former corrections can be derived in a standard fashion (i.e. we would calculate these terms in the same way as is done more traditional gravity wave methodologies [2, 3]), however, these corrections are treated below to keep this discussion self-contained. On the other hand, the derivation of the latter corrections are native to the EFT approach.

The corrections to the form (130) arise from higher order terms in the multipole expansion. The first of such corrections arises from third term in (123)

$$S = -\frac{1}{2M_{pl}} \int dt d^3x \frac{1}{2} x_k x_l (\partial_k \partial_l h_{ij}) T_{ij}. \tag{139}$$

We may decompose this term into irreducible representations of $SO(3)$ as follows

$$T^{ij} x^k x^l \sim \mathbf{2} \otimes (\mathbf{1} \otimes \mathbf{1})_S \sim \mathbf{2} \otimes (\mathbf{2} \oplus \mathbf{0})_S = \mathbf{4} \oplus \mathbf{3} \oplus \mathbf{2}_M \oplus \mathbf{2}_C \oplus \mathbf{1} \oplus 0, \tag{140}$$

consisting of a 16-pole mass moment ($\mathbf{4}$), a current octopole moment ($\mathbf{3}$), correction terms to mass and current quadrupoles ($\mathbf{2}_{M,C}$) as well as corrections to non-radiating moments transforming as $\mathbf{1}, \mathbf{0}$. The corrections to the quadrapoles arise from the traces which are subtracted out from the higher dimensional representations. As opposed to the leading order case, where the traces can not radiate, in this case the traces can contribute since there are a sufficient number of indices that the trace (i.e. δ_{ab}) need not lead to a contraction which vanishes by Einstein's equations (131). The form of the mass quadrapole correction follows by performing the appropriate decomposition of the tensor product and is given by

$$Q_{ij}^{(2)} = \int d^3\mathbf{x} \left[\frac{4}{21} T^{aa} x^i x^j - \frac{2}{7} \left(x^i x^a T^{ja} + x^j x^a T^{ia} \right) + \frac{11}{21} T^{ij} \mathbf{x}^2 \right]. \tag{141}$$

We will not go into the algebra needed to derive this result, and refer the reader to [51] where the form multipoles is given to all orders. Instead we will concentrate on the corrections to T_{00}.

T_{00} couples to h_{00}, as such, we may extract T_{00} by calculating the expectation value $\langle h_{00} \rangle$. Formally we should derive this by introducing the notion of the effective action, which is a idea familiar to readers who have taken an introductory quantum field theory course.

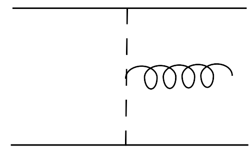

FIG. 8. A contribution to the one point function leading to a potential energy piece of the stress tensor.

However, we will side step this formalism, and instead will rely on some simple handwaving arguments. We begin by noting that if we were not to integrate out any modes whatsoever, then by definition we would have

$$\int d^4x T_{\mu\nu}(x) = -2M_{pl}\frac{\delta S}{\delta h_{\mu\nu}}. \tag{142}$$

That is, if we linearize the action in the metric then we can directly read off the value of the stress energy tensor. It is simply the coefficient of the metric in the action. As a trivial example let's use this technique to calculate the leading order contribution to T_{00} in the PN expansion. The term linear in h in the action is

$$S_{LO}^{int} = -\sum_a \frac{M_a}{2M_{pl}}\int dt h_{00} = -\sum_a \frac{M_a}{2M_{pl}}\int dt \int d^4y h_{00}(y)\delta^{(4)}(y - x(t)) \tag{143}$$

which leads to the elementary result

$$T_{00}(y) = \sum_a M_a \int dt\delta^{(4)}(y - x(t)). \tag{144}$$

We can see that for tadpole contributions, going through this exercise overly pedantic, and we can simply read off $T_{\mu\nu}$ from the Lagrangian. Subleading tad-pole contributions are easily read off from the action as well. For instance, the kinetic contribution to T_{00} will arise from the v^2 term in (130). However, in addition we have contributions to stress tensor that arise from interactions of the metric with the potential. Such a contribution is shown in figure (8). Before tackling such contributions to the stress tensor, note that what we are really interested in are the *moments* of the stress tensor, which determine the multipoles, and are most easily calculated in momentum space. In this space we may extract the moments by comparing the Taylor expansion of the one point function with [30]

$$T^{\mu\nu}(x^0,\mathbf{q}) = \sum_{n=0}^{\infty} \frac{i^n}{n!}\left[\int d^3\mathbf{x} T^{\mu\nu}(x^0,\mathbf{x}) x^{i_1}\cdots x^{i_n}\right] q_{i_1}\cdots q_{i_n}. \tag{145}$$

For tadpole diagrams this is trivial since the dependence on the external momentum (k) arises only from the wave function of the external state. That is, after truncating the propagator in momentum space [31] the exponential $e^{i\vec{k}\cdot\vec{x}}$ remains. Let us be highly explicit by redoing our rudimentary example for the leading order T_{00} which arises from the tadpole, i.e. the one graviton absorption diagram. At leading order this comes from one insertion of word-line coupling to h_{00}.

$$T_{00}(q) \;=\; (2iM_{pl})\frac{-iM_a}{2M_{pl}}\int dt\sum_a {}_{out}\langle 0\mid h_{00}(x_a,t)\mid q\rangle_{in} = \sum_a M_a\int dt e^{i\vec{x}_a(t)\cdot\vec{q}}e^{-itq_0} \tag{146}$$

and thus

$$T_{00}(\vec{q},t) \;=\; \sum_a M_a e^{i\vec{x}_a(t)\cdot\vec{q}}, \tag{147}$$

where in eq.(146) we have inserted the leading order interaction with the worldlines. A factor of $i2M_{pl}$ has been included (see the definition (142)), Then by comparing to (145) we find the trivial moments

$$\int d^3x T_{00} x^{i_1}x^{i_2}..x^{i_n} = \sum_a M_a x_a^{i_1}x_a^{i_2}..x_a^{i_n} \tag{148}$$

So that the tadpole contribution to the second moment is not surprisingly given by given by

$$\int d^3x T_{00}x^i x^j = \sum_a (M_a + \frac{1}{2}M_a v_a^2)x_a^i x_a^j \tag{149}$$

Where the $1PN$ corrections from eq. (23), has been included. Now we'd like to calculate the 1PN contribution to the moments of T_{00} stemming from the potential energy. In this case we cant simply read off the results from the action but have to integrate out the potential mode.

[30] With this phase (sign) convention, this corresponds to the absorption of an incoming graviton. i.e the field convention is $h \sim \int d^3k(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x})$. The expansion is normalized such that $\langle 0\mid h(x)\mid q\rangle = e^{-iq\cdot x}$. The state $|q\rangle_{in}$ corresponds to an incoming graviton of momentum q annihilated by a.

[31] For readers with less background in quantum field theory, this just means that we take the result for the diagram and multiply by k^2, where k the is four-momentum of the external graviton. The exponential which remains after truncation is just the exponential weight of the Fourier transform.

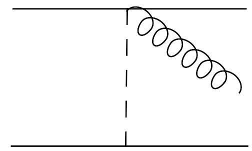

FIG. 9. A contribution to the one point function leading to a potential energy piece of the stress tensor. The dashed/curly line is a potential/radiation graviton.

We follow the same line of reasoning as above and calculate the expectation value of $h(x,t)$. We know that the result should be trivial, i.e. there should be an additional contribution of the form $\frac{G_N M_1 M_2}{|x_1-x_2|}$. However, doing the explicit calculation in a turn the crank fashion is an excellent exercise. The potential contributions to the mass-energy come the diagrams such as (8) and (9). Let us begin with the simpler of the two cases, namely figure (9). The mixed potential-radiation vertex can be read off from the quadratic part of (23), after making the substitution (20). The result for this Feynman diagram is then

$$T_{00}^{fig10}(t,\vec{q}) \;=\; \sum_a e^{i\vec{q}\cdot\vec{x}_a}(2iM_{pl})\left(\frac{i2M_1}{8M_{pl}^2}\right)\left(\frac{-iM_2}{2M_{pl}}\right)\int[d^3k]\frac{ie^{-\vec{i}k\cdot(\vec{x}_1-\vec{x}_2)}}{-2\vec{k}^2}. \tag{150}$$

Truncating the external graviton line leaves

$$\begin{aligned} T_{00}^{fig10}(t,\vec{q}) \;&=\; \sum_a e^{i\vec{q}\cdot\vec{x}_a(t)}\left(\frac{M_1M_2}{8M_{pl}^2}\right)\int[d^3k]\frac{e^{-i\vec{k}\cdot(\vec{x}_1(t)-\vec{x}_2(t))}}{\vec{k}^2} \\ &=\; -\sum_a e^{i\vec{q}\cdot\vec{x}_a(t)}V \end{aligned} \tag{151}$$

where

$$V = -\frac{M_1M_2}{32\pi M_{pl}^2}\frac{1}{r} = -\frac{GM_1M_2}{r} \tag{152}$$

and use of been made of the generalized integral in eq. (30) Note there is no factor of two which would arise from putting the external line on either world line, since we are summing over worldlines. Also one must be careful in general to assign the correct coordinate to the radiation graviton wave function. For this particular diagram it is obvious that the graviton wave function should be evaluated at x_a, i.e. the line from which it was emitted.

This diagram leads to the following contribution to the moment

$$\int d^3x T_{00}^{fig10} x^i x^j = \frac{GM_1M_2}{r} \sum_a x_a^i x_a^j \tag{153}$$

Now let us consider the contribution from diagram (8). This diagram is more complicated due to the algebraic complexity of the three graviton vertex which is derived by expanding out the Einstein-Hilbert action, along with the background field gauge fixing term to third order in the fields and is a straightforward yet tedious exercise [32]. However, let us pause here to power count the three graviton vertex. Two of the gravitons are potential and one is radiation. In addition, since the vertex arises from the bulk curvature term the vertex involves two derivatives each of which scales as v/R and the vertex comes with a factor for $1/M_{pl}$. The bulk measure will scale, according to the rules developed in the previous chapters, as the inverse of $dk_0 d^3k$ [33] where dk_0 scales as v/r and $dk \sim 1/r$. Thus the vertex scales as

$$S_{3g} \sim \frac{1}{M_{pl}}(r^4/v)(v^2 M_{pl}/\sqrt{L})(v^{5/2}M_{pl}/\sqrt{L})(v/r)^2. \tag{154}$$

The result for this diagram gives

$$T_{00}(q) = \frac{M_1M_2}{M_{pl}^2} \int dt_1 dt_2 \int \frac{[d^dk_1]}{k_1^2}\frac{[d^dk_2]}{k_2^2}(2\pi)^4\delta^d(k_1+k_2+q)e^{ik_1\cdot x_1+ik_2\cdot x_2}F(k_1,k_2,q) \tag{155}$$

where F is a second order polynomial in derivatives, to be given below. Care must be taken to include the proper symmetry factor for this diagram. The bulk vertex is second order in the potential and linear in radiation fields, so there is a factor of two that is included for the number of Wick contractions.

Notice that (155) is manifestly symmetric under $1 \leftrightarrow 2$, but performing one of the momentum integrals will temporarily obscure this symmetry. The result for $T_{00}(t,\vec{q})$ is

$$T_{00}(t,\vec{q}) \;=\; -\frac{3}{8}\frac{M_1M_2}{M_{pl}^2}\int \frac{[d^3k]}{(\vec{k}^2)(\vec{k}+\vec{q})^2}e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}e^{i\vec{q}\cdot\vec{x}_2}(\vec{k}^2+\vec{k}\cdot\vec{q}+...) \tag{156}$$

[32] The required manipulations are readily performed in a standard tensor symbolic manipulations package.
[33] For those who have not read the previous chapters, this is equivalent to determining the scaling from the delta function arising the spatial integral.

[34] Only terms up to quadratic order in q have been kept, since those are the relevant pieces for the second moment. Subleading terms i.e. proportional to k_0 have been dropped as have terms proportional to $\vec{q}^2$, as they will contribute terms proptional to δ_{ij} in the, trace free, quadrapole moment. Expanding the exponent to second order in $\vec{q}$ and performing the k integration gives

$$\int d^3x T_{00}^{fig9} x^i x^j = -\frac{3}{2}\frac{GM_1M_2}{r}\sum_a x_a^i x_a^j. \tag{157}$$

Adding in the results for figure (10) gives for the net contribution to the 1PN radiative correction to the leading order form of the quadrapole

$$\int d^3x T_{00}^{1PN} x^i x^j = \sum_{a\neq b}(M_a + \frac{1}{2}M_a v_a^2 - \frac{GM_aM_b}{2r})[x_a^i x_a^j]_{TF} \tag{158}$$

where $[TF]$ serves to remind the reader of the trace free condition. The other 1PN correction to the quadrapole comes from the change in the form given by (141).

B. Calculating the Power Loss

As discussed in the previous chapter we may calculate the power loss by either calculating the imaginary part of the energy or, perhaps more simply, by directly calculating the semi-classical amplitude for graviton radiation. That is, we calculate the matrix element for the emission of one radiation graviton, and then square it to get an emission rate. In terms of Feynman diagrams this means we calculate all the diagrams with one external radiation graviton leg hanging off. We can save ourselves a lot of time by using the fact that the mass and angular momentum are conserved and thus do not radiate, so that the leading order PN contribution will stem from the emission of one graviton off of the quadrapole moment. Although this contribution will be leading order it will scale with a non-negative power of v, which is consistent with fact that there are no Newtonian gravitons.

The matrix element for one (on-shell $k_0^2 - \vec{k}^2 = 0$) graviton emission off of the quadrapole

[34] This result follows the choice of all incoming momenta at the vertex. Also the three graviton vertex has been symmetrized, so there is no need count possible Wick contractions.

moment is given by

$$
\begin{aligned}
iA(k) &= \frac{i}{2}\langle 0 \mid \int dt Q_{ij}(t) E^{ij}(x(t)) \mid k\rangle \\
&= -\frac{i}{4M_{pl}} \int dt Q_{ij}(t) k_0^2 \epsilon_{ij}(k) e^{-ik\cdot x(t)} \\
&= -\frac{i}{4M_{pl}} \int dt \int d\omega \tilde{Q}_{ij}(-\omega) e^{-i(\omega - k_0)t} k_0^2 \epsilon_{ij}(k) \\
&= -\frac{i}{4M_{pl}}\ \tilde{Q}_{ij}(k_0) k_0^2 \epsilon^h_{ij}(k)
\end{aligned}
\tag{159}
$$

where took our multipole expansion around the center of mass $x^\mu(t), (t, \vec{0})$ and included only the traverse part of the metric in the definition of E since that is the only piece that radiates. The h superscript is the polarization index. The graviton emission rate is determined by squaring the amplitude and weighting it by the phase space factor

$$
d\Gamma(k) = \frac{1}{T} \sum_{pol} \mid A(k) \mid^2 \frac{[d^3k]}{2k}, \tag{160}
$$

where T is the emission time, which will drop out of our final result. The power is determined by weighting the graviton emission by the energy such that

$$
P = \int k d\Gamma(k). \tag{161}
$$

Performing the index contractions using the polarization sum (100) leads to the time honored result

$$
\begin{aligned}
P &= \frac{G_N}{5\pi T} \int_0^\infty dk k^6 |Q_{ij}(k)|^2 \\
&= \frac{G_N}{5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle,
\end{aligned}
\tag{162}
$$

where the dots denote time derivatives, and the brackets denote a time average. Of course, this equation is just the usual quadrupole radiation formula. The power loss from one graviton emission due to other (higher) moments follows in a similar fashion.

Exercise n.11 Show that the leading order power loss scales as

$$P_{rad} \sim \frac{v^{10}}{G_N} \tag{163}$$

then use (108) to show that the power loss for a black hole due to absorption scales as

$$P_{abs} \sim \frac{v^{18}}{G_N}. \tag{164}$$

Definining the radiation reaction and absoprtion force via $P \sim Fv$ where $F_N = GM^2/r^2 \sim v^4/G_N$, we find the ratio $F_{rad}/F_N \sim v^5$.

Exercise n.12 Using the boost invariance of the effective action for a binary system in the post-Newtonian limit show that Noethers' theorem imples that the center of mass position $X^i \equiv \int d^3x x^i T_{00}$ is a constant of the motion. Then calculate the 1PN correction to X^i. To accomplish this calculate the one point function of g_{00} and follow the reasoning sub-section A. If we did not choose the center of mass position to vanish, what would it couple to in eq. (115)? See problem (6) below.

1. *The Tail Effect*

One certainly does not need the EFT formalism to perform the simple calculations involved in the last section. Where the EFT formalism helps, is when we have processes involving both radiation and potential modes. The tail effect, is such a situation, where a radiation graviton scatters off of the background of the binary. The relevant Feynman diagram is shown in figure (11). Relative to the single quadrapole emission, this diagram includes one mass insertion and one three graviton insertion. Now it is important to recall here that the potential mode is not the same potential mode that was integrated out in our previous discussion. As such, the scaling of this potential differs from that listed in table (1). The potential mode again carries no energy, but its momentum no longer scales as $1/R$. Recall that this second EFT only knows about that scale $1/R$ via the matching coefficients (multipole moments). Given that the only energy or momentum scale in the problem is now the radiation energy-momentum $p \sim v/R$, the potential itself must carry momentumn of the

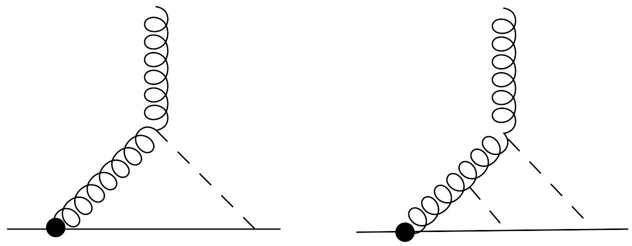

FIG. 10. The contributions to the tail. The heavy dot is an insertion of a quadrapole which radiates and scatters of a potential mode which is generated by a mass at the vertex. The dotted line corresponds to a static graviton which is part of the Schwarzchild background of the total binary system. There are an infinite number of diagrams of this form with more and more mass insertions. The sum of the divergence part generates the Coulomb phase.

same order, i.e. v/R and using our usual arguments we find that this potential field scales identically to the radiation, i.e. $H/M_{pl} \sim v^{5/2}/\sqrt{L}$.

Do determine the scaling of the tail effect we need to power count the three graviton vertex which now scales differently then in the case of the EIH calculation. This vertex scales as

$$S_{3g} \sim \int d^4x \frac{1}{M_{pl}} h(\partial h)^2 \sim v^{5/2}. \tag{165}$$

A quick calculation shows that the mass insertion now scales as $\sqrt{v}$ thus the leading tail piece is a 1.5 PN correction to the leading order radiation result.

In the infra-red region of the loop integral the integrand for this diagram has the form

$$I = \frac{\int [d^3k]}{\vec{k}^2} \frac{1}{\vec{q} \cdot \vec{k}} \tag{166}$$

which is infra-red divergent. This should not be surprising as the Newtonian potential falls off as $1/r$ and thus one can scatter off the background at arbitrarily long distances. This is just the Coulomb phase, which drops out when we calculate the square of the amplitude. Indeed it is straightforward to resum the appropriate set of diagrams (the first two of which are shown in figure (11))that lead to the exponentiation of the phase (see [52, 53]).

2. Summing Logs in the Power: The Renormalization Group Trajectory

There is another very interesting piece of this particular calculation that deserves comment. Namely the second diagram in figure (11) while being IR divergent also generates a

UV divergence which is not associated with the Coulomb phase. In addition, there are two other diagrams, without the tail topology, that have UV divergences as well, and the reader is encouraged to determine the relevant topologies as an exercise. The two loop diagrams are logarithmically divergent, leading to renormalization group running. We will not go into the details of the two loop calculations which can be found in the appendix of [53]. Here we will focus on the UV divergence since that is the piece which contains the information about the running. Normalizing to the leading order radiation[35] amplitude ($A_0 = \frac{\vec{k}^2}{4M_{pl}}\epsilon^{\star}_{ij}Q_{ij}$) the result for the amplitude is [36] of the form

$$\frac{A}{A_0} = (\omega G_N m)^2 \frac{107}{105}(\frac{1}{\epsilon} - Log(\vec{k}^2/\mu^2) + ...) \tag{167}$$

thus we can absorb the divergence into $Q(\omega, \mu)$. The reader should find this slightly discomforting since Q is a short distance matching coefficient whose physics is determined by wavelengths of order of the size of the system which is parametrically smaller than the inverse frequency by a factor of the velocity. Indeed it seems that perhaps we should really be absorbing the divergence into a counter-term for an operators of the form $Q\ddot{E}$, in which case the ω^2 contribution arises from the long distance contribution of the matrix element of the operator. The renormalization would then correspond to operator mixing, which is a subject that was discussed in the earlier chapters of this book. Thus to keep the classical discussion self contained we will stick to the technique whereby the matching coefficient depends upon ω. Needless to say, both methods yield the same result.

The μ independence of the (net) amplitude leads to the differential equation

$$\mu \frac{d}{d\mu} Q_{ij}(\mu, \omega) = -2 \times (\omega G_N m)^2 \frac{107}{105} Q_{ij}(\mu, \omega) \tag{168}$$

whose solution is given by

$$Q_{ij}(\mu, \omega) = (\mu/\mu_0)^{-\frac{214}{105}(\omega G_N m)^2} Q_{ij}(\mu_0, \omega). \tag{169}$$

μ_0 should be chosen to be the natural scale for the quadrapole. i.e. the size of the bound state (r) while μ should be chosen to minimize the logarithms in the amplitude (167), which

[35] Recall this is on-shell so $\omega^2 = \vec{k}^2$.

[36] This diagram also has an infrared divergence which is purely imaginary and will not contribute once we square the amplitude, as it represents an unphysical phase much, as in Coulomb scattering. We will discuss this in more details when we calculate the wave form.

is accomplished by taking $\mu \sim \omega$. We can expand the result to expose the complete series of logarithms that we have resummed. At the level of at the amplitude squared we have [53]

$$| \frac{A}{A_0} |^2 = 1 - \frac{428}{105}(G_n m\omega)^2 Log(\omega/\mu_0) + \frac{91592}{11025}(G_n m\omega)^4 Log^2(\omega/\mu_0) - \frac{32901376}{3472875}(G_n m\omega)^6 Log^3(\omega/\mu_0) + ... \tag{170}$$

The existence of the UV log divergence in the effective one body theory is telling us that had we calculated this diagrams in the "full" theory, we wold generate logs of the form $\log(\omega r)$. That is, the size of the state cuts off the apparent divergence in the full theory. Diagrammatically we can see how this arises by opening up and dissecting the quadrapole interaction in the effective theory. This is illustrated in figure (12) where now the single lines opens up into two lines corresponding to the constituents of the binary. By choosing $\mu_0 = 1/r$ we eliminate any large logs ($Log(v)$) from the quadruple moment.

It is important to realize that resumming the logs does not improve the accuracy of our prediction. This is as opposed to the case of the renormalization group in a quantum field theory. The reason is that each power of a log is accompanied by a factor of v^2, so we are not resumming a set of order one contributions. In the quantum field theory case we resummed the product of a large log and a coupling such that the product was order one. However, there is no limit in which $v^2 Log(v) \sim 1$. Furthermore, we can not be assured that at a given order in v we have captured *all* of the logs. In this case however, the terms in the series do indeed capture the leading order contribution at each order in $\log(v)$. In fact, the logs in (170) have been compared to the analytic solution derived by solving the wave equation in the extreme mass ratio limit (one body is taken as a test mass) and agreement has been found [56]. At each power of $\log(v)$ there are sub-leading corrections which arise which have additional powers of v in front. For instance, diagrams involving two quadrapole insertions renormalize the mass and lead to sub-leading log contributions [57] of the form $v^4(Gm\omega)^{2n+1}\log^n(v)$.

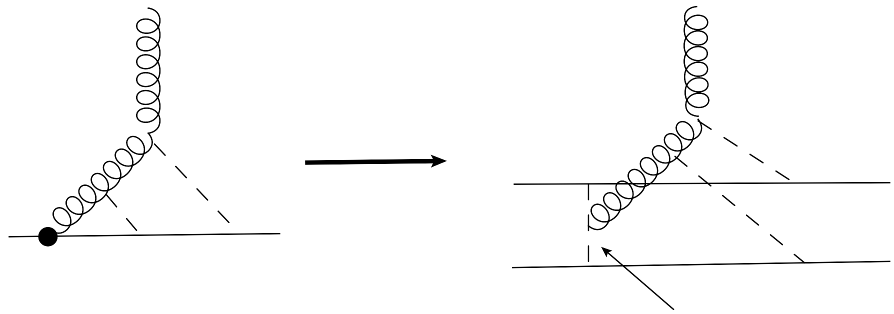

FIG. 11. This diagram shows how if one opens up the quadrapole insertion (heavy dot) in the effective theory, into the full theory diagrams (on the RHS) the UV divergence is cut-off by the size of the binary. The arrow points to the propagator which softens the UV behavior of the integral.

C. Radiation Reaction[37]

1. *The World-Line Gravitational In-In Formalism*

In this previous chapter we calculated the force on finite size spheres due to radiation back-reaction. The gravitational analogue of this force plays an important role in gravity wave phenomenology. As we discussed to properly account for such dissipative forces we worm in the in-in formalism. However, as we will see, building an action which is invariant under all the relevant symmetires takes a little more work than in the case of E+M.

Here we will discuss this force in the context of the PN expansion [38]. The strategy is completely analogous to the electromagnetic case and was first performed in [26]. There is however, a key difference between the two cases. In the electromagnetic case we can accelerate a point partice via an external forces (say a spring) that has no electromagentic properties, whereas in the gravitational case every source will have stress energy so the point particle is not a closed system. Instead in gravity we will be considering a bound orbit of two particles which will act as a closed system.

Since we are working in the radiation theory, we start with the action (115) and calculate an effective action by integrating out (using retarded propagators) the radiation. The leading contribution, in the PN expansion, arises from the self energy diagram (see figure (7) of the previous chapter) where the vertices are now quadrapole insertions as dictated by the

[37] This section assumes the reader has read section (IIB) of the previous chapter.
[38] For a discussion in the context of the extreme mass ratio limit see [**?**].

Feynman rules generated by (115). The amplitude for this diagram is given by

$$iM = -\frac{1}{2}\int d\tau d\tau' \langle E_{\mu\nu}(\tau) E_{\rho\sigma}(\tau')\rangle Q^{\mu\nu}(\tau) Q_{\rho\sigma}(\tau'). \tag{171}$$

Next we use the linearized form of Electric Weyl tensor (129), keeping in mind that in the radiation theory the time and space derivatives scale identically. The retarded (harmonic gauge) propagator is given by

$$D^R_{\mu\nu\alpha\beta} = \frac{P_{\mu\nu\alpha\beta}}{2\pi}\theta(x^0(\tau) - \theta(x^0(\tau'))\delta([x(\tau) - x(\tau')]^2), \tag{172}$$

where

$$P_{\mu\nu\alpha\beta} = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}). \tag{173}$$

By choosing the affine parameter to be $\sqrt{(x(\tau) - x(\tau'))^2}$, varying this effective action with respect to the latest time parameter, and using the leading order result for the quadruple moment (107) leads to the the Burke-Thorn result [58]

$$\dddot{x}^i(t) = -\frac{2}{5}G_N \frac{d^5 Q^{ij}(t)}{dt^5} x^j. \tag{174}$$

There are two sources of higher order corrections: Non-linear interactions in the bulk as well as higher order multipole moments on the world-line. Note that non-linear terms on the world-line themselves will be quantum in nature and will thus be suppressed. To see this one notes that such diagrams will always lead to at least one closed graviton loop.

VII. OTHER DEGREES OF FREEDOM: SPIN

We can generalize the theory further by allowing for the other degrees of freedom. The most obvious generalization would be the inclusion of spin which is highly relevant for the phenomenology of gravity waves. As in the case of absorption, we must allow for a new variable to live on the world line. The challenge is to introduce the spin vector $\vec{S}$ in a covariant way. We will follow the ideas presented in [59] but will not go into all of the technical details regarding the theory of constrained systems. We begin by studying the relativistic spherical top in flat space [60]. One first introduces a field on the world-line,

$\Lambda_A^{\ \mu}(\tau)$, which is an element of the Lorentz group $SO(3,1)$

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{A}(\tau)\Lambda^{\nu}{}_{B} = \eta_{AB}, \tag{175}$$

which transforms the set of inertial frames to the co-rotating (body fixed) frame[39] (whose components are denoted by capital Roman letters.).

The quantity

$$\Omega^{\mu\nu} \equiv \Lambda^{\mu A}\dot{\Lambda}^{\nu}{}_{A} = -\Omega^{\mu\nu}, \tag{176}$$

is the relativistic analog of the usual angular velocity ω^{ij} of a rotating frame

$$\frac{dx^i}{dt} = x^j\omega^{ji}. \tag{177}$$

The anti-symmetric nature follows from differentiating eq.(175). The spin $S_{\mu\nu}$ is defined as being the momentum conjugate to Λ_μ^A

$$S_{\mu\nu} \equiv -2\frac{\delta L}{\partial\Omega_{\mu\nu}}. \tag{178}$$

The two is chosen for future notational convenience. We will have to generalize this result once we go to a curved space.

It is clear that the phase space variables are redundant, since we know that there really should only be three spin degrees of freedom. However, we can see this redundancy without the need for this non-relativistic crutch. Our phase space variables are (x^μ, p^μ) and $(\Lambda_\mu^A, S_{\mu\nu})$. The redundancy in the coordinate pair we know follows from the reparameterization invariance which forces the action to be homogenous of degree one in the affine parameter ($L = \sqrt{\dot{x}^2}$), which in turn implies that action is singular [40] and one can not solve for the velocities in terms of the momenta. We could try to follow this line of reasoning and construct the most general RPI action that includes the Ω. However, we are immediately met with a large space of possible terms (up to four powers of Ω, by the Cayley-Hamitlon theorem), which would lead to some unwieldy algebra. Instead the redundancy in the spin variables is made apparent by noticing that the time direction of the body fixed frame $\Lambda^\mu_{A=0}$, is fixed by p^μ and thus is a redundant variable conjugate to to S_{0i}. In fact the choice of S_{0i}

[39] Think of this frame as a coordinate system painted on the object.

[40] Meaning the Hessian vanishes, $Det(\partial^2 L/\partial\dot{x}^\mu\partial\dot{x}^\nu) = 0$.

has a simple physical interpretation in terms of the choice of reference world lines which is needed to define the angular momentum. In the non-relativistic limit we would take this worldline to correspond to the center of mass which is unique, but once we include relativistic effects it becomes observer dependent. To understand this consider a spinning mass moving in some inertial frame. The top and bottom hemispheres will have differing velocity and hence will have differing relativistic masses and the object develops an observer dependent mass dipole which we identify with S_{0i} pointing normal to the direction of motion and is a measure of displacement, due to relativistic effects, of the true (rest frame) center of mass. So a choice of reference points, is equivalent to imposing a constraint on $S_{\mu\nu}$ (see [61] for a discussion). It is possible to fix the constraint which generates the shift in reference points (gauge transformations) ([62, 63]), and thereby build a guage invariant action. While this is a useful and interesting formal exercise, it will not be necessary for our purposes. Instead we will calculating by choosing a reference point by imposing a constraint on the spin tensor.

The classic work by Pryce [64] on this subject gives three useful choices for reference points

- The rest frame center of momentum:

$$\vec{x}_{rf} = \frac{\sum_i E_0^i \vec{x}_i}{\sum_i E_0^i} \tag{179}$$

  where E_i is the energy of the ith component particle computed when the total momentum is zero. The corresponding constraint is covariant

$$S_{rf}^{\mu\nu} p_\nu = 0. \tag{180}$$

  In this rest frame, the mass dipole will vanish, $S_{0i} = 0$, so that the reference point is the center of mass.

- Generalized center of momentum:

$$\vec{x}_g = \frac{\sum_i E^i \vec{x}_i}{\sum_i E^i} \tag{181}$$

  where E^i is the energy as measured by an arbitrary observer. The corresponding constraint is

$$S_g^{0\mu} = 0. \tag{182}$$

- The Newton-Wigner coordinate: This is a hybrid of the above two choices given by

$$\vec{x}_{NW} = \frac{M\vec{x}_{rf} + E\vec{x}_g}{M + E}. \tag{183}$$

where M is the total invariant mass of the system. The corresponding constraint is

$$MS_{NW}^{0\mu} - S_{NW}^{\mu\nu} p_\nu = 0. \tag{184}$$

Each one of these definitions generates the necessary number (three) of constraints and leads to a different set of Poisson brackets (or quantum mechanically, commutation relations). Indeed the Newton-Wigner coordinates are chosen such that the position operator commutes with itself. For the other choices of coordinates the algebra is non-canonical and will have to be treated with care. We will choose to work with the covariant constraint (180) as it proves convenient when working in a curved space which we now discuss.

In a general space-time we can define a local orthogonal frame via the vierbein such that

$$g_{\mu\nu} = e_\mu^a e_\nu^b = \eta^{ab}. \tag{185}$$

Then we can relate the co-rotating frame (body fixed) labelled by capital Roman letters to the local orthogonal frame (lower case Roman letters) via the introduction of e_I^μ such that

$$e_a^\mu = \Lambda_a{}^I e_I^\mu. \tag{186}$$

Note that e_I^ν is a function of τ whereas e_a^ν which encodes information about the bulk space-time and is not an explicit function of τ. To reiterate, we have established three frames: The global, sometimes called PN, frame labelled (Greek letters), the local orthogonal frame (small Roman letters), and the co-rotating frame (capital Roman letters). In the flat space limit, the PN and local orthogonal frame can be chosen to be identical, and the co-rotating frame is related to them by a generalized boost Λ.

The generalized coordinates and velocities which describe the particle are $(x^\mu, u^\mu, e_\mu^I, \dot{e}_\mu^I)$.

The flat space angular velocity is now generalized to $(\Omega_{\mu\nu})$ which is defined via

$$u \cdot \nabla e^{\mu}_{I}(\tau) = -\Omega^{\mu\nu} e_{I\nu}. \tag{187}$$

The indices of e^I_μ are raised and lowered by the space-time metric $g_{\mu\nu}$ and the local flat metric η_{IJ}.

Now we would like to write down a first order action for the spin that is both coordinate invariant as well as RPI. Given this latter symmetry, this implies that the Hamiltonian should vanish and therefore

$$S = -\frac{1}{2}\int d\tau S_{\mu\nu}(\tau)\Omega^{\mu\nu}(\tau). \tag{188}$$

where we note that $S_{\mu\nu}$ is RPI by dint of the fact that it is a conjugate momentum which are by definition RPI ($p \sim \delta L/\delta \dot{q}$ and L must scale as $1/\lambda$). This is the generalization of the usual $p\dot{q}$ action. The action written in terms of the local orthogonal frame variables is then given by

$$S = -\frac{1}{2}\int d\tau S^{ab}(\Omega_{ab} + u_\alpha \omega^\alpha_{ab}) \tag{189}$$

where the normalization has been fixed in order to reproduce the correct non-relativisitic flat space limit. ω^α_{ab} is the spin connection defined via

$$\omega^\alpha_{ab} = (\nabla^\alpha e^\mu_a) e_{\mu b}. \tag{190}$$

Notice, that up to this point the choice of constraints has not played a role. We will return to this issue in a moment.

> **Exercise n.13** Derive eq. (189) by expanding out Ω in eq, (188) and writing the spin in terms of the local flat coordinates.

Given that S is a conjugate momentum, and the object of interest, we utilize a Routhian description of the system, whereby the Routhian R acts as a Hamiltonian for the spin degrees of freedom and a Lagrangian for the world line position

$$R = -\sum_i \left(M_i \sqrt{u_i^2} + \frac{1}{2} S_i^{ab} \omega_{ab\mu} u_i^\mu \right). \tag{191}$$

We have not included the term which is independent of the graviton $S_{ab}\Omega^{ab}$ as this term will not contribute to any potentials.

The equations of motion then follow from

$$\frac{\delta}{\delta x^\mu} \int R \, d\lambda = 0, \quad \frac{dS^{ab}}{d\lambda} = \{S^{ab}, R\}. \tag{192}$$

Obviously to get the equations of motion for the spin we will need to know the algebra for the phase space variable $(x^a, p^b, S_{ab}, e^I_\mu)$, where p is the canonical (i.e. not mechanical) momentum. The spin generates rotations obey the $SO(1,3)$ algebra [60] and thus obey

$$\{S_{ab}, S_{cd}\} = \eta_{ac} S_{bd} + \eta_{bd} S_{ac} - \eta_{ad} S_{bc} - \eta_{bc} S_{ad}. \tag{193}$$

These relations obviously include the redundant degrees of freedom which we could choose to eliminate, leading to the reduced phase space algebra [41]. However, for the sake of simplicity it is better not impose the SSC until the end of our calculation [65]. Note that the algebra applies to the spin in the local orthogonal frame.

If we are to choose the covariant SSC then we need to make sure that the constraint condition is conserved in time i.e.

$$u \cdot \nabla (p_\mu S^{\mu\nu}) = 0. \tag{194}$$

We can ensure this by adding the proper couplings to the action as shall now see.

We have at our disposal the equations of motion

$$u \cdot \nabla (S_{\mu\sigma}) = S_{\rho\mu} \Omega^\rho{}_\sigma - S_{\mu\nu} \Omega^\nu{}_\sigma \tag{195}$$

$$u \cdot \nabla (p^\gamma) = -\frac{1}{2} R^\gamma_{\rho\alpha\beta} S^{\alpha\beta} u^\rho. \tag{196}$$

[41] It is in this reduced phase space that the coordinates no longer have a non-vanishing Poisson bracket with themselves.

Exercise n.14 Prove eq (195) by varying the action

$$\delta S(x,\dot{x},e^I_\mu,\dot{e}^I_\mu) = \int d\lambda \frac{\partial L}{\partial \Omega^{\mu\nu}}\delta\Omega^{\mu\nu}, \tag{197}$$

and using

$$\delta\Omega^{\mu\nu} = -\delta(\eta^{IJ} e^\nu_I \frac{De^\mu_J}{d\lambda}). \tag{198}$$

Then prove (196) by varying the action with respect to x and $\dot{x}$

$$\delta S_x = \frac{dL}{dx^\mu}\delta x^\mu = \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial \Omega^{\mu\nu}}\frac{\partial \Omega^{\mu\nu}}{\partial x^\rho}\delta x^\rho = \frac{\partial L}{\partial x^\mu}\delta x^\mu - \frac{1}{2}S_{\mu\nu}\frac{\partial \Omega^{\mu\nu}}{\partial x^\rho}\delta x^\rho \tag{199}$$

$$\delta S_{\dot{x}} = \frac{dL}{d\dot{x}^\mu}\delta \dot{x}^\mu = \frac{\partial L}{\partial \dot{x}^\mu}\delta \dot{x}^\mu + \frac{\partial L}{\partial \Omega^{\mu\nu}}\frac{\partial \Omega^{\mu\nu}}{\partial \dot{x}^\rho}\delta \dot{x}^\rho = -p_\mu\delta \dot{x}^\mu - \frac{1}{2}S_{\mu\nu}\frac{\partial \Omega^{\mu\nu}}{\partial \dot{x}^\rho}\delta \dot{x}^\rho \tag{200}$$

Furthermore, by writing out the most general Lagrangian[42] in terms of $(\Omega, \dot{x})$ one can derive [60] the Mathisson-Papapetrou (MP equations)

$$u \cdot \nabla(S_{\mu\sigma}) = p^\mu u^\sigma - p^\sigma u^\mu. \tag{201}$$

We may use these equations to determine p, given that we wish to impose (194). A simple calculation leads to the conclusion that

$$p^\alpha = \frac{1}{\sqrt{u^2}}\left(mu^\alpha + \frac{1}{2m}R_{\beta\nu\rho\sigma}S^{\alpha\beta}S^{\rho\sigma}u^\nu\right) + ... \tag{202}$$

The Routhian is then given by

$$\mathcal{R} = -\sum_i \left(m_i\sqrt{u_i^2} + \frac{1}{2}S_i^{ab}\omega_{ab\mu}u_i^\mu + \frac{1}{2m_i}R_{deab}(x_i)S_i^{cd}S_i^{ab}\frac{u_i^e u_{ic}}{\sqrt{u_i^2}} + ...\right) \tag{203}$$

where the dots represent curvature terms that account for the higher order terms in the relation between u and p.

In (202) we have dropped terms which will be highly suppressed in the PN expansion. We will justify and quantify this approximation once we have set up the power counting for

[42] One writes down a irreducible set of Poincare invariant quantities and allows the Lagrangian to be arbitrary function of this set. The MP equations then follow by by varying the action.

spin. Also in deriving the result (202) we used the approximate version of the SSC, $v \cdot S = 0$. The extra term needed to preserve the SSC is quadratic in spin and will not play a role in inter-spin potentials proportional to $S_1 \cdot S_2$. However, before going on to calculate these potentials let us first pause to discuss the finite size spin effects.

Unlike the non-spinning case, we expect there to be world line operators linear in the metric which are induced by finite size effects. The spin generates a quadruple moment which then effects the one point function (i.e. the solution to Einstein's equation is sensitive to the spin) so as to generate the Kerr solution. Following our usual logic, the lowest dimensional operator that we can write down that this consistent with the symmetries is

$$L_{ES^2} = \frac{C_1}{2mM_{pl}} \frac{E_{ab}}{\sqrt{u^2}} S^a{}_c S^{cb}. \tag{204}$$

The velocity factor is there to preserve RPI. When calculating potentials proportional to S^2 one needs to include the effects of (204) as well as (202).

A. Power Counting and Feynman Rules for Spin

For a black hole the natural length scale is $r_s \sim m/M_{pl}^2$ so the moment of inertia scales as $I \sim m^3/M_{pl}^4$. Whereas for a neutron star the radius is an independent parameter that will depend upon the equation of state. Thus the spin scales as

$$S = I\omega \sim \frac{m^3}{M_{pl}^4} \frac{\tilde{v}}{r_s}. \tag{205}$$

For generic black holes, the case of interest for phenomenology, the holes are believed to be nearly maximally rotating $\tilde{v} = \omega/r_s \sim 1$, thus

$$S \sim m^2/M_{pl}^2 \sim Lv. \tag{206}$$

Given this scaling along with the results tabulated in table 1, we are prepared to derive the scaling of vertices in the action. First we must expand out the terms in (191) in terms of the metric fluctuations.

We may re-write the spin dependent part of the Routhian (191) as

$$R = \frac{1}{2} S_{LM} e^L_\mu e^M_\nu \eta^{IJ} e^\nu_I (u^\alpha \nabla_\alpha e^\mu_J = u^\alpha (\partial_\alpha e^\mu_J + \Gamma^\mu_{\alpha\beta} e^\beta_J)) \tag{207}$$

Imposing the relation

$$\eta_{IJ} e^I_\mu e^J_\nu = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_p}. \tag{208}$$

We find the relations

$$e^J_\mu = \Lambda^J_\mu + \frac{h^\nu_\mu \Lambda^J_\nu}{2M_{pl}} - \frac{h^\rho_\mu h_{\rho\sigma} \Lambda^{\sigma I}}{8M^2_{pl}}, \tag{209}$$

$$e^{J\nu} = \Lambda^{J\nu} - \frac{h^\nu_\rho \Lambda^{J\rho}}{2M_{pl}} + \frac{h^\rho_\sigma h^\nu_\rho \Lambda^{J\sigma}}{4M^2_{pl}}. \tag{210}$$

$$e^\nu_J = \Lambda^\nu_J - \frac{h^\nu_\rho \Lambda^\rho_J}{2M_{pl}} + \frac{3h^\rho_\sigma h^\nu_\rho \Lambda^\sigma_J}{8M^2_{pl}}. \tag{211}$$

Here we have included terms quadratic in the metric though we will only utilize the linear terms, since we will only explicitly calculate the leading order spin dependent potential.

Working at linear order we may use the relation

$$S_{LM} \Lambda^M_\mu = S_{L\mu}, \tag{212}$$

since the local orthogonal frame and the global frame are the same to leading order.

$$R = \frac{1}{2} S_\mu{}^J (-u^\alpha \delta^\rho_J \partial_\alpha \frac{h^\mu_\rho}{2M_{pl}} + u^\alpha \Gamma^\mu_{\alpha\beta} \delta^\beta_J)). \tag{213}$$

The first term vanishes due to anti-symmetry of the spin tensor. Now using the result

$$\Gamma^{(1)\mu}_{\alpha\beta} = \frac{1}{2M_{pl}} (h^\mu_{\alpha,\beta} + h^\mu_{\beta,\alpha} - h_{\alpha\beta}{}^{;\mu}). \tag{214}$$

We have

$$R = -\frac{1}{2M_{pl}} S_{\mu\beta} u^\alpha h_\alpha{}^{\beta,\mu}. \tag{215}$$

Note this is all in the global (PN) frame, and the constraint as well as the spin algebra are written in the local frame so one must be careful when working at higher orders.

We are now in position to calculate the leading order potential. However, before doing

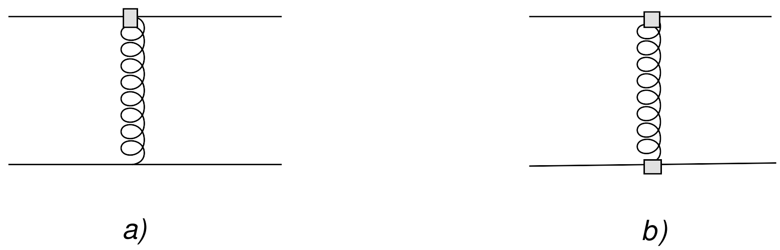

FIG. 12. Diagrams generating the leading order spin dependent potentials. The square vertex represents a spin insertion. Diagram a) is the spin-orbit interaction which is order 1.5 PN while diagram b) is the spin1-spin2 potential which is order 2 PN.

so let us power count this interaction. Expanding we find

$$R = -\frac{1}{2M_{pl}}(S_{0\beta}h_0{}^{\beta,0} + S_{i\beta}h_0{}^{\beta,i} + S_{0\beta}v_i h^{i\beta,0} + S_{j\beta}v_i h^{i\beta,j}). \tag{216}$$

Now given that we are interested in potentials, time derivatives scale as v^2 while spatial derivatives are linear in v. Furthermore, given our covariant constraints we have $S_{0i} \sim vS_{ij}$. Thus the leading order terms are

$$R = -\frac{1}{2M_{pl}}(S_{ij}H_0{}^{j,i} + S_{i0}H_0{}^{0,i} + S_{jk}v_i h^{ik,j}). \tag{217}$$

Where, following our notation, are now considering the potential graviton (H) since we're interested in calculating potentials. Using Table (1) we can read off the scaling of this interaction as $v^2\sqrt{L}$.

We are now prepared to calculate the spin dependent potentials. The leading order potential will come from interactions involving one insertion of the spin interaction connecting to a mass insertion on the other world line. Naiviely, we would think this potential is order v^2 (1PN), however, given that the interaction (217) is proportional to H_{0i}, the mass insertion on the other line must couple to the interaction in (28), because the graviton propagator (27) does not connect h_{00} to h_{0i}. The Feynman diagram responsible for this potential is shown in figure (13a). Performing the contraction between the vertices leads to the result

$$V = \frac{GM_2}{r^2}(S_{ij}^{(1)}n_i v_{1j} - 2S_{ij}^{(1)}n_i v_{2j} + S_{i0}^{(1)}n_i) + (1 \leftrightarrow 2). \tag{218}$$

The equations of motion are then given by [43]

$$\frac{dS_{ij}}{d\tau} = \{S_{ij}, -V\}, \tag{219}$$

from which is follows that

$$\frac{dS_{ab}}{d\tau} \;=\; -\frac{GM_2}{r^2}\left[(n_i v_{1j} - 2n_i v_{2j})\{S_{ab}, S_{ij}\}) + n_i\{S_{ab}, S_{i0}\}\right] \tag{220}$$

and

$$\frac{d\vec{s}_1}{dt} = 2\left(1 + \frac{M_2}{M_1}\right)\frac{\mu G_N}{r^2}(\vec{n}\times\vec{v})\times\vec{s}_1 + \frac{M_2 G_N}{r^2}(\vec{s}_1\times\vec{n})\times\vec{v}_1 \tag{221}$$

where we have written the result in terms of the spin vector defined via

$$S_{ij} = \epsilon_{ijk} s_k, \tag{222}$$

and n is the normal vector in the direction of $\vec{r}$. This is not the complete result for this diagram. We must recall that the SSC is defined in the local frame, not in the PN frame in which the observer lives. We can relate the velocity in the local from (L) to the PN frame via the tetrad

$$\begin{aligned} v_L^j \;&\approx\; e^j_\mu v^\mu = v^\mu(\delta^j_\mu + \frac{1}{2M_{pl}} h_{\rho\mu}\eta^{j\rho}) = v^j + \frac{v^\mu}{2M_{pl}} h^j_\mu \\ v_{Lj} \;&=\; v_j + \frac{1}{2M_{pl}} h_{j0} \end{aligned} \tag{223}$$

The field value (one point function) generated by the second spin is easily calculated and is given by

$$\begin{aligned} \langle h_{j0}\rangle \;&=\; \frac{iS^{(2)}_{ab}}{2M_{pl}}\int d\lambda \langle h_{0j}(x) h_{0b,a}(y)\rangle. \\ &=\; \frac{iS^{(2)}_{ab}}{2M_{pl}}\frac{-in_a}{8\pi r^2}\delta_{bj} \end{aligned} \tag{224}$$

[43] The apparent extra minus sign is due to our choice of spin algebra (193) conventions.

Now let us reconsider the piece (220) proportional to S_{0i} in (220)

$$\begin{aligned}\frac{dS_{ab}}{dt} &= -\frac{GM_2}{r^2}n_i\{S_{ab}, S_{i0}\}\\ &= -\frac{GM_2}{r^2}n^i(-\delta_{ai}S_{b0} + \delta_{bi}S_{a0})\\ &= \frac{G^2M_2}{r^4}(\vec{n}\times((\vec{n}\times\vec{s}_2)\times\vec{s}_1)\end{aligned} \tag{225}$$

Note that this piece is bi-linear in spin, so one might not wish to consider this part of the spin orbit coupling, but this is just a matter of nomenclature. There are of course, other contribution to the spin1-spin2 potential coming from direct contribution. The leading order contribution is shown to in figure (13b), the resulting 2PN potential is given by

$$V^{ss}_{2PN} = -\frac{G_N}{r^3}(\vec{s}_1\cdot\vec{s}_2 - 3\frac{\vec{r}\cdot\vec{s}_1\vec{r}\cdot\vec{s}_2}{r^2}), \tag{226}$$

In addition at 2PN we have the contribution stemming from s_i^2 terms, which corresponds to a quadarpole-monopole potential. This contribution is generated by the contraction of a mass insertion on one side and a spin squared (finite size) insertion from (204) on the other. The spin squared term in the Routhian that is needed to preserve the SSC is higher order and not relevant at 2PN. The resulting potential is given by

$$V^{so}_{2PN} = -\frac{C_1G_NM_2}{2M_1r^3}\left(\vec{s}_1\cdot\vec{s_1} - 3(\vec{n}\cdot\vec{s}_1)^2\right) + \ 1\leftrightarrow 2, \tag{227}$$

where C_1 is the matching coefficient for the finite size operator (204). This coefficient is easily matched by calculating the field value (one point function) due to the finite size operator and matching it onto the Kerr solution, with the result being $C_1 = 1$ [59].

Exercise n.15 Calculate the leading order contribution to the spin-orbit potential which corresponds to one graviton exchange with a spin insertion on one line and a mass insertion on the other.

VIII. CALCULATING THE WAVE FORM

In addition to the power loss, which leads to the phase of the wave, the other relevant observable is the wave form itself, the calculation of which follows simply via the techniques we have developed so far. The important distinction being that we are now interested in real time expectation values of the field, $\langle h(x,t)\rangle$, and not in-out scattering matrix elements. Thus the path integral technology that we have developed must be modified.

In quantum field theory the technology for calculating time sensitive quantities is called the in-in formalism [54, 55]. Going into this formalism would take us too far afield. Furthermore, the in-in formalism is a large hammer for our relatively small nail, since we are not interested in quantum effects. Indeed, for non-radiative moments calculating the field value at some a space-time point (x,t) simply follows by convolving the source multipole (Q) with a retarded Greens function (D_R) and has the generic form

$$h(x,t) \sim \int d^4y \partial..\partial D_R(x-y)Q(y). \tag{228}$$

The derivatives arise due to the form of the couplings of the E and B fields to the moments in (115). The Greens function (in our harmonic gauge) is given by (172

We project onto the the physical transverse traceless piece

$$h_{ij}^{TT} = \frac{1}{M_{pl}}\Lambda_{ij,kl}h_{kl}. \tag{229}$$

The M_{pl} is put there to make h dimensionless and is standard normalization. The projector is given by

$$\Lambda_{ij,kl} = (\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l) - \frac{1}{2}(\delta_{ij} - n_i n_j)(\delta_{kl} - n_k n_l), \tag{230}$$

where n is the unit vector in the direction of $\vec{r}$, which connects the source to the observer.

The direct contribution of the multipoles to the wave form then follows from the coupling of the multipoles to E and B in (115)

$$h_{ij}^{TT}(t,\mathbf{x}) = -\frac{4G}{|\mathbf{x}|}\Lambda_{ij,kl}[\frac{1}{2}\frac{d^2Q^{kl}}{dt^2} - \epsilon^{ab(k}\frac{2}{3}\frac{d^2J^{l)b}}{dt^2}n_a + ...] \tag{231}$$

where we have kept only the leading order mass and current (electric and magnetic) contributions for the sake of illustration and the parenthesis around the indices indicate symmetrization. The procedure for calculating the multipole moments was discussed in section (VI A).

The radiative moments are formally more interesting and illustrate utility of the EFT approach. Once we have integrated out the potentials, we are working with a single particle endowed with dynamical multipole moments. The only scales around are the wavelength of the radiation and the distance to the detector, which is, for all intents and purposes, infinite.

Let us consider the tail effect previously discussed in the context of the power radiated. The leading order contribution to the wave form coming from scattering off of the effective Schwarzschild background of the one body is shown in figure (11). The in-in formalism implies that the only change needed, relative to our usual Feynman diagram calculation, is that we should use causal propagators. This change could have been guessed on physical grounds. For gravitons which couple to the time independent mass this is moot since they are instantaneous anyway, but for those which couple to time dependent moments, using the causal (retarded) propagator implies that we must change the $i\epsilon$ prescription as follows

$$\frac{1}{k_0^2 - \vec{k}^2 + i\epsilon} \to \frac{1}{(k_0 + i\epsilon)^2 - \vec{k}^2}. \tag{232}$$

In position space this prescription simply ensures that emission predates scattering.

The insuing integral for this diagram, which will be left as an exercise for the reader[44], is infrared divergent and is regulated by utilizing dimensional regularization,

$$iA_{Q^{ij}M}(\omega, \omega\mathbf{n}) = iA_{Q^{ij}} \times (iGM\omega)\left[-\frac{(\omega+i\epsilon)^2}{\pi\mu^2}e^{\gamma_E}\right]^{(d-4)/2} \times \left[\frac{2}{d-4} - \frac{11}{6}\right] \tag{233}$$

where $A_{I^{ij}} = \frac{i}{4m_p}\omega^2\epsilon^*_{ij}Q^{ij}(\omega)$ and ϵ_{ij} is the polarization tensor. After expanding d around $4-2\epsilon$ we can read off the contribution from the tail to the radiative quadrupole moment (we absorb the factor of π into μ from now on),

$$Q^{ij}_{\rm rad}(\omega) = Q^{ij}_0(\omega)\left\{1 + GM\omega\left(\mathrm{sign}(\omega)\pi + i\left[\frac{2}{\epsilon_{\rm IR}} + \log\frac{\omega^2}{\mu^2} + \gamma_E - \frac{11}{6}\right]\right)\right\}. \tag{234}$$

Furthermore, and crucially, one can show by summing the diagrams with further mass inser-

[44] The calculation of this diagram is greatly simplified by the fact that we may take the external graviton to be on-shell, i.e. $k^2 = 0$. This is permissible because we are interested in the field at infinity, and off-shellness leads to a vanishing field at asymptotia.

tions [53] that the IR divergences exponentiate into a pure phase in the emission amplitude, i.e. $\mathcal{A} = \mathcal{A}_{\mathrm{IR-finite}}\ e^{2iGM\omega/\epsilon_{\mathrm{IR}}}$.

The divergence arises because we are scattering off of a $1/r$ potential. This may seem puzzling at first since we are interested in measuring this wave form and any measurable quantity must obviously be finite. But if we were to calculate in position space, then we would see that the phase is sensitive to arbitrary long times, since we are not solving the initial value problem. That is, we can't expect to be able to predict the value of the phase since we have assumed that the source is eternal. However, we can predict differences in the phase such that once we measure the phase at some time, then we can predict it at a later time.

From the above expressions the contribution from the $Q^{ij}M$ coupling to the GW amplitude becomes

$$\left(h^{TT}_{ij}\right)_{I^{ij}+I^{ij}M}(\tilde{t},\mathbf{x}) = -\frac{2G}{|\mathbf{x}|}\Lambda_{ij,kl}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\left[-\omega^2 e^{i\omega\tilde{t}_{\mathrm{ret}}(\mu)}e^{i\theta_{\mathrm{tail}}(\omega,\mu)}\left(1+GM|\omega|\pi\right)\right]Q_0^{ij}(\omega), \tag{235}$$

with

$$\theta_{\mathrm{tail}}(\omega,\mu) \equiv GM\omega\left(\log\frac{\omega^2}{\mu^2}+\gamma_E-\frac{11}{6}\right), \tag{236}$$

and we have absorbed the $1/\epsilon_{\mathrm{IR}}$ into a time redefinition, namely

$$t_{\mathrm{bare}} \to \tilde{t} - 2GM/\epsilon_{\mathrm{IR}}. \tag{237}$$

(235) is precisely (up to an unphysical rescaling of μ) the result previously obtained in [66] using traditional methods.

$\tilde{t}$ represents the time variable used by the experimentalists. This is in accord with our previous discussion, in the sense that one must choose a time origin for the signal, which is of course arbitrary. Notice that even after this shift there will still be a dependence upon the arbitrary reference scale μ_0 whose variation, $\mu_0 \to \lambda\mu_0$ can be cancelled out by a simultaneous transformation $\tilde{t}(\lambda\mu_0) \to \tilde{t}(\mu_0) + 2GM\log\lambda$. Hence the GW amplitude is independent of μ.

Phases arising from other multipole-mass interactions will also appear and, while the logarithm (and Euler constant γ_E) is universal, the rational numbers which appear will not be and thus the difference between these numbers will be physical, see for instance [67].

Exercise n.15 Show that after multipole expanding the world line action, one generates a coupling of the angular momentum of the binary system to the spin connection ω^{μ}_{ab}. First show that at linear order $\omega^{0}_{ij} = \frac{1}{2}\left(\partial_j \bar{h}_{0i} - (i \leftrightarrow j)\right)$. To do this start with the relation

$$\omega^{\alpha}_{ab} = (\nabla^{\alpha} e^{\mu}_{a}) e_{\mu b} \tag{238}$$

and use the relations (85) to write e^{μ}_{a} in term of $h_{\mu\nu}$.

Prove the result (189). To do so utilize (187) to solve for $\Omega_{\mu\nu}$. Show that it gives the correct non-relativistic limit where the action is $\frac{1}{2} I \omega^2$, where I is the moment of inertia $\vec{S} = I\vec{\omega}$.

Exercise n.16: This problem has to do with the *gravitational* Van Der Waals interaction between strings. First let us show that the classical force between such objects vanishes, when we take the tension (mass per unit length) of the objects to be small compared to the Planck scale M_{pl}. This approximation allows us to expand the metric around flat space, such that $g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{pl}}$. The action for the strings are given by $S = -\sum_i \tau_i \int d^4x \sqrt{g^i}\delta^{(2)}(x - x_i)$, where τ_i and g_i are the tension and induced metric of the i'th string. For simplicity we will take the string to lie along one coordinate axis and we will choose the local space-time coordinates on the string to coincide with the global coordinate system. In this way the induced metric is just the value of the bulk metric on the string restricted to the $0, 1$ coordinates. We expand the action using

$$\sqrt{g^i} \approx 1 + \frac{h_a^b}{2M_{pl}} + \frac{h_a^a h_b^b}{8M_{pl}^2} - \frac{h_a^b h_a^b}{4M_{pl}^2} + ... \tag{239}$$

where the indices are now summed only over the two dimensional sub-space. The classical force will then arise from the one graviton exchange diagram. Using the background field method discussed in this chapter, where the graviton propagator takes the form

$$D_{\mu\nu,\rho\sigma}(q) = -i\frac{\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\nu\sigma}\eta_{\nu\rho} - \frac{2}{d-1}\eta_{\mu\nu}\eta_{\rho\sigma}}{\vec{q}^2} \tag{240}$$

show that the classical force vanishes. To prove that the classical force vanishes, to all orders in the tension, one must solve the full einstein equations to account for the curving of the space. However, as was shown in [**?**], co-dimension two objects leave space uncurved. The net effect of the tension is to remove a deficit angle in the space-time. That is, the space is conical. Thus there is no classical force to all orders in the tension.

To find the first correction due to the tension on the force, we consider the analog of figure (**??**). Using the action (**??**), show that this diagram generates the potential per unit length

$$V = -\frac{\tau_1\tau_2}{64\pi^3 R^2 M_{pl}^4} \tag{241}$$

where R is the distance between the strings. Note that in gravity we have additional contributions beyond those in diagram (**??**) arising from graviton self interactions. For the discussion of the complete force including these effects see [**?**].

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