Cosmic Inflation [ICTP summer school 2026]

classical/ background : rapid expansion of space time → homogeneity on large scales

(quantum) fluctuations : quantum fluctuation become stretched → classical → seeds for inhomogeneities (CMB, Galaxies)

↓ lectures by Marko & Matthew

Plan

1) Motivation: Big Bang Puzzles
2) Classical background dynamics: slow roll inflation
3) Quantum fluctuations I: density perturbation
4) Quantum fluctuations II: gravitational waves
5) Tests of cosmic inflation

Literature

- Daniel Baumann, 0907.5424, TASI lectures, "Inflation" (+ "Advanced Cosmology")
- Scott Dodelson "Modern Cosmology"
- Julien Lesgourgues, 1302.4640, "Cosmological perturbation"
- Michele Maggiore, gr-qc/9909001, "Gravitation Waves", Vol I

1) FRW Cosmology

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$ Einstein's equations

metric ⇄ matter

homogeneity & isotropy

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

$$T^{\mu}_{\ \nu} = \mathrm{diag}(\rho, -p, -p, -p)$$

perfect fluid

Friedmann equations: $[M_p = 1]$

$$(F1) \quad \left(\frac{\dot a}{a}\right)^2 = H^2 = \frac{1}{3}\rho - \frac{k}{a^2} \qquad \dot{} = \frac{d}{dt}$$

$$(F2) \quad \left(\frac{\ddot a}{a}\right) = \dot H + H^2 = -\frac{1}{6}(\rho + 3p)$$

$$\hookrightarrow \dot\rho = -3H(\rho + p)$$ continuity equation

$$\omega \equiv p/\rho$$ equation of state parameter

$$\Gamma \ (\dot{F1}) \to \dot\rho = \frac{d}{dt}\left(3H^2 + 3k/a^2\right)$$

$$= 6H\dot H - 2\frac{3k}{a^2}H \qquad \dot H = -\frac{1}{6}(\rho + 3p) - \underbrace{H^2}_{\frac{1}{3}\rho - \frac{k}{a^2}}$$

$$= H\left[-\rho - 3p - 2\rho + \cancel{\frac{6k}{a^2}} - \cancel{\frac{6k}{a^2}}\right]$$

$$= -3H(\rho + p) \qquad \blacksquare \ \lrcorner$$

$$\rightarrow \frac{d\rho}{da} = \frac{d\rho}{dt}\frac{1}{\dot a} = -3\underbrace{\frac{H}{\dot a}}_{1/a}(\rho+p) = -\frac{3\rho}{a}(1+w)$$

if covered by MoM

$$\rightarrow \frac{d\ln\rho}{d\ln a} = -3\,(1+w)$$

$$\rightarrow \rho \propto a^{-3(1+w)}$$

- $w=0 \quad (p=0) \rightarrow \rho = a^{-3}$ matter
- $w = 1/3 \rightarrow \rho = a^{-4}$ radiation
- $w = -1 \rightarrow \rho = \text{const.}$ cc / inflation

"hot big bang": $k=0,\ w = 0, 1/3$

$\hookrightarrow a(t) = t^{\frac{2}{3(1+w)}}$

$\rightarrow$ singularity "big bang" at $a(t) = 0$

2) Horizon problem

Why is the CMB so isotropic?

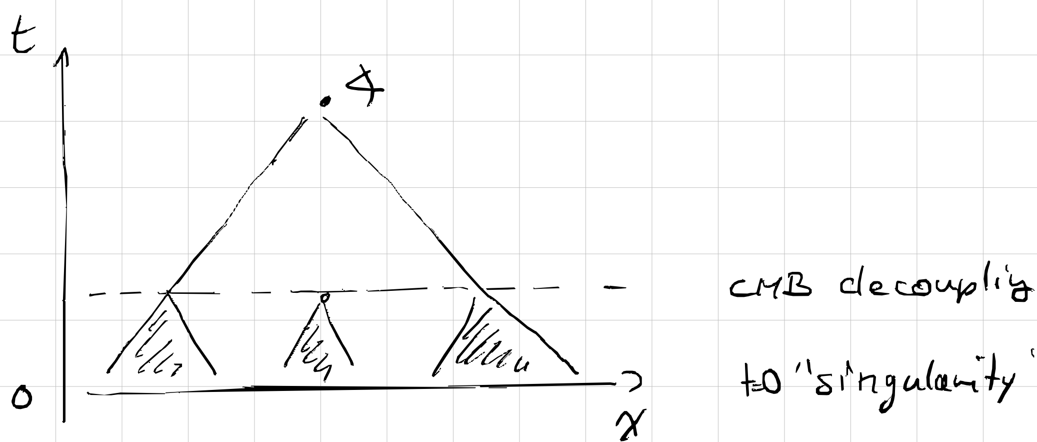
past light-cone, region of causal contact

$$\chi = \int_0^t \frac{dt'}{a(t')} \underset{H=\frac{\dot a}{a}}{=} \int_0^a \frac{da}{Ha^2} = \int_0^a d\ln a \underbrace{\underbrace{\frac{1}{aH}}_{1/\dot a}}_{\text{comoving Hubble horizon}}$$

(F2): $\ddot a = -\frac{1}{6} a \rho (1+3w) < 0$ for $w > -1/3$

$\rightarrow t \nearrow \quad \dot a \searrow \quad \frac{1}{\dot a} = \frac{1}{aH} \nearrow$ for $w > -1/3$

$\rightarrow$ fraction of universe in causal contact increases with time



CMB as observed today $\simeq 10^5$ regions which were never in causal contact

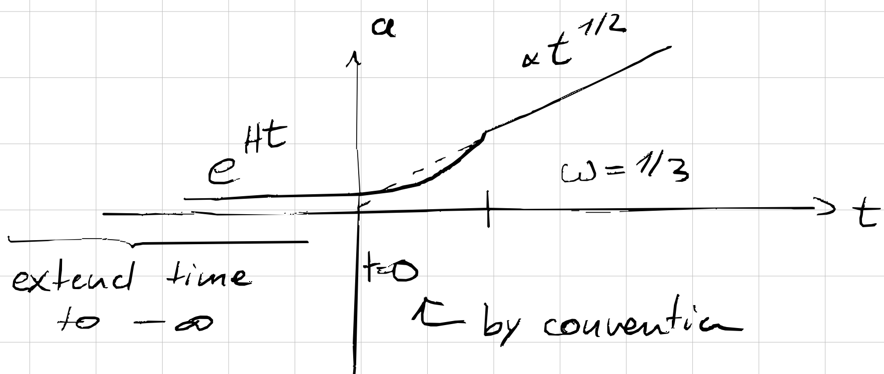
$\rightarrow$ why do they all have the same T?

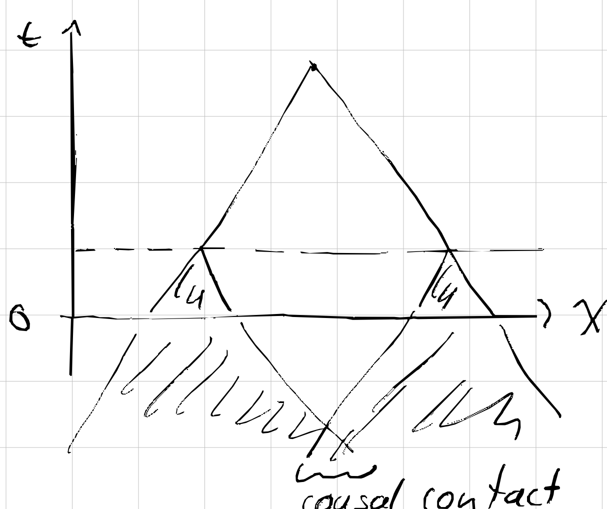
resolution

need $\omega < -1/3$: $1/aH$ decreases w. time, later observer sees "less" than early observer

e.g. $\omega = -1 \rightarrow \rho \propto a^{-3(1+\omega)} = \text{const} \overset{k=0}{=} 3H^2$

$$\rightarrow \frac{\dot{a}}{a} = H = \text{const} \rightarrow a \propto e^{Ht}$$





3) Flatness problem

$k = \{-1, 0, 1\}$

$$(F1) \rightarrow 1 - \underbrace{\frac{\rho(a)}{3H^2(a)}}_{\equiv \Omega(a)} = -k \underbrace{\frac{1}{(aH)^2}}_{\substack{\text{grows w. time} \\ \text{for } \omega > -1/3}}$$

Today, $\Omega(a_0) \simeq 1$ (observed)

↳ unstable point for $k \neq 0$

↳ $\Omega(a_{BBN}) - 1 \leq \mathcal{O}(10^{-16})$

→ why is the universe so flat?
$k = 0$ or fine-tuning?

Also solved by decreasing comoving horizon in the past, i.e. $\omega < -1/3$.

- negative pressure ?
- how do we achieve this ?

Bonus: also explains characteristic structure of CMB fluctuations.

4) <u>slow-roll inflation</u>

- Consider real scalar field ϕ

$$S = \int d^4x \sqrt{-g}\left(\tfrac{1}{2}R - \tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right)$$

$$= S_{EH} + S_\phi$$

$\delta\sqrt{-g}/\delta g^{\mu\nu}$

$$\rightarrow T_{\mu\nu}^{(\phi)} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\tfrac{1}{2}\partial^\sigma\phi\,\partial_\sigma\phi - V(\phi)\right)$$

$(-,+,+,+)$ $\quad -\partial_0\phi\,\partial_0\phi + \partial_i\phi\,\partial_i\phi$

Assume $\phi(\vec{x},t) = \phi(t)$ homogeneous, $\partial_i\phi = 0$

$\hookrightarrow T_{\mu\nu}^{(\phi)} = \mathrm{diag}(\rho, -p, -p, -p)$ with

$$\rho_\phi = \tfrac{1}{2}\dot{\phi}^2 + V(\phi), \quad p_\phi = \tfrac{1}{2}\dot{\phi} - V(\phi)$$

$$\rightarrow w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{1/2\,\dot{\phi}^2 - V(\phi)}{1/2\,\dot{\phi}^2 + V(\phi)} \xrightarrow[\dot{\phi}^2/2 \ll V]{} -1 < -1/3 \quad \boxed{}$$

"slow roll"