Lecture II: Slow-roll inflation

Recap: Horizon & Flatness problem

$\rightarrow$ solved $\omega = P/\rho < -1/3$ in the past $\left(\begin{array}{l}\frac{1}{aH} \text{ decreases,} \\ \ddot{a} < 0\end{array}\right)$

Slow-roll inflation:

$$S = \int d^4x \sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right), \quad \phi(x,t) = \phi(t)$$

$$\hookrightarrow \quad \omega_\phi = \frac{P_\phi}{\rho_\phi} = \frac{1/2\,\dot{\phi}^2 - V(\phi)}{1/2\,\dot{\phi}^2 + V(\phi)} \xrightarrow[\dot{\phi}^2/2 \ll V]{} -1 < -1/3$$

"slow roll"

1) equation of motion:

$$\frac{\partial S}{\partial \phi} - \partial_\mu \frac{\partial S}{\partial \phi_{,\mu}} = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, \partial^\mu \phi\right) + V_{,\phi} = 0$$

$\downarrow$ $\phi(t,\vec{x}) = \phi(t)$, FRW metric

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \quad \& \quad H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)}$$

cont. equation $\rightarrow$

$\rho = a^{-3(1+\omega)} \rightarrow$ const for $\omega = -1$

(F1): $H = \dot{a}/a = \sqrt{\rho/3} =$ const. $\rightarrow a(t) \propto e^{Ht}$

recall

$\rightarrow$ exponentially expanding universe, dominated by potential energy of scalar field.

2) Slow roll approximation

$\varepsilon \equiv -\dot{H}/H^2 \overset{F1+F2}{=} \frac{1}{2}\frac{\dot{\phi}}{H^2} < 1$ $\quad\leftrightarrow$ accelerated expansion $\equiv$ inflation ($\hat{=}\ \omega < -\frac{1}{3}$)

$\eta \equiv -\ddot{\phi}/(H\dot{\phi}) < 1$ $\quad\leftrightarrow$ ε changes slowly $\rightarrow$ sustained inflation

$\rightarrow$ slow-roll inflation: $\varepsilon, |\eta| < 1$.

"Potential slow-roll parameters": $\varepsilon_V = \frac{M_p^2}{2}\left(\frac{V_{\phi}}{V}\right)^2$, $\eta_V = M_p^2 \frac{V_{,\phi\phi}}{V}$

$\varepsilon_V, |\eta_V| \ll 1 \rightarrow \varepsilon \simeq \varepsilon_V$, $\eta \simeq \eta_V - \varepsilon_V$

$$\Rightarrow \boxed{3H\dot{\phi} + V_{,\phi} = 0, \quad H^2 \simeq \frac{1}{3}V \simeq \text{const}, \quad a(t) \sim e^{Ht}}$$

slow-roll eom

More convenient time coordinates (X)

- conformal time : $d\tau = dt/a \rightarrow \tau = -\frac{1}{aH}$ ↖ <0 during inflation

  $(\rightarrow ds^2 = a^2(-d\tau^2 + d\vec{r}^{\,2}))$

- e-folds : $dN = -H\,dt$ (elapsed Hubble times)

  $= -\frac{da}{a}\frac{dt}{dt} = -d\ln a$ linear measure of exp. expansion

$$N(\phi) = \ln(a_{end}/a) = \int_t^{t_{end}} H\,dt = \int_\phi^{\phi_{end}} \frac{H}{\dot\phi}\,d\phi = \int_{\phi_{end}}^{\phi} \frac{1}{\sqrt{2\varepsilon}}\,d\phi$$

$\varepsilon = 1$

Solving horizon & flatness problem requires $N \gtrsim 60$

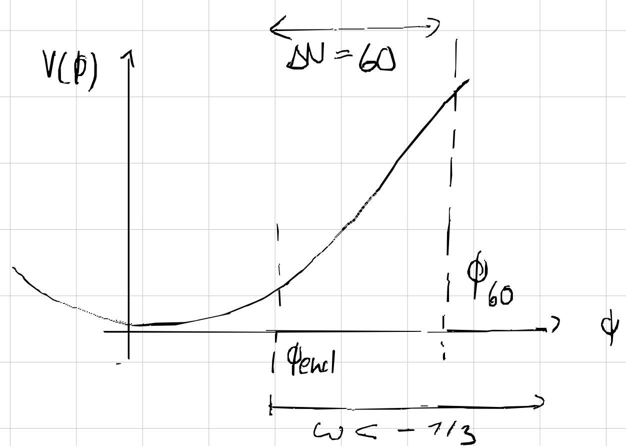
3) An example: $V(\phi) = \frac{1}{2}m^2\phi^2$

$$\varepsilon_V = \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 = 2\left(\frac{M_P}{\phi}\right)^2 \overset{!}{<} 1 \rightarrow \phi_{end}(\varepsilon = 1) = \sqrt{2}\,M_P$$

(inflation for $\phi > \phi_{end}$)

$$N(\phi) = \frac{1}{M_P}\int_{\phi_{end}}^{\phi} \frac{1}{\sqrt{2\varepsilon}}\,d\phi = \frac{\phi^2}{4M_P^2} - \frac{1}{2} \overset{!}{=} 60 \rightarrow \phi_{60} = 15\,M_P$$

(start of observable inflation)



→ working inflation model

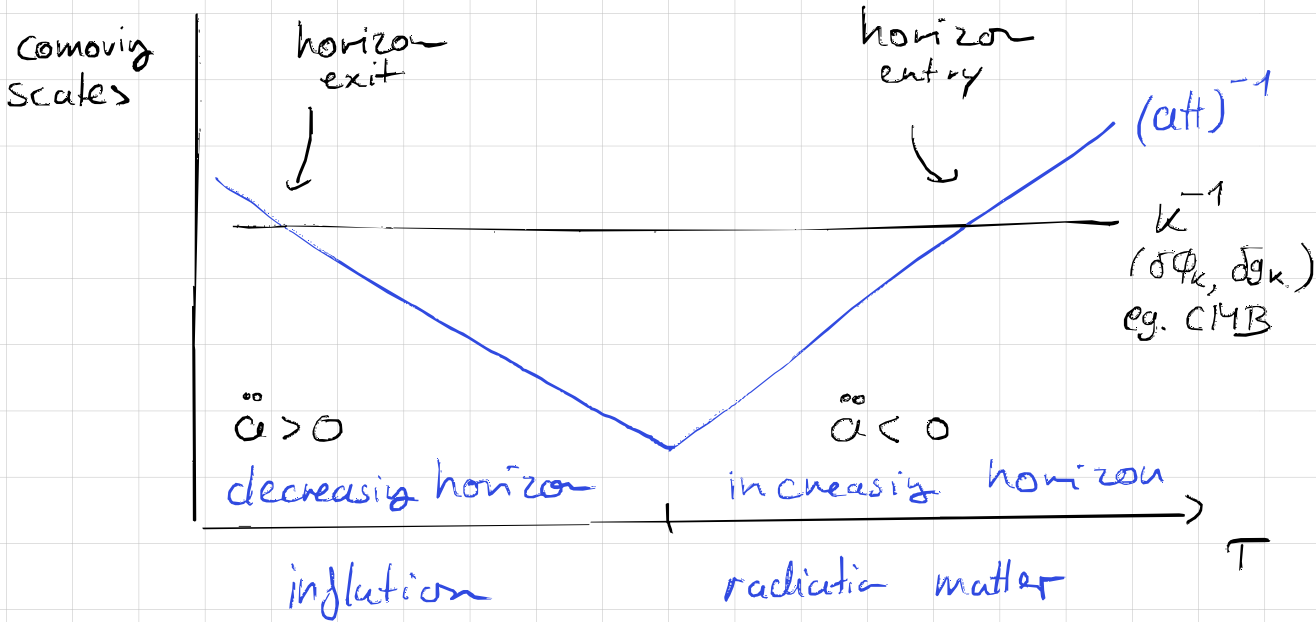
4) Scales and horizons

$$\phi(x,t) = \bar{\phi}(t) + \delta\phi(x,t)$$

$$g_{\mu\nu}(x,t) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x,t)$$

homogeneous background, evolves under slow-roll eom

small perturbations, to first order, evolve linearly → Fourier transform $\phi_{\vec{k}}$, $h_{\vec{k}}$



this lecture

↳ see Marko & Mathew's lectures

SVT decomposition and gauge invariance (GR)

$\delta\phi(x,t)$ → scalar dof

$\delta g_{\mu\nu}(x,t)$ → 4 scalars, 2 vectors, 1 tensor

↓ 2 can be gauged away under coordinate transformation $x^\mu \to x^\mu + \varepsilon^\mu$

↓ decay

↓ lecture 4

define gauge invariant scalar

e.g. $\mathcal{R} = \psi + \frac{H}{\dot\phi}\delta\phi$

$$g_{ij} = a^2[(1+2\psi)\delta_{ij} + \dots]$$

"comoving curvature perturbation"

= spatial curvature on constant-ϕ hypersurface

choose a gauge:

e.g. Newtonian gauge (Mathews lecture)

or: $\delta\phi = 0$

$$g_{ij} = a^2[(1-2\mathcal{R})\delta_{ij} + h_{ij}]$$

"comoving gauge"

↓

one dynamical scalar $\mathcal{R}(\vec{x},t)$, h_{ij} = pure tensor

↓

lecture 3

Notes :

slow roll parameters.

$$\omega = \frac{x-1}{x+1} \simeq -1+2x = -1+\frac{2}{3}\varepsilon$$

$$x = \frac{1/2\,\dot{\phi}^2}{V_{\leftarrow 3H^2}} = \frac{1}{3}\varepsilon \qquad \rightarrow \omega < -1/3 \;\hat{=}\; \varepsilon < 1$$