Lecture 3 - scalar perturbations

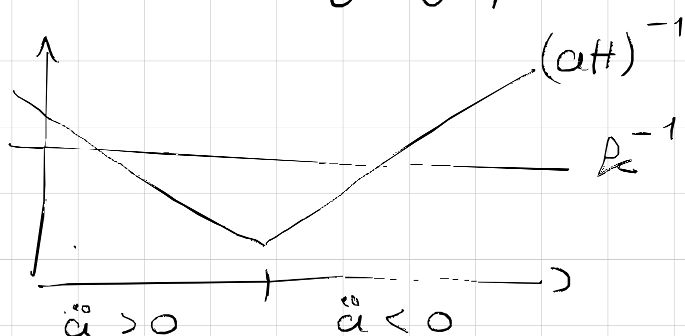
Recap:

- $\phi(x,t) = \phi(t) + \delta\phi, \quad g_{\mu\nu}(x,t) = g_{\mu\nu}(t) + \delta g_{\mu\nu}(t)$

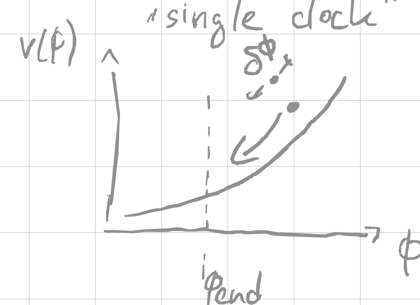
  $\delta\phi, \delta g \longrightarrow \mathcal{R}, h_{ij}$

- horizon crossing of perturbations

$dN = \frac{1}{\sqrt{2\epsilon}} d\phi$



"single clock"



- Strategy

  a) $(S_\phi + S_{EH})^{(2)} \rightarrow$ eom for $\mathcal{R}$ $[\mathcal{O}(\mathcal{R})]$

  b) initial conditions: quantum fluctuatio

  c) solve a+b $\rightarrow$ i.c. for CMB, LSS

a) $(S_\phi + S_{EH})^{(2)} = \ldots$

$$= \frac{1}{2}\int d\tau\, d^3x \left[v'^2 + (\partial_i v)^2 + \underbrace{\frac{z''}{z} v^2}_{\text{"mass" term}} \right]$$

$z = \frac{a\dot{\phi}}{H}$

$v = z\mathcal{R}$

Mukhanov variable

1) Equation of motion

D. Baumann, App B

$$(S_\phi + S_{EH})^{(2)} = \ldots = \frac{1}{2}\int d^4x\, a^3 \frac{\dot\phi^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i \mathcal{R})^2\right]$$

$z = \frac{a\dot\phi}{H}$ $\longrightarrow$

$$= \frac{1}{2}\int d\tau\, d^3x \left(\underbrace{a^2 z^2 \dot{\mathcal{R}}^2}_{(v' - z'\mathcal{R})^2} - \underbrace{z^2(\partial_i \mathcal{R})^2}_{(\partial_i v)^2}\right)$$

$\dot{} = \frac{d}{dt}$

dt/a

$v = z\mathcal{R}$

Mukhanov Variable

$' = \frac{d}{d\tau}$

$\mathcal{R} = v/z$

$$= \frac{1}{2}\int d\tau\, d^3x \left[v'^2 + (\partial_i v)^2 \underbrace{- 2v'v\frac{z'}{z} - \left(v\frac{z'}{z}\right)^2}_{\underbrace{-\frac{d}{d\tau}\left[\frac{z'}{z}v^2\right]}_{\text{total derivative} \to 0 \text{ under } d\tau} + v^2\frac{z''}{z}}\right]$$

$$= \frac{1}{2}\int d\tau\, d^3x \left[v'^2 + (\partial_i v)^2 + \underbrace{\frac{z''}{z}v^2}_{\text{"mass" term}}\right]$$

Equation of motion (Euler Lagrange),
Fourier expansion (linear perturbation theory)

$$v(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, v_{\vec{k}}(\tau)\, e^{i\vec{k}\vec{x}}$$

$$\rightarrow \boxed{v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0} \quad (M)$$

Mukhanov equation

$\vec{k} \to k$ because no directional dependence

$\frac{z''}{z} \approx 2a^2H^2$ slow-roll

mass term set by inverse Hubble horizon

→ 'sub-horizon' ($k \gg aH$) : free oscillator

'super-horizon' ($k \ll aH$) : $\frac{z''}{z}(\tau)$ important

2) Quantization and initial conditions

$$v \to \hat{v} = \int \frac{d^3k}{(2\pi)^3} \left[v_k(\tau)\, \hat{a}_{\vec{k}}\, e^{i\vec{k}\vec{x}} + v_k^*(\tau)\, \hat{a}^+_{\vec{k}}\, e^{-i\vec{k}\vec{x}} \right]$$

with $[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = (2\pi)^3 \delta(\vec{k}-\vec{k}')$

canonical commutation relation

$$[\hat{v}(\vec{x},\tau), \hat{v}'(\vec{y},\tau)] = i\delta(\vec{x}-\vec{y}) \longrightarrow \frac{i}{\hbar}\left(v_k^* v_k' - v_k'^* v_k\right) = 1 \quad (*)$$

$\hookrightarrow$ FT, @ $\vec{k} = -\vec{k}'$

(fixes 1 out 2 i.c.)

In the far past $k \gg aH \xrightarrow{(M)} v_k'' + k^2 v_k = 0$

$\hookrightarrow$ harmonic oscillator

$\hookrightarrow$ unique state which minimizes energy

$$\boxed{\lim_{\tau \to -\infty} v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}}$$

Bunch Davis vacuum

only positive frequency modes due to $(*)$

$\to$ initial conditions fully fixed

3) Solution and power spectrum

Solve (H) with these i.c. in de Sitter ($H = \text{const.}$)

$$\tau = -\frac{1}{aH} \to v_k'' + \left(k^2 - \frac{2}{\tau^2}\right) v_k = 0$$

$$\to v_k = \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) \qquad (\ast\ast)$$

sub-horizon: $-k\tau = \frac{k}{aH} \gg 1 \to$ Bunch Davies vacuum

super-horizon: $-k\tau = \frac{k}{aH} \ll 1 \to v_k'' - \frac{2}{\tau^2} v_k \cong 0$

$$\to v_k = A\tau^2 + B/\tau$$

$$\to \mathcal{R} \propto \frac{v_k}{a} = \underbrace{-AH\tau^3}_{\substack{\text{decays} \\ \text{as } \tau \to 0}} \underbrace{-BH}_{\substack{\text{constant} \\ \hookrightarrow \text{"freeze-out"}}}$$

$\Rightarrow$ compute $\mathcal{R}_k(\tau)$ at horizon exit,
stays constant until horizon re-entry

Power spectrum $\langle R_{\vec{k}} R_{\vec{k}'} \rangle$

- statistical homogeneity & isotropy (SHI):

$$\langle R(t,\vec{x})\, R(t,\vec{x}') \rangle \equiv \xi(t,\vec{x},\vec{x}') = \xi(t, \underbrace{|\vec{x}-\vec{x}'|}_{\text{homogeneity}})$$

isotropy

in Fourier space:

$$\langle R_{\vec{k}}(t)\, R^{*}_{\vec{k}'}(t) \rangle \overset{SHI}{=} (2\pi)^3 \underbrace{\delta(\vec{k}-\vec{k}')}_{\text{homogeneity}} P_R(k)$$

$\vec{k} \to k$ isotropy

power spectrum

also,

$$\int \frac{d^3k}{(2\pi)^3} P_R(k) = \int d\ln k \underbrace{\frac{k^3}{2\pi^2} P_R(k)}_{\Delta^2_R(k)}$$

- insert our solution for $R = \frac{v}{z} = \frac{vH}{a\dot{\phi}}$:

$$\langle R_{\vec{k}} R_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k}+\vec{k}') \left(\frac{H}{\dot{\phi}}\right)^2 \underbrace{\frac{|v_k(\tau)|^2}{a^2}}_{(**)}$$

$$\frac{1}{a^2}\frac{1}{2k}\left(1+\frac{1}{k^2\tau^2}\right)$$

$$= \underbrace{\frac{H^2}{2k^3}}_{\substack{\text{const,}\\ \text{no } \tau\text{-dependence}}} \left(1 + \underbrace{k^2\tau^2}_{\to\, 0 \text{ on super-horizon scales}}\right)$$

$\langle R^2 \rangle_{k_* = a_* H_*}$ horizon crossing τ_k

const after horizon crossing

$$\rightarrow \langle R_{\vec{k}} R_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k}+\vec{k}') \left(\frac{H_*}{\dot{\phi}_*}\right)^2 \frac{H_*^2}{2k^2} \quad \forall \tau \text{ with } aH > k$$

$$\rightarrow \boxed{\Delta_R^2 = \frac{H_*^2}{(2\pi)^2} \underbrace{\frac{H_*^2}{\dot{\phi}_*^2}}_{k = a_* H_*}}$$

$H, \dot{\phi} \simeq$ const

$\rightarrow$ approximately scale invariant spectrum of perturbations

"inflation model"

Connecting to slow-roll parameters.

ansatz: $\Delta_R^2 = A_s \left(\frac{k}{k_{ref}}\right)^{\underbrace{n_s - 1}_{\text{tilt}} + \underbrace{\frac{1}{2}\alpha_s \ln \frac{k}{k_{ref}}}_{\text{running}} + \dots}$

$$A_s = \frac{H_*^2}{(2\pi)^2} \frac{H_*^2}{\underbrace{\dot{\phi}^2}_{1/2\varepsilon}} \simeq \frac{V(\phi)}{24\pi^2 \varepsilon_V} \Bigg|_{\phi = \phi_{CMB}}$$

$$n_s - 1 = \frac{d \ln \Delta_R^2}{d \ln k} = \frac{d \ln \Delta_R^2}{d\phi} \underbrace{\frac{d\phi}{dN}}_{-\frac{1}{\sqrt{2\varepsilon_V}} d\phi} \underbrace{\frac{dN}{d \ln k}}_{1+\varepsilon} \qquad k = aH = k_0 e^{N - N_0}$$

$$= -6\varepsilon_V - 2\eta_V$$

$$\Rightarrow \boxed{V(\phi) \rightarrow A_s, n_s}$$

Notes

$dt = a\,d\tau$

$$\tau = \int_0^t \frac{dt}{a} = \int_0^t \underbrace{\frac{dt}{da}}_{H}\frac{da}{a} = \int_0^a \frac{da}{(aH)}$$

inflation: $a = a_0 e^{Ht}$

$$\hookrightarrow \int_0^t \frac{dt}{a_0} e^{-Ht} = -\frac{1}{a_0 H}\left[e^{-Ht}\right]_0^t$$

$$= -\frac{1}{aH} + \underbrace{\frac{1}{a(0)H}}_{\simeq 0 \text{ at late times}} \simeq -\frac{1}{aH}$$

Wronskian condition:

$$[\hat{\phi}(\vec{x},t), \hat{\pi}(x,t)] = i\delta(\vec{x}-\vec{y}) \qquad (\hbar = 1)$$

$$\xrightarrow[FT]{} [\underbrace{\phi_{\vec{k}'}(\tau)}_{=\phi^*_{\vec{k}'}(\tau)}, \pi_{\vec{k}}(\tau)] = i(2\pi)^3\delta(\vec{k}'+\vec{k})$$

$\vec{k}' = -\vec{k}$

$\phi \to v$

$\pi \to v'$

$$i(2\pi)^3\delta(0) = [(v_k^* \hat{a}^+_{\vec{k}} e^{-i\vec{k}\vec{x}} + v_k \hat{a}_{\vec{k}} e^{i\vec{k}\vec{x}}), (v_k' \hat{a}_{\vec{k}} e^{i\vec{k}\vec{x}} + v_k'^* \hat{a}^+_{\vec{k}} e^{-i\vec{k}\vec{x}})]$$

$$= v_k^* v_k' \underbrace{[\hat{a}^+_{\vec{k}}, \hat{a}_{\vec{k}}]}_{-(2\pi)^3\delta(0)} + v_k v_k' \underbrace{[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}}]}_{(2\pi)^3\delta(0)}$$

$$\Rightarrow -v_k^* v_k' + v_k v_k'^* = i$$