Inverse model applications

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7th Summer School on Theory, Mechanisms and Hierarchical Modelling of Climate Dynamics
Estimating Ocean Transports: Single Sections, Box Models and Reanalysis Products
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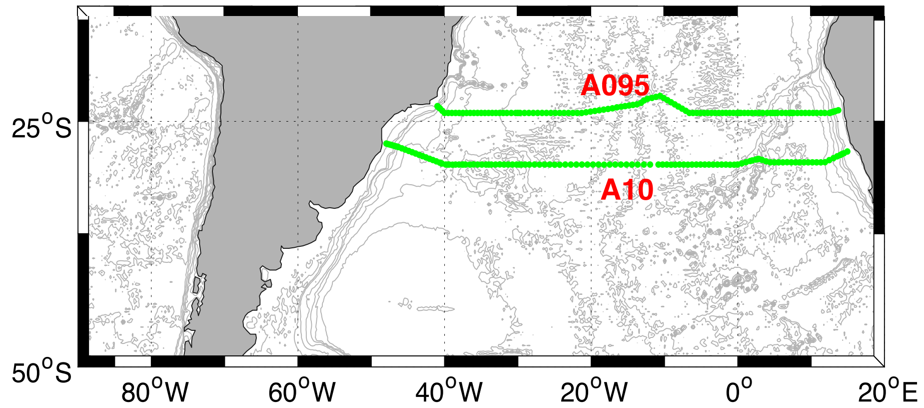
To download today’s files onto your computer:

www.oceanografia.es/Vero2.zip

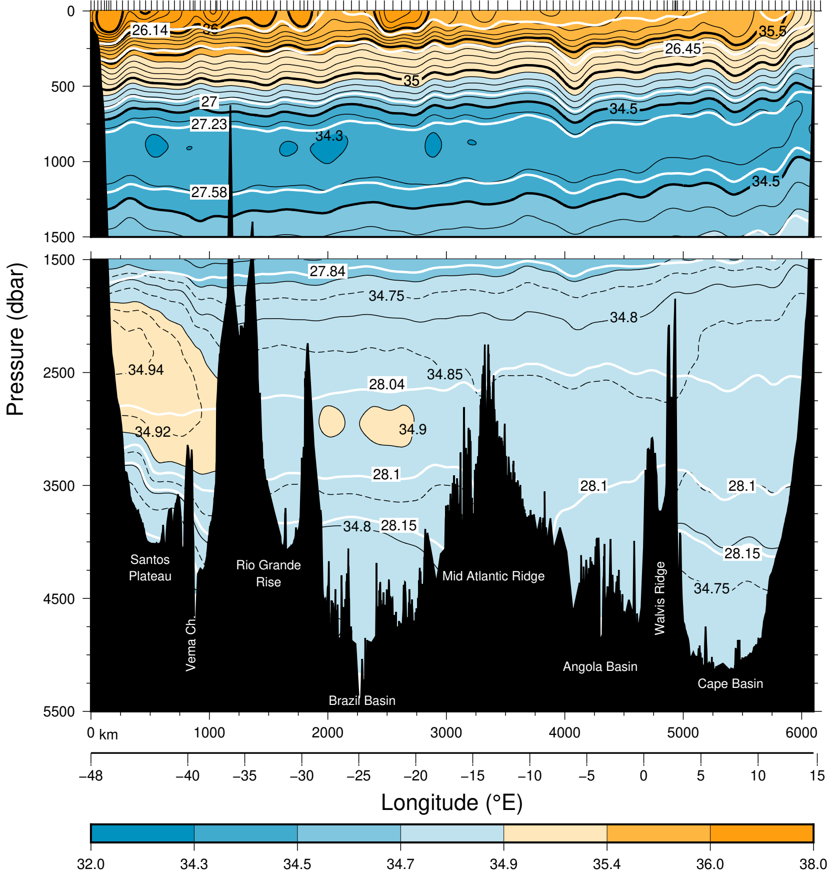
www.oceanografia.es/Vero.zip

Area of study – Example: South Atlantic

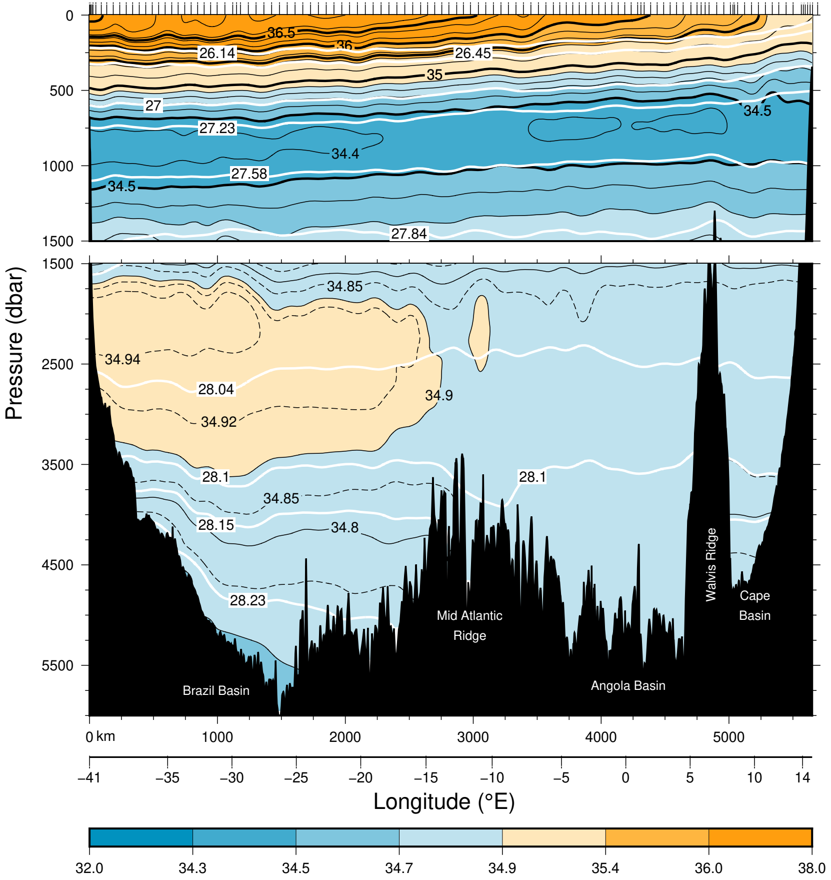
GO-SHIP sections



2011 A10 (30°S)



2018 A09.5 (24°S)



Inverse model formulation: single box with mass conservation

$$\boldsymbol{E}\,\boldsymbol{x} + \boldsymbol{n} = \boldsymbol{y}$$

$$
\left(\begin{array}{r|cccccccc}
 & & & & & & & \multicolumn{2}{c}{\overbrace{\hphantom{1\quad 1}}^{\text{Ekman transport applied in first layer}}} \\
\text{Total conservation for the box (both sections: A+B)} \rightarrow & e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\
\text{Total conservation for section A} \rightarrow & e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\
\text{Regional constraints for section A} \rightarrow & e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\
\text{Total conservation for section B} \rightarrow & 0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\
\text{Regional constraints for section B} \rightarrow & 0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\
 & e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\
 & e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\
\text{Conservation for each layer for} & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\
\text{the box (both sections A+B)} \Bigg\{ & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\
 & e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0
\end{array}\right)
\begin{pmatrix} b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}} \end{pmatrix}
=
\begin{pmatrix}
y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\
y_{A_t} + T_{A_{Ek}} \\
y_{A_{reg}} \\
y_{B_t} + T_{B_{Ek}} \\
y_{B_{reg}} \\
y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\
y_{A_2} + y_{B_2} \\
\vdots \\
\vdots \\
y_{A_{q-1}} + y_{B_{q-1}} \\
y_{A_q} + y_{B_q}
\end{pmatrix}
$$

e: mass
y: mass transport

b: reference velocities
ΔT_{Ek}: Ekman correction
T_{Ek}: Ekman transport

n: number of pair of stations for section A
m: number of pair of stations for section B
q: number of layers (11)

Inverse model formulation: single box with mass conservation

1) Create a local copy of these two folders from the root:

Path for the original folders:

/home/esp-shared-a/Distribution/Workshops/OceanTransports2026/example_inverse_model

/home/esp-shared-a/Distribution/Workshops/OceanTransports2026/ictpSS26_toolboxes

2) Initial check

Run this script:

example_inverse_model/model_example/Main_inverse_model_example.m

You should see 9 figures

Inverse model formulation: single box with mass conservation

We will be working with mass transport, not volume transport

Volume transport units: 1 Sverdrup (Sv) = 10^6 m^3/s

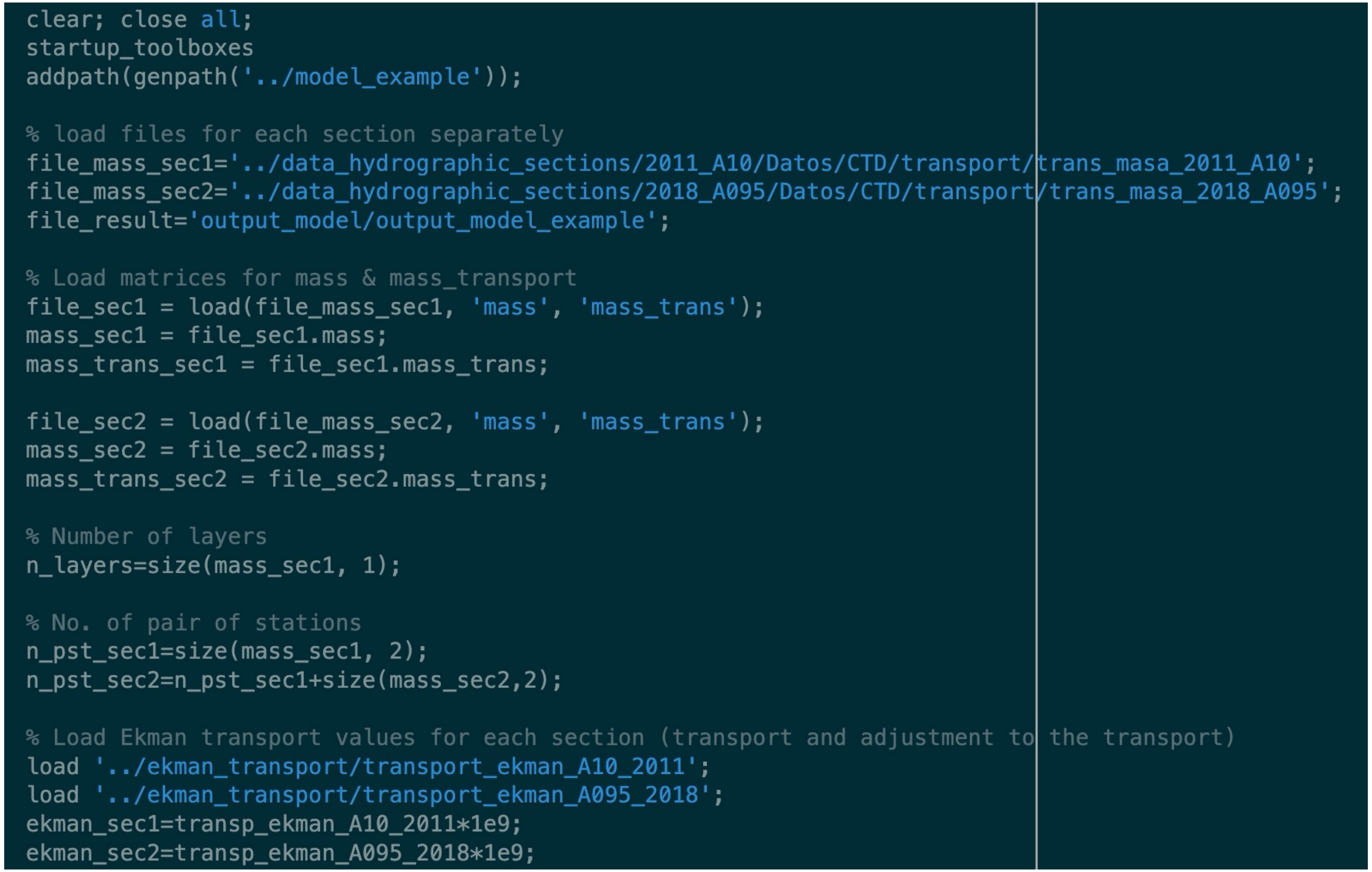
Multiply by density (kg/m^3)

Mass transport units: 10^9 kg/s

Inverse model formulation: single box with mass conservation

Open code for inverse model computation in your local folder:

example_inverse_model/model_example/Main_inverse_model_example.m



```
clear; close all;
startup_toolboxes
addpath(genpath('../model_example'));

% load files for each section separately
file_mass_sec1='../data_hydrographic_sections/2011_A10/Datos/CTD/transport/trans_masa_2011_A10';
file_mass_sec2='../data_hydrographic_sections/2018_A095/Datos/CTD/transport/trans_masa_2018_A095';
file_result='output_model/output_model_example';

% Load matrices for mass & mass_transport
file_sec1 = load(file_mass_sec1, 'mass', 'mass_trans');
mass_sec1 = file_sec1.mass;
mass_trans_sec1 = file_sec1.mass_trans;

file_sec2 = load(file_mass_sec2, 'mass', 'mass_trans');
mass_sec2 = file_sec2.mass;
mass_trans_sec2 = file_sec2.mass_trans;

% Number of layers
n_layers=size(mass_sec1, 1);

% No. of pair of stations
n_pst_sec1=size(mass_sec1, 2);
n_pst_sec2=n_pst_sec1+size(mass_sec2,2);

% Load Ekman transport values for each section (transport and adjustment to the transport)
load '../ekman_transport/transport_ekman_A10_2011';
load '../ekman_transport/transport_ekman_A095_2018';
ekman_sec1=transp_ekman_A10_2011*1e9;
ekman_sec2=transp_ekman_A095_2018*1e9;
```

Loads QC example data from GO-SHIP sections

Define our q (number of layers) and m and n (number of stations)

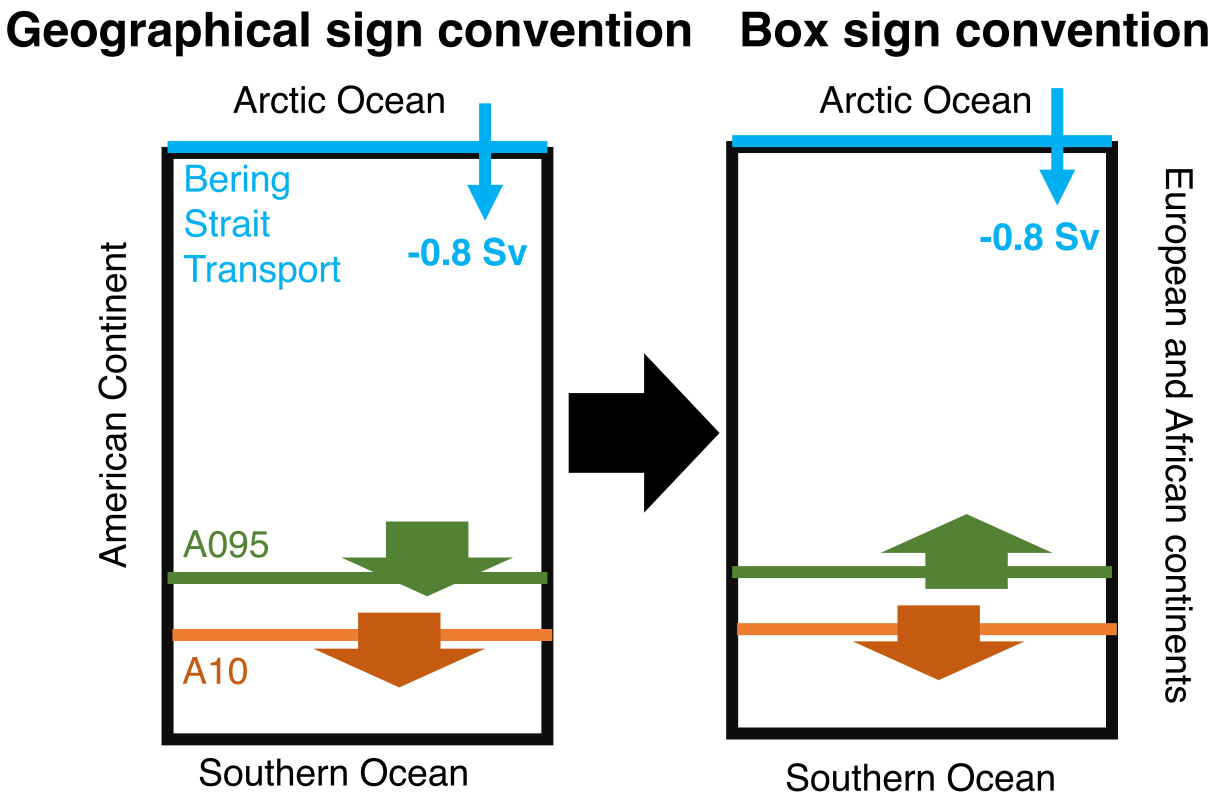
Load the initial estimates of Ekman transports

Inverse model formulation: single box with mass conservation

Defining the constraints for our inverse model – what is our ‘truth’?

$$\begin{pmatrix} e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\ e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\ e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\ 0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\ e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\ e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\ e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}} \end{pmatrix} = \begin{pmatrix} y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_t} + T_{A_{Ek}} \\ y_{A_{reg}} \\ y_{B_t} + T_{B_{Ek}} \\ y_{B_{reg}} \\ y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_2} + y_{B_2} \\ \vdots \\ \vdots \\ y_{A_{q-1}} + y_{B_{q-1}} \\ y_{A_q} + y_{B_q} \end{pmatrix}$$

Total conservation for the box (both sections: A+B)



Mass conservation

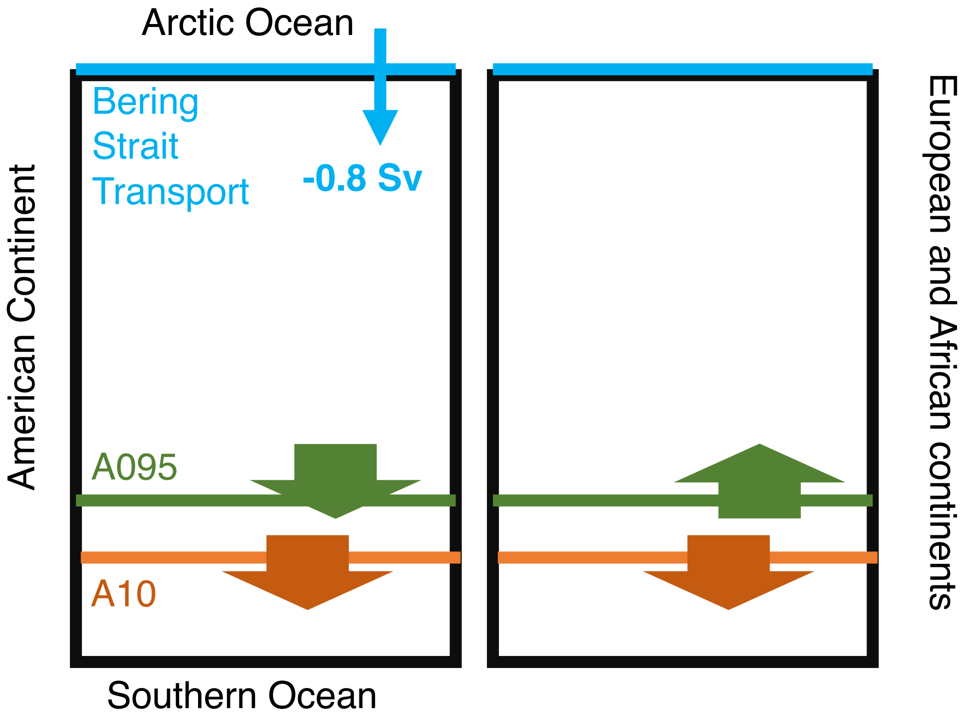
Sec1 (A10) + Sec2 (A095) = 0

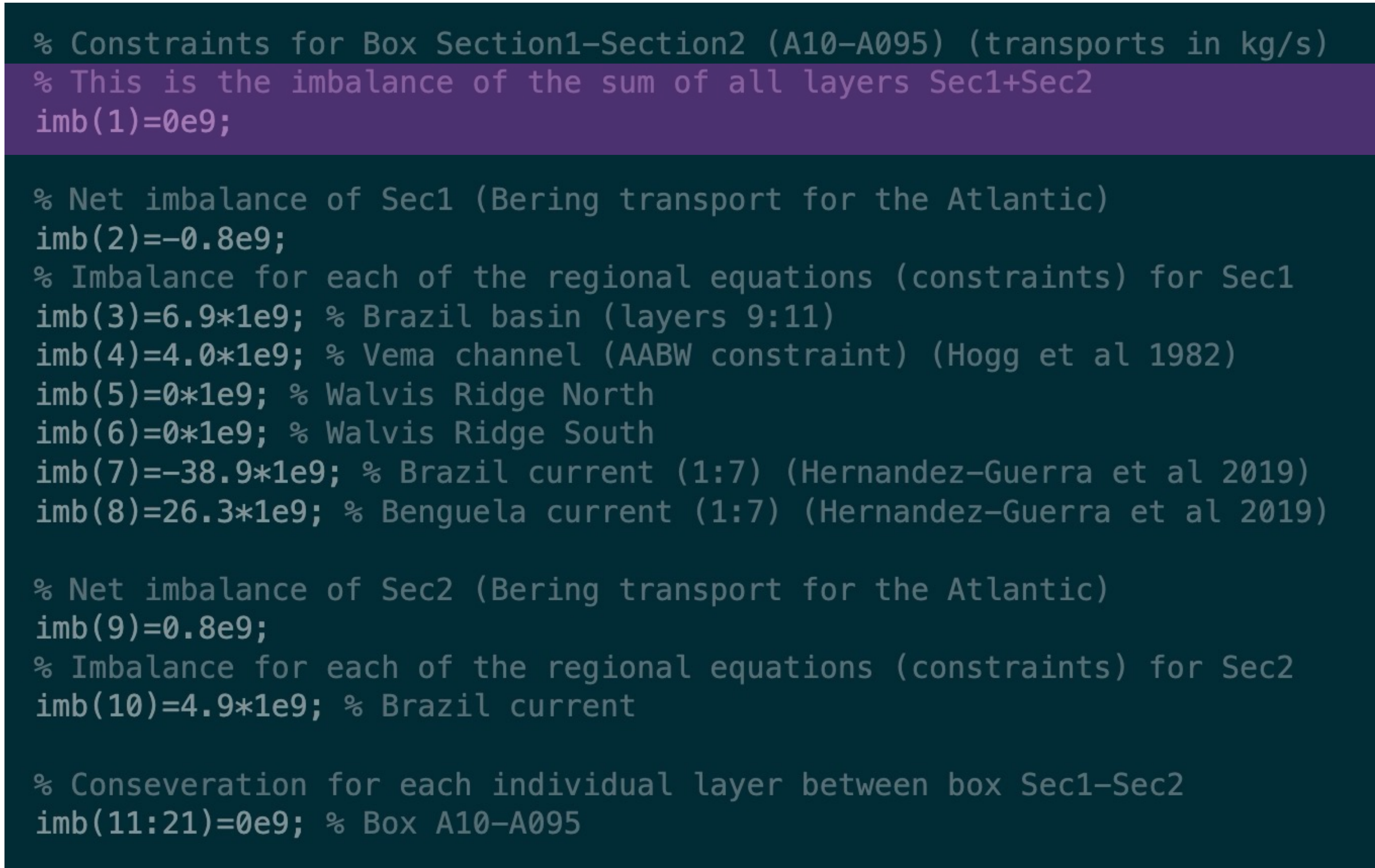
Inverse model formulation: single box with mass conservation

Defining the constraints for our inverse model – what is our ‘truth’?

$$\begin{pmatrix} e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\ e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\ e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\ 0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\ e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\ e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\ e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}} \end{pmatrix} = \begin{pmatrix} y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_t} + T_{A_{Ek}} \\ y_{A_{reg}} \\ y_{B_t} + T_{B_{Ek}} \\ y_{B_{reg}} \\ y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_2} + y_{B_2} \\ \vdots \\ \vdots \\ y_{A_{q-1}} + y_{B_{q-1}} \\ y_{A_q} + y_{B_q} \end{pmatrix}$$

Total conservation for the box (both sections: A+B)





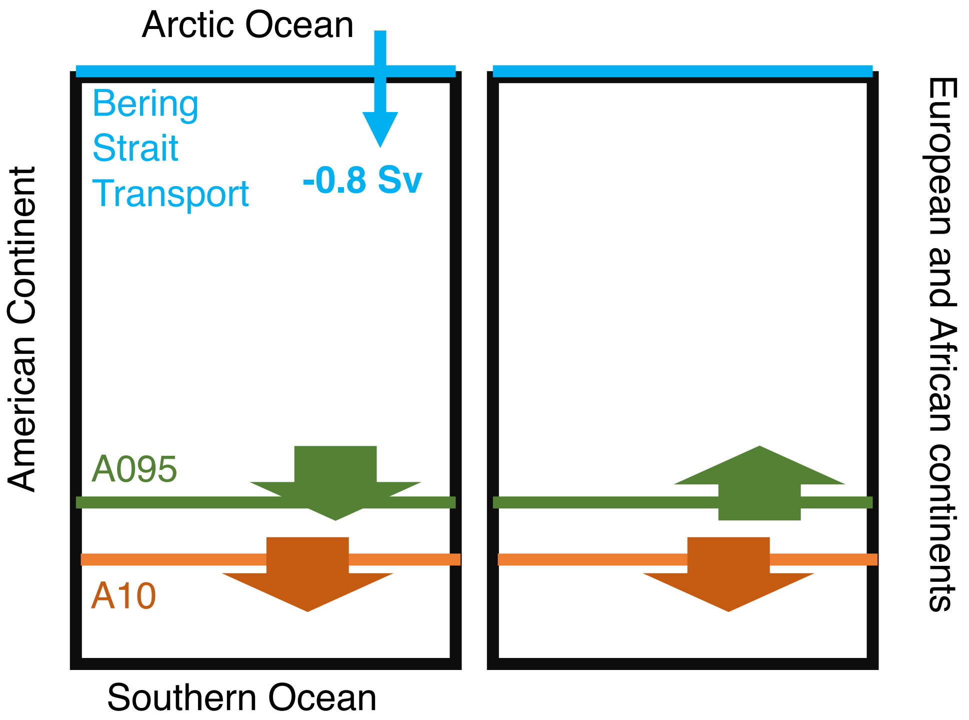
```
% Constraints for Box Section1-Section2 (A10-A095) (transports in kg/s)
% This is the imbalance of the sum of all layers Sec1+Sec2
imb(1)=0e9;

% Net imbalance of Sec1 (Bering transport for the Atlantic)
imb(2)=-0.8e9;
% Imbalance for each of the regional equations (constraints) for Sec1
imb(3)=6.9*1e9; % Brazil basin (layers 9:11)
imb(4)=4.0*1e9; % Vema channel (AABW constraint) (Hogg et al 1982)
imb(5)=0*1e9; % Walvis Ridge North
imb(6)=0*1e9; % Walvis Ridge South
imb(7)=-38.9*1e9; % Brazil current (1:7) (Hernandez-Guerra et al 2019)
imb(8)=26.3*1e9; % Benguela current (1:7) (Hernandez-Guerra et al 2019)

% Net imbalance of Sec2 (Bering transport for the Atlantic)
imb(9)=0.8e9;
% Imbalance for each of the regional equations (constraints) for Sec2
imb(10)=4.9*1e9; % Brazil current

% Conseveration for each individual layer between box Sec1-Sec2
imb(11:21)=0e9; % Box A10-A095
```

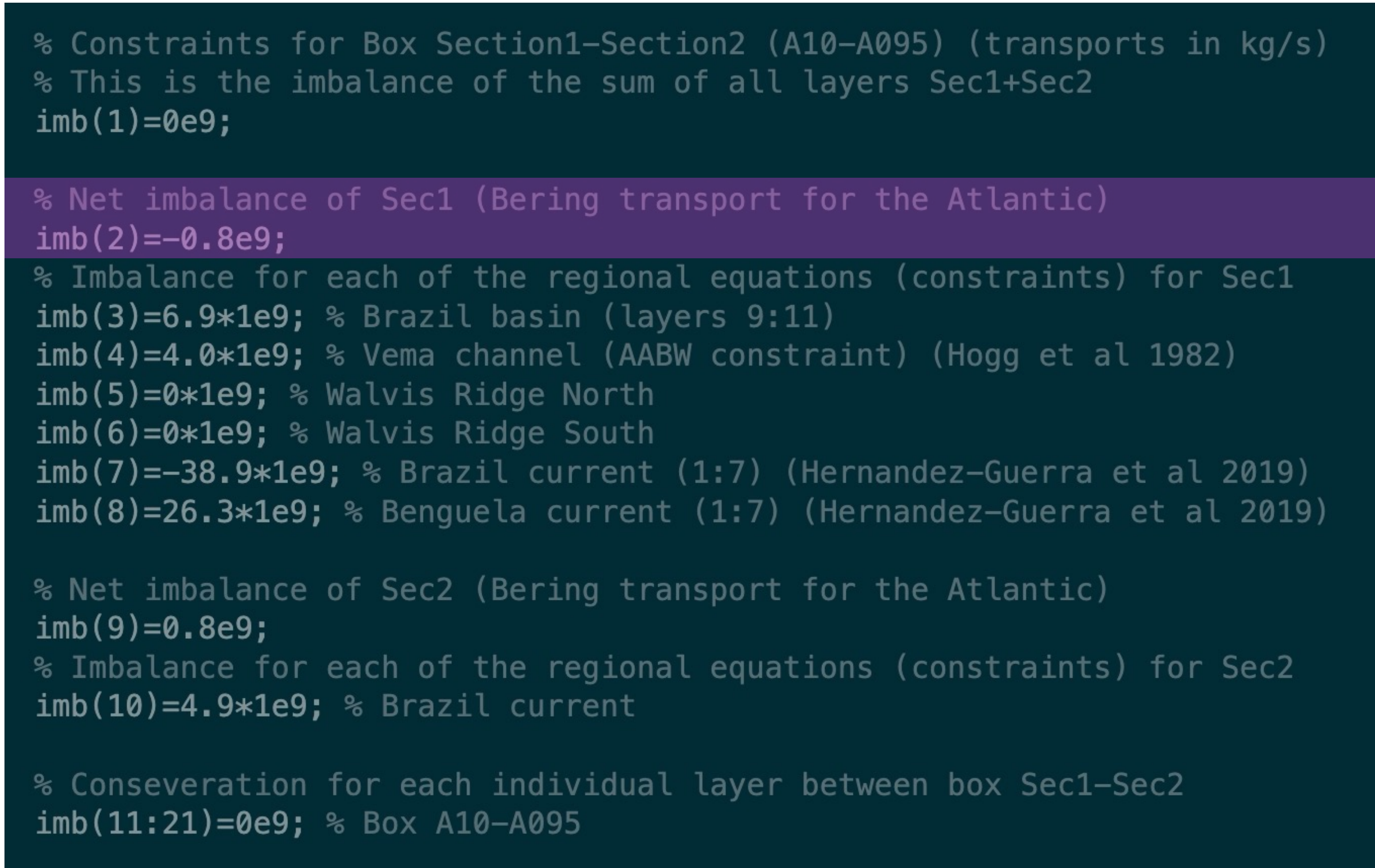
Inverse model formulation: single box with mass conservation

Defining the constraints for our inverse model – what is our ‘truth’?

$$\begin{pmatrix} e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\ e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\ e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\ 0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\ e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\ e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\ e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}} \end{pmatrix} = \begin{pmatrix} y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_t} + T_{A_{Ek}} \\ y_{A_{reg}} \\ y_{B_t} + T_{B_{Ek}} \\ y_{B_{reg}} \\ y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_2} + y_{B_2} \\ \vdots \\ \vdots \\ y_{A_{q-1}} + y_{B_{q-1}} \\ y_{A_q} + y_{B_q} \end{pmatrix}$$

Total conservation for section A





```
% Constraints for Box Section1-Section2 (A10-A095) (transports in kg/s)
% This is the imbalance of the sum of all layers Sec1+Sec2
imb(1)=0e9;

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imb(6)=0*1e9; % Walvis Ridge South
imb(7)=-38.9*1e9; % Brazil current (1:7) (Hernandez-Guerra et al 2019)
imb(8)=26.3*1e9; % Benguela current (1:7) (Hernandez-Guerra et al 2019)

% Net imbalance of Sec2 (Bering transport for the Atlantic)
imb(9)=0.8e9;
% Imbalance for each of the regional equations (constraints) for Sec2
imb(10)=4.9*1e9; % Brazil current

% Conseveration for each individual layer between box Sec1-Sec2
imb(11:21)=0e9; % Box A10-A095
```
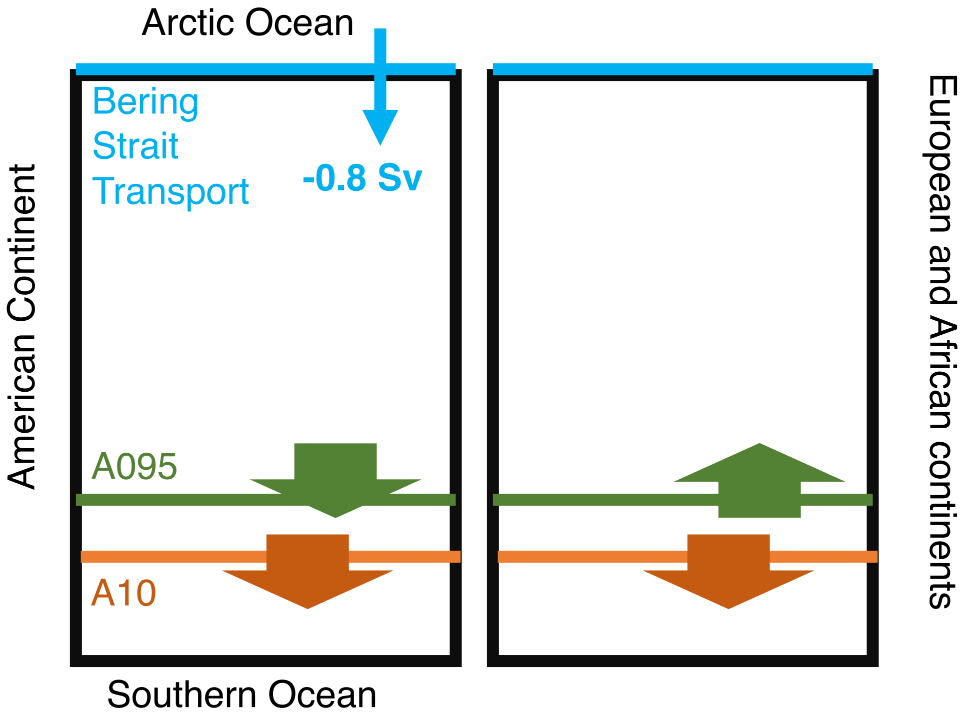
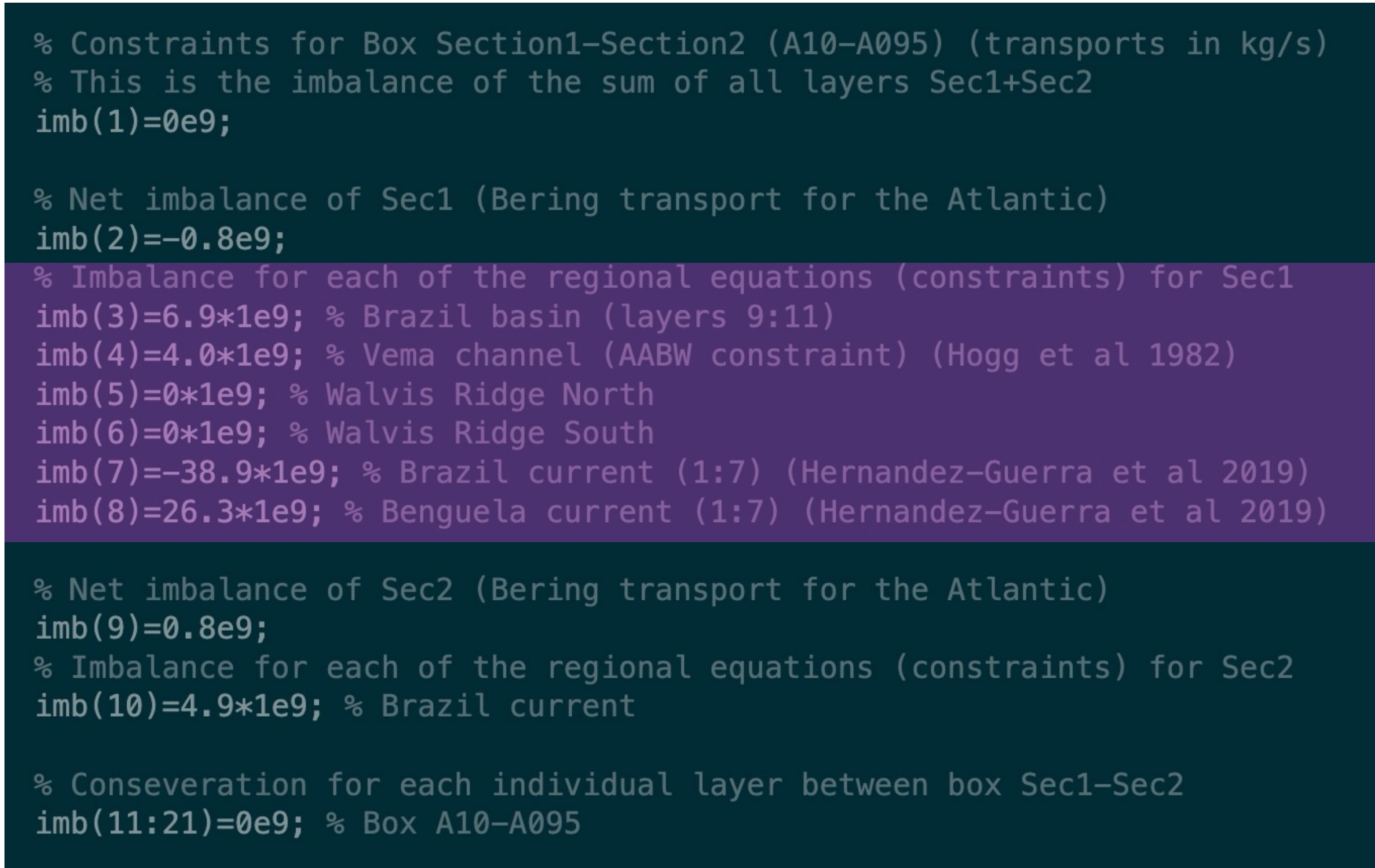

Inverse model formulation: single box with mass conservation

Defining the constraints for our inverse model – what is our ‘truth’?

$$
\begin{pmatrix}
e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\
e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\
e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\
0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\
e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\
e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\
e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}}
\end{pmatrix}
=
\begin{pmatrix}
y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\
y_{A_t} + T_{A_{Ek}} \\
y_{A_{reg}} \\
y_{B_t} + T_{B_{Ek}} \\
y_{B_{reg}} \\
y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\
y_{A_2} + y_{B_2} \\
\vdots \\
\vdots \\
y_{A_{q-1}} + y_{B_{q-1}} \\
y_{A_q} + y_{B_q}
\end{pmatrix}
$$

Regional constraints for section A





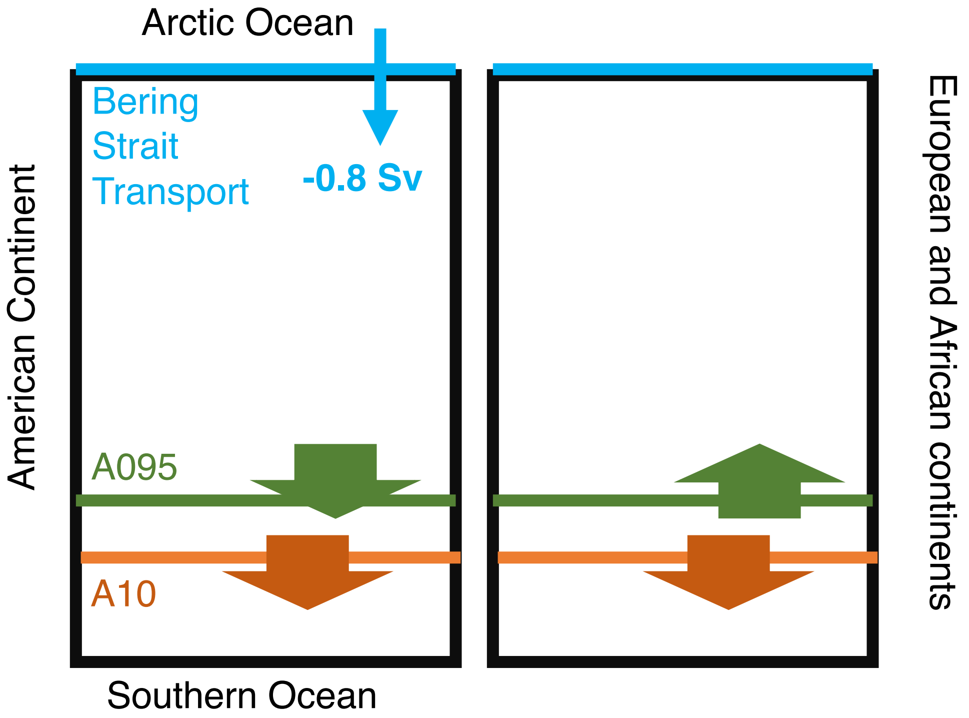
```
% Constraints for Box Section1-Section2 (A10-A095) (transports in kg/s)
% This is the imbalance of the sum of all layers Sec1+Sec2
imb(1)=0e9;

% Net imbalance of Sec1 (Bering transport for the Atlantic)
imb(2)=-0.8e9;
% Imbalance for each of the regional equations (constraints) for Sec1
imb(3)=6.9*1e9; % Brazil basin (layers 9:11)
imb(4)=4.0*1e9; % Vema channel (AABW constraint) (Hogg et al 1982)
imb(5)=0*1e9; % Walvis Ridge North
imb(6)=0*1e9; % Walvis Ridge South
imb(7)=-38.9*1e9; % Brazil current (1:7) (Hernandez-Guerra et al 2019)
imb(8)=26.3*1e9; % Benguela current (1:7) (Hernandez-Guerra et al 2019)

% Net imbalance of Sec2 (Bering transport for the Atlantic)
imb(9)=0.8e9;
% Imbalance for each of the regional equations (constraints) for Sec2
imb(10)=4.9*1e9; % Brazil current

% Conseveration for each individual layer between box Sec1-Sec2
imb(11:21)=0e9; % Box A10-A095
```

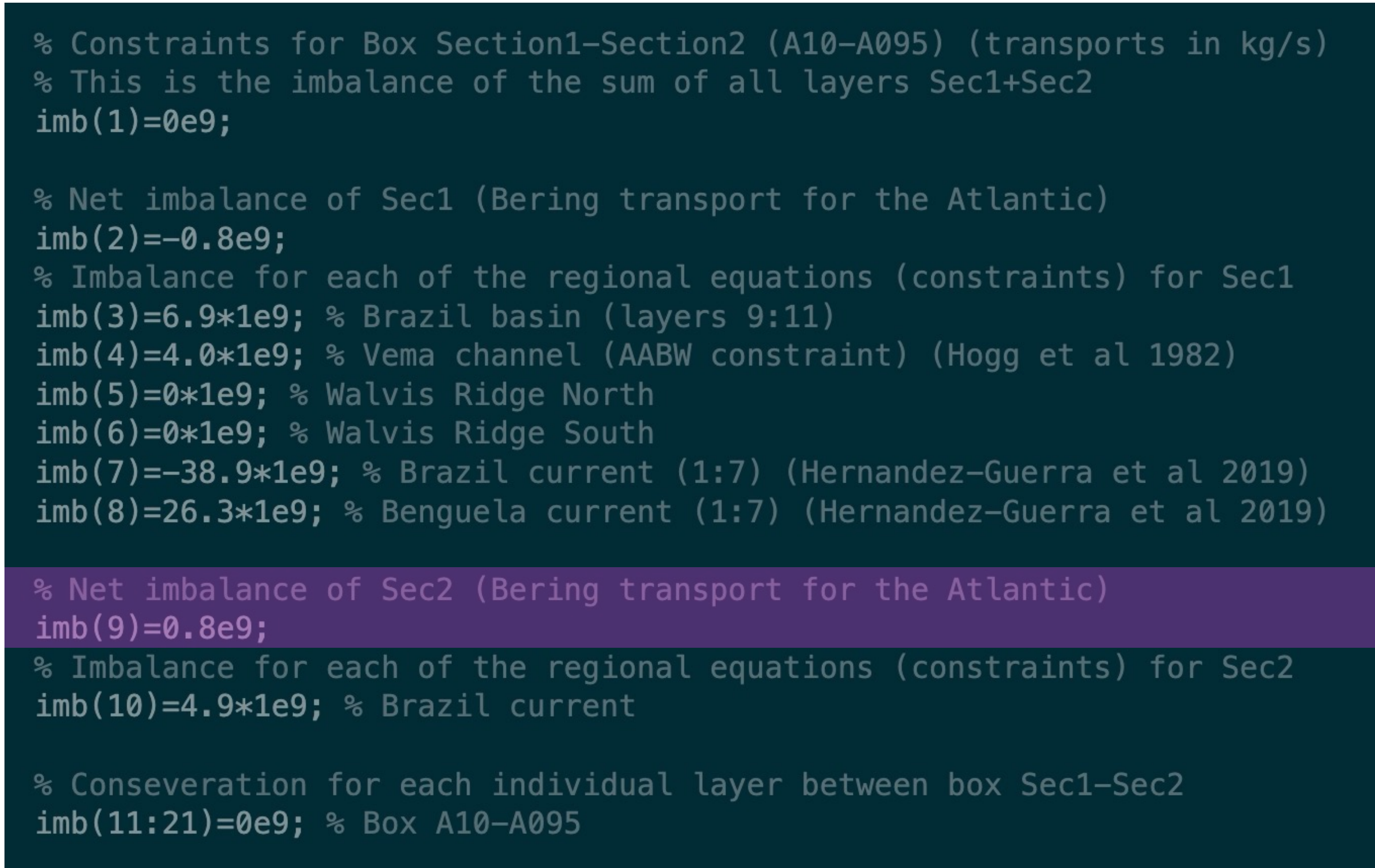
Inverse model formulation: single box with mass conservation

Defining the constraints for our inverse model – what is our ‘truth’?

$$\begin{pmatrix} e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\ e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\ e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\ 0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\ e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\ e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\ e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}} \end{pmatrix} = \begin{pmatrix} y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_t} + T_{A_{Ek}} \\ y_{A_{reg}} \\ y_{B_t} + T_{B_{Ek}} \\ y_{B_{reg}} \\ y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_2} + y_{B_2} \\ \vdots \\ \vdots \\ y_{A_{q-1}} + y_{B_{q-1}} \\ y_{A_q} + y_{B_q} \end{pmatrix}$$

Total conservation for section B





```
% Constraints for Box Section1-Section2 (A10-A095) (transports in kg/s)
% This is the imbalance of the sum of all layers Sec1+Sec2
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% Net imbalance of Sec2 (Bering transport for the Atlantic)
imb(9)=0.8e9;
% Imbalance for each of the regional equations (constraints) for Sec2
imb(10)=4.9*1e9; % Brazil current

% Conseveration for each individual layer between box Sec1-Sec2
imb(11:21)=0e9; % Box A10-A095
```
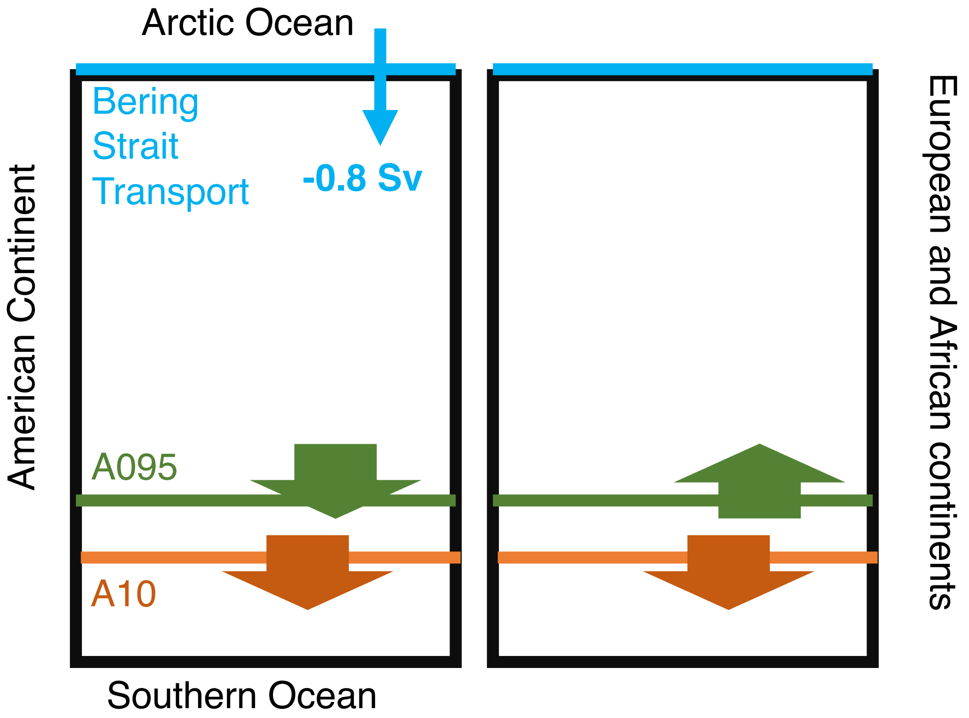
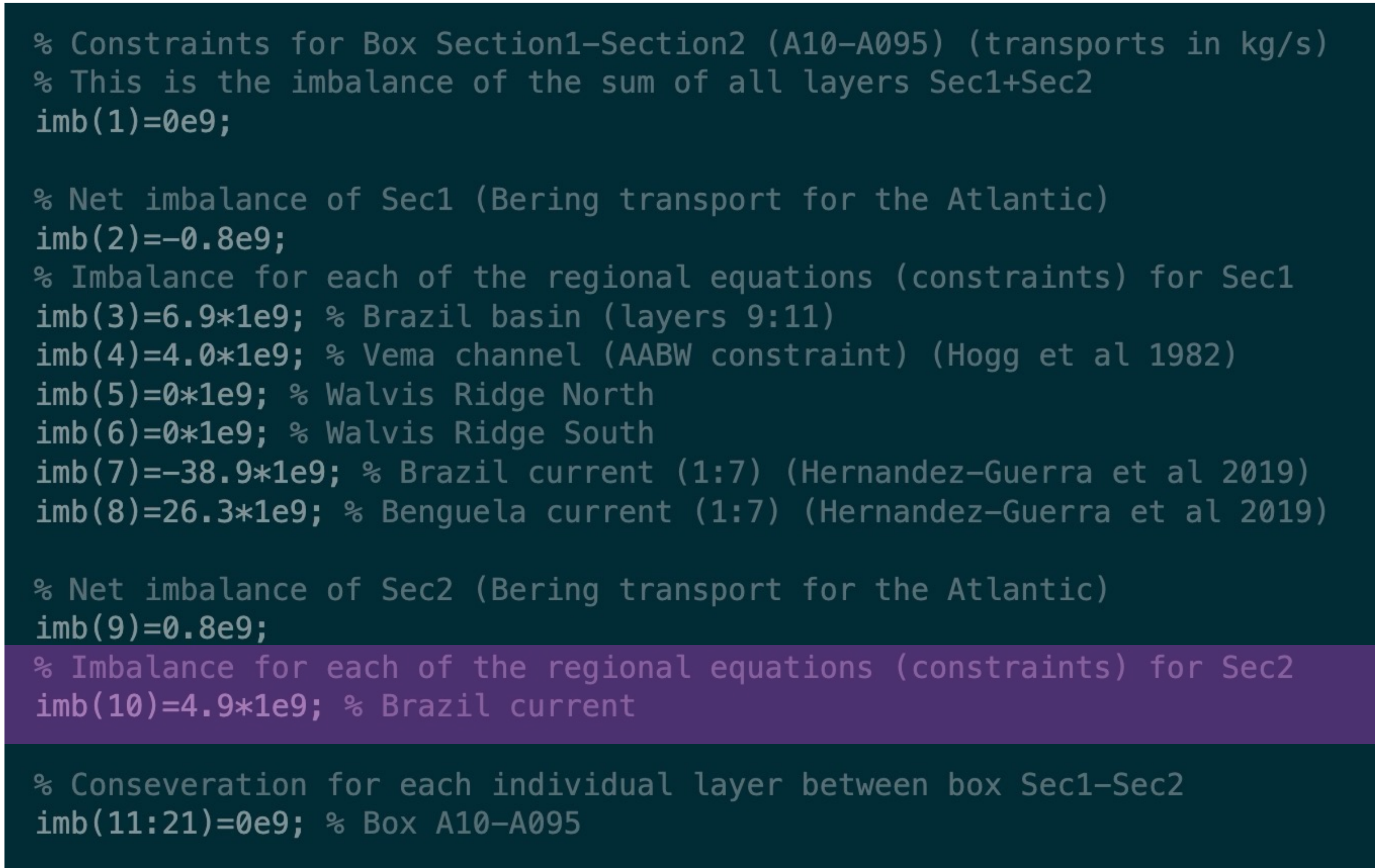

Inverse model formulation: single box with mass conservation

Defining the constraints for our inverse model – what is our ‘truth’?

$$\begin{pmatrix} e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\ e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\ e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\ 0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\ e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\ e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\ e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}} \end{pmatrix} = \begin{pmatrix} y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_t} + T_{A_{Ek}} \\ y_{A_{reg}} \\ y_{B_t} + T_{B_{Ek}} \\ y_{B_{reg}} \\ y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_2} + y_{B_2} \\ \vdots \\ \vdots \\ y_{A_{q-1}} + y_{B_{q-1}} \\ y_{A_q} + y_{B_q} \end{pmatrix}$$

Regional constraints for section B





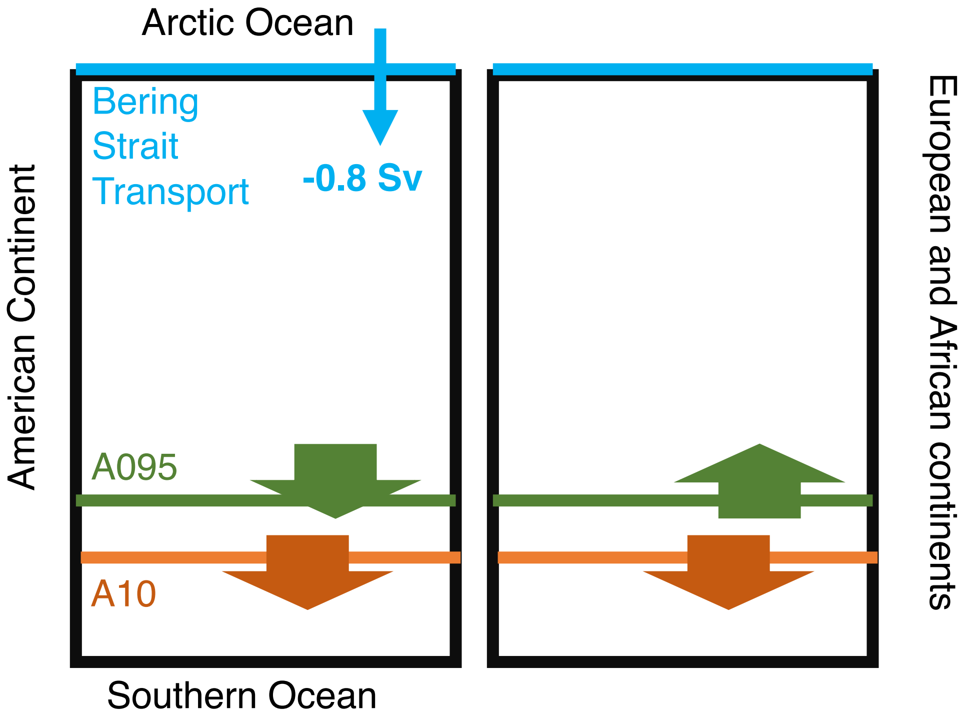
```
% Constraints for Box Section1-Section2 (A10-A095) (transports in kg/s)
% This is the imbalance of the sum of all layers Sec1+Sec2
imb(1)=0e9;

% Net imbalance of Sec1 (Bering transport for the Atlantic)
imb(2)=-0.8e9;
% Imbalance for each of the regional equations (constraints) for Sec1
imb(3)=6.9*1e9; % Brazil basin (layers 9:11)
imb(4)=4.0*1e9; % Vema channel (AABW constraint) (Hogg et al 1982)
imb(5)=0*1e9; % Walvis Ridge North
imb(6)=0*1e9; % Walvis Ridge South
imb(7)=-38.9*1e9; % Brazil current (1:7) (Hernandez-Guerra et al 2019)
imb(8)=26.3*1e9; % Benguela current (1:7) (Hernandez-Guerra et al 2019)

% Net imbalance of Sec2 (Bering transport for the Atlantic)
imb(9)=0.8e9;
% Imbalance for each of the regional equations (constraints) for Sec2
imb(10)=4.9*1e9; % Brazil current

% Conseveration for each individual layer between box Sec1-Sec2
imb(11:21)=0e9; % Box A10-A095
```

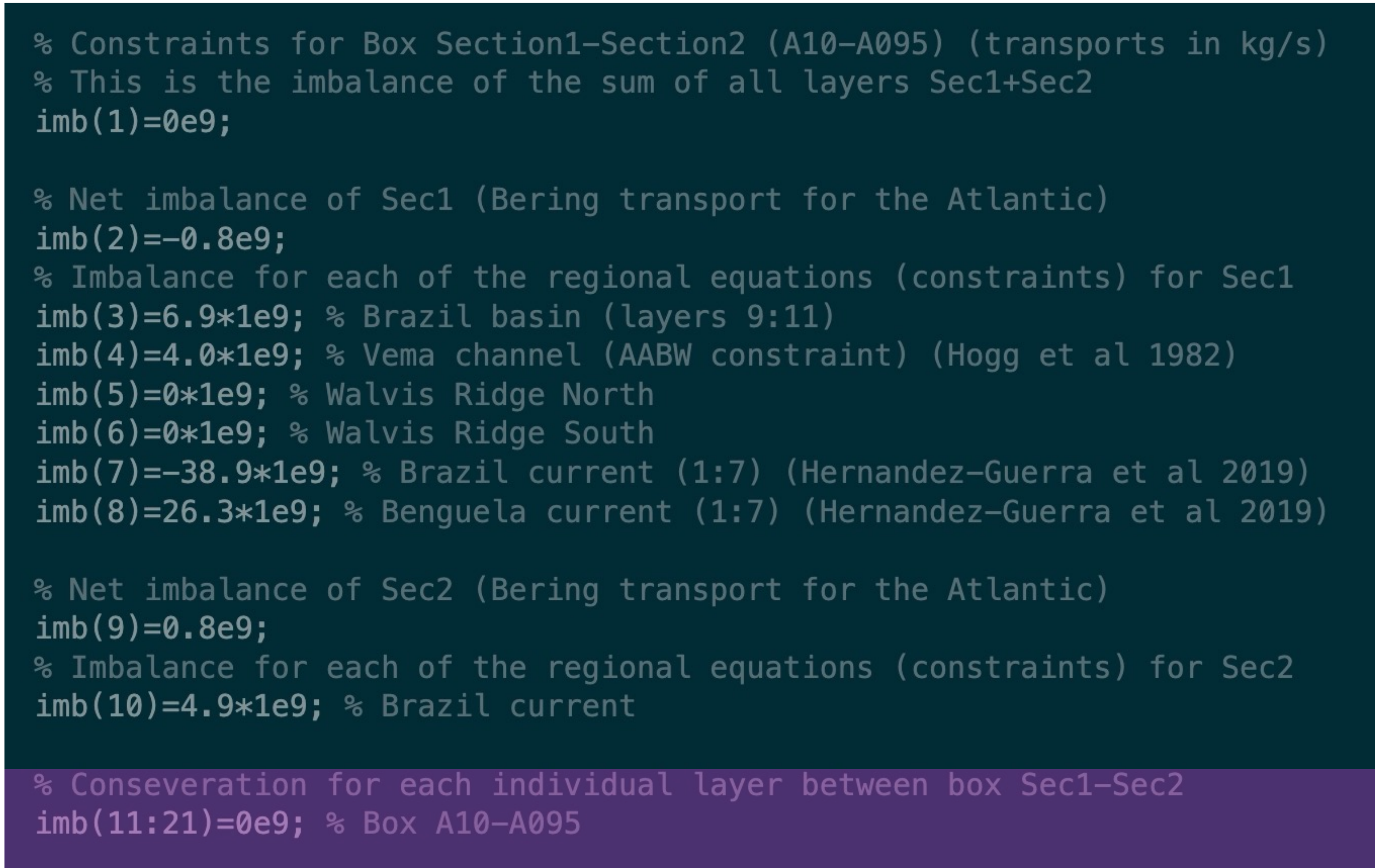
Inverse model formulation: single box with mass conservation

Defining the constraints for our inverse model – what is our ‘truth’?

$$\begin{pmatrix} e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\ e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\ e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\ 0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\ e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\ e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\ e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}} \end{pmatrix} = \begin{pmatrix} y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_t} + T_{A_{Ek}} \\ y_{A_{reg}} \\ y_{B_t} + T_{B_{Ek}} \\ y_{B_{reg}} \\ y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_2} + y_{B_2} \\ \vdots \\ \vdots \\ y_{A_{q-1}} + y_{B_{q-1}} \\ y_{A_q} + y_{B_q} \end{pmatrix}$$

Conservation for each layer for the box (both sections A+B)

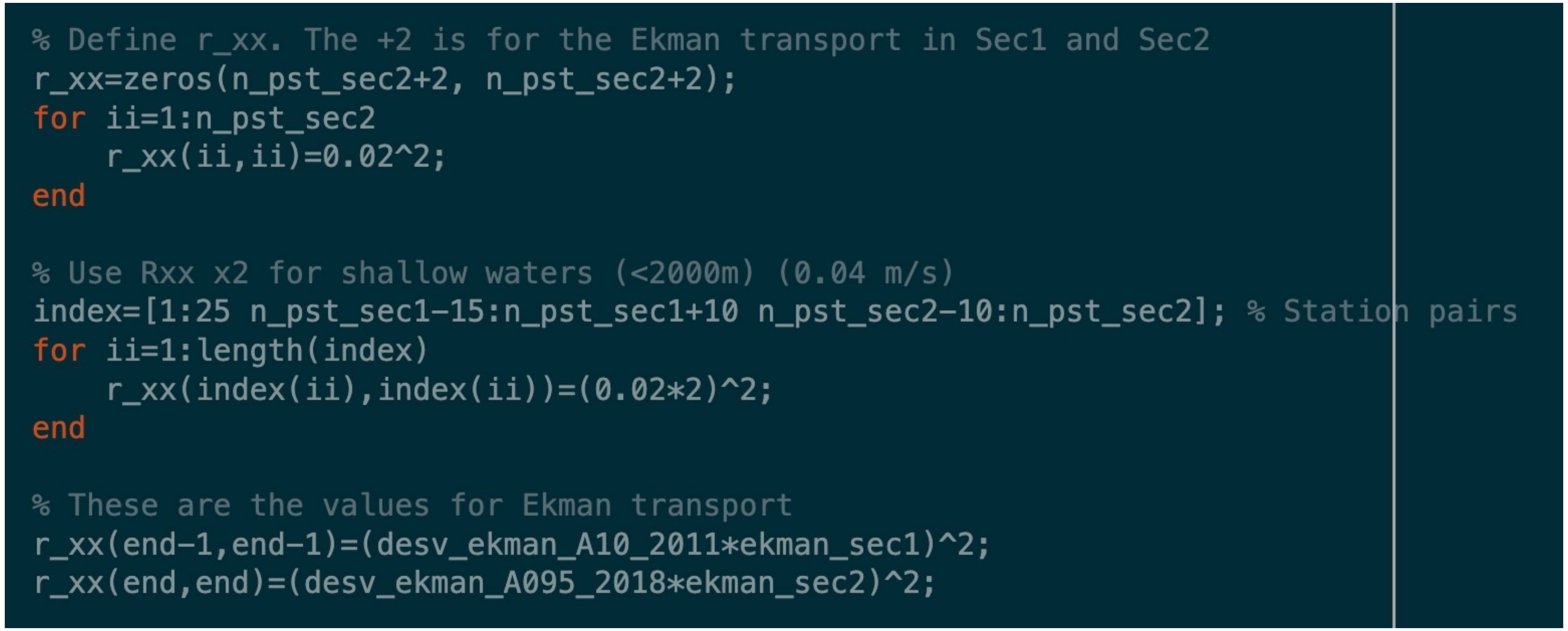




```
% Constraints for Box Section1-Section2 (A10-A095) (transports in kg/s)
% This is the imbalance of the sum of all layers Sec1+Sec2
imb(1)=0e9;

% Net imbalance of Sec1 (Bering transport for the Atlantic)
imb(2)=-0.8e9;
% Imbalance for each of the regional equations (constraints) for Sec1
imb(3)=6.9*1e9; % Brazil basin (layers 9:11)
imb(4)=4.0*1e9; % Vema channel (AABW constraint) (Hogg et al 1982)
imb(5)=0*1e9; % Walvis Ridge North
imb(6)=0*1e9; % Walvis Ridge South
imb(7)=-38.9*1e9; % Brazil current (1:7) (Hernandez-Guerra et al 2019)
imb(8)=26.3*1e9; % Benguela current (1:7) (Hernandez-Guerra et al 2019)

% Net imbalance of Sec2 (Bering transport for the Atlantic)
imb(9)=0.8e9;
% Imbalance for each of the regional equations (constraints) for Sec2
imb(10)=4.9*1e9; % Brazil current

% Conseveration for each individual layer between box Sec1-Sec2
imb(11:21)=0e9; % Box A10-A095
```

Inverse model formulation: single box with mass conservation

Define the uncertainty associated with the a priori estimate (R_{xx})

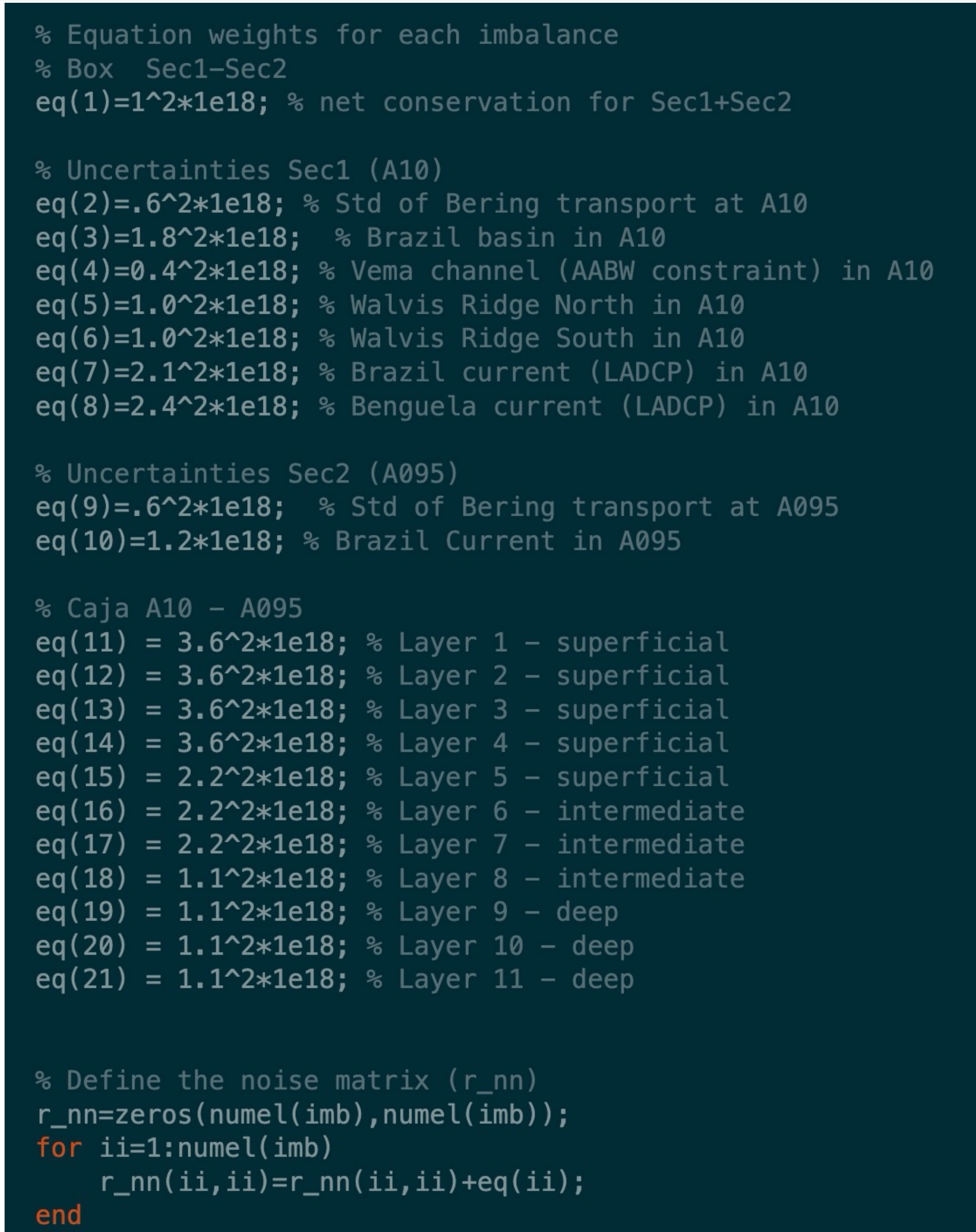


```
% Define r_xx. The +2 is for the Ekman transport in Sec1 and Sec2
r_xx=zeros(n_pst_sec2+2, n_pst_sec2+2);
for ii=1:n_pst_sec2
    r_xx(ii,ii)=0.02^2;
end

% Use Rxx x2 for shallow waters (<2000m) (0.04 m/s)
index=[1:25 n_pst_sec1-15:n_pst_sec1+10 n_pst_sec2-10:n_pst_sec2]; % Station pairs
for ii=1:length(index)
    r_xx(index(ii),index(ii))=(0.02*2)^2;
end

% These are the values for Ekman transport
r_xx(end-1,end-1)=(desv_ekman_A10_2011*ekman_sec1)^2;
r_xx(end,end)=(desv_ekman_A095_2018*ekman_sec2)^2;
```

Inverse model formulation: single box with mass conservation

Define the uncertainty associated with the observations for each of the equations (R_{nn})

$$
\begin{pmatrix}
e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\
e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\
e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\
0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\
e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\
e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\
e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}}
\end{pmatrix}
=
\begin{pmatrix}
y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\
y_{A_t} + T_{A_{Ek}} \\
y_{A_{reg}} \\
y_{B_t} + T_{B_{Ek}} \\
y_{B_{reg}} \\
y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\
y_{A_2} + y_{B_2} \\
\vdots \\
\vdots \\
y_{A_{q-1}} + y_{B_{q-1}} \\
y_{A_q} + y_{B_q}
\end{pmatrix}
$$



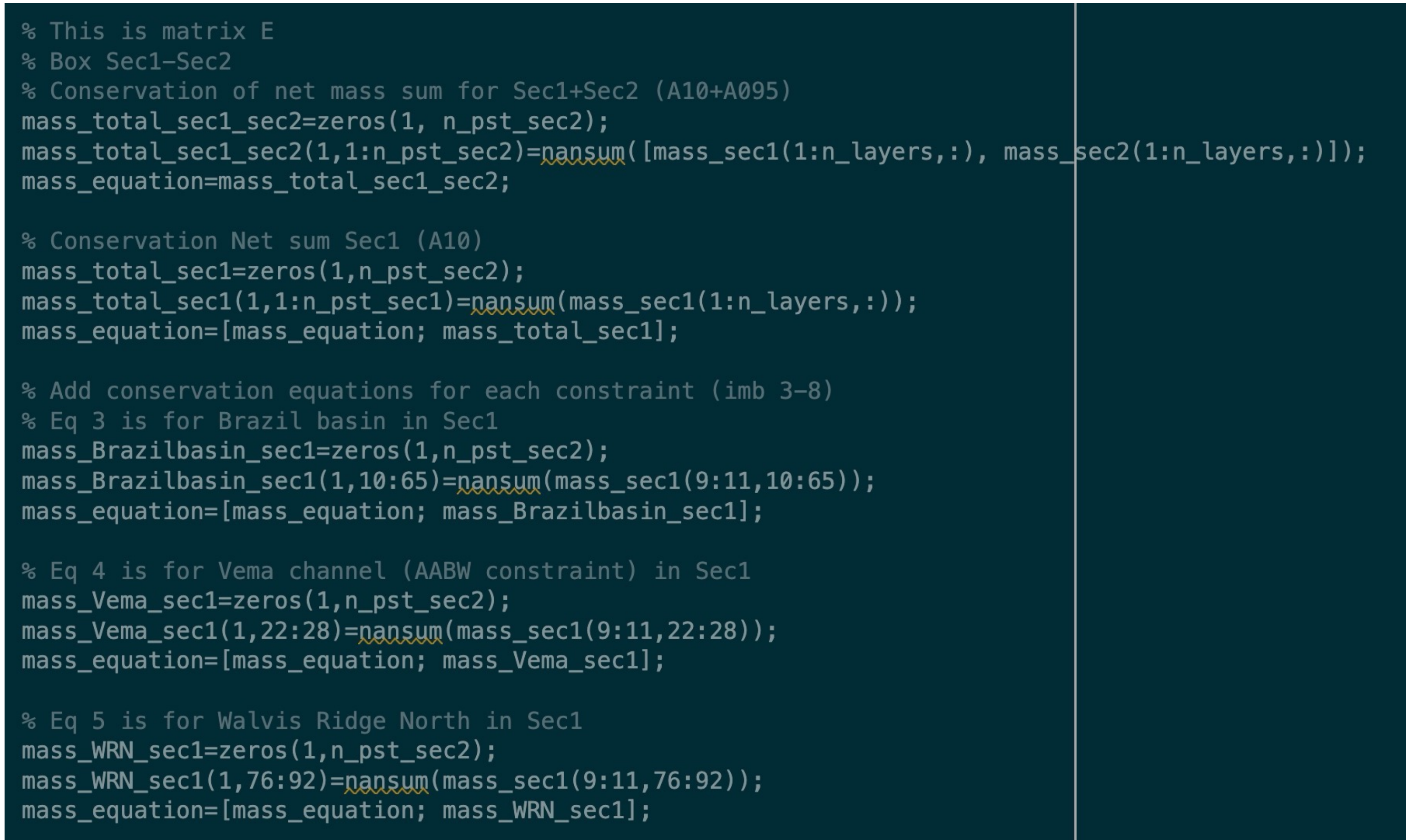
```
% Equation weights for each imbalance
% Box  Sec1-Sec2
eq(1)=1^2*1e18; % net conservation for Sec1+Sec2

% Uncertainties Sec1 (A10)
eq(2)=.6^2*1e18; % Std of Bering transport at A10
eq(3)=1.8^2*1e18;  % Brazil basin in A10
eq(4)=0.4^2*1e18; % Vema channel (AABW constraint) in A10
eq(5)=1.0^2*1e18; % Walvis Ridge North in A10
eq(6)=1.0^2*1e18; % Walvis Ridge South in A10
eq(7)=2.1^2*1e18; % Brazil current (LADCP) in A10
eq(8)=2.4^2*1e18; % Benguela current (LADCP) in A10

% Uncertainties Sec2 (A095)
eq(9)=.6^2*1e18;  % Std of Bering transport at A095
eq(10)=1.2*1e18; % Brazil Current in A095

% Caja A10 - A095
eq(11) = 3.6^2*1e18; % Layer 1 - superficial
eq(12) = 3.6^2*1e18; % Layer 2 - superficial
eq(13) = 3.6^2*1e18; % Layer 3 - superficial
eq(14) = 3.6^2*1e18; % Layer 4 - superficial
eq(15) = 2.2^2*1e18; % Layer 5 - superficial
eq(16) = 2.2^2*1e18; % Layer 6 - intermediate
eq(17) = 2.2^2*1e18; % Layer 7 - intermediate
eq(18) = 1.1^2*1e18; % Layer 8 - intermediate
eq(19) = 1.1^2*1e18; % Layer 9 - deep
eq(20) = 1.1^2*1e18; % Layer 10 - deep
eq(21) = 1.1^2*1e18; % Layer 11 - deep


% Define the noise matrix (r_nn)
r_nn=zeros(numel(imb),numel(imb));
for ii=1:numel(imb)
    r_nn(ii,ii)=r_nn(ii,ii)+eq(ii);
end
```

Inverse model formulation: single box with mass conservation

Call for the inverse model function ***inverse_model_example***

$$\begin{pmatrix} e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\ e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\ e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\ 0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\ e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\ e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\ e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}} \end{pmatrix} = \begin{pmatrix} y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_t} + T_{A_{Ek}} \\ y_{A_{reg}} \\ y_{B_t} + T_{B_{Ek}} \\ y_{B_{reg}} \\ y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_2} + y_{B_2} \\ \vdots \\ \vdots \\ y_{A_{q-1}} + y_{B_{q-1}} \\ y_{A_q} + y_{B_q} \end{pmatrix}$$

Define the layers and stations associated with each equation for mass (matrix E)



```
% This is matrix E
% Box Sec1-Sec2
% Conservation of net mass sum for Sec1+Sec2 (A10+A095)
mass_total_sec1_sec2=zeros(1, n_pst_sec2);
mass_total_sec1_sec2(1,1:n_pst_sec2)=nansum([mass_sec1(1:n_layers,:), mass_sec2(1:n_layers,:)]);
mass_equation=mass_total_sec1_sec2;

% Conservation Net sum Sec1 (A10)
mass_total_sec1=zeros(1,n_pst_sec2);
mass_total_sec1(1,1:n_pst_sec1)=nansum(mass_sec1(1:n_layers,:));
mass_equation=[mass_equation; mass_total_sec1];

% Add conservation equations for each constraint (imb 3-8)
% Eq 3 is for Brazil basin in Sec1
mass_Brazilbasin_sec1=zeros(1,n_pst_sec2);
mass_Brazilbasin_sec1(1,10:65)=nansum(mass_sec1(9:11,10:65));
mass_equation=[mass_equation; mass_Brazilbasin_sec1];

% Eq 4 is for Vema channel (AABW constraint) in Sec1
mass_Vema_sec1=zeros(1,n_pst_sec2);
mass_Vema_sec1(1,22:28)=nansum(mass_sec1(9:11,22:28));
mass_equation=[mass_equation; mass_Vema_sec1];

% Eq 5 is for Walvis Ridge North in Sec1
mass_WRN_sec1=zeros(1,n_pst_sec2);
mass_WRN_sec1(1,76:92)=nansum(mass_sec1(9:11,76:92));
mass_equation=[mass_equation; mass_WRN_sec1];
```

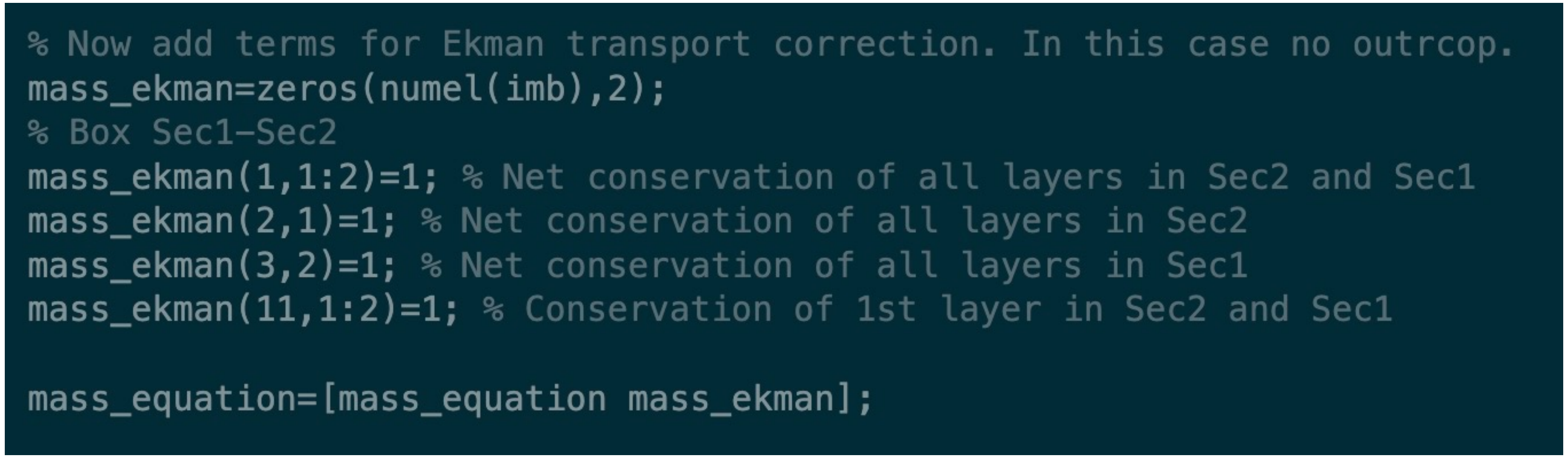
Inverse model formulation: single box with mass conservation

Call for the inverse model function ***inverse_model_example***

$$\begin{pmatrix} e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\ e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\ e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\ 0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\ e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\ e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\ e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}} \end{pmatrix} = \begin{pmatrix} y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_t} + T_{A_{Ek}} \\ y_{A_{reg}} \\ y_{B_t} + T_{B_{Ek}} \\ y_{B_{reg}} \\ y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_2} + y_{B_2} \\ \vdots \\ \vdots \\ y_{A_{q-1}} + y_{B_{q-1}} \\ y_{A_q} + y_{B_q} \end{pmatrix}$$

Add the values for the Ekman correction (0 or 1)

Only in the first layer if no outcrop – if there is outcrop, we have to add the portion of Ekman associated to the amount from each layer that reaches the surface.



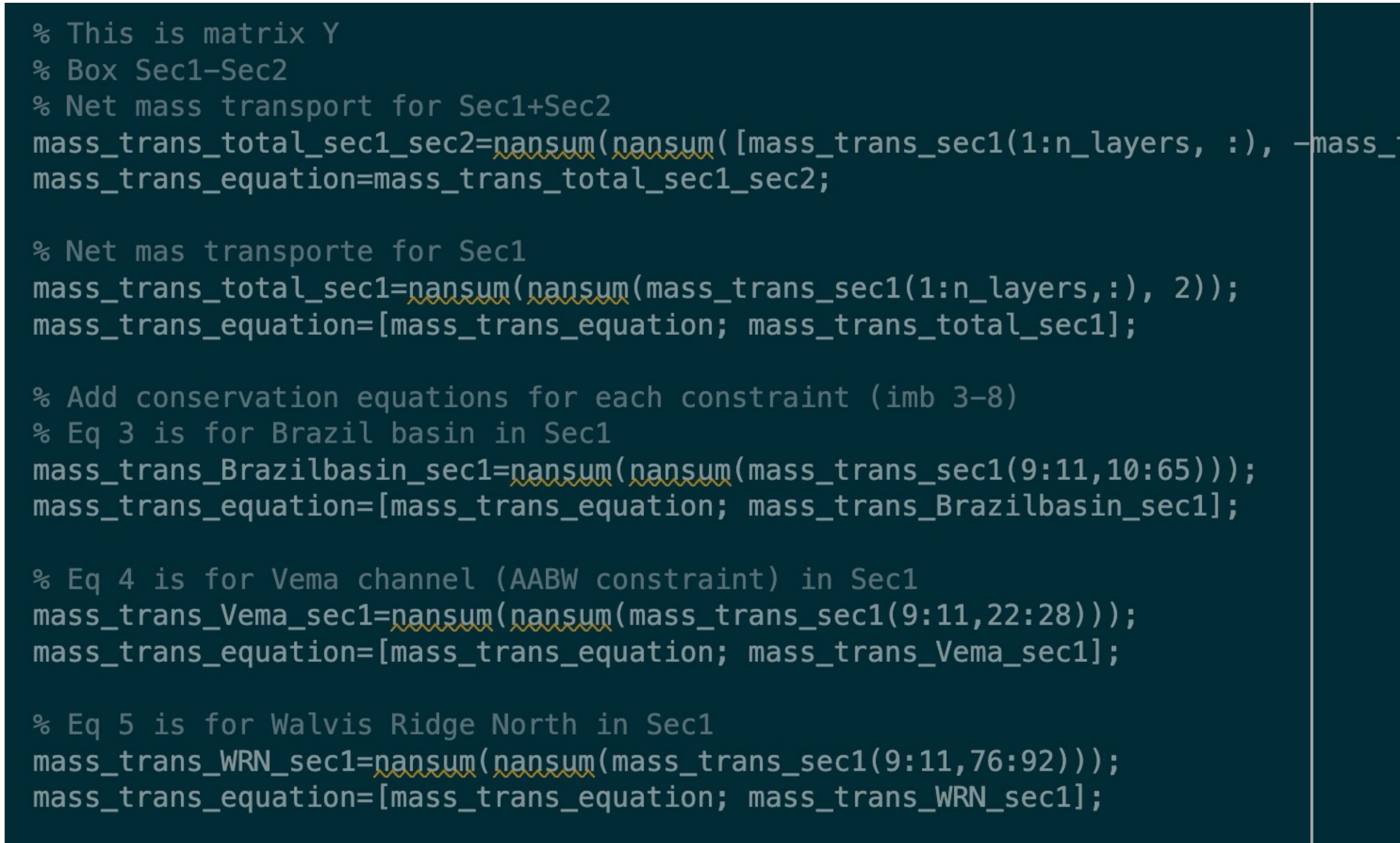
```
% Now add terms for Ekman transport correction. In this case no outrcop.
mass_ekman=zeros(numel(imb),2);
% Box Sec1-Sec2
mass_ekman(1,1:2)=1; % Net conservation of all layers in Sec2 and Sec1
mass_ekman(2,1)=1; % Net conservation of all layers in Sec2
mass_ekman(3,2)=1; % Net conservation of all layers in Sec1
mass_ekman(11,1:2)=1; % Conservation of 1st layer in Sec2 and Sec1

mass_equation=[mass_equation mass_ekman];
```

Inverse model formulation: single box with mass conservation

Call for the inverse model function ***inverse_model_example***

$$\begin{pmatrix} e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\ e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\ e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\ 0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\ e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\ e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\ e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}} \end{pmatrix} = \begin{pmatrix} y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_t} + T_{A_{Ek}} \\ y_{A_{reg}} \\ y_{B_t} + T_{B_{Ek}} \\ y_{B_{reg}} \\ y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_2} + y_{B_2} \\ \vdots \\ \vdots \\ y_{A_{q-1}} + y_{B_{q-1}} \\ y_{A_q} + y_{B_q} \end{pmatrix}$$

Define the layers and stations associated with each equation for mass transport (matrix Y)



```
% This is matrix Y
% Box Sec1-Sec2
% Net mass transport for Sec1+Sec2
mass_trans_total_sec1_sec2=nansum(nansum([mass_trans_sec1(1:n_layers, :), -mass_
mass_trans_equation=mass_trans_total_sec1_sec2;

% Net mas transporte for Sec1
mass_trans_total_sec1=nansum(nansum(mass_trans_sec1(1:n_layers,:), 2));
mass_trans_equation=[mass_trans_equation; mass_trans_total_sec1];

% Add conservation equations for each constraint (imb 3-8)
% Eq 3 is for Brazil basin in Sec1
mass_trans_Brazilbasin_sec1=nansum(nansum(mass_trans_sec1(9:11,10:65)));
mass_trans_equation=[mass_trans_equation; mass_trans_Brazilbasin_sec1];

% Eq 4 is for Vema channel (AABW constraint) in Sec1
mass_trans_Vema_sec1=nansum(nansum(mass_trans_sec1(9:11,22:28)));
mass_trans_equation=[mass_trans_equation; mass_trans_Vema_sec1];

% Eq 5 is for Walvis Ridge North in Sec1
mass_trans_WRN_sec1=nansum(nansum(mass_trans_sec1(9:11,76:92)));
mass_trans_equation=[mass_trans_equation; mass_trans_WRN_sec1];
```

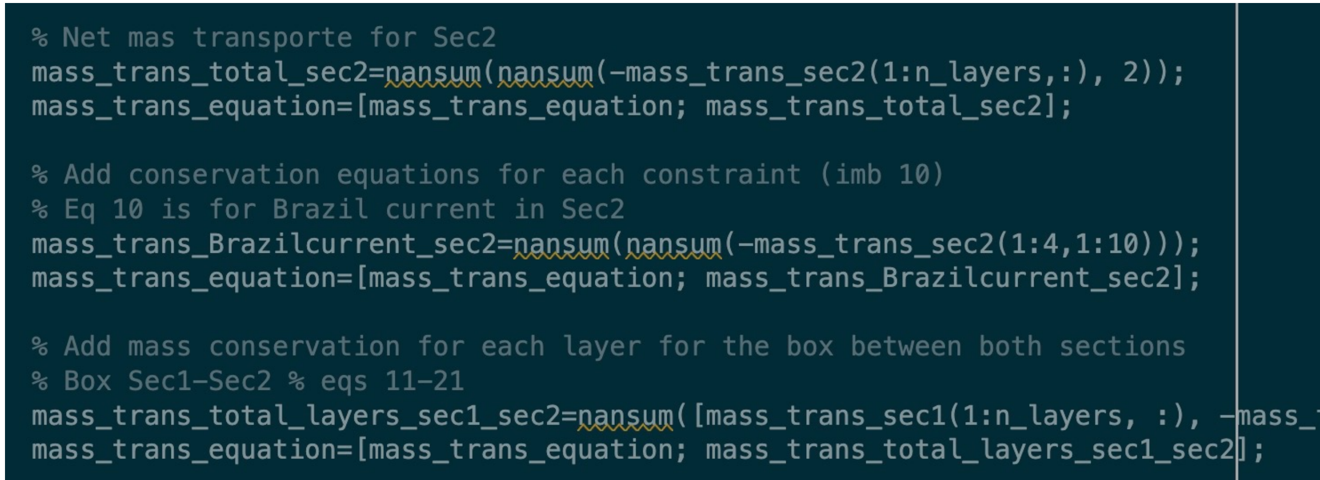
Inverse model formulation: single box with mass conservation

Call for the inverse model function ***inverse_model_example***

$$\begin{pmatrix} e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\ e_{A_{t,1}} & \cdots & e_{A_{t,n}} & 0 & \cdots & 0 & 1 & 0 \\ e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 0 & 1 \\ 0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\ e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\ e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\ e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}} \end{pmatrix} = \begin{pmatrix} y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_t} + T_{A_{Ek}} \\ y_{A_{reg}} \\ y_{B_t} + T_{B_{Ek}} \\ y_{B_{reg}} \\ y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\ y_{A_2} + y_{B_2} \\ \vdots \\ \vdots \\ y_{A_{q-1}} + y_{B_{q-1}} \\ y_{A_q} + y_{B_q} \end{pmatrix}$$

Define the layers and stations associated with each equation for mass transport (matrix Y)

Watch out for the sign – Section 2 has a different sign to account for the box sign rather than geographical sign



```
% Net mas transporte for Sec2
mass_trans_total_sec2=nansum(nansum(-mass_trans_sec2(1:n_layers,:), 2));
mass_trans_equation=[mass_trans_equation; mass_trans_total_sec2];

% Add conservation equations for each constraint (imb 10)
% Eq 10 is for Brazil current in Sec2
mass_trans_Brazilcurrent_sec2=nansum(nansum(-mass_trans_sec2(1:4,1:10)));
mass_trans_equation=[mass_trans_equation; mass_trans_Brazilcurrent_sec2];

% Add mass conservation for each layer for the box between both sections
% Box Sec1-Sec2 % eqs 11-21
mass_trans_total_layers_sec1_sec2=nansum([mass_trans_sec1(1:n_layers, :), -mass_
mass_trans_equation=[mass_trans_equation; mass_trans_total_layers_sec1_sec2];
```

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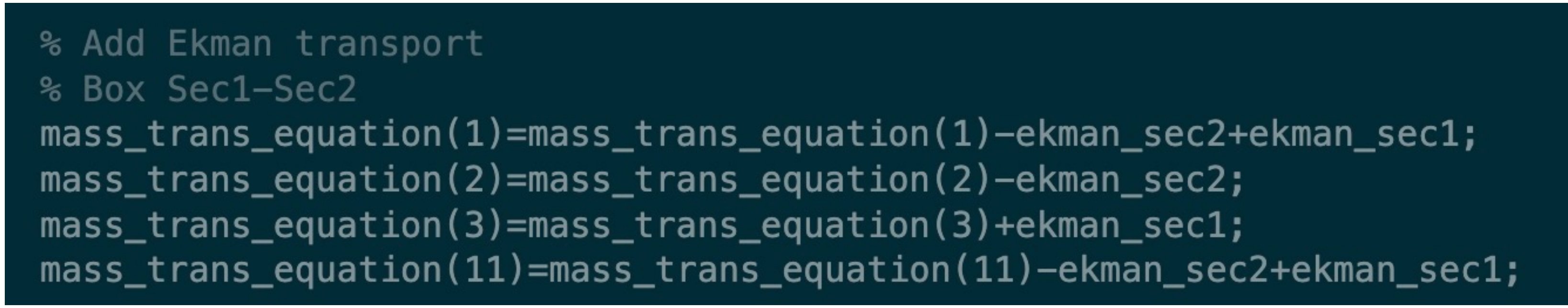
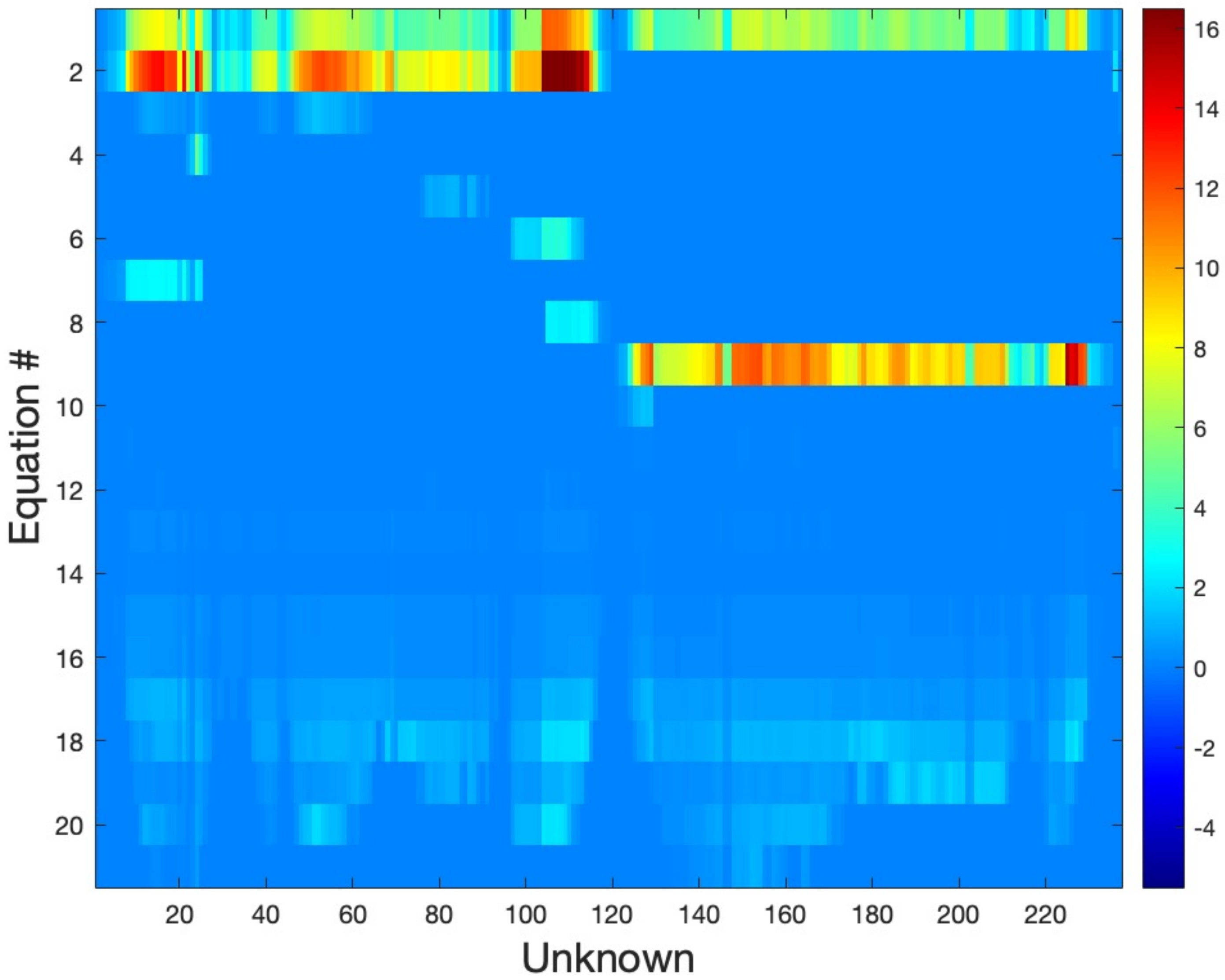
Add the values of Ekman transport to the equations containing the first layer (if no outcropping)



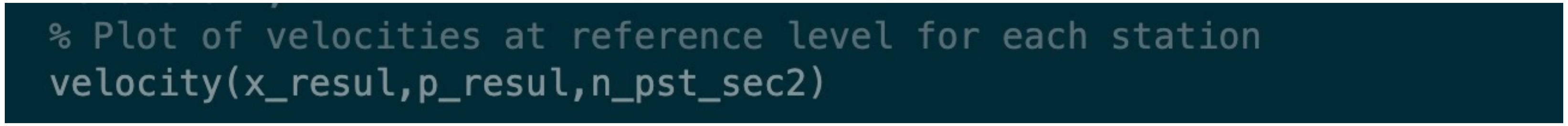
```
% Add Ekman transport
% Box Sec1-Sec2
mass_trans_equation(1)=mass_trans_equation(1)-ekman_sec2+ekman_sec1;
mass_trans_equation(2)=mass_trans_equation(2)-ekman_sec2;
mass_trans_equation(3)=mass_trans_equation(3)+ekman_sec1;
mass_trans_equation(11)=mass_trans_equation(11)-ekman_sec2+ekman_sec1;
```

Inverse model formulation: single box with mass conservation

Diagnostic plots from the inverse model solution

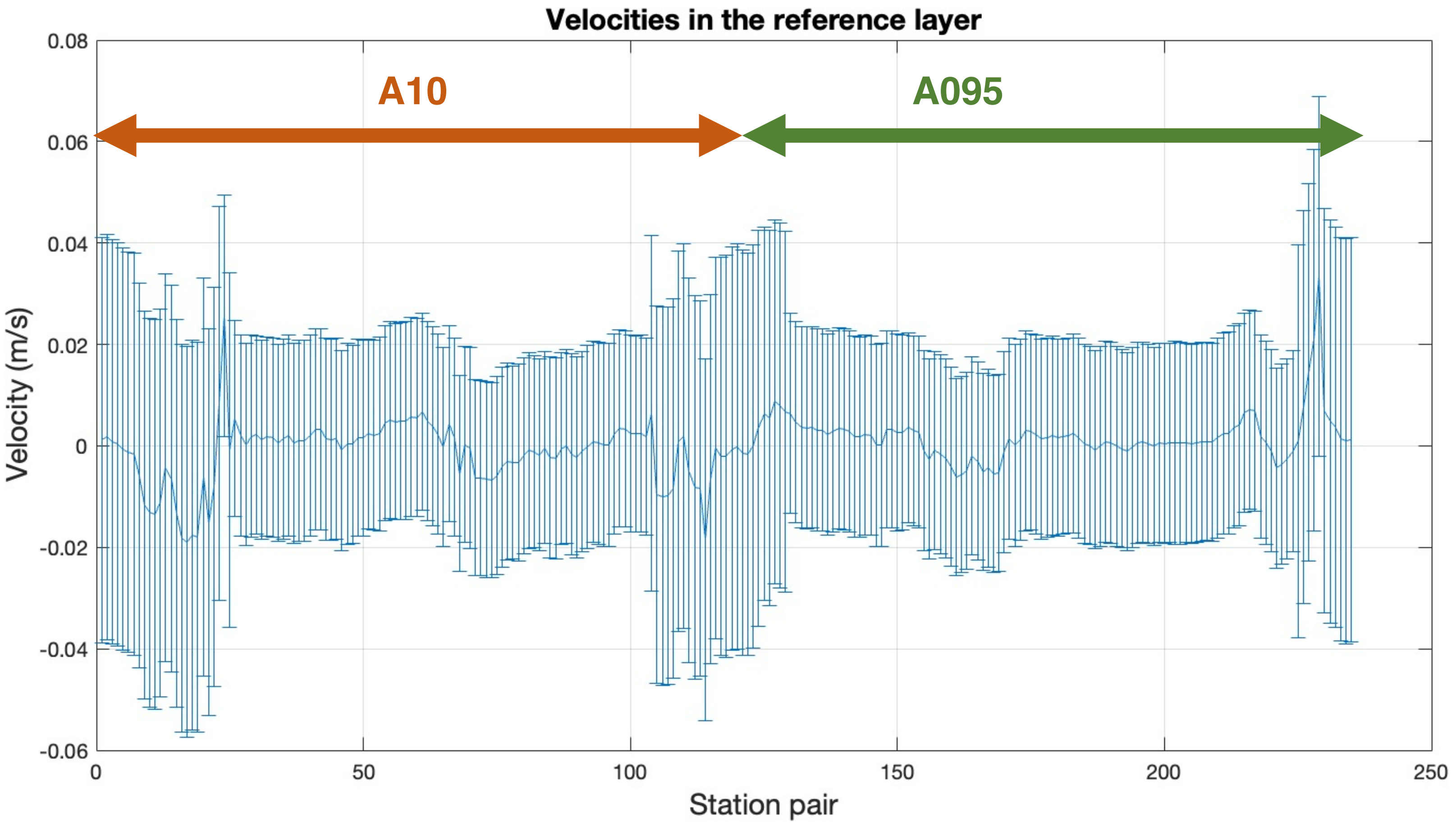


Inverse model formulation: single box with mass conservation

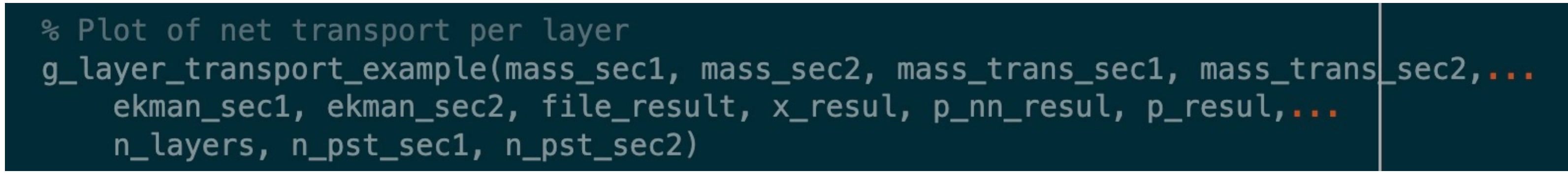


```
% Plot of velocities at reference level for each station
velocity(x_resul,p_resul,n_pst_sec2)
```

Results from the inverse model solution:
VELOCITY AT REFERENCE LEVEL FOR EACH STATION



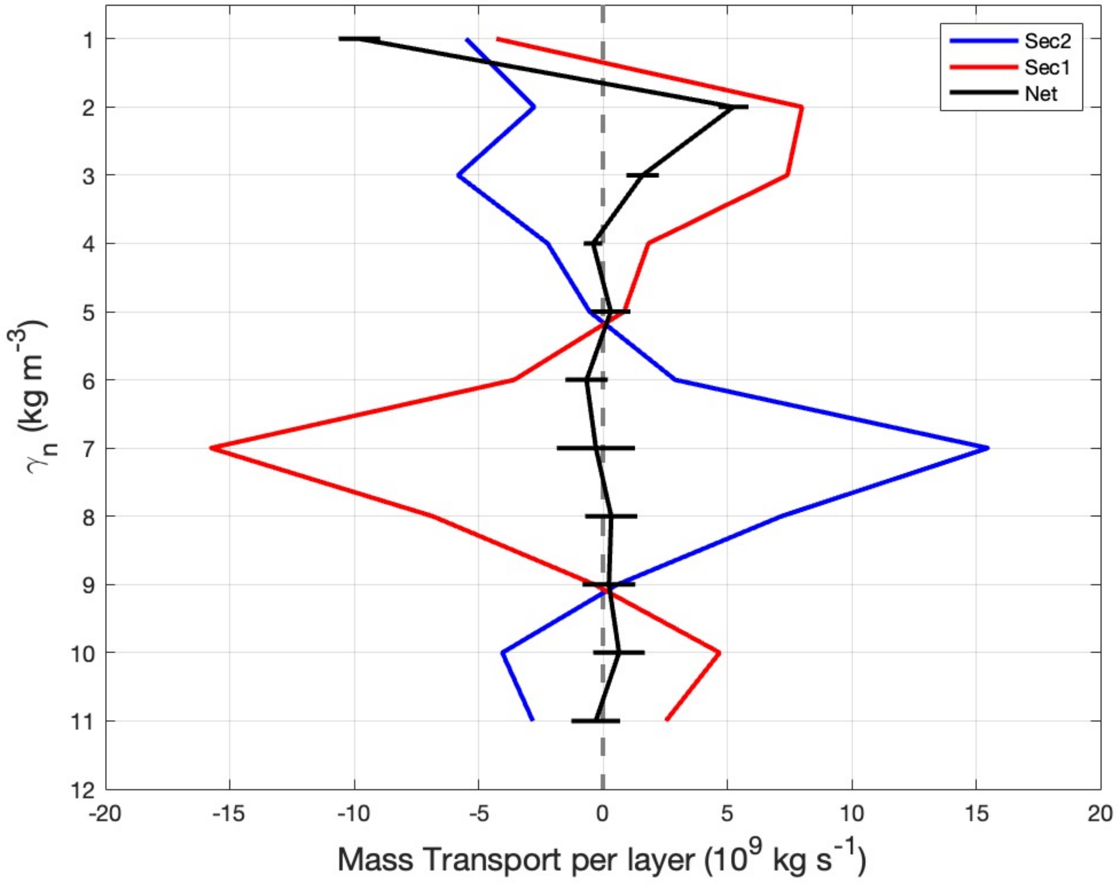
Inverse model formulation: single box with mass conservation



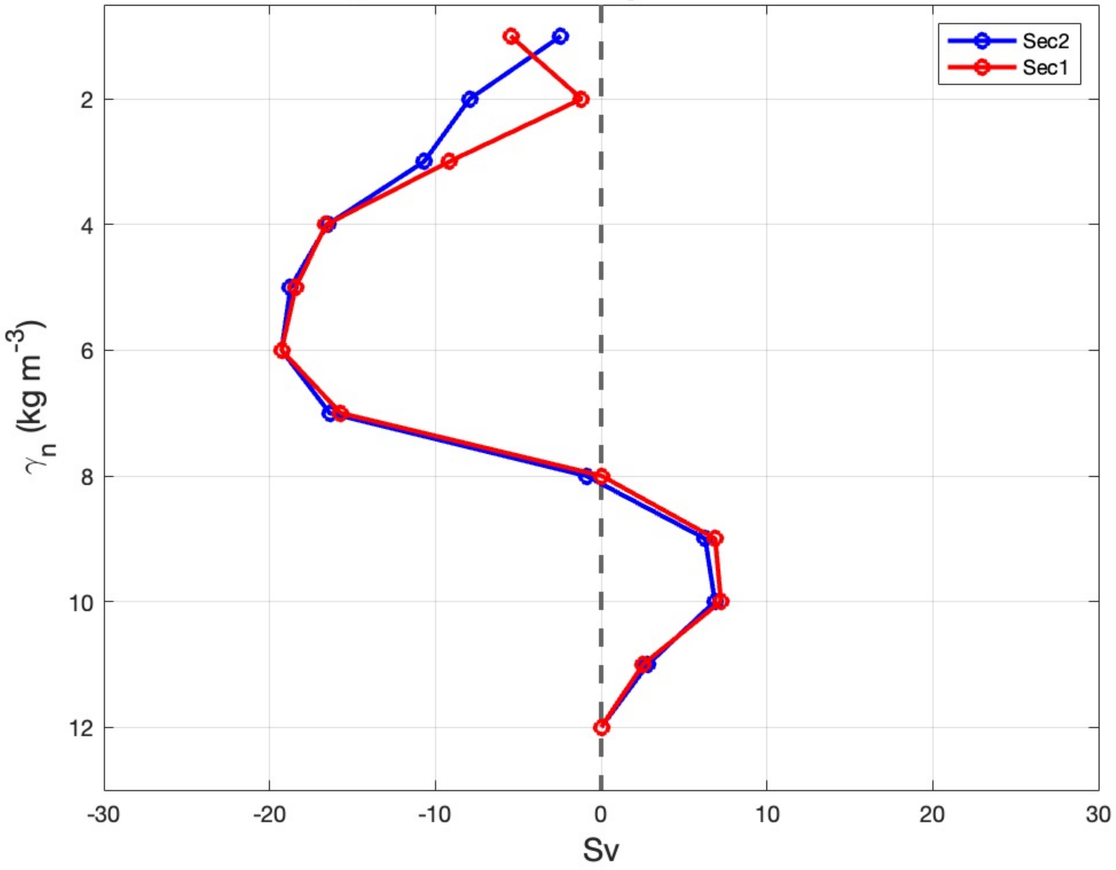
```
% Plot of net transport per layer
g_layer_transport_example(mass_sec1, mass_sec2, mass_trans_sec1, mass_trans_sec2,...
    ekman_sec1, ekman_sec2, file_result, x_resul, p_nn_resul, p_resul,...
    n_layers, n_pst_sec1, n_pst_sec2)
```

Results from the inverse model solution:
MASS TRANSPORT PER LAYER AND OVERTURNING STREAMFUNCTION

Net layer transports - MASS CONSERVATION



Adjusted Overturning Streamfunction

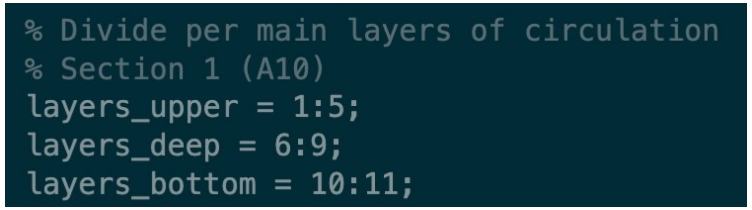


Watch out with the sign convention!

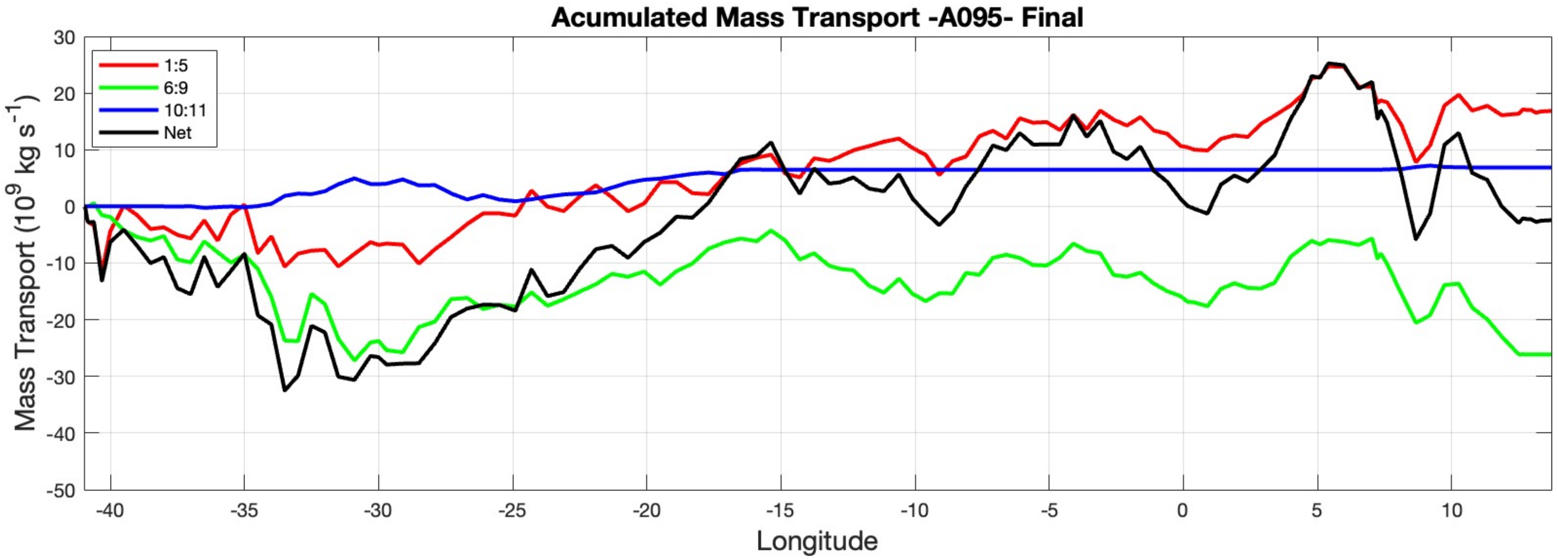
Inverse model formulation: single box with mass conservation

```
% Grafico de transporte acumulado
g_accumulated_transport_example(file_result);
```

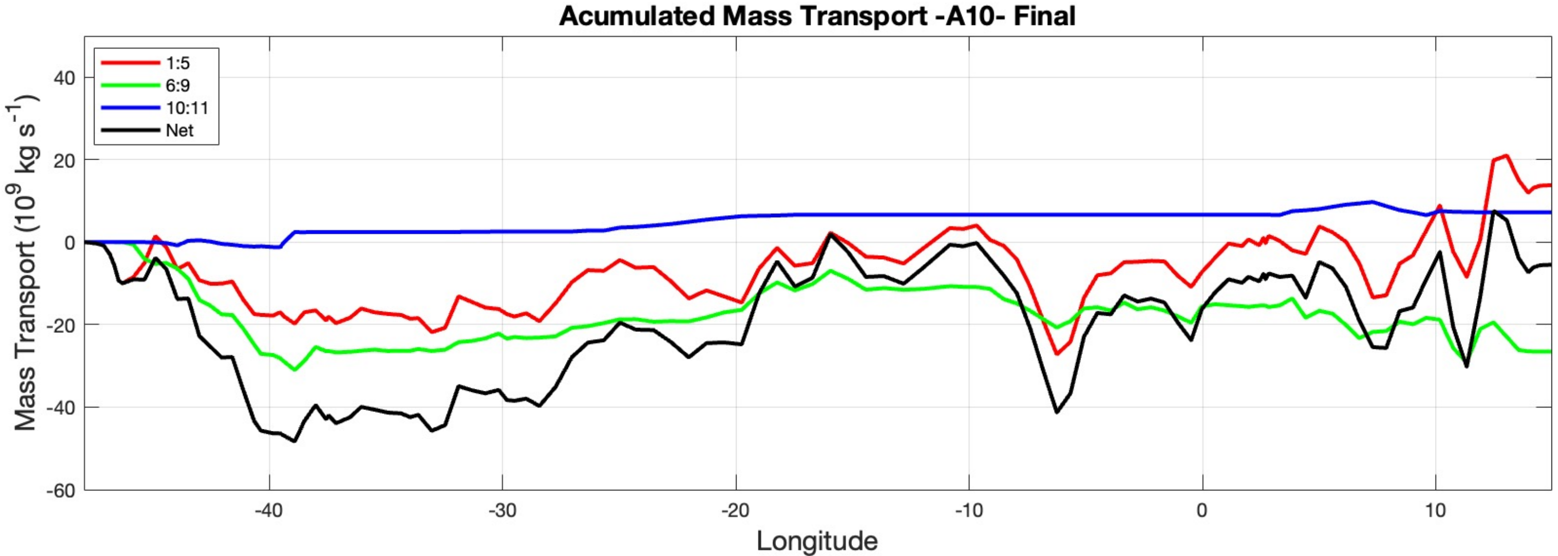
Results from the inverse model solution:
HORIZONTAL ACCUMULATED TRANSPORT



```
% Divide per main layers of circulation
% Section 1 (A10)
layers_upper = 1:5;
layers_deep = 6:9;
layers_bottom = 10:11;
```



Acumulated Mass Transport -A095- Final



Acumulated Mass Transport -A10- Final

Inverse model formulation: single box with mass conservation

EXERCISE: modify some part of the inverse model computation and answer these questions

1) What have you changed and how?
2) What effects are you expecting from this change and how will you be able to diagnose them?
3) Have you found these changes in the results? If not, what might be the reason?

Examples:

- Changing the weight of the equations: what would happen if we didn't prioritize the net transport across the section?
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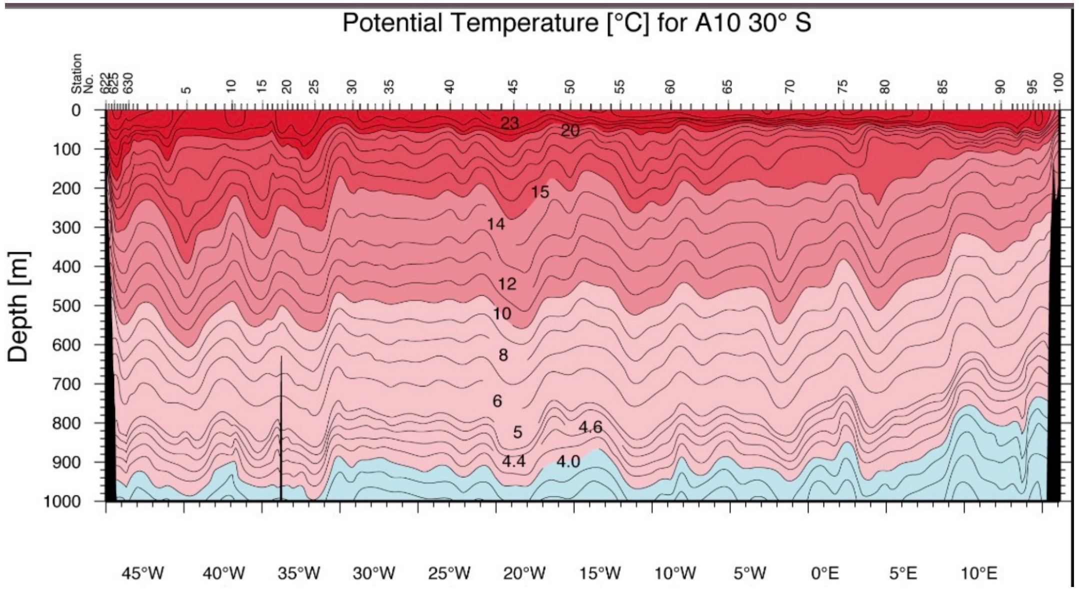
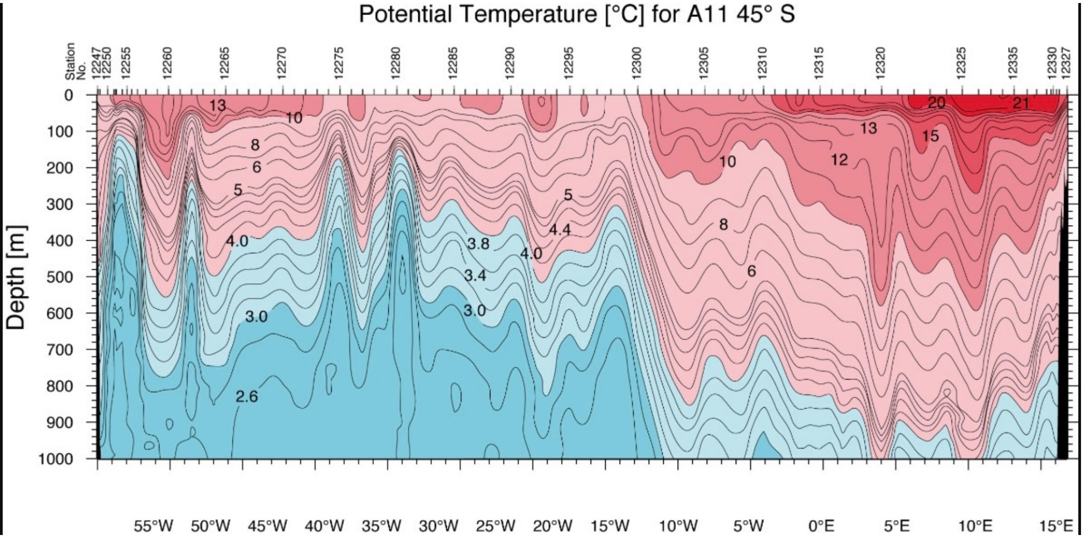
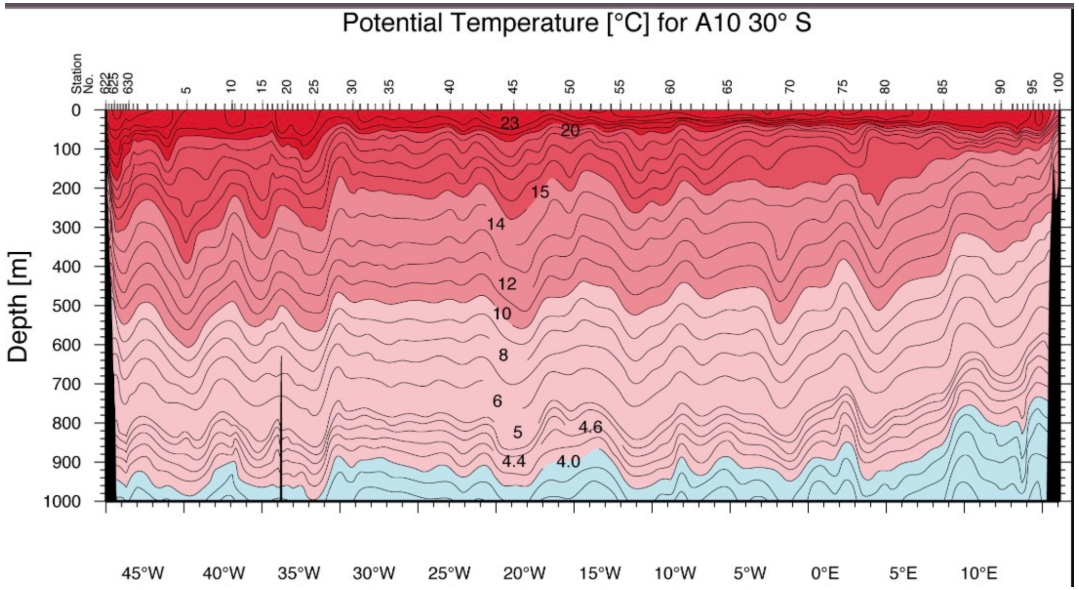
- What would happen if used the conservation of salt flux across Bering Strait for each single section instead of mass?

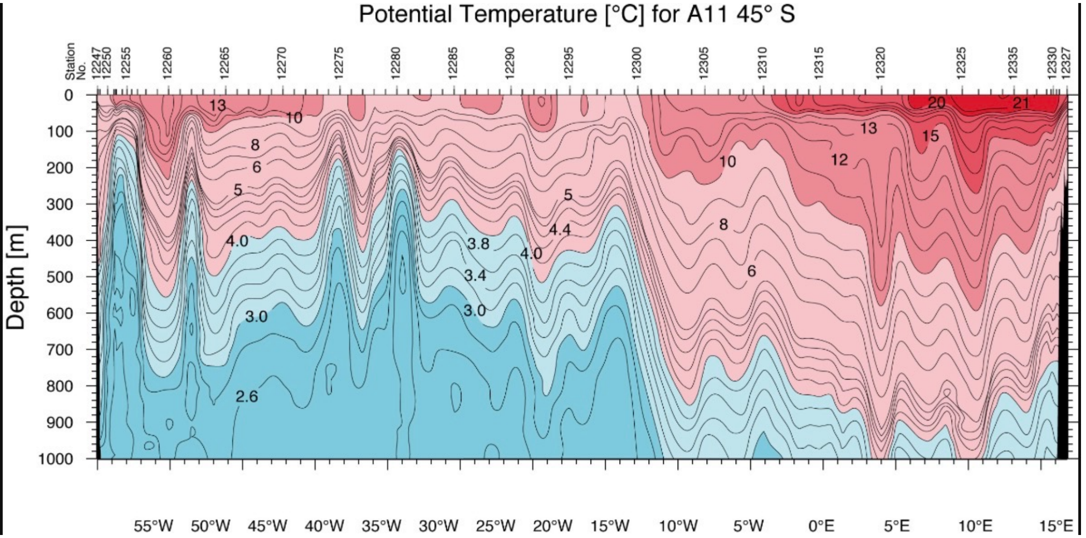
Ekman transport outcropping


Potential Temperature [°C] for A10 30° S

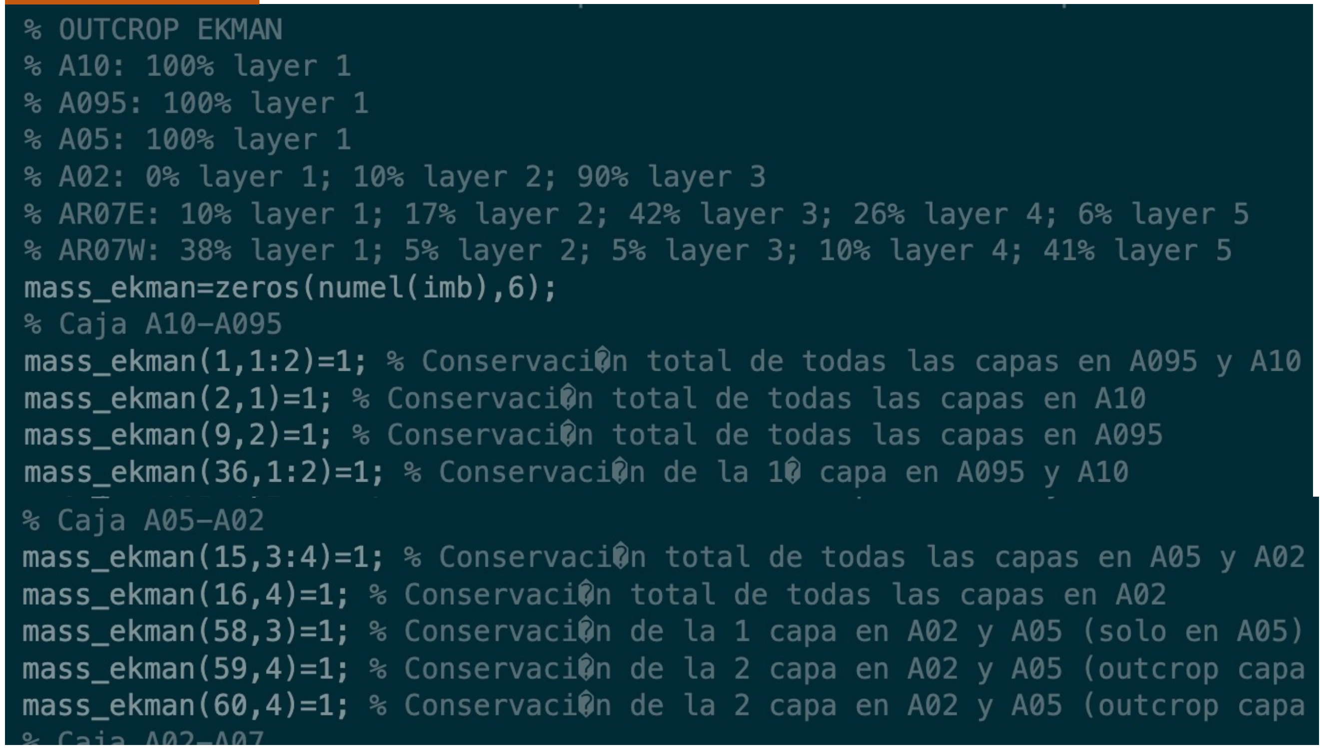

Potential Temperature [°C] for A11 45° S

Ekman transport outcropping



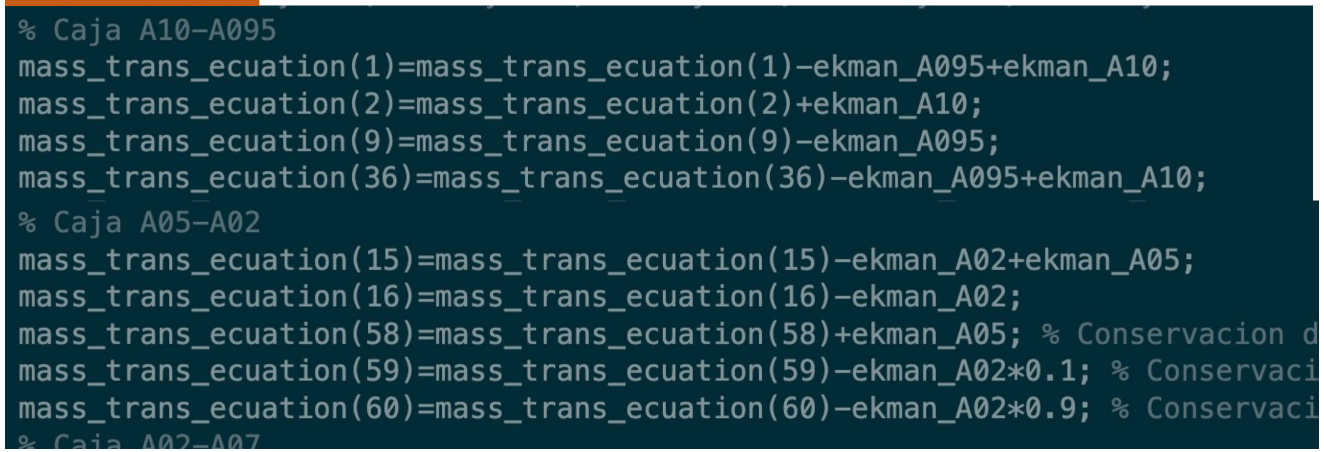


Matrix E



```
% OUTCROP EKMAN
% A10: 100% layer 1
% A095: 100% layer 1
% A05: 100% layer 1
% A02: 0% layer 1; 10% layer 2; 90% layer 3
% AR07E: 10% layer 1; 17% layer 2; 42% layer 3; 26% layer 4; 6% layer 5
% AR07W: 38% layer 1; 5% layer 2; 5% layer 3; 10% layer 4; 41% layer 5
mass_ekman=zeros(numel(imb),6);
% Caja A10-A095
mass_ekman(1,1:2)=1; % Conservaci�n total de todas las capas en A095 y A10
mass_ekman(2,1)=1; % Conservaci�n total de todas las capas en A10
mass_ekman(9,2)=1; % Conservaci�n total de todas las capas en A095
mass_ekman(36,1:2)=1; % Conservaci�n de la 1� capa en A095 y A10
% Caja A05-A02
mass_ekman(15,3:4)=1; % Conservaci�n total de todas las capas en A05 y A02
mass_ekman(16,4)=1; % Conservaci�n total de todas las capas en A02
mass_ekman(58,3)=1; % Conservaci�n de la 1 capa en A02 y A05 (solo en A05)
mass_ekman(59,4)=1; % Conservaci�n de la 2 capa en A02 y A05 (outcrop capa
mass_ekman(60,4)=1; % Conservaci�n de la 2 capa en A02 y A05 (outcrop capa
% Caja A02-A07
```

Matrix Y



```
% Caja A10-A095
mass_trans_ecuation(1)=mass_trans_ecuation(1)-ekman_A095+ekman_A10;
mass_trans_ecuation(2)=mass_trans_ecuation(2)+ekman_A10;
mass_trans_ecuation(9)=mass_trans_ecuation(9)-ekman_A095;
mass_trans_ecuation(36)=mass_trans_ecuation(36)-ekman_A095+ekman_A10;
% Caja A05-A02
mass_trans_ecuation(15)=mass_trans_ecuation(15)-ekman_A02+ekman_A05;
mass_trans_ecuation(16)=mass_trans_ecuation(16)-ekman_A02;
mass_trans_ecuation(58)=mass_trans_ecuation(58)+ekman_A05; % Conservacion d
mass_trans_ecuation(59)=mass_trans_ecuation(59)-ekman_A02*0.1; % Conservaci
mass_trans_ecuation(60)=mass_trans_ecuation(60)-ekman_A02*0.9; % Conservaci
% Caja A02-A07
```

Inverse modelling - MASS + SALT

$$
\begin{array}{l}
\text{Total conservation for each box (both sections)} \\
\text{Regional constraints for each section} \\
\\
\text{Conservation for each layer (both sections)} \\
\\
\\
\\
\\
\\
\text{Total conservation for each section} \\
\\
\end{array}
\begin{pmatrix}
e_{A_{t,1}} & \cdots & e_{A_{t,n}} & e_{B_{t,1}} & \cdots & e_{B_{t,m}} & 1 & 1 \\
e_{A_{reg}} & \cdots & e_{A_{reg}} & 0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & e_{B_{reg}} & \cdots & e_{B_{reg}} & 0 & 0 \\
e_{A_{1,1}} & \cdots & e_{A_{1,n}} & e_{B_{1,1}} & \cdots & e_{B_{1,n}} & 1 & 1 \\
e_{A_{2,1}} & \cdots & e_{A_{2,n}} & e_{B_{2,1}} & \cdots & e_{B_{2,n}} & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
e_{A_{q-1,1}} & \cdots & e_{A_{q-1,n}} & e_{B_{q-1,1}} & \cdots & e_{B_{q-1,n}} & 0 & 0 \\
e_{A_{q,1}} & \cdots & e_{A_{q,n}} & e_{B_{q,1}} & \cdots & e_{B_{q,n}} & 0 & 0 \\
s_{A_{t,1}} & \cdots & s_{A_{t,n}} & 0 & \cdots & 0 & \overline{\left(\frac{s_{A_1}}{e_{A_1}}\right)} & 0 \\
0 & \cdots & 0 & s_{B_{t,1}} & \cdots & s_{B_{t,m}} & 0 & \overline{\left(\frac{s_{B_1}}{e_{B_1}}\right)}
\end{pmatrix}
\begin{pmatrix}
b_{A_1} \\ \vdots \\ b_{A_n} \\ b_{B_1} \\ \vdots \\ b_{B_m} \\ \Delta T_{A_{Ek}} \\ \Delta T_{B_{Ek}}
\end{pmatrix}
=
\begin{pmatrix}
y_{A_t} + y_{B_t} + T_{A_{Ek}} + T_{B_{Ek}} \\
y_{A_{reg}} \\
y_{B_{reg}} \\
y_{A_1} + y_{B_1} + T_{A_{Ek}} + T_{B_{Ek}} \\
y_{A_2} + y_{B_2} \\
\vdots \\
\vdots \\
y_{A_{q-1}} + y_{B_{q-1}} \\
y_{A_q} + y_{B_q} \\
z_t + T_{A_{Ek}} \cdot \overline{\left(\frac{s_{A_1}}{e_{A_1}}\right)} \\
z_{B_t} + T_{B_{Ek}} \cdot \overline{\left(\frac{s_{B_1}}{e_{B_1}}\right)}
\end{pmatrix}
$$

e: mass
s: salt

y: mass transport
z: salt transport

b: reference velocities
ΔT_{Ek}: Ekman correction
T_{Ek}: Ekman transport

n: number of pair of stations for section A
m: number of pair of stations for section B
q: number of layers (11)

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Send your slides to: vcainzo1@ictp.it