Periodic Homogenization of Permeability for Upscaling Topography in Ocean Models

4th COMMODORE Workshop, Trieste, September 2026

Nicholas Kevlahan
Department of Mathematics and Statistics, McMaster University

SCIENCE
Department of
Mathematics & Statistics
McMaster
University
NSERC
CRSNG

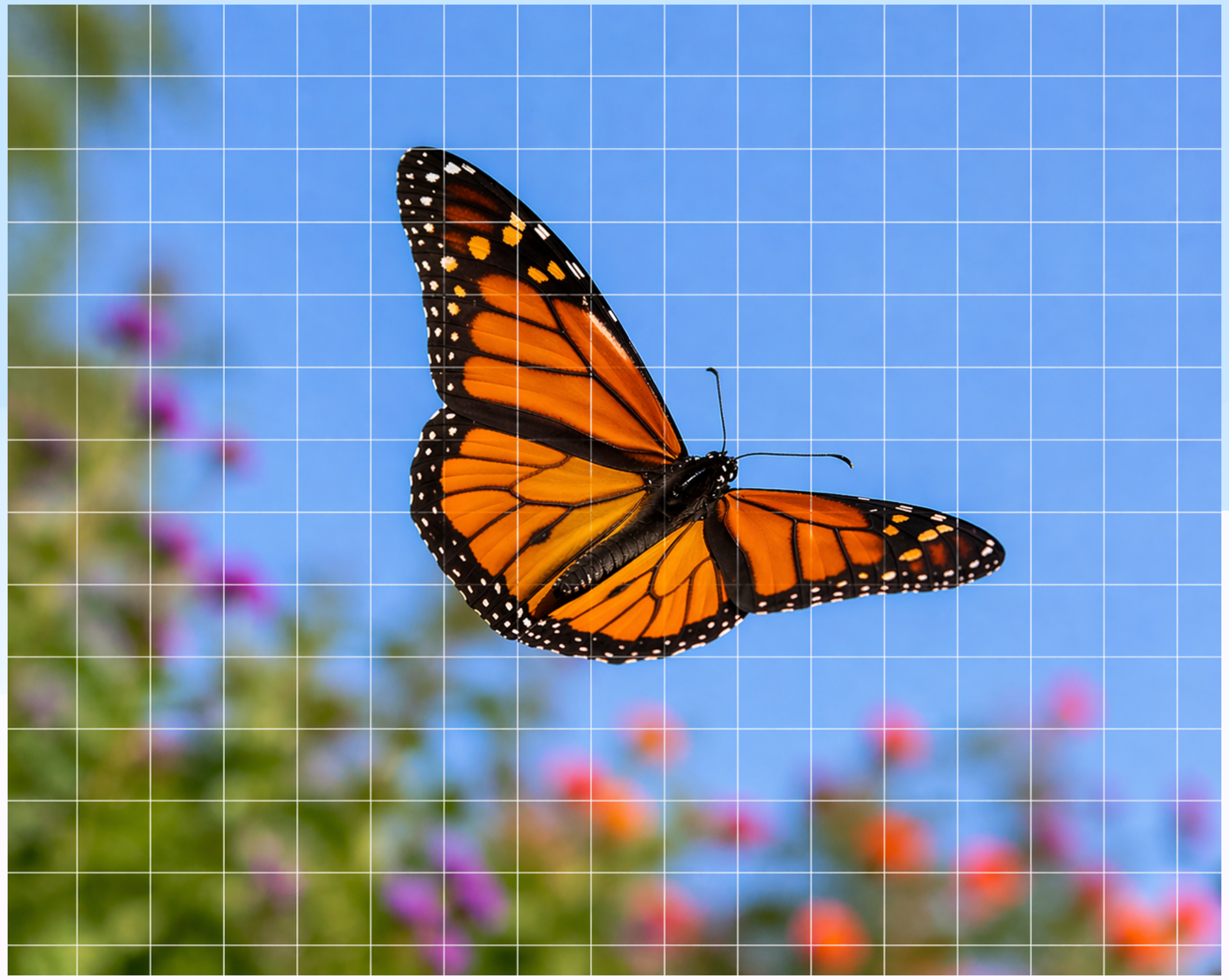
Collaborators

Lisa Patrascu
Institute for Aerospace Studies, University of Toronto, Canada
former MSc student at McMaster University

Laurent Debreu
INRIA, Université Grenoble-Alpes, France

Patrick Marchesiello
LEGOS, IRD, Toulouse, France

The challenge of complex geometries



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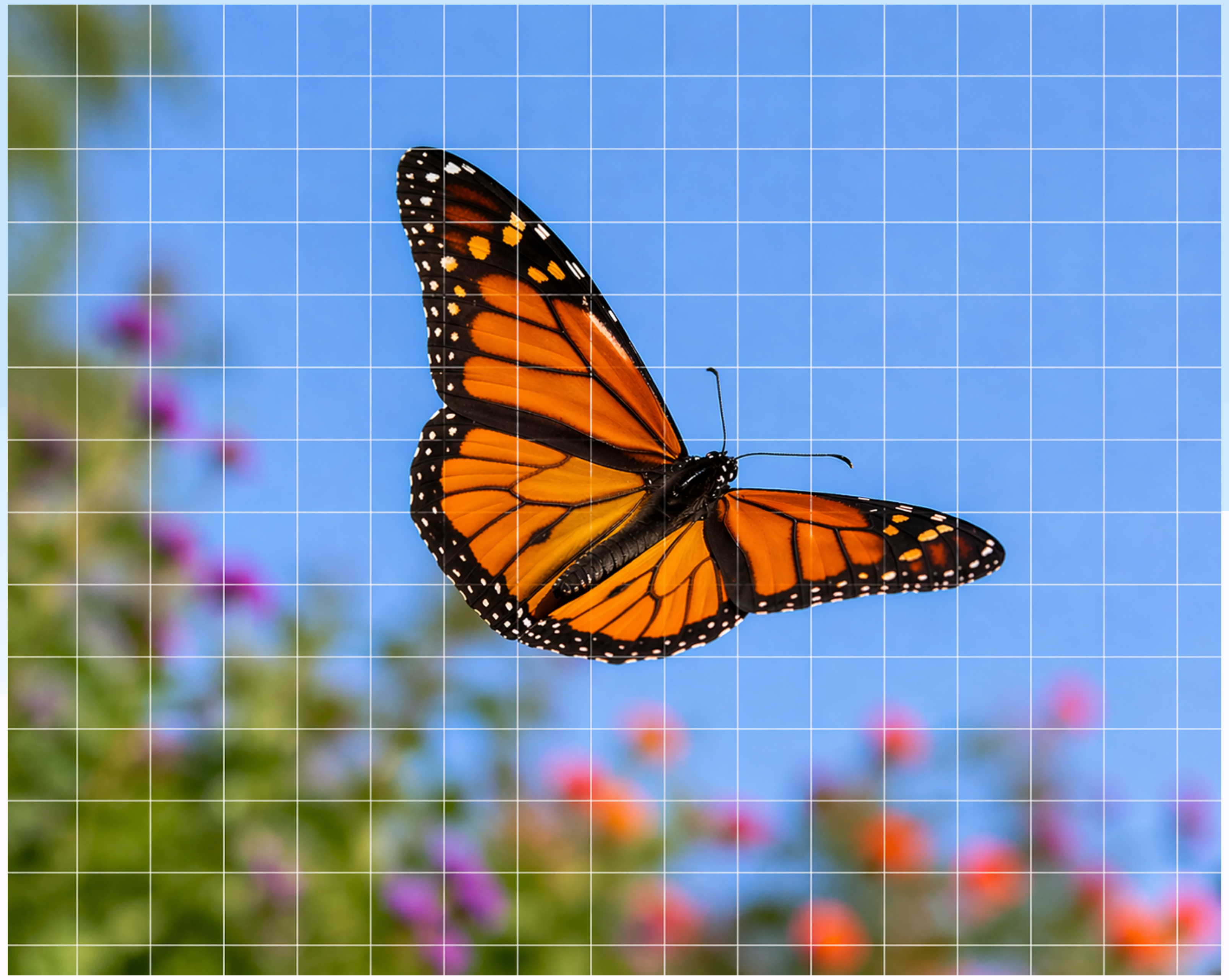

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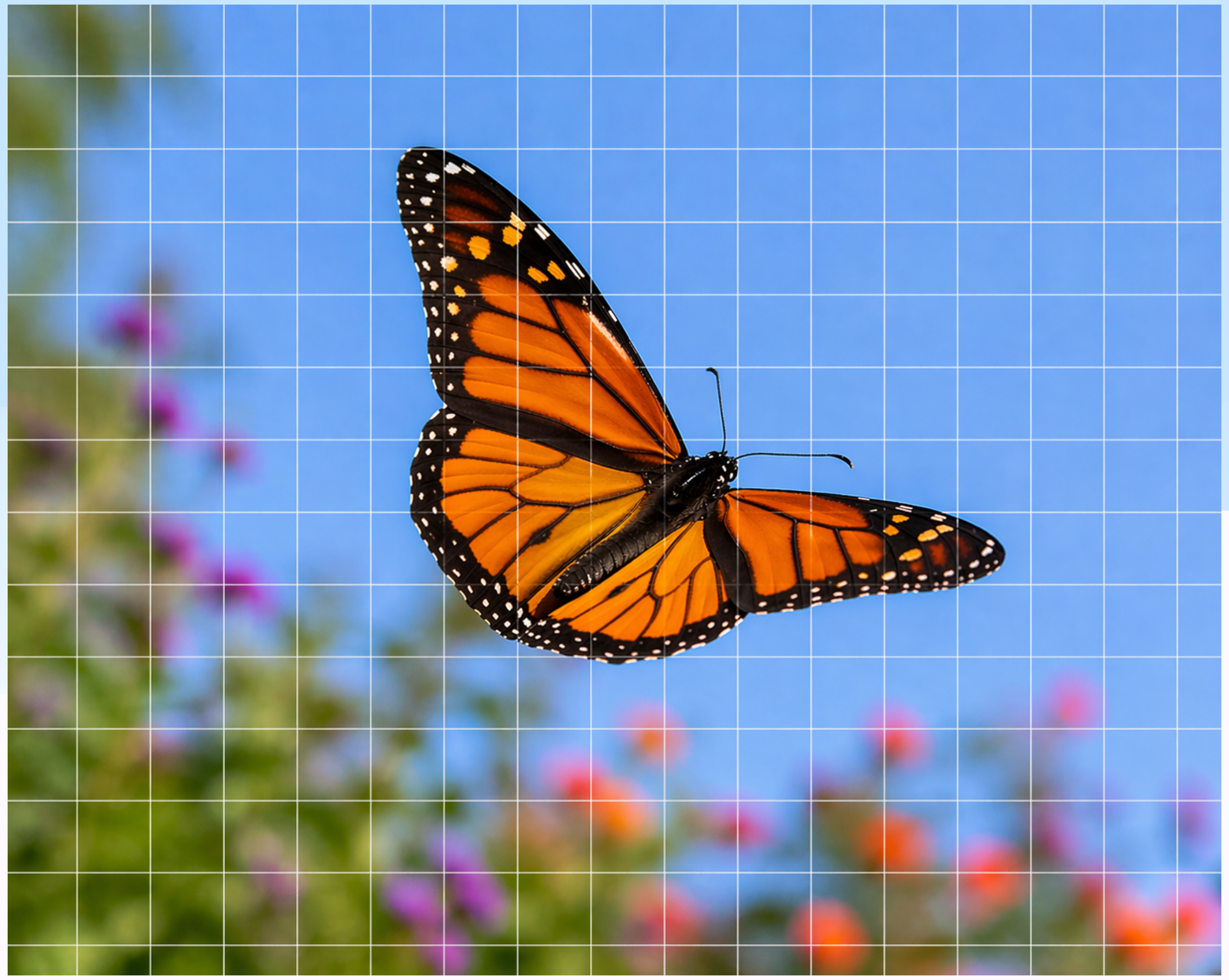

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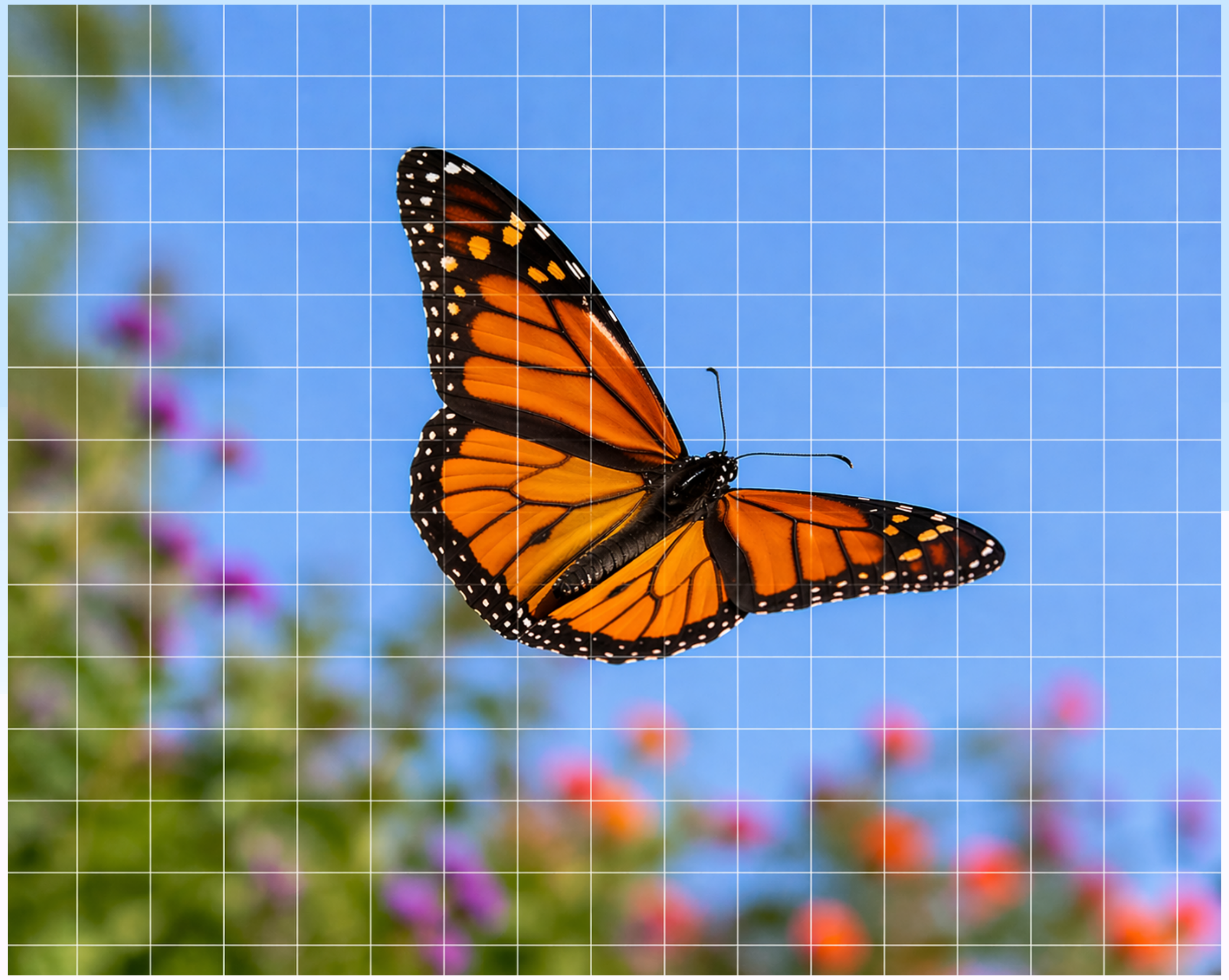

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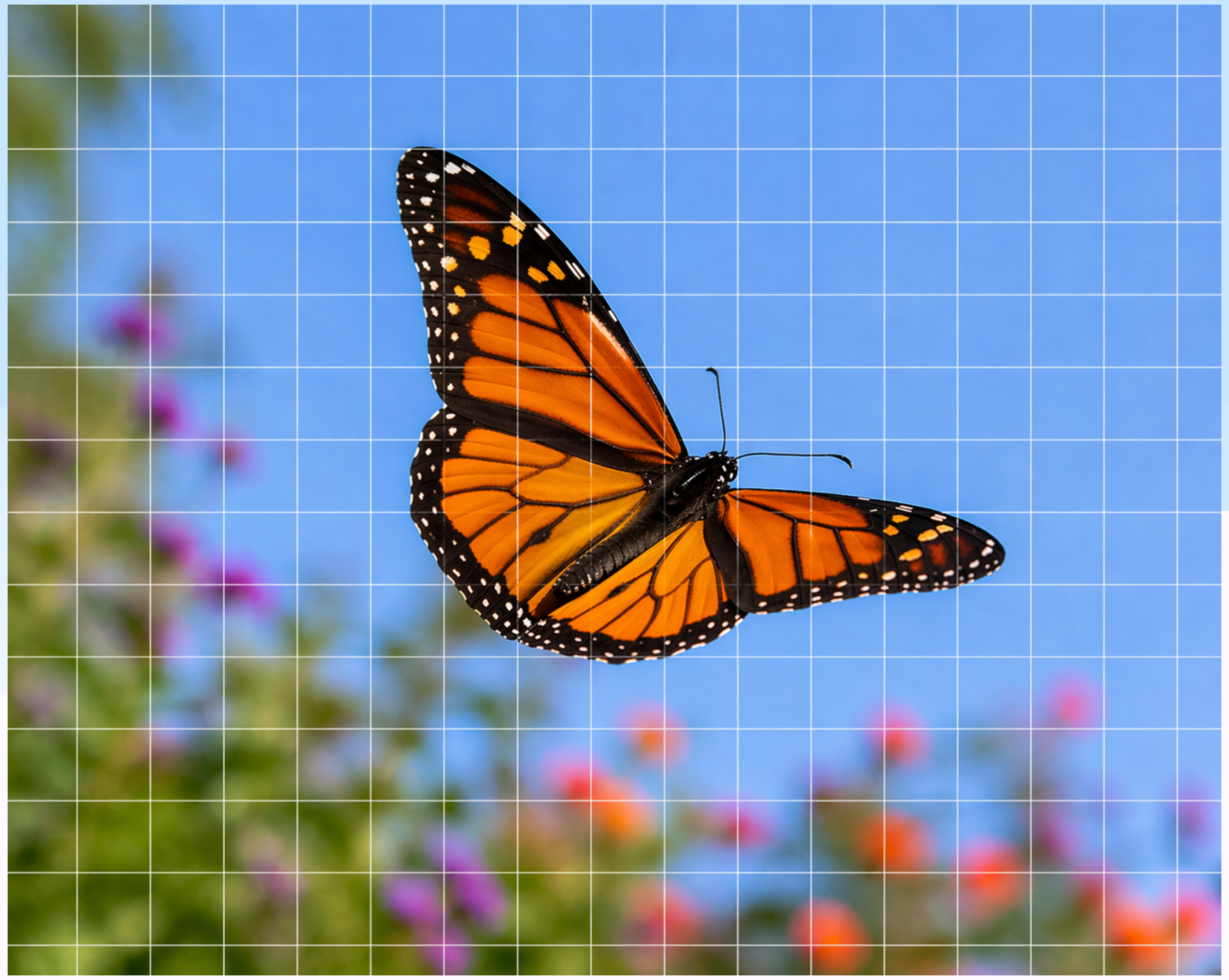

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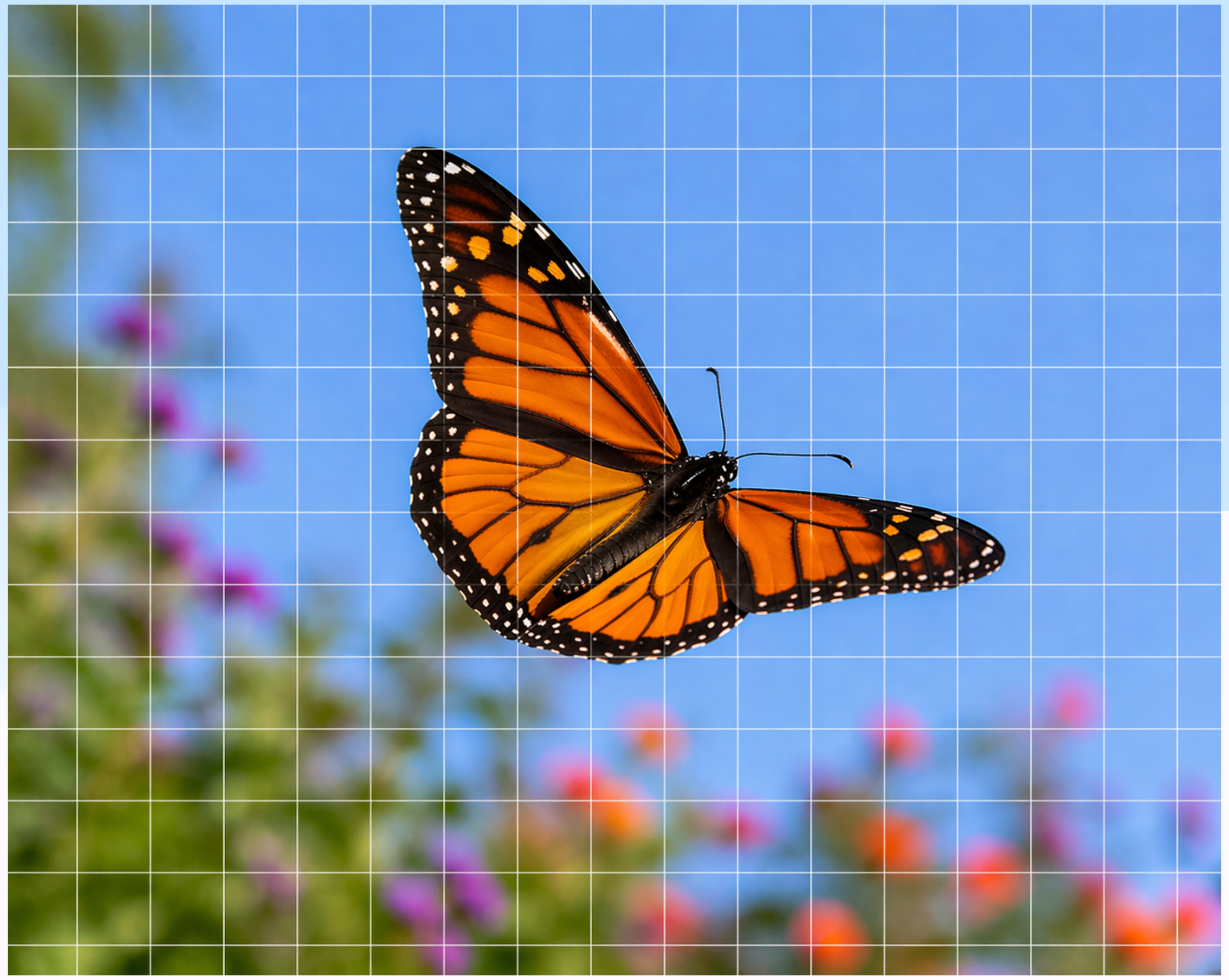

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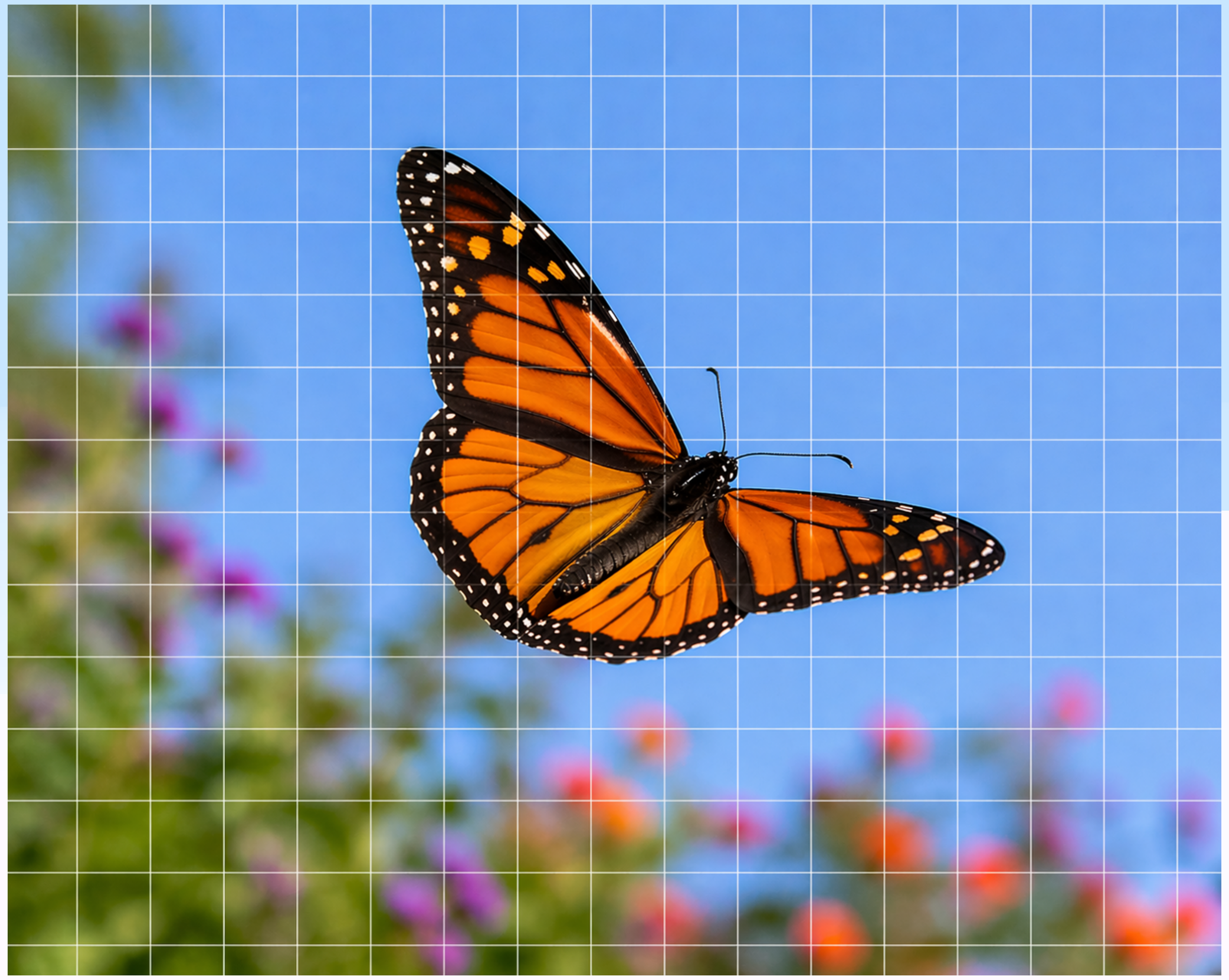

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Hard!

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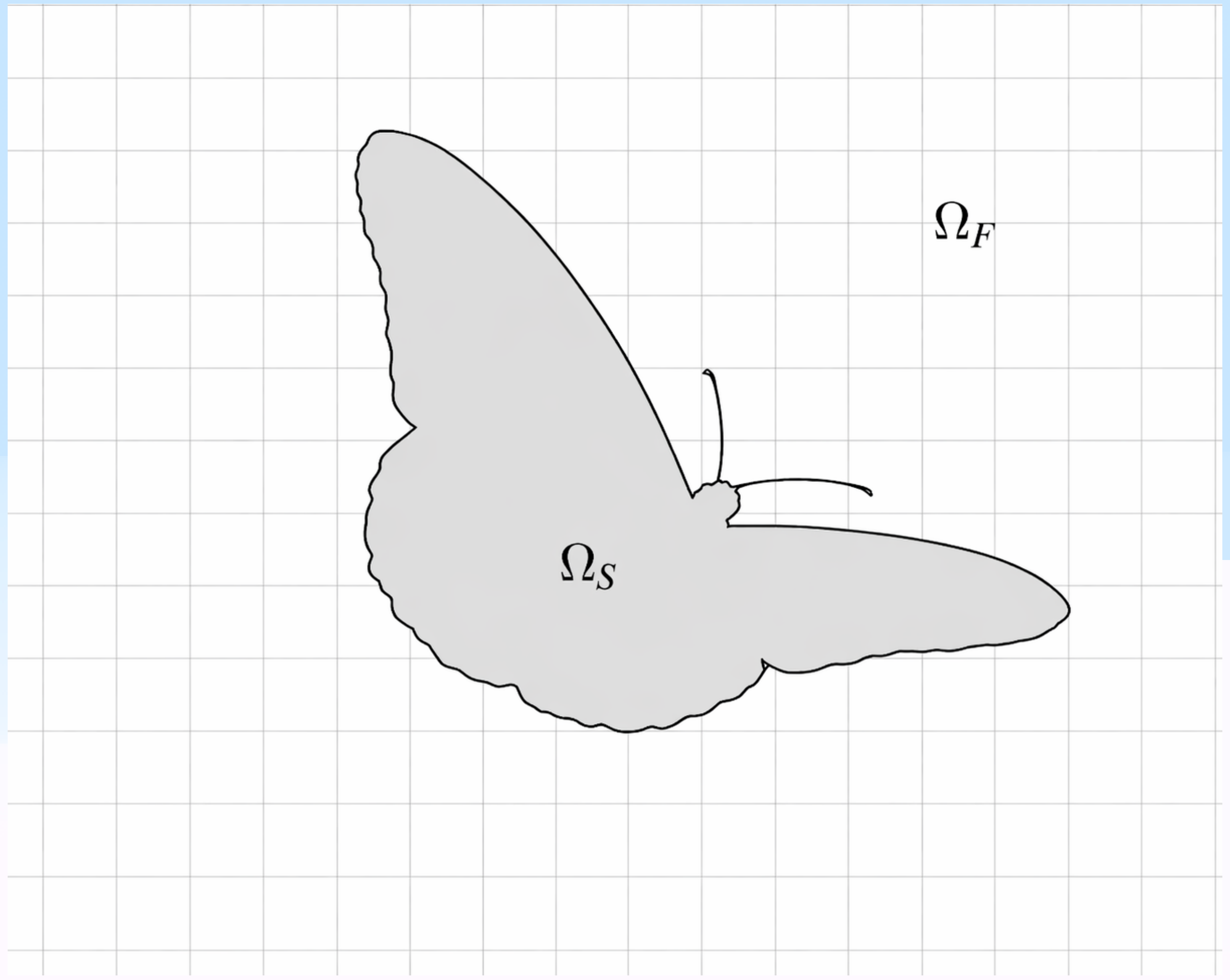

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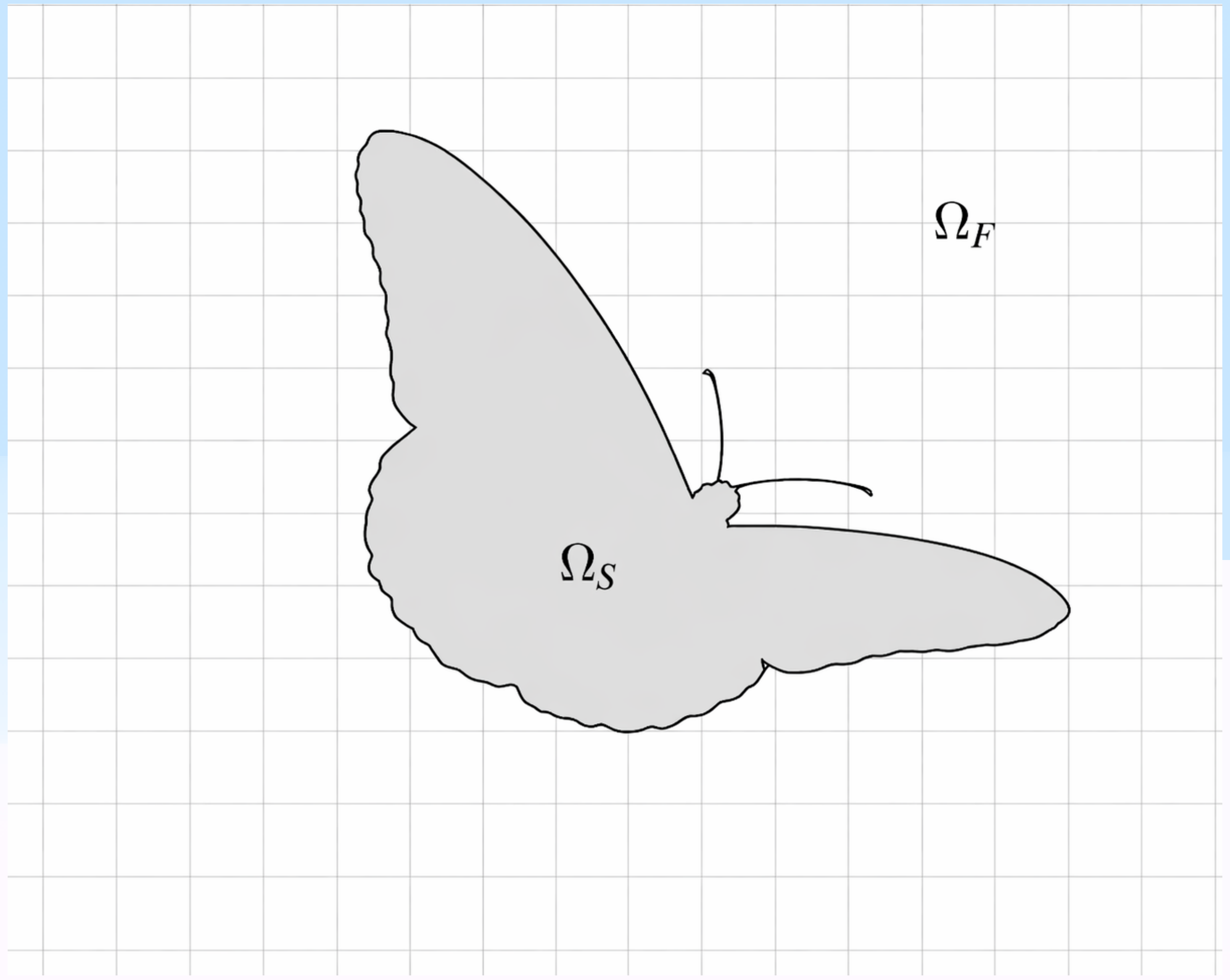

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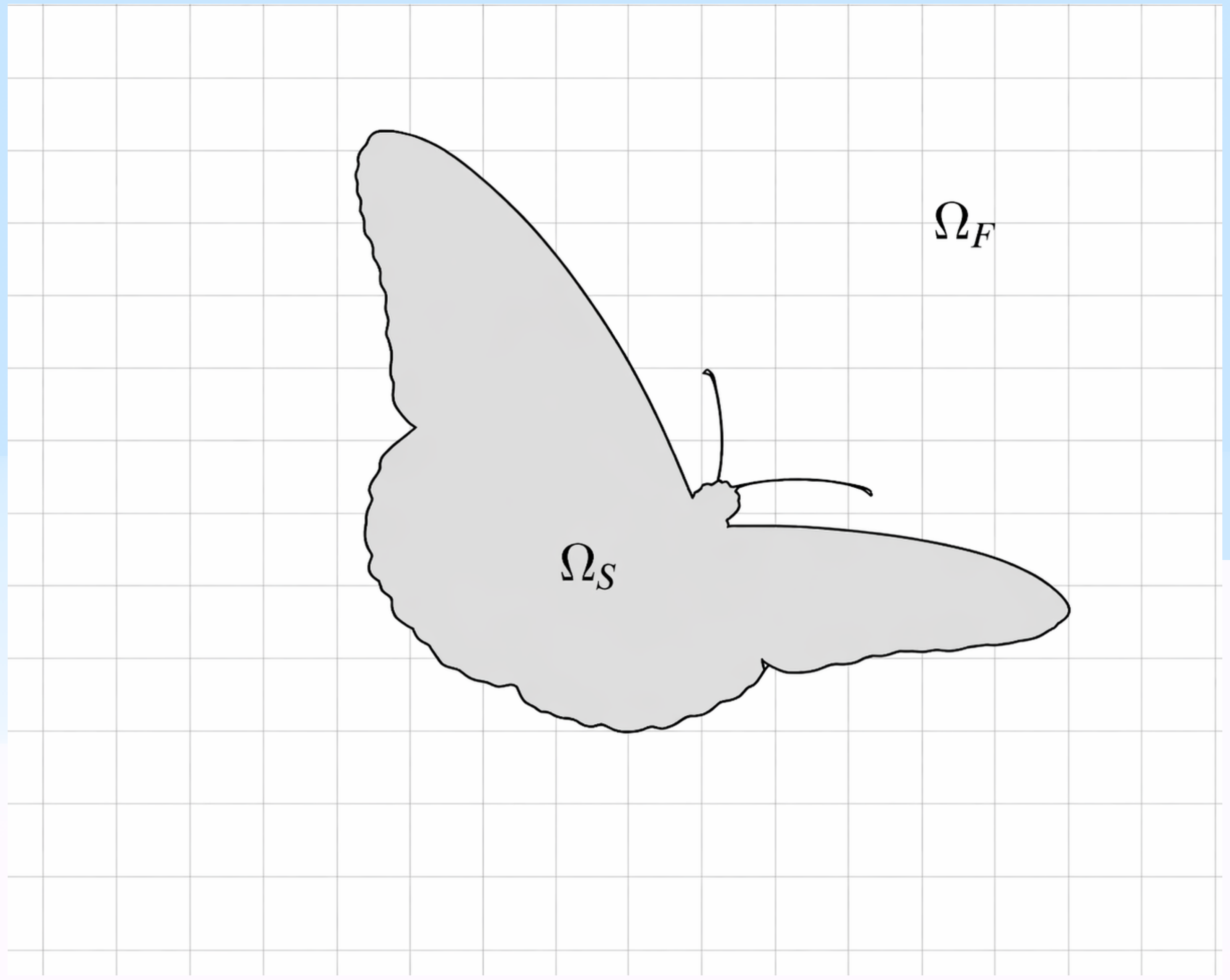

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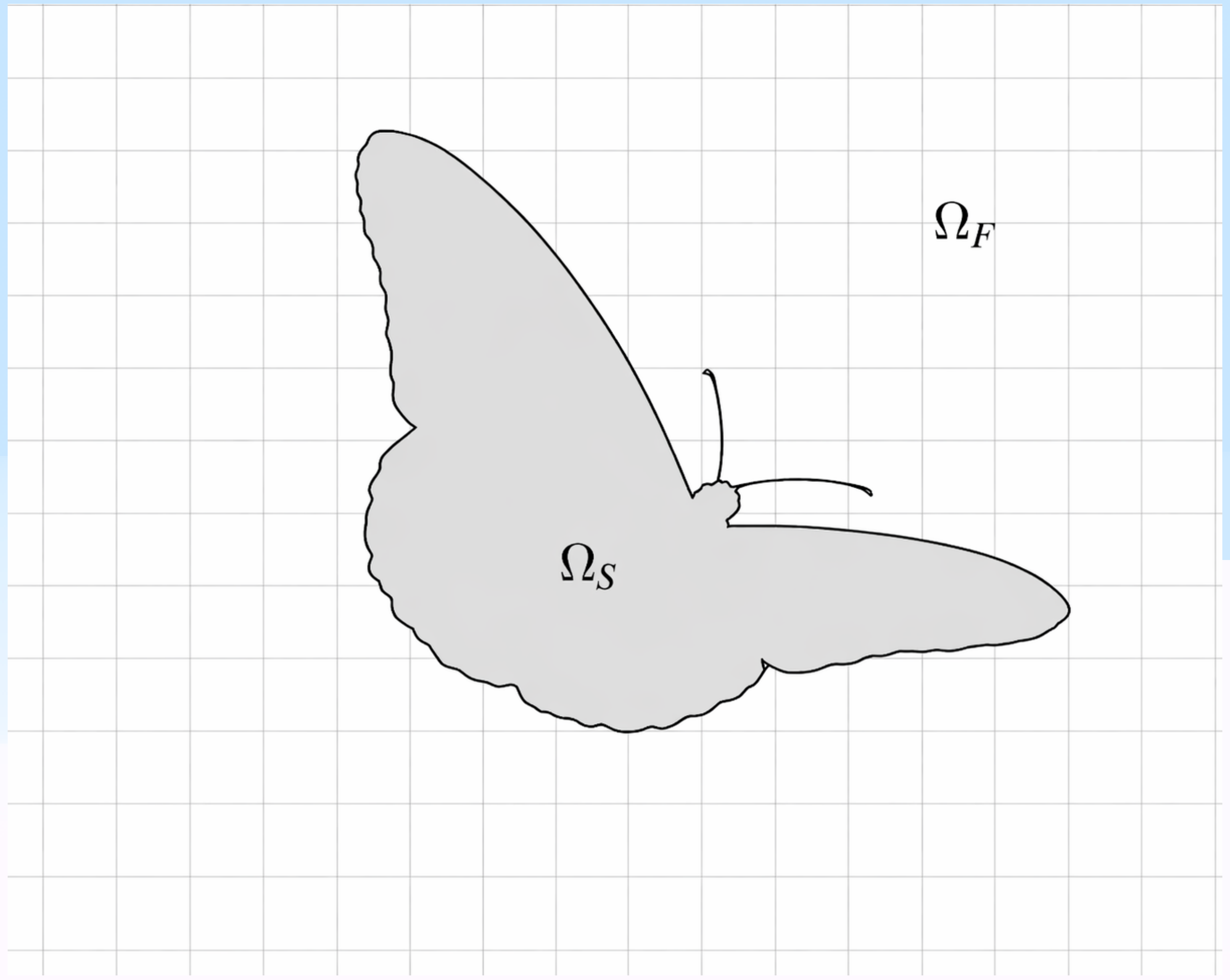

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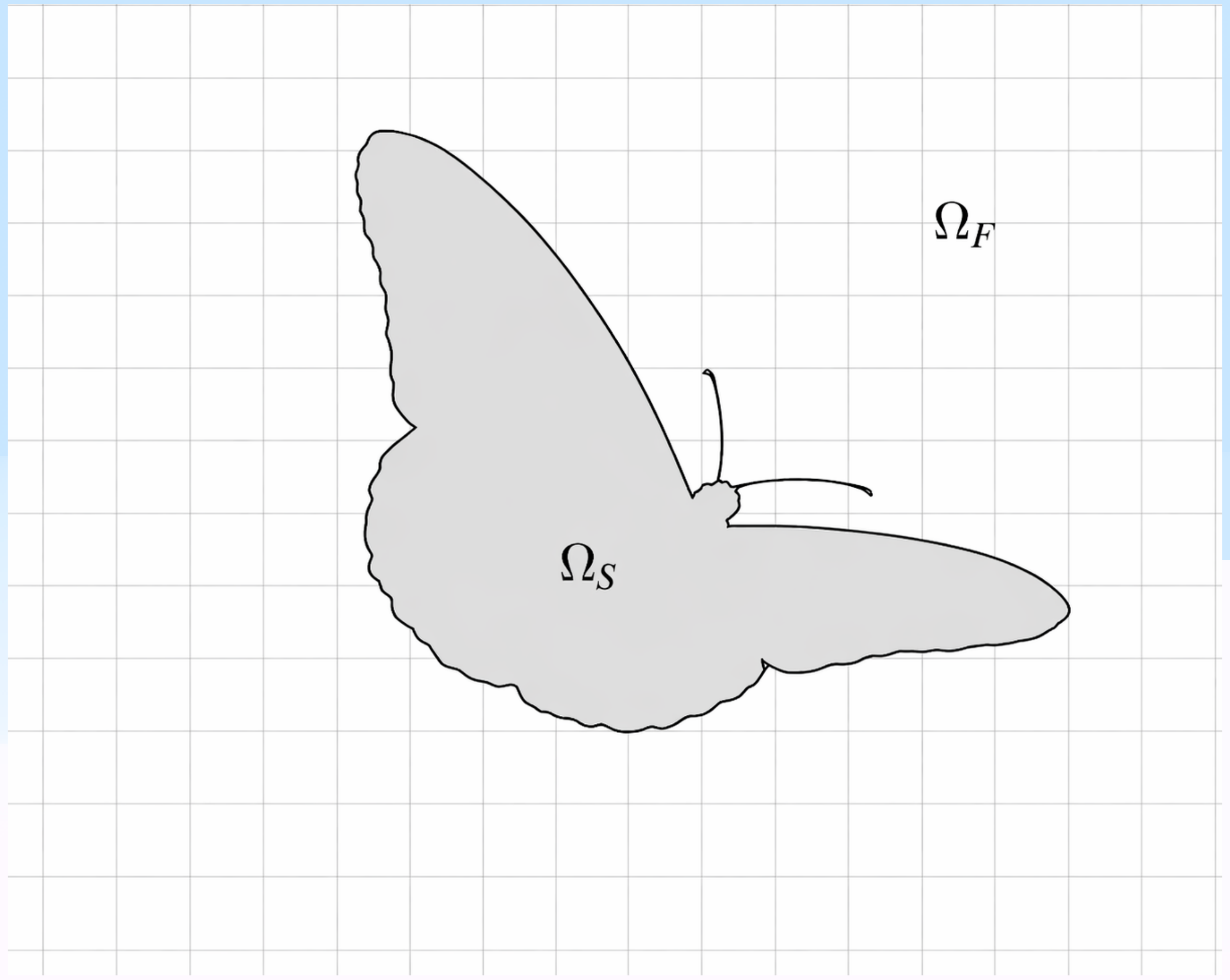

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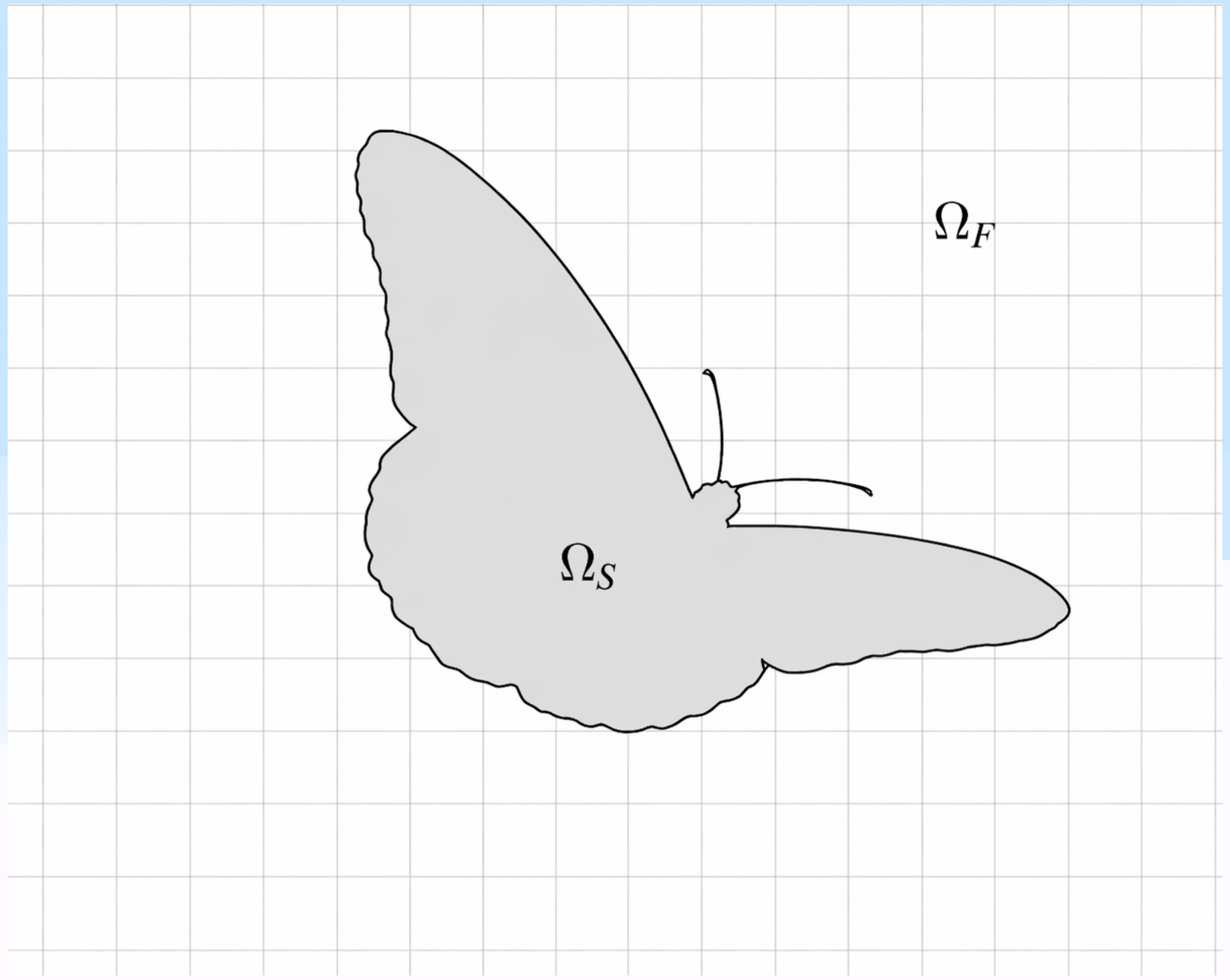

Example: Brinkman volume penalization.

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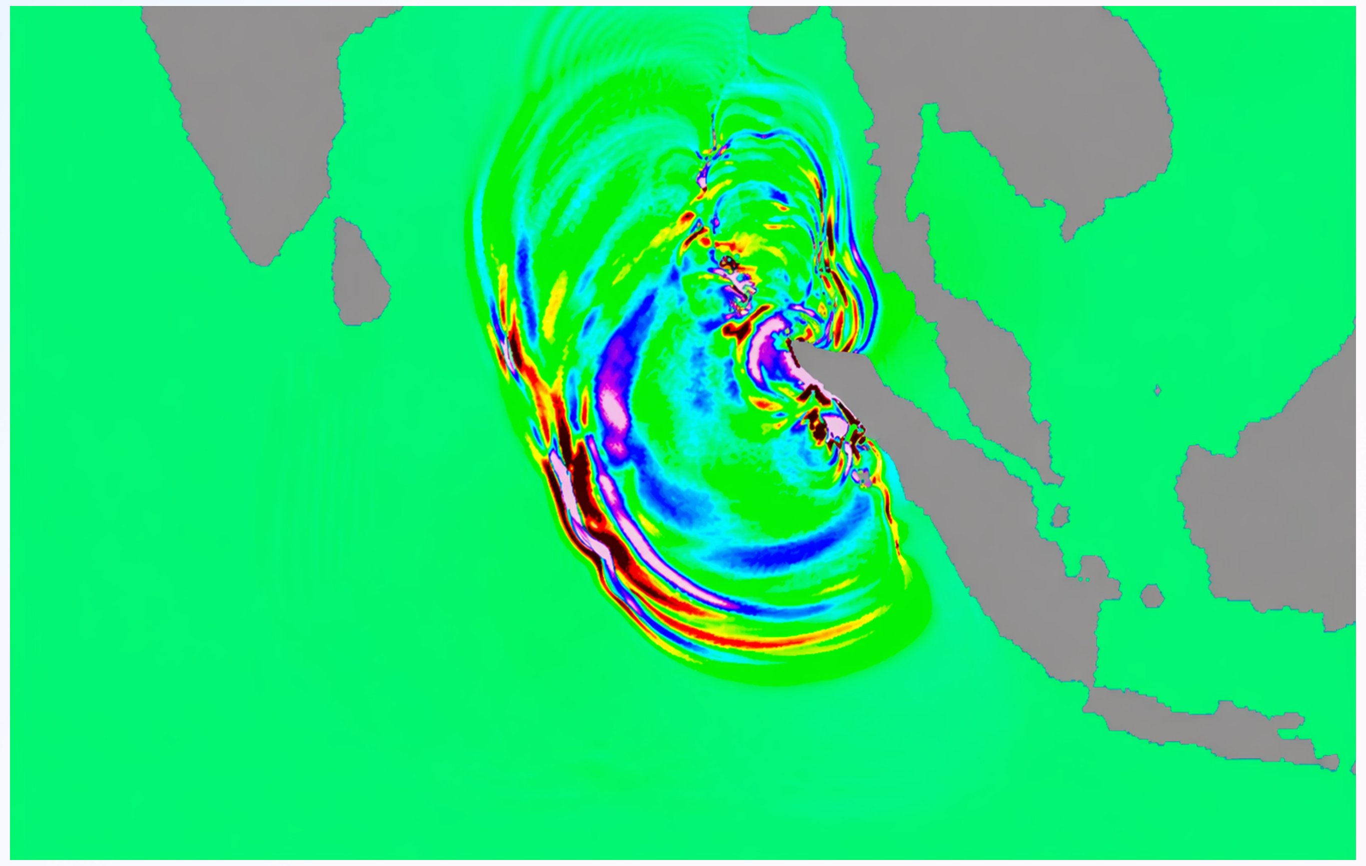
- **Shallow water equations**: tsunami propagation
  Kevlahan, Dubos, Aechtner (2015)



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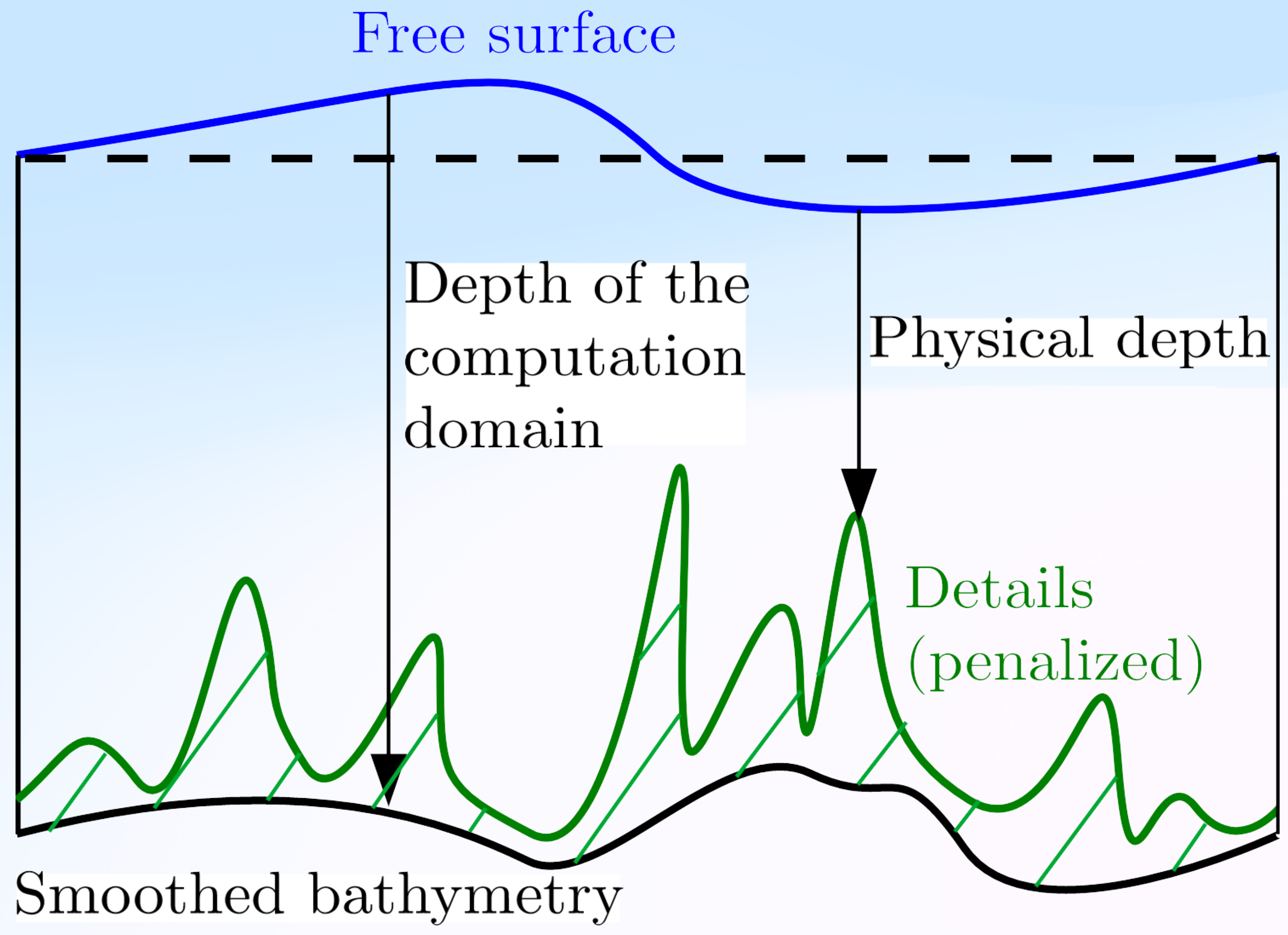

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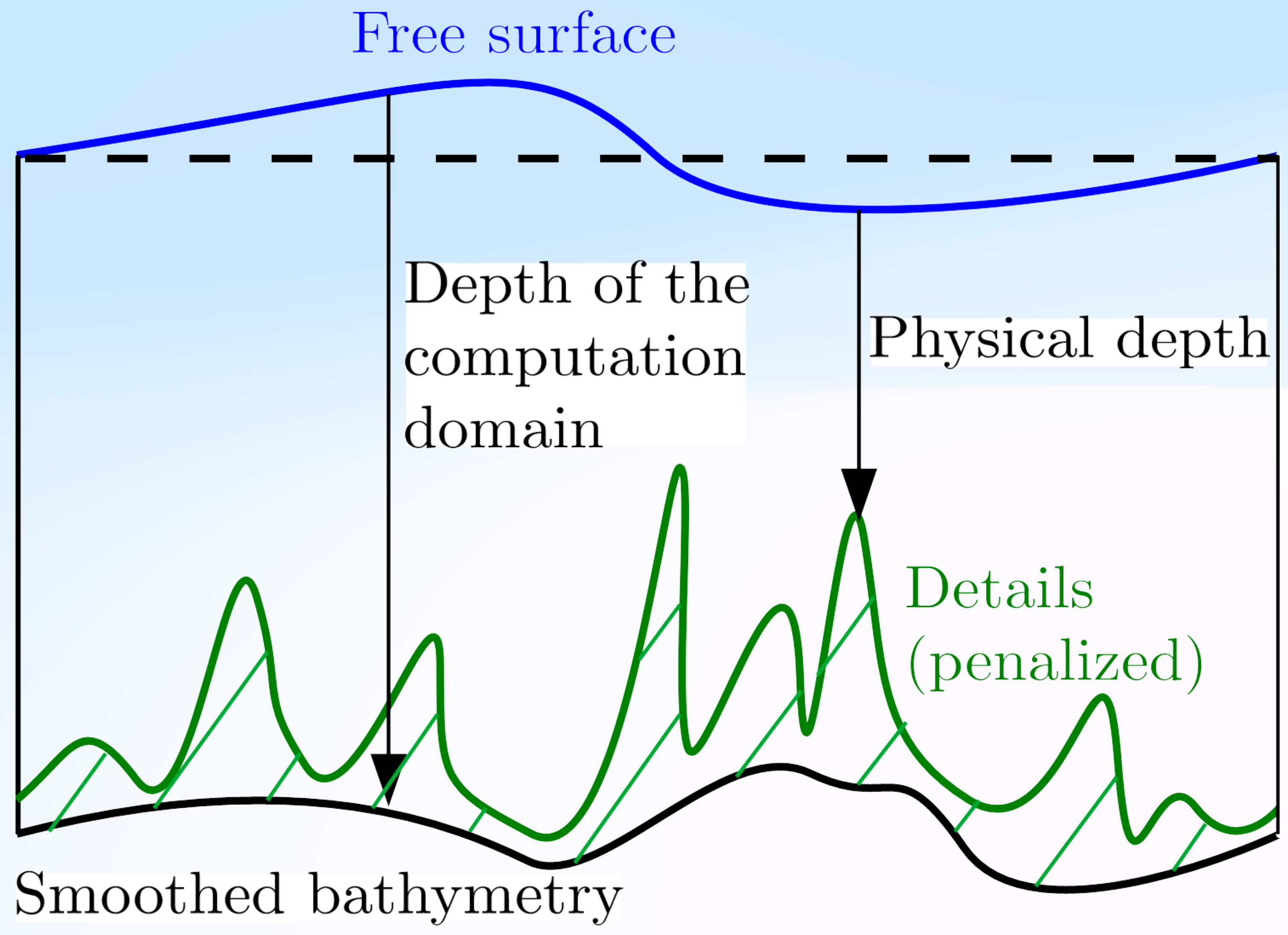

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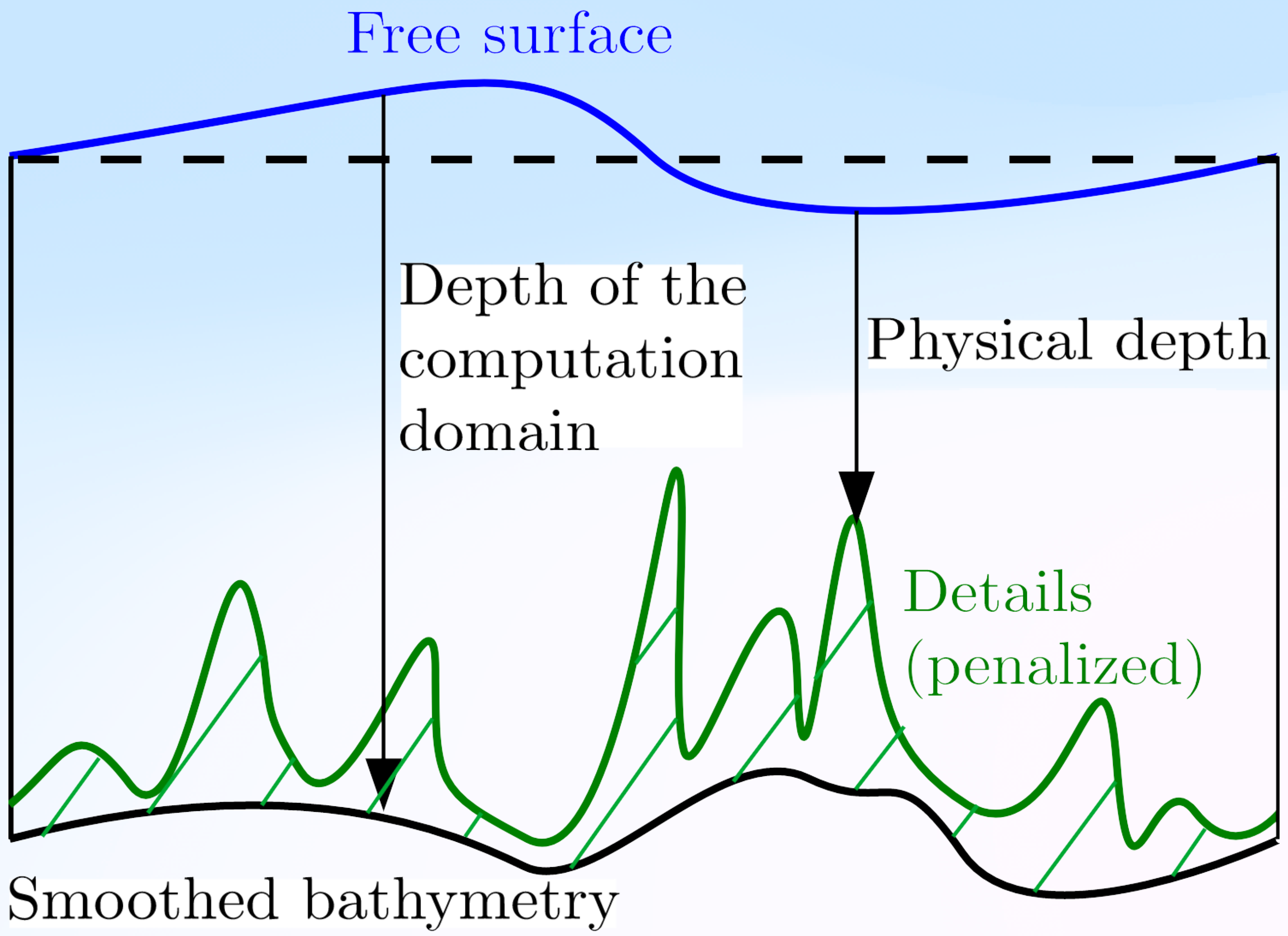

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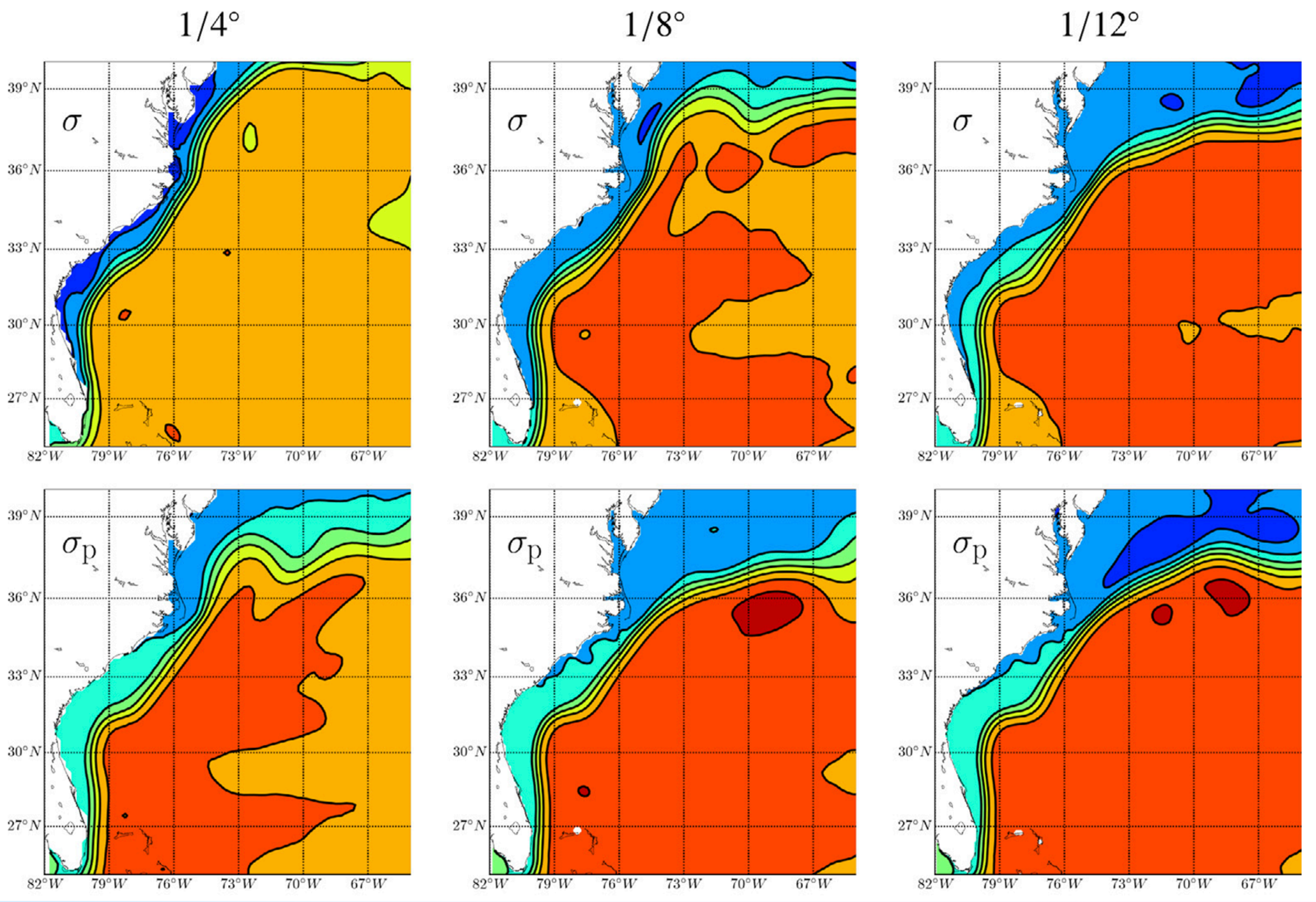
- Control parameters for enforcing no-slip boundary conditions are *porosity* $\phi \ll 1$ and *permeability* $k \ll 1$.
  Error $O(\phi k^{1/2})$.

Brinkman volume penalization for Gulf Stream

Multilayer shallow water ocean model (CROCO) *(Debreu, Kevlahan & Marchesiello 2020, 2022)*

Non-penalized
σ

Penalized
σ_P



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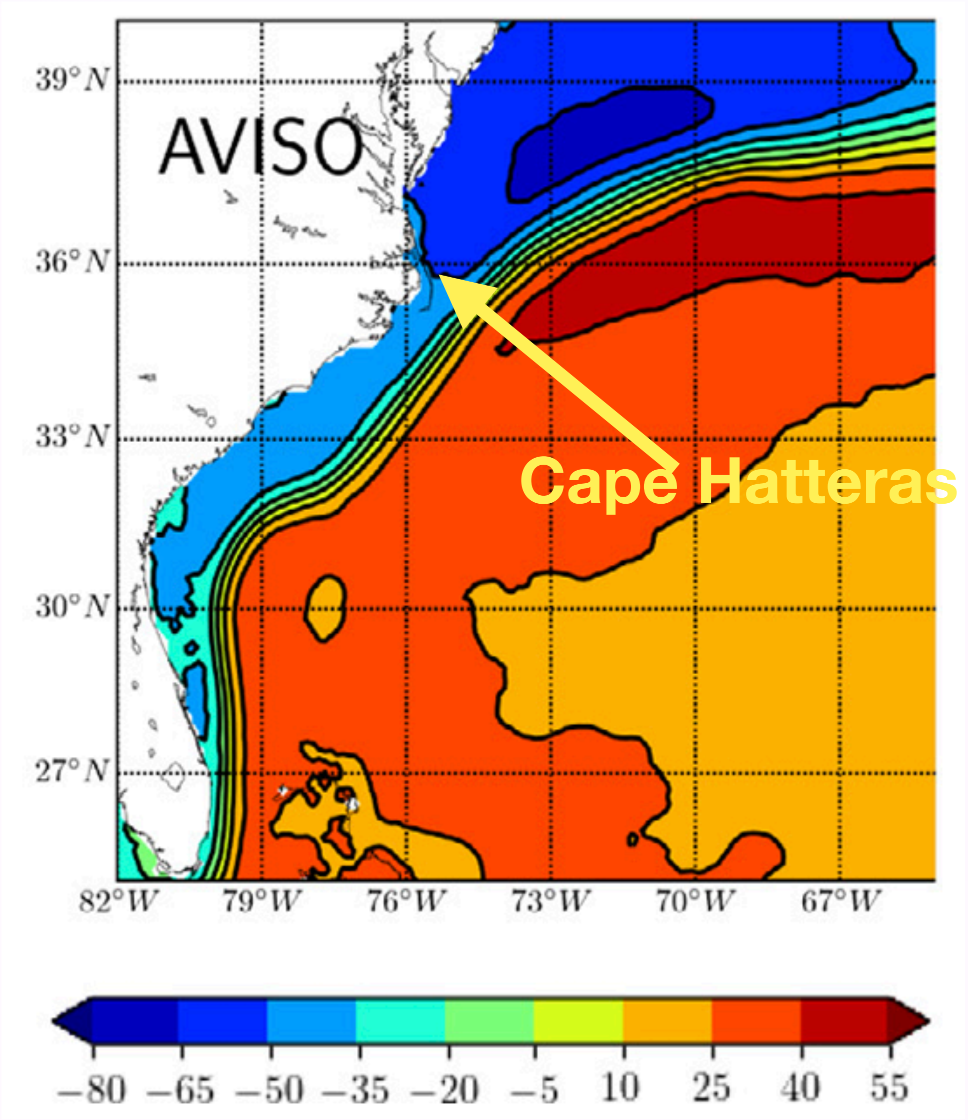

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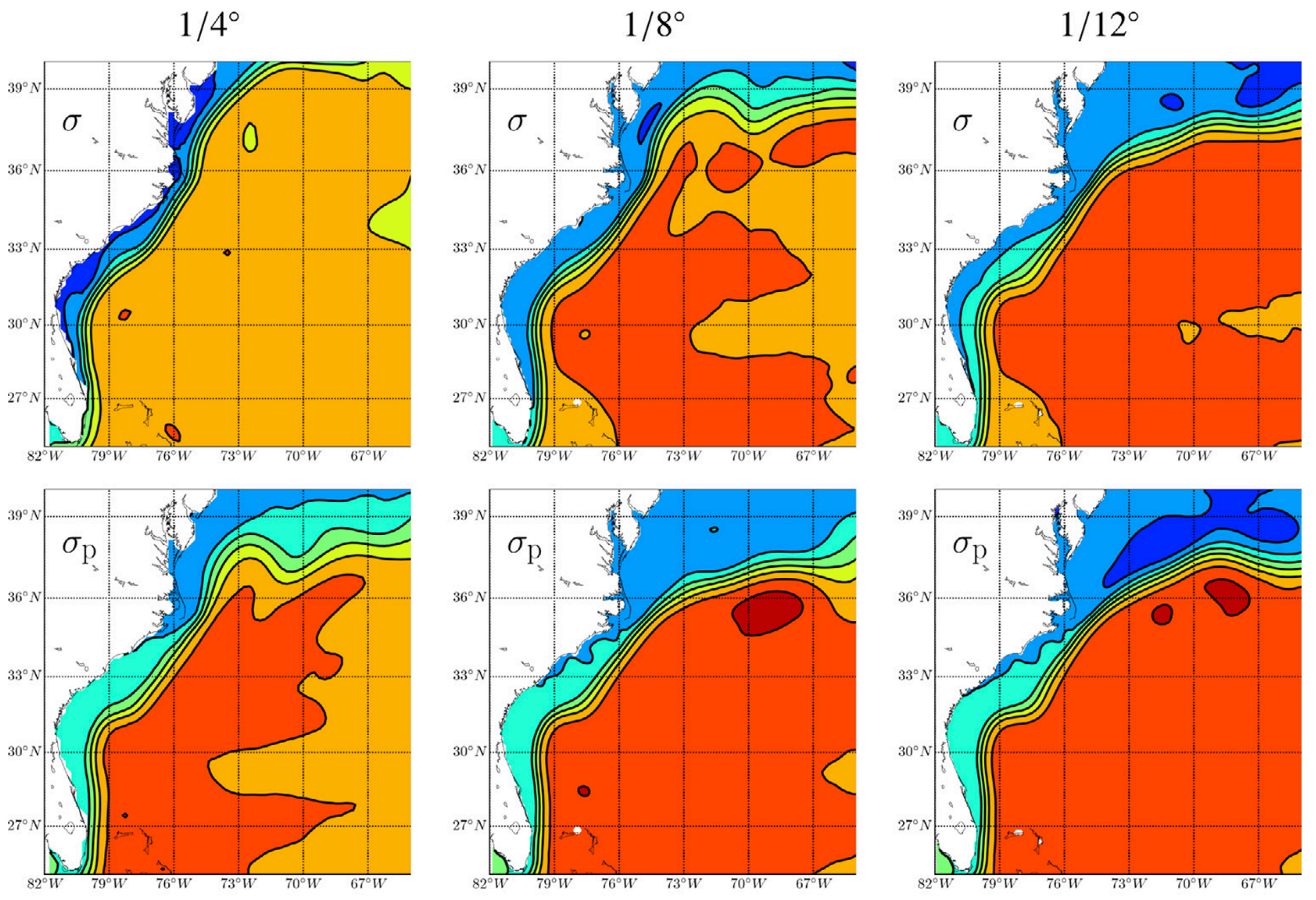

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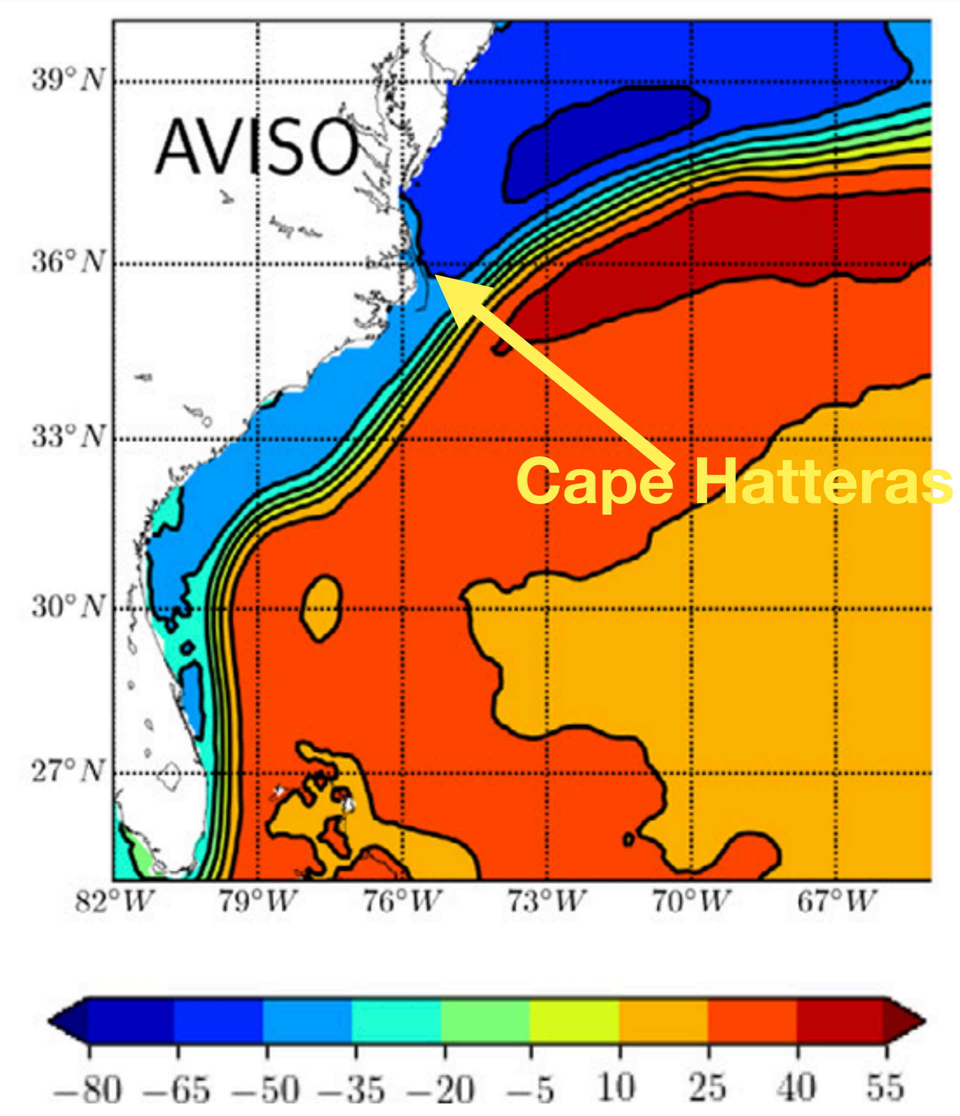

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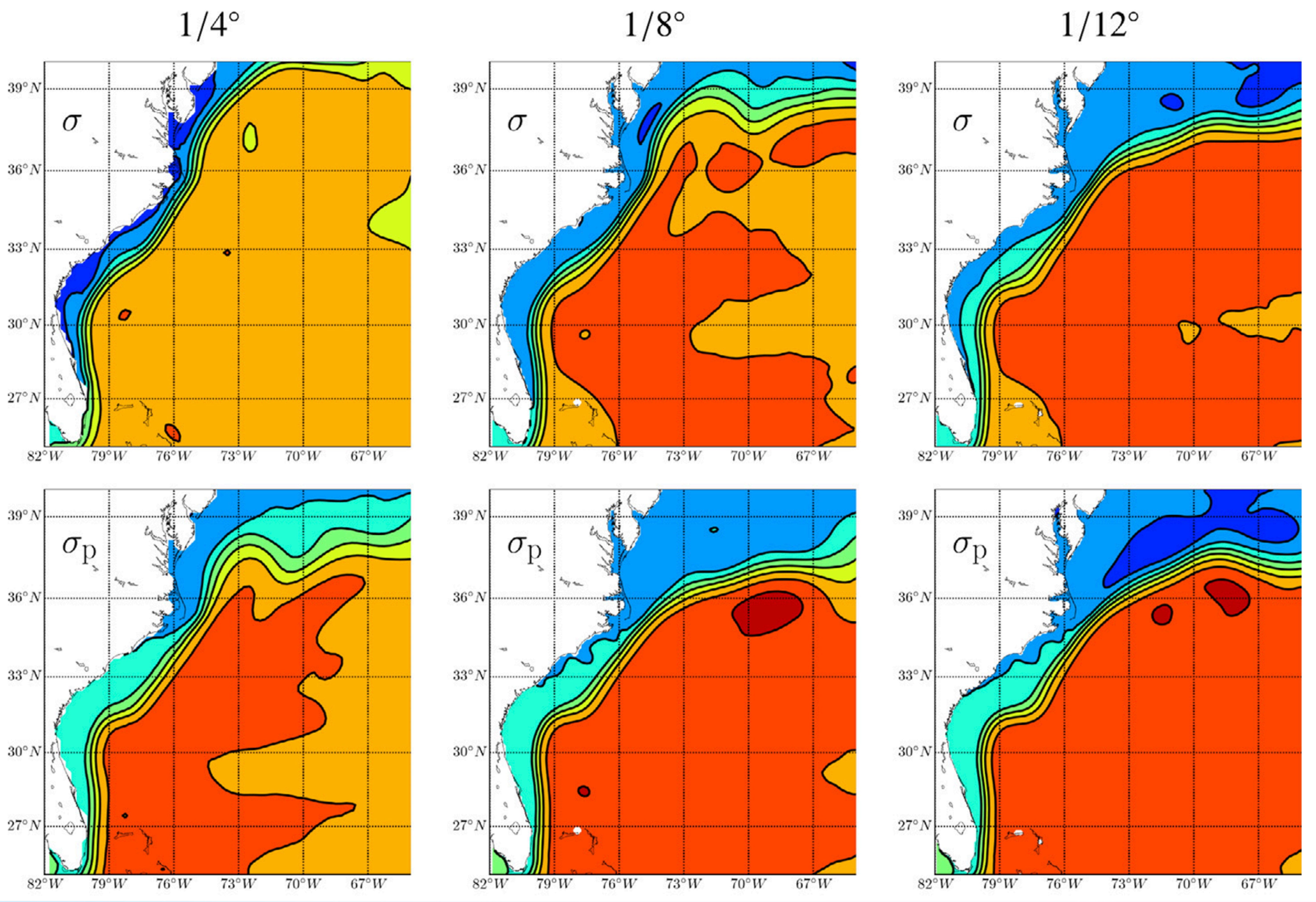

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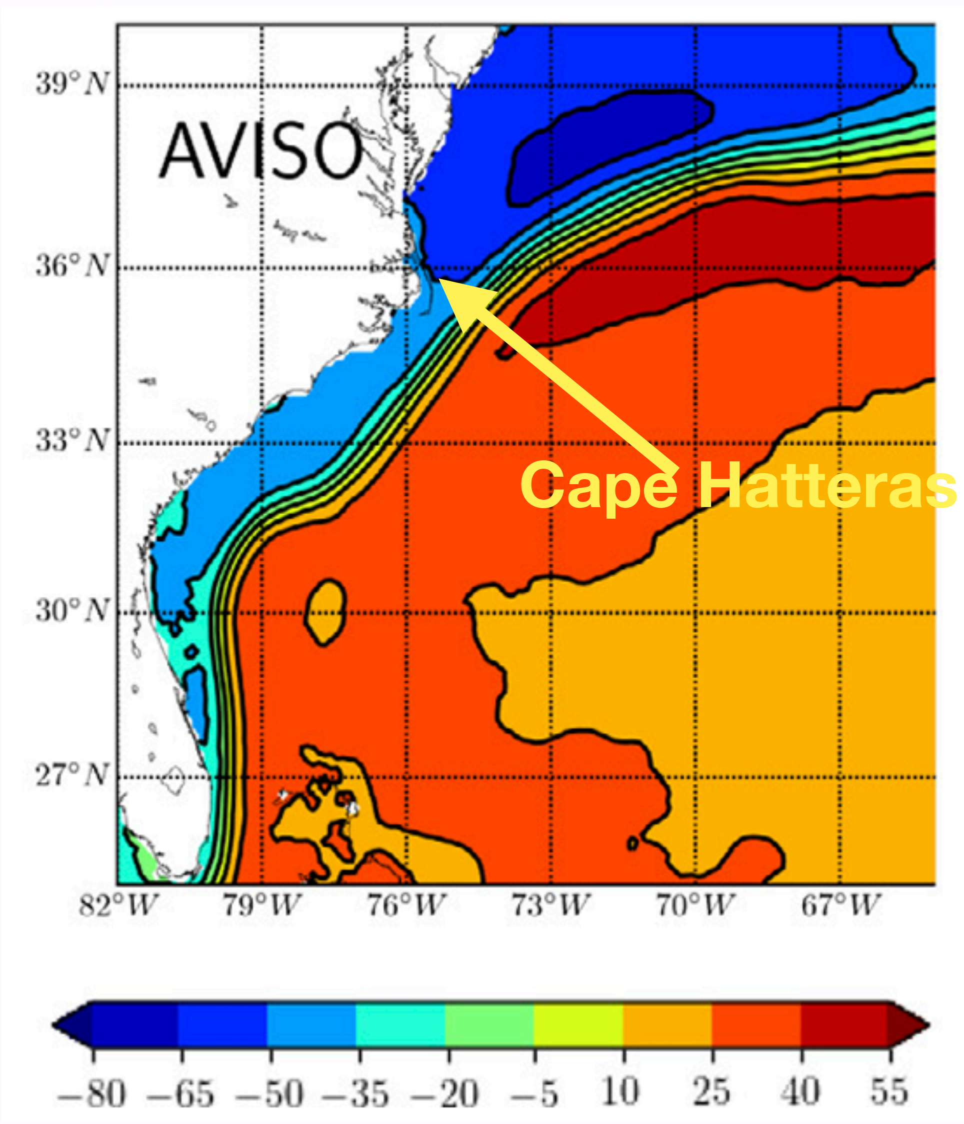

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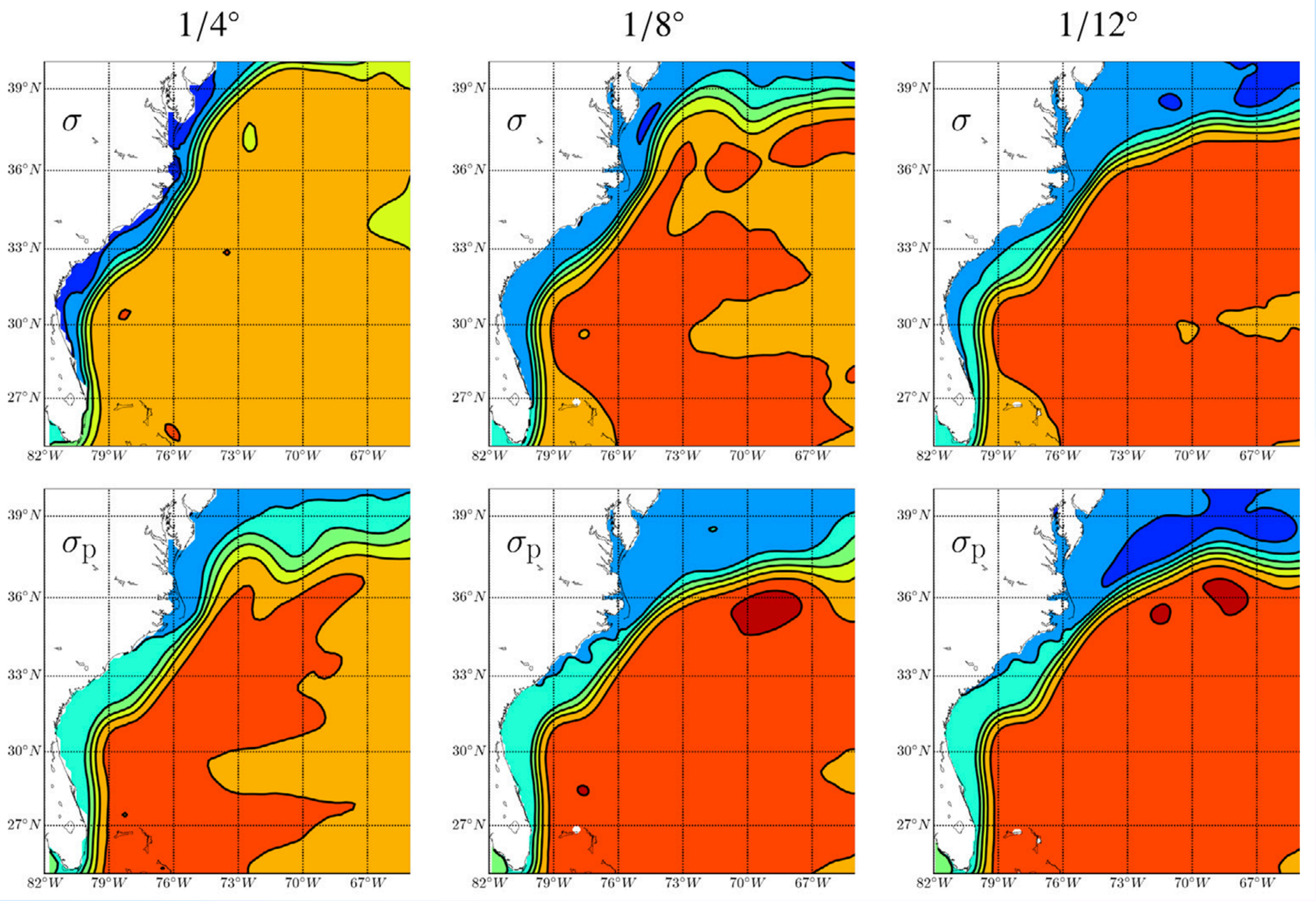

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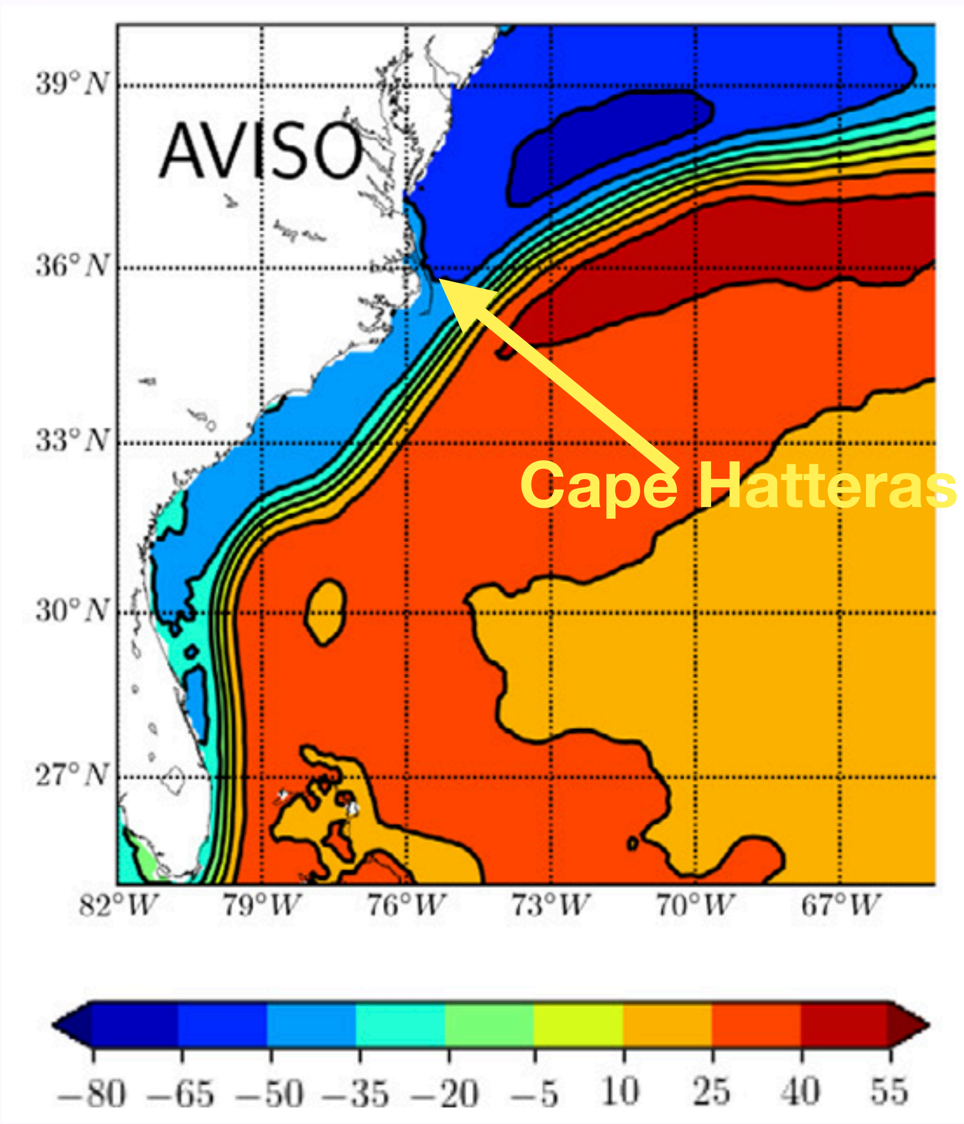

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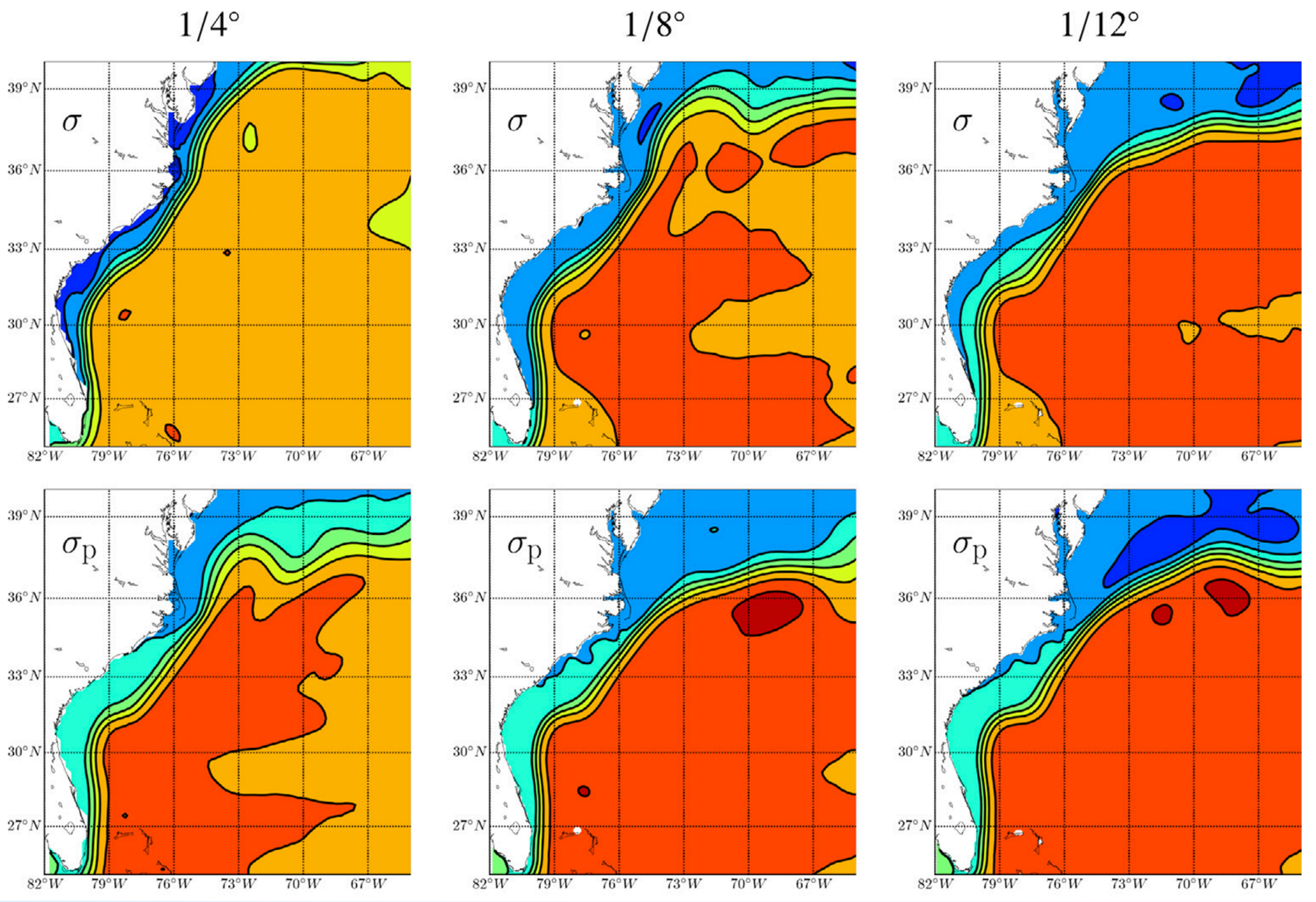

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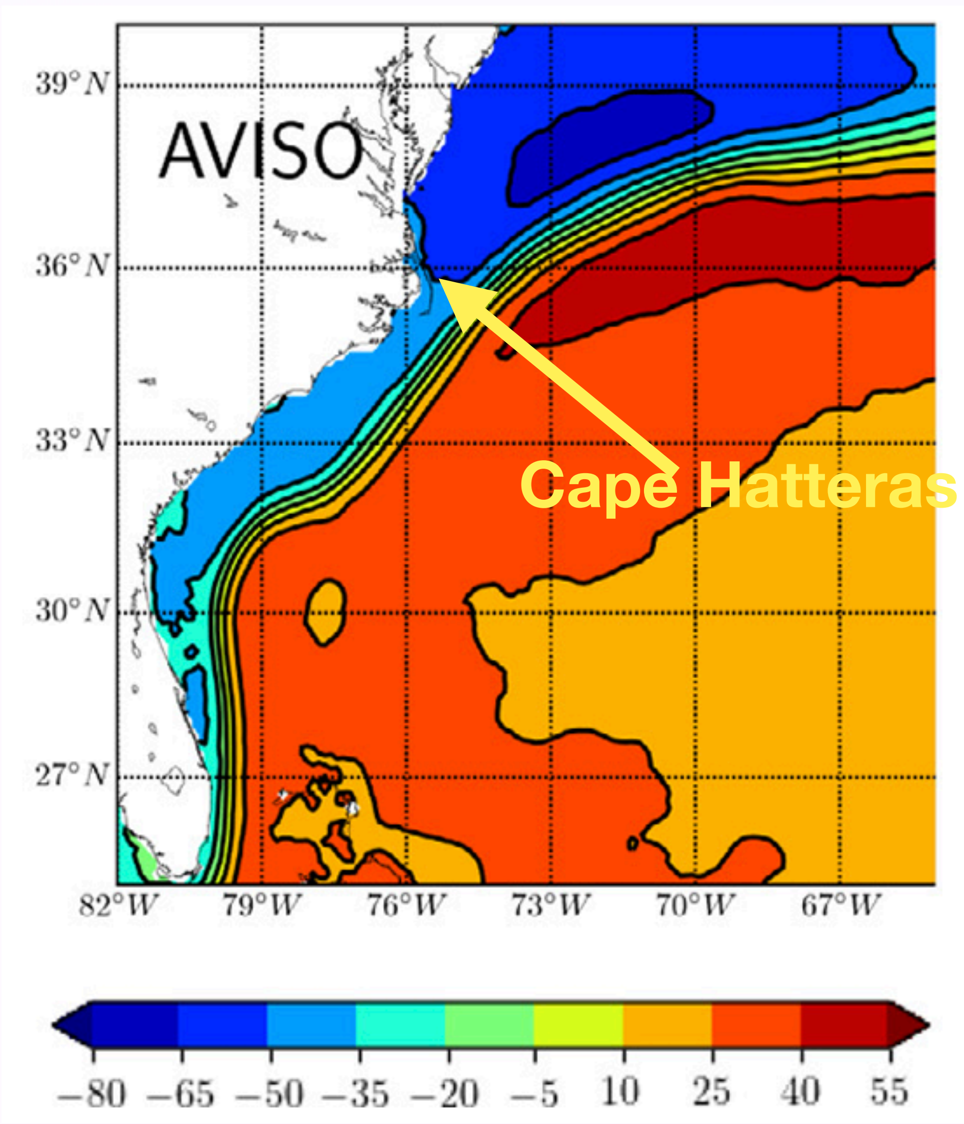

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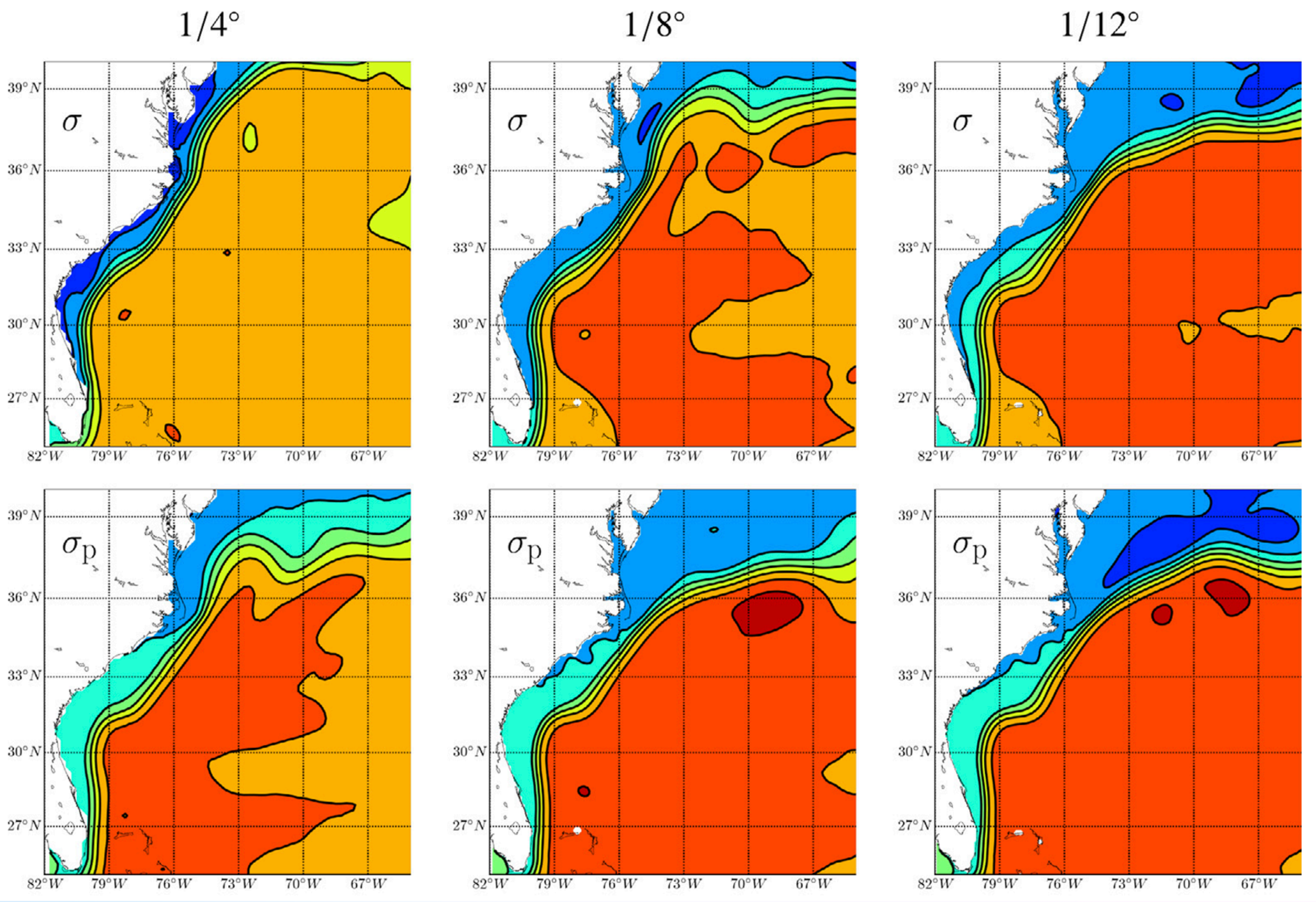

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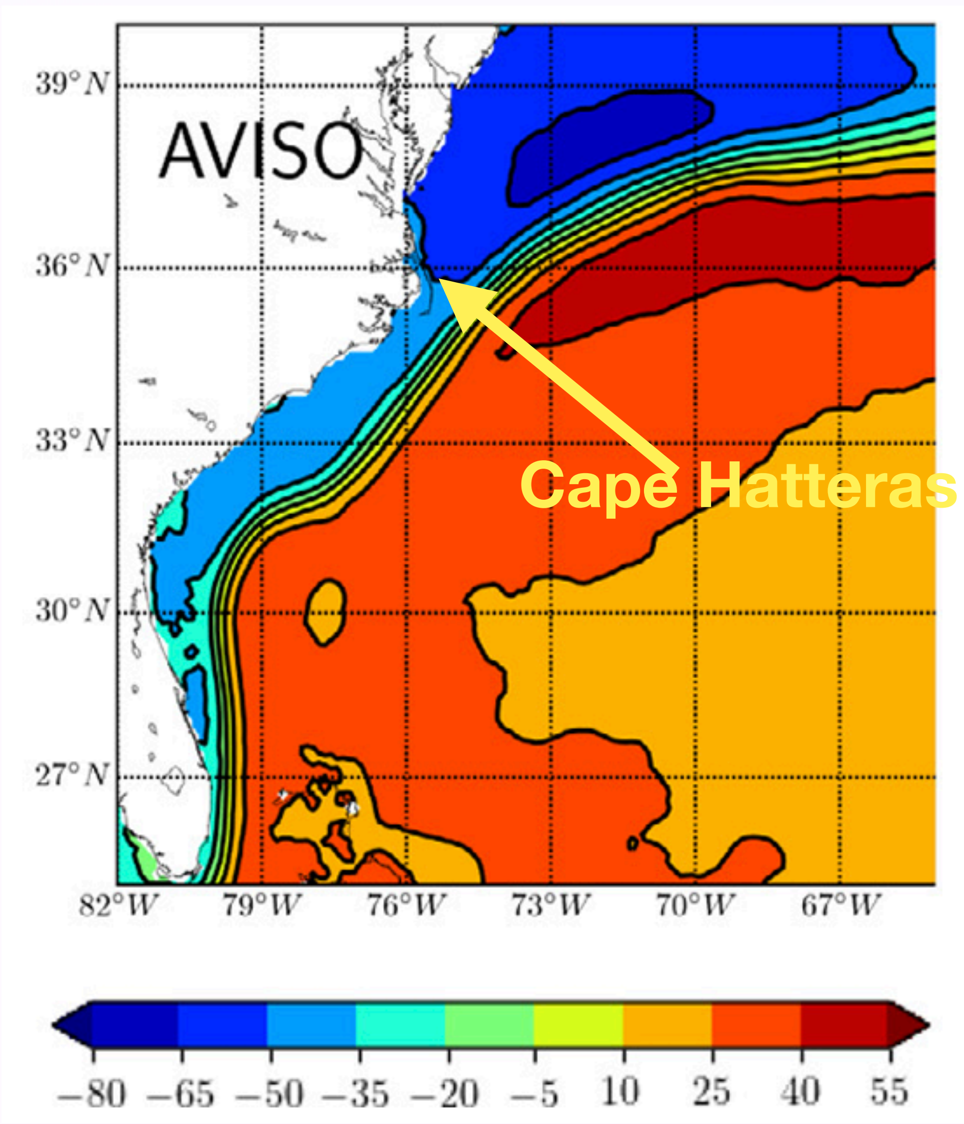

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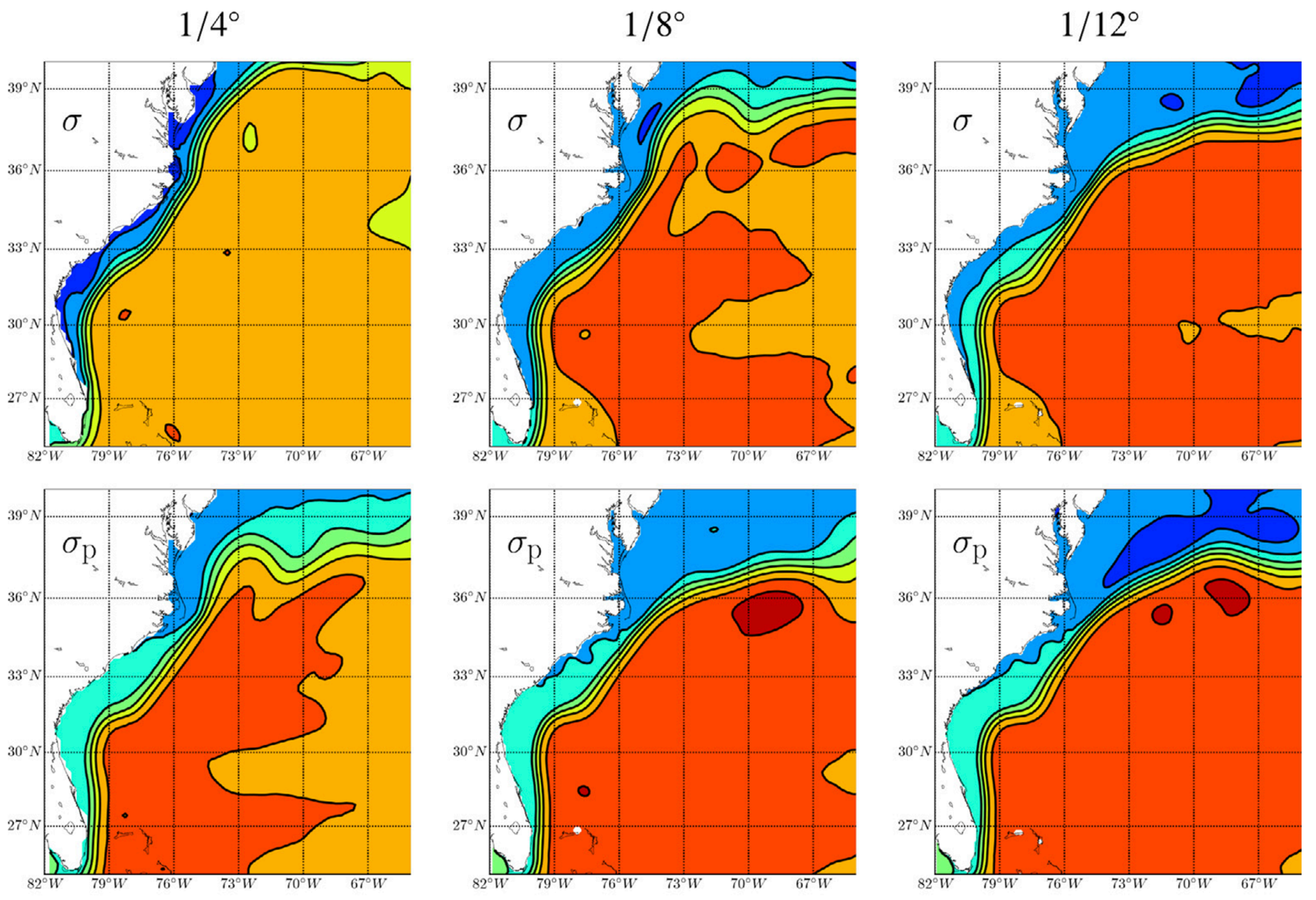

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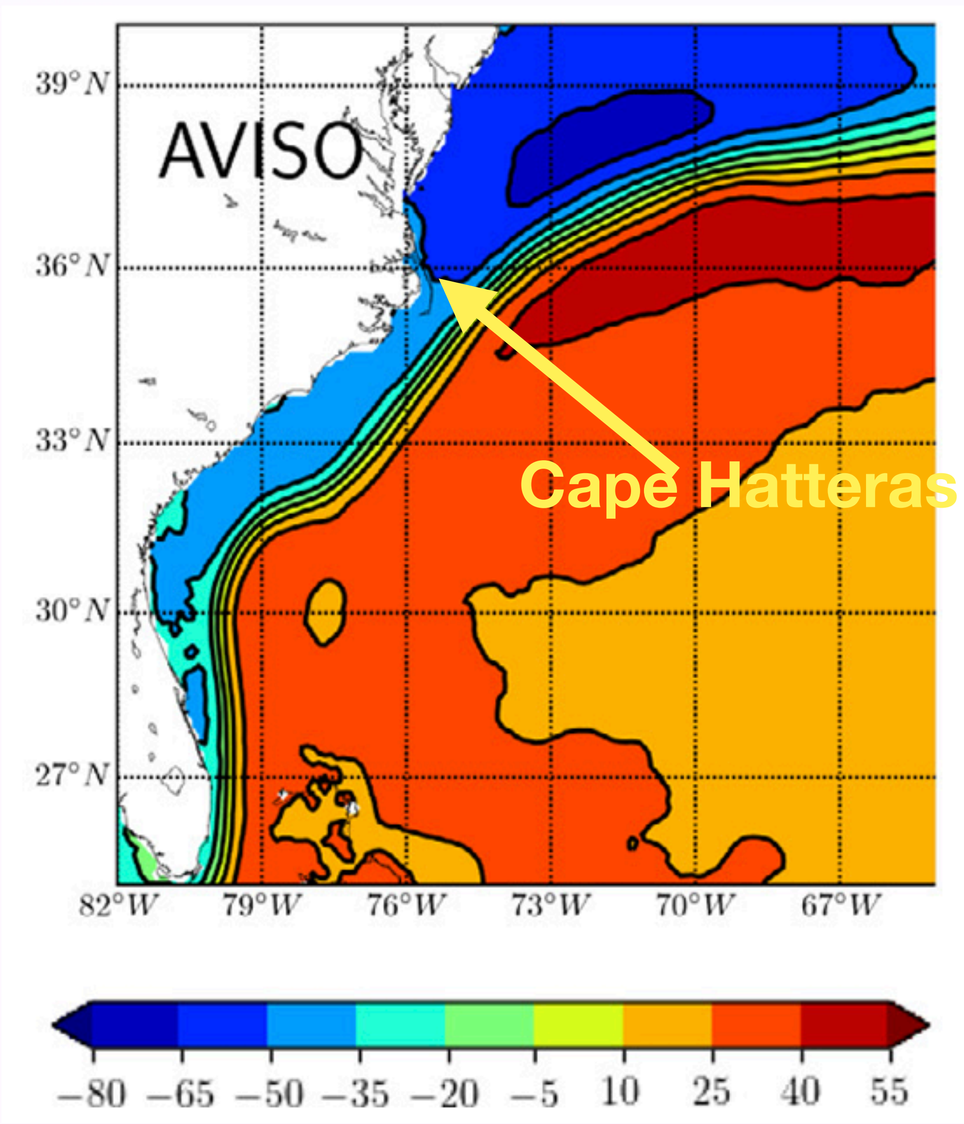

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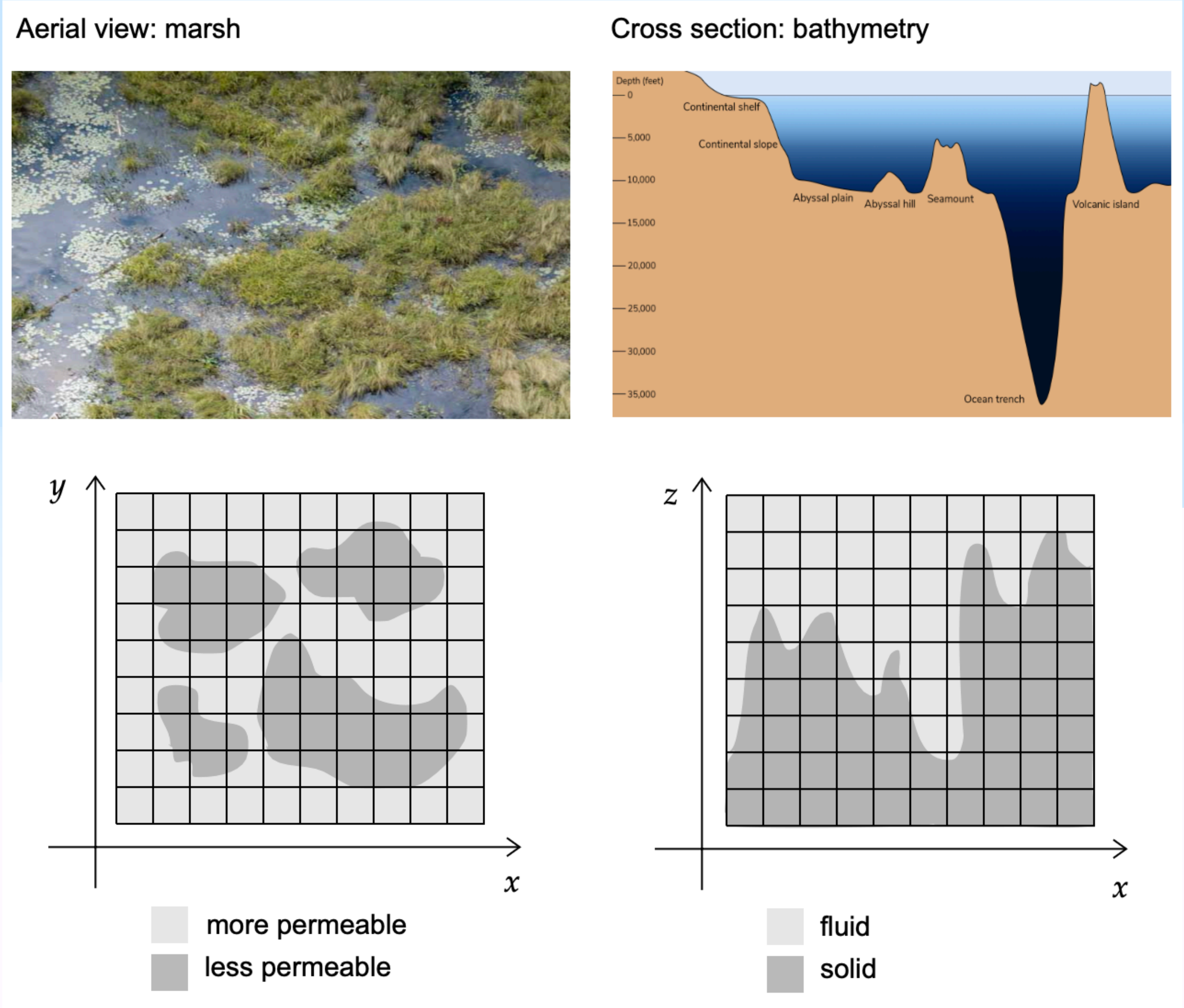

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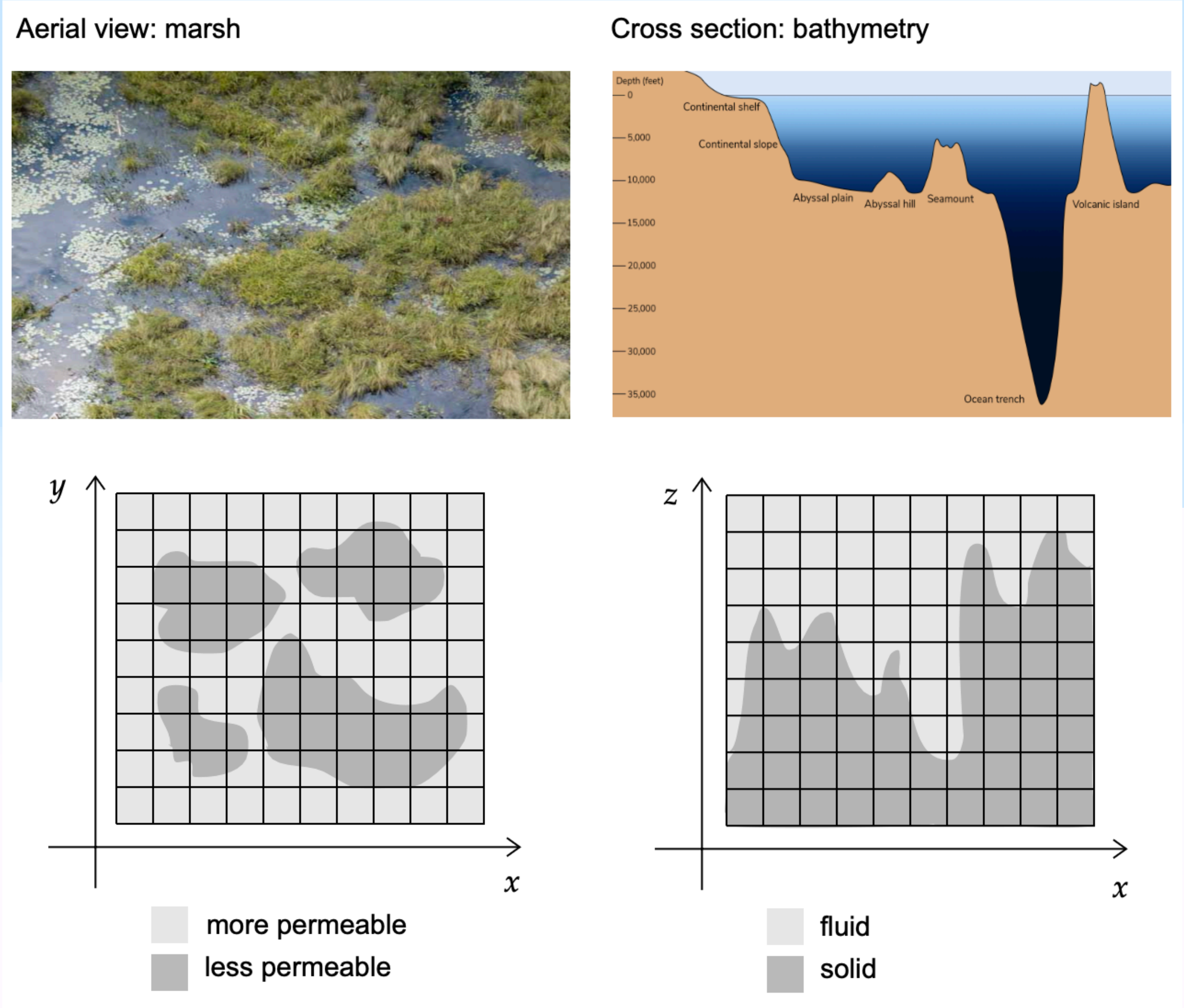

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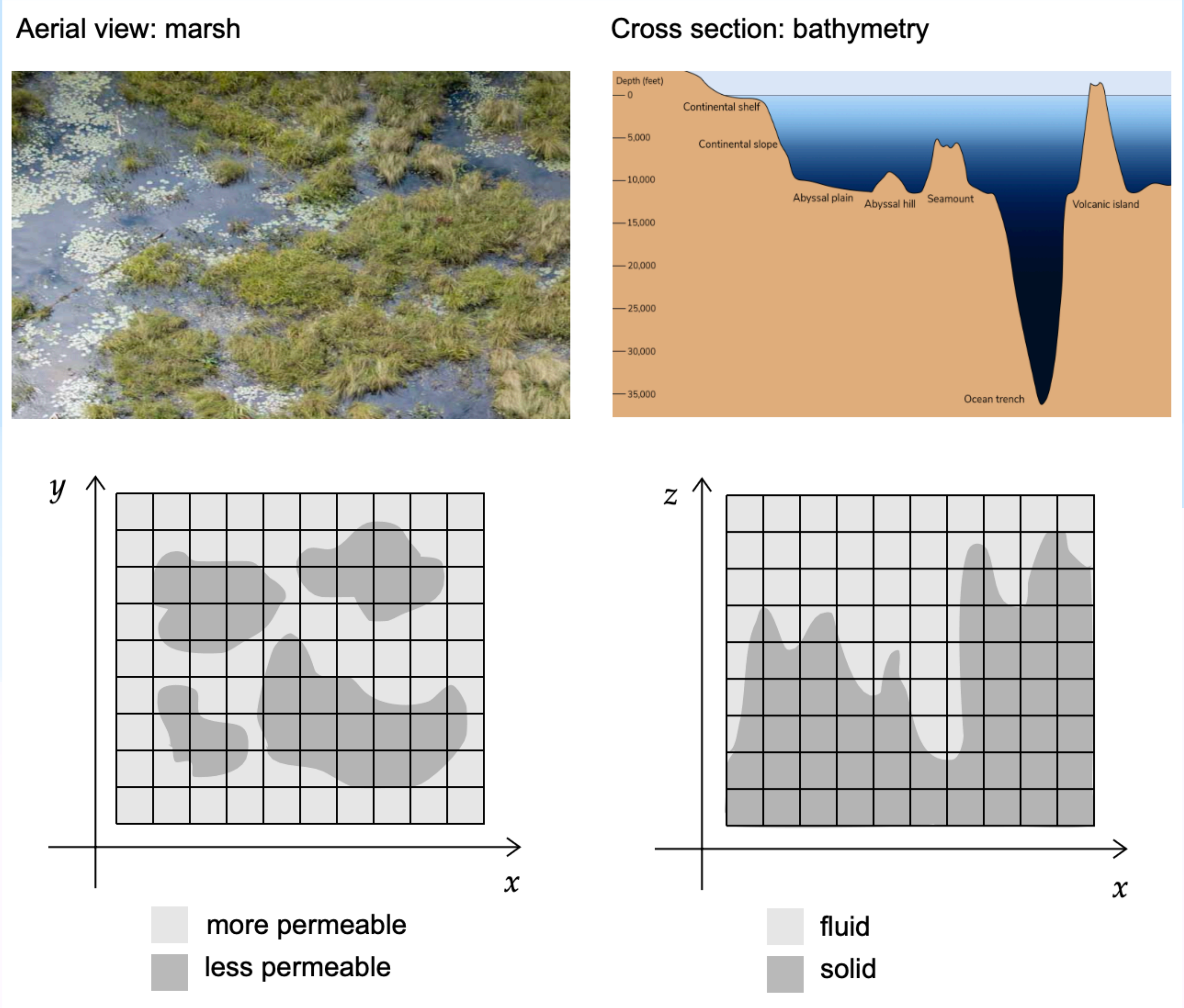

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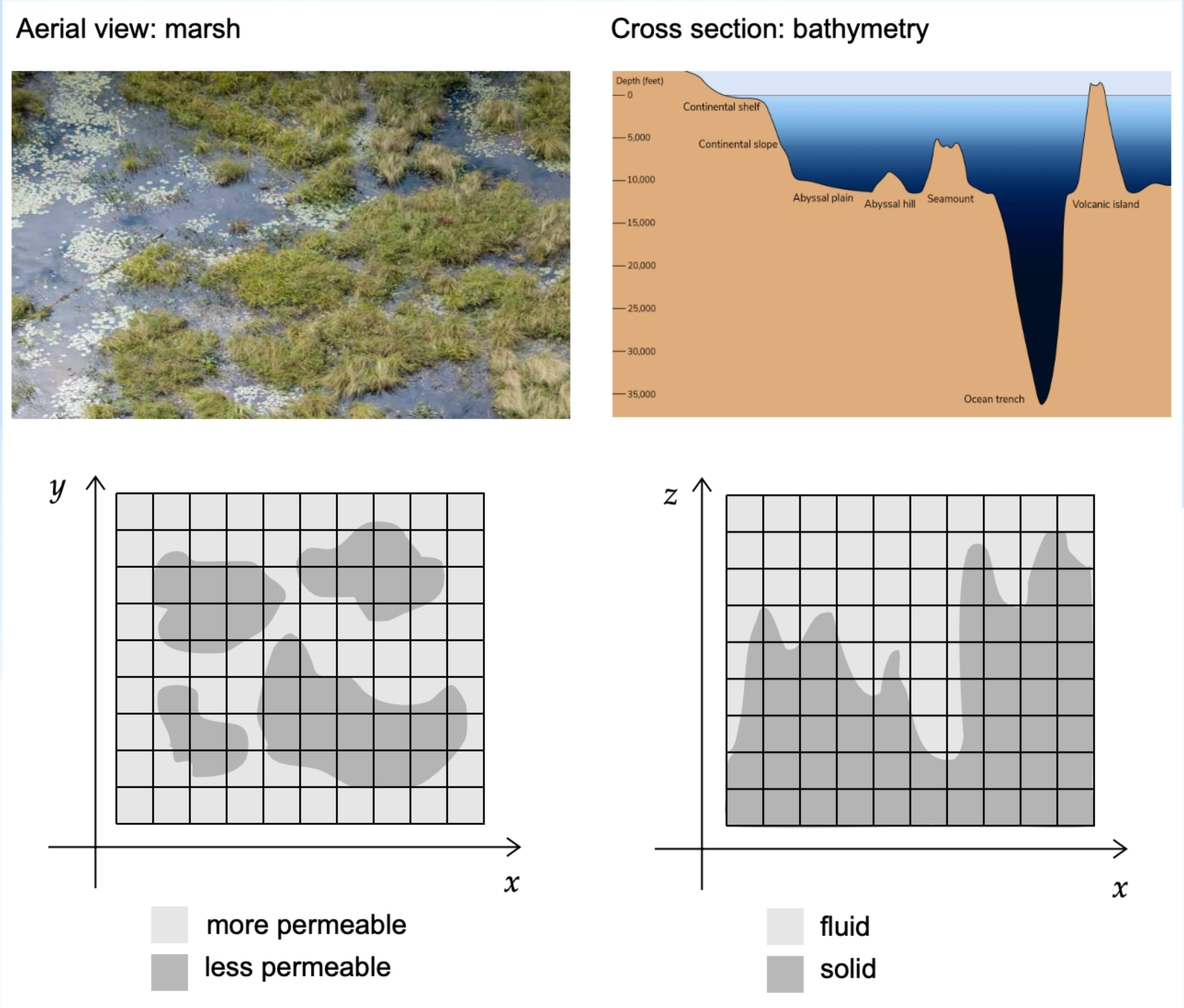

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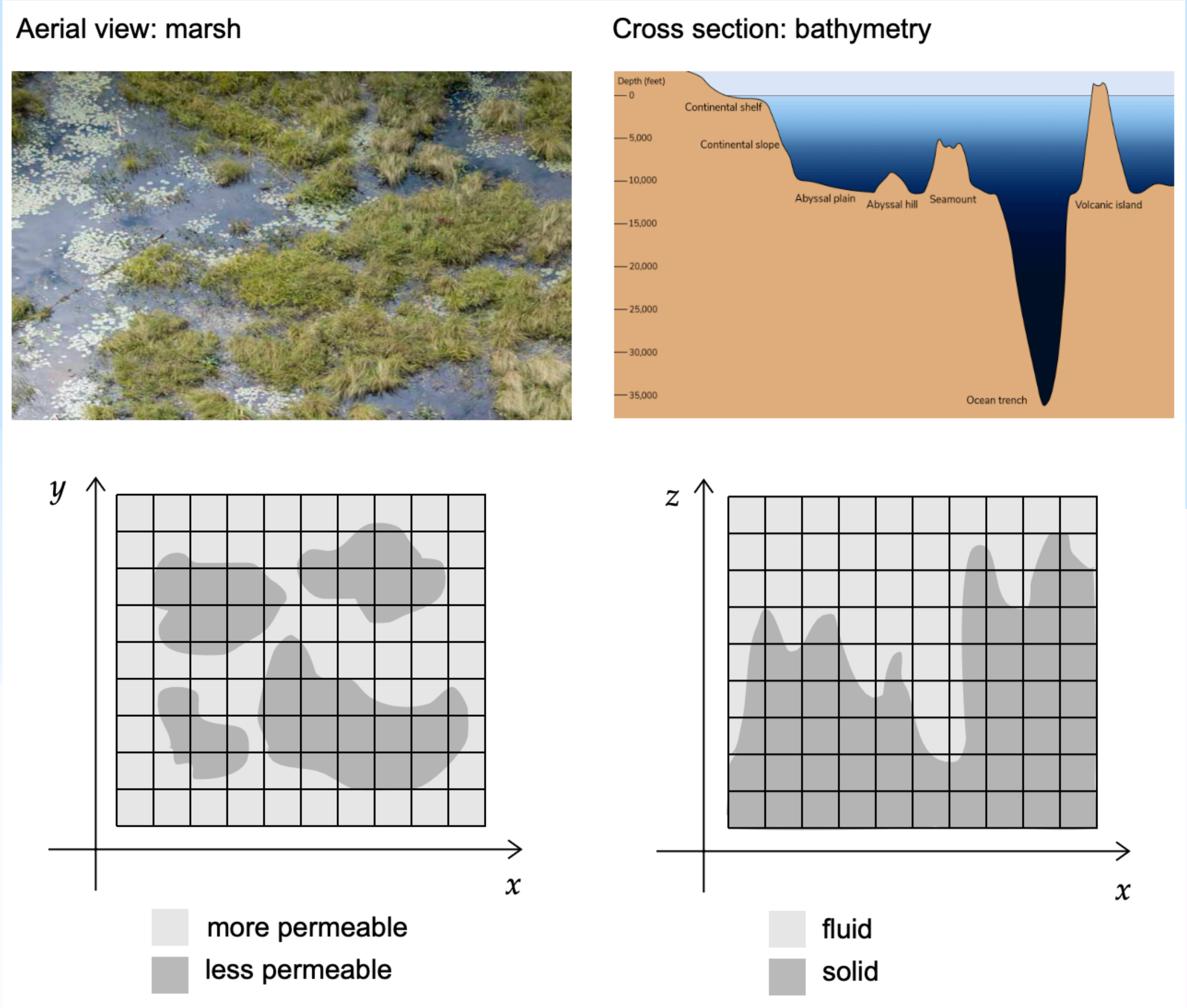

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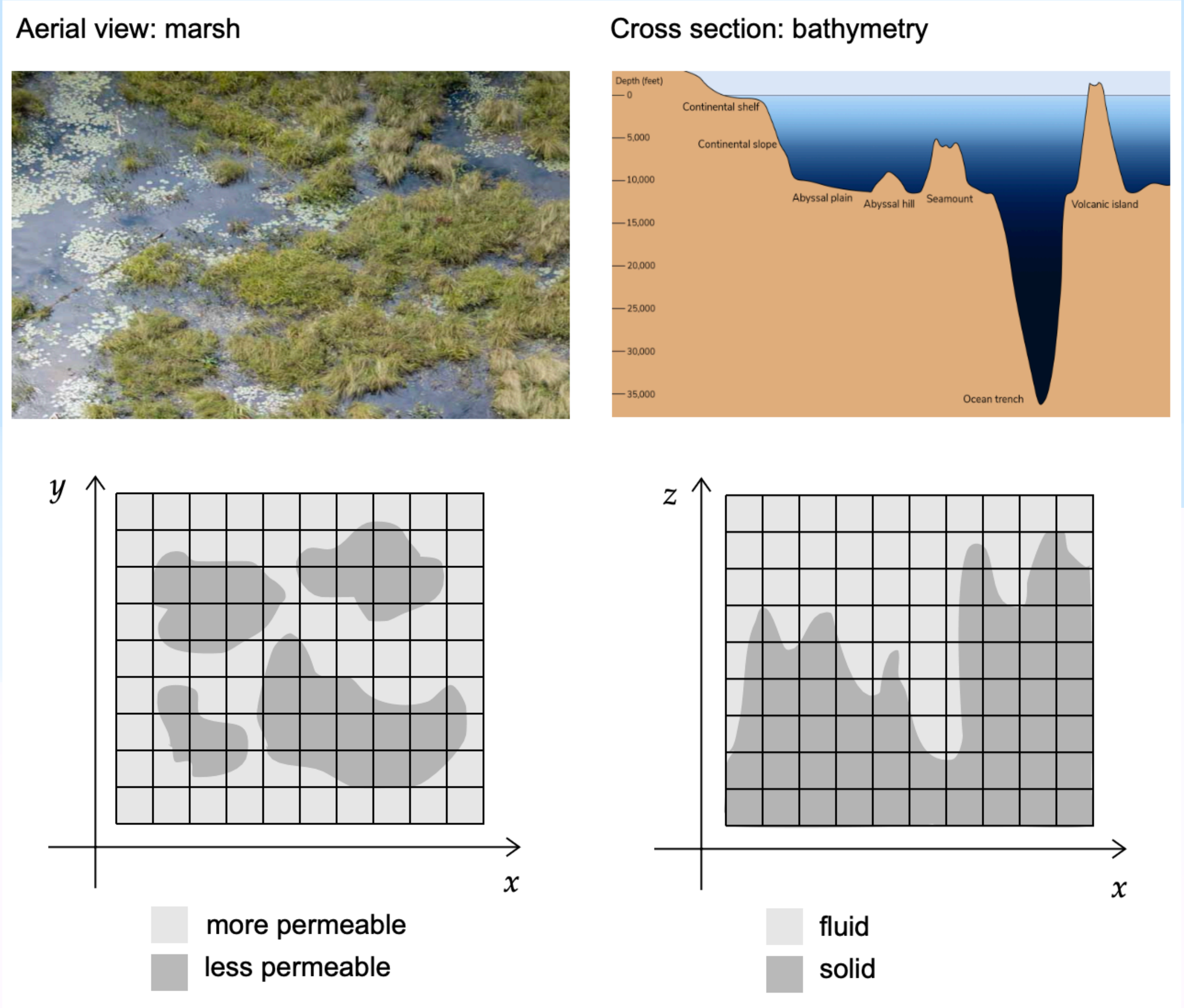
- **Marsh or estuary**
  Porosity and permeability of SGS cells given by data.



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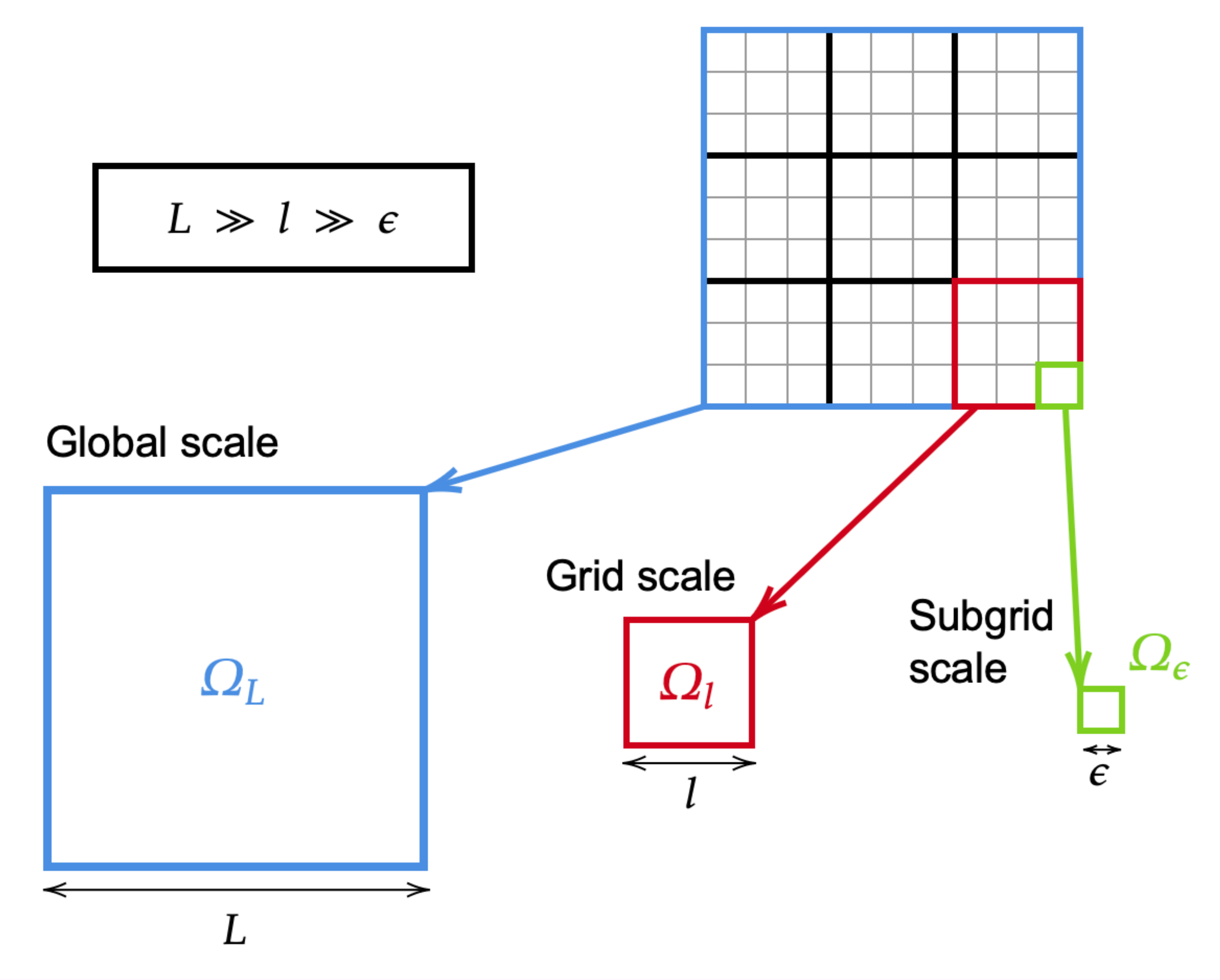

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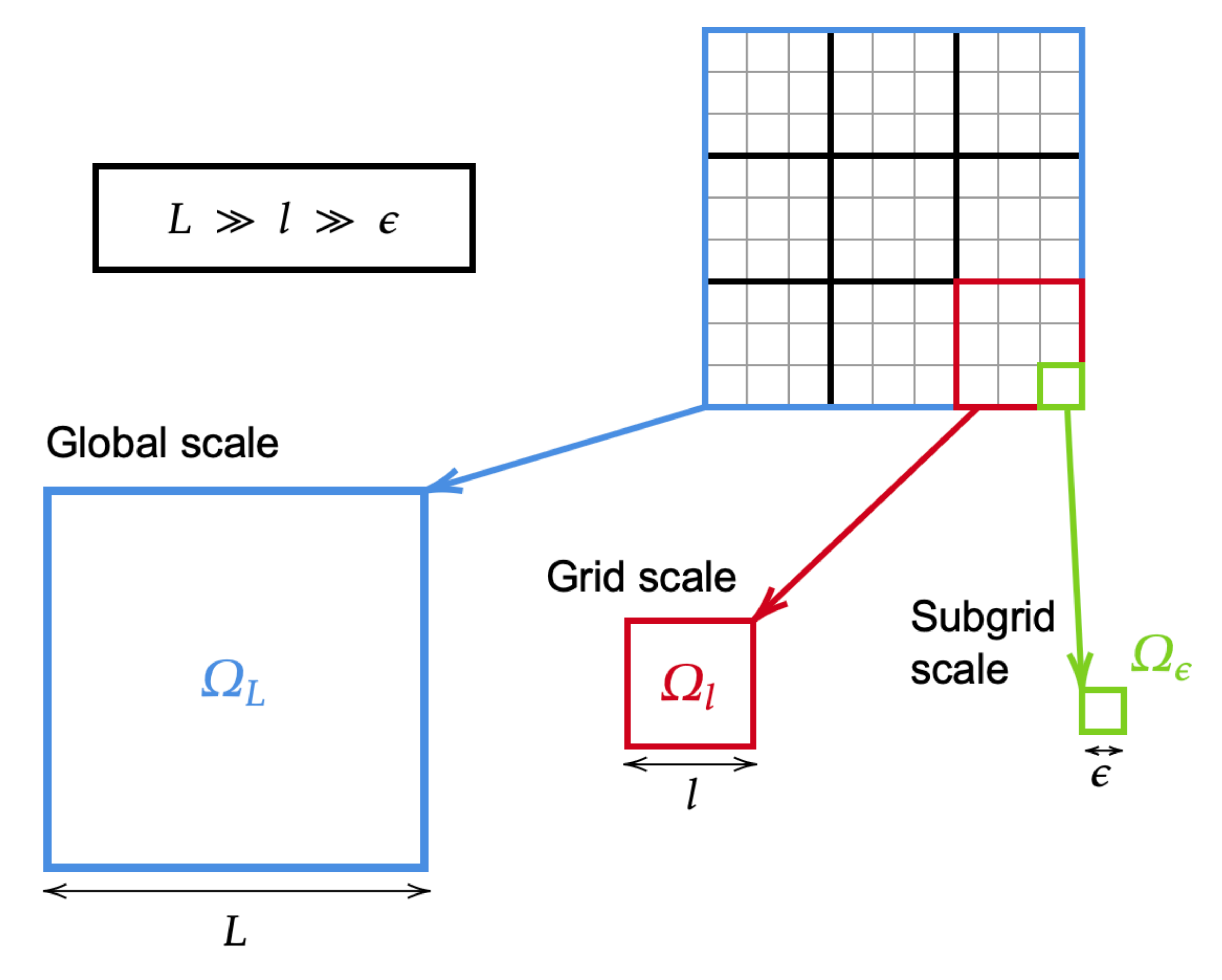

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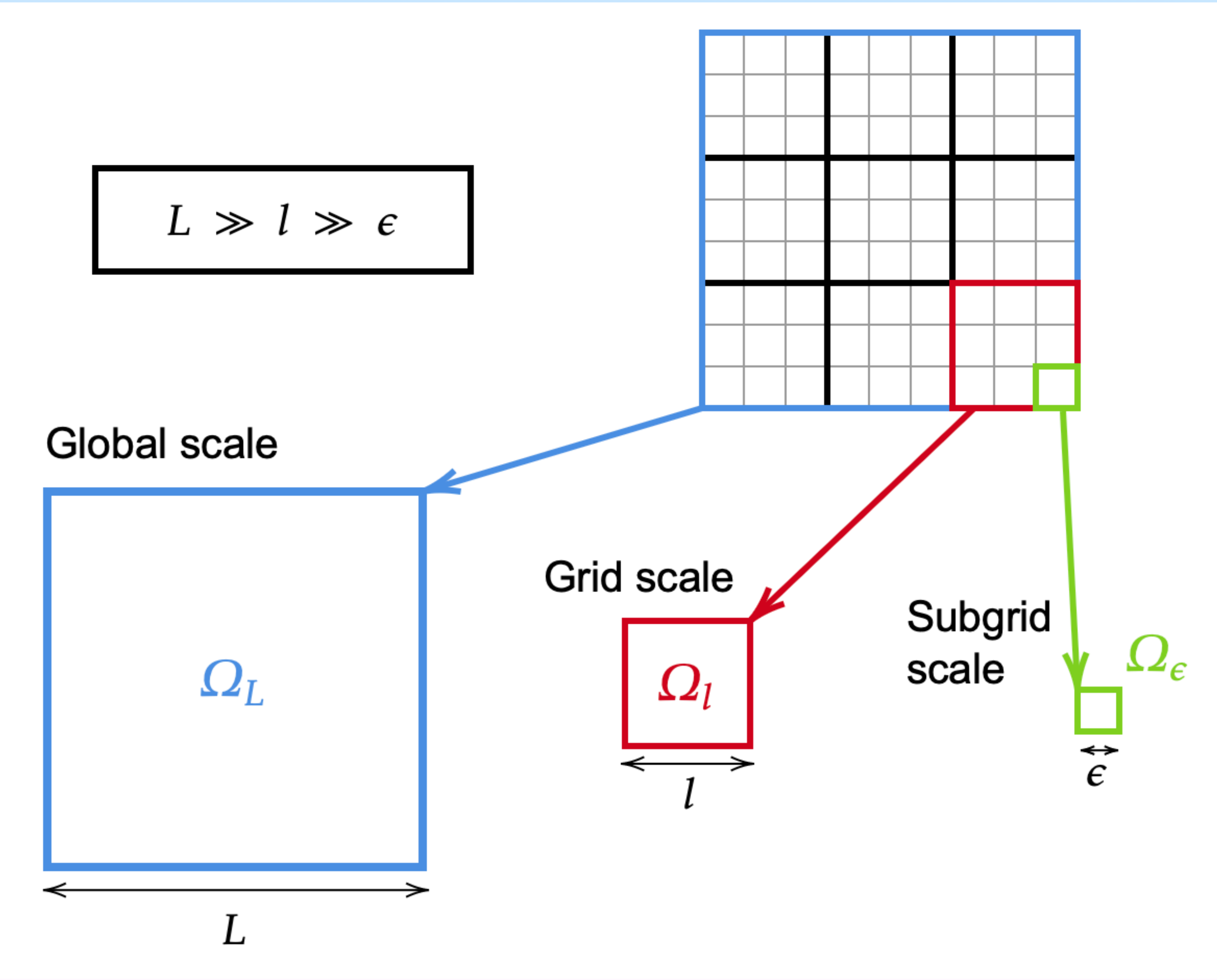

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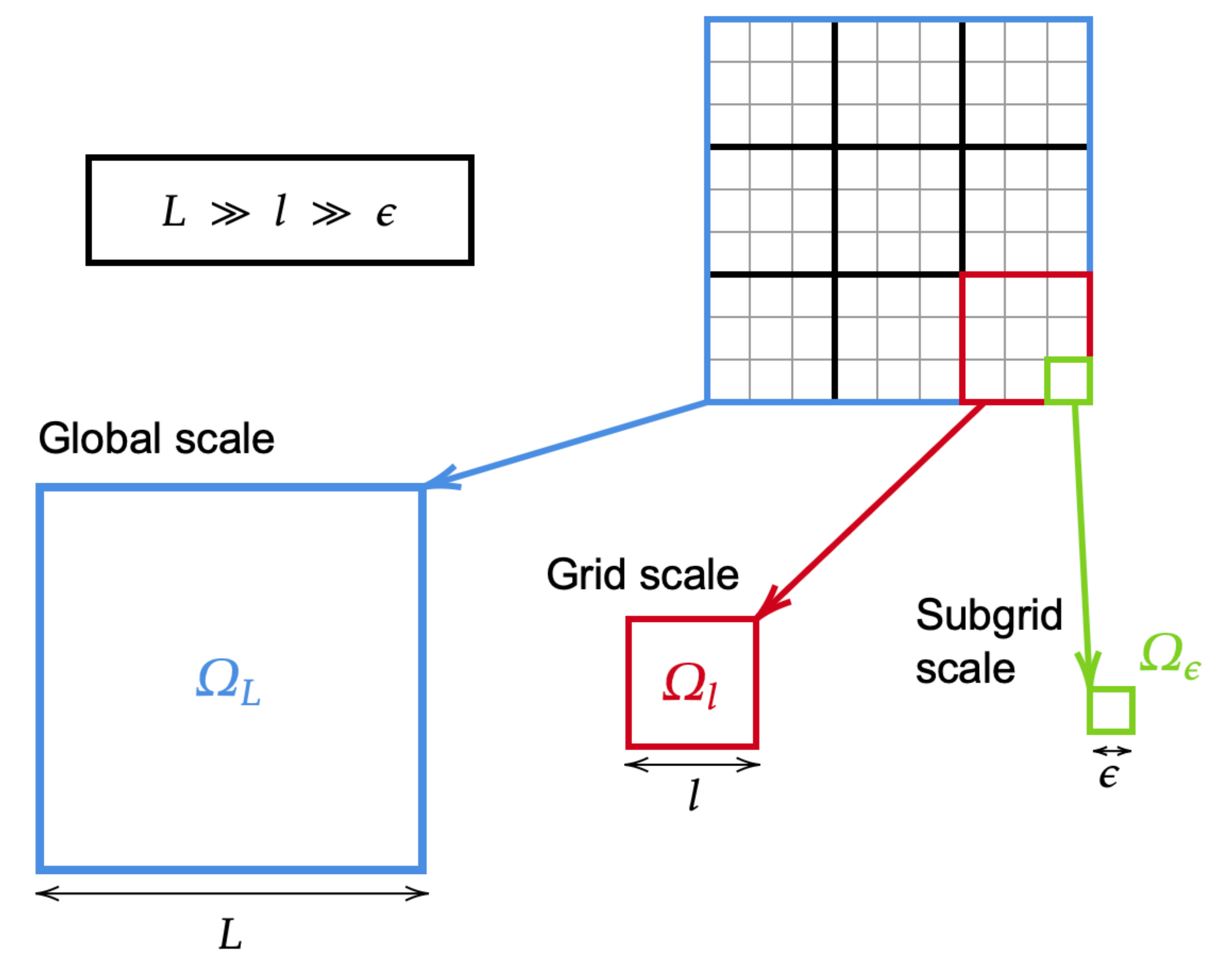

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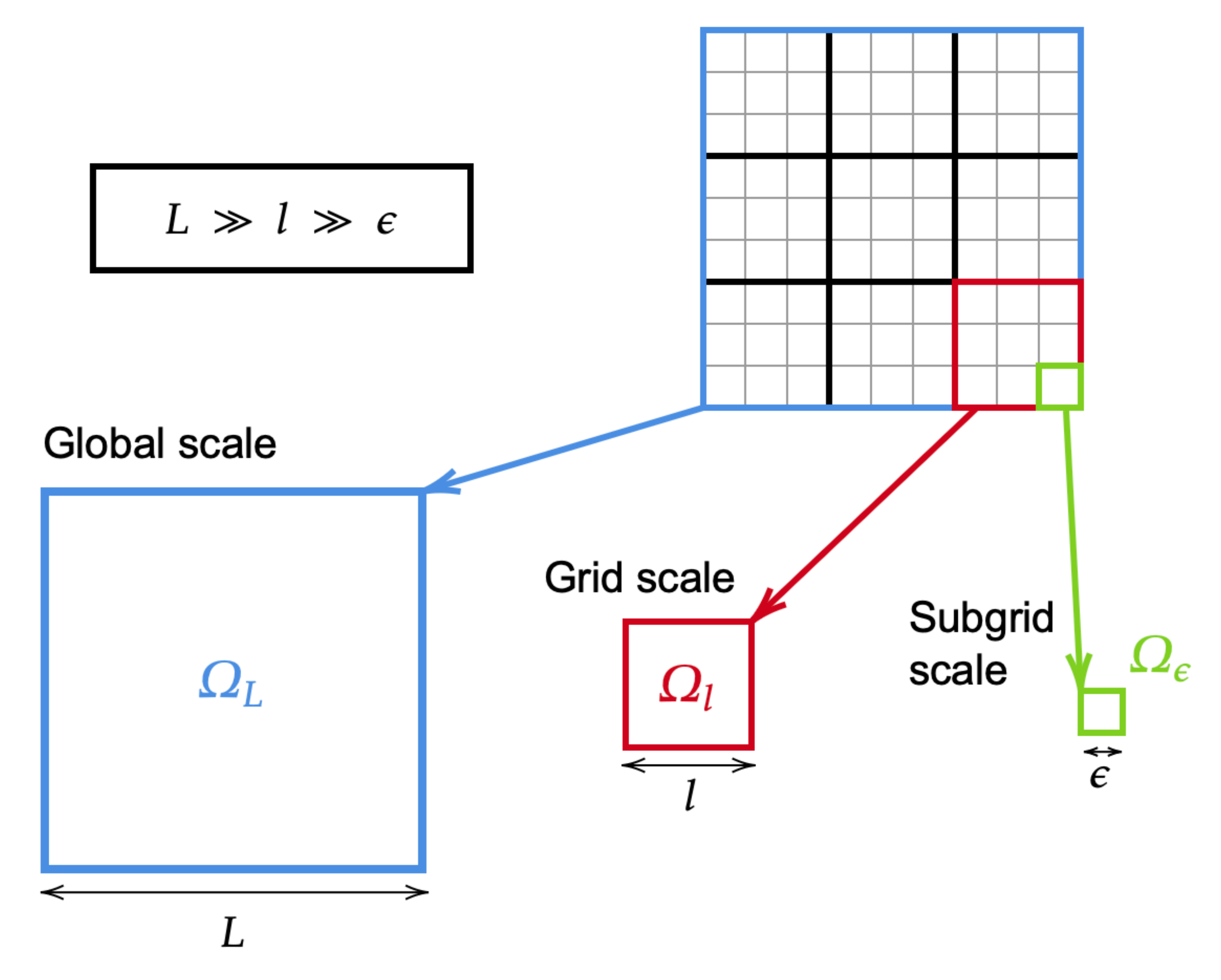

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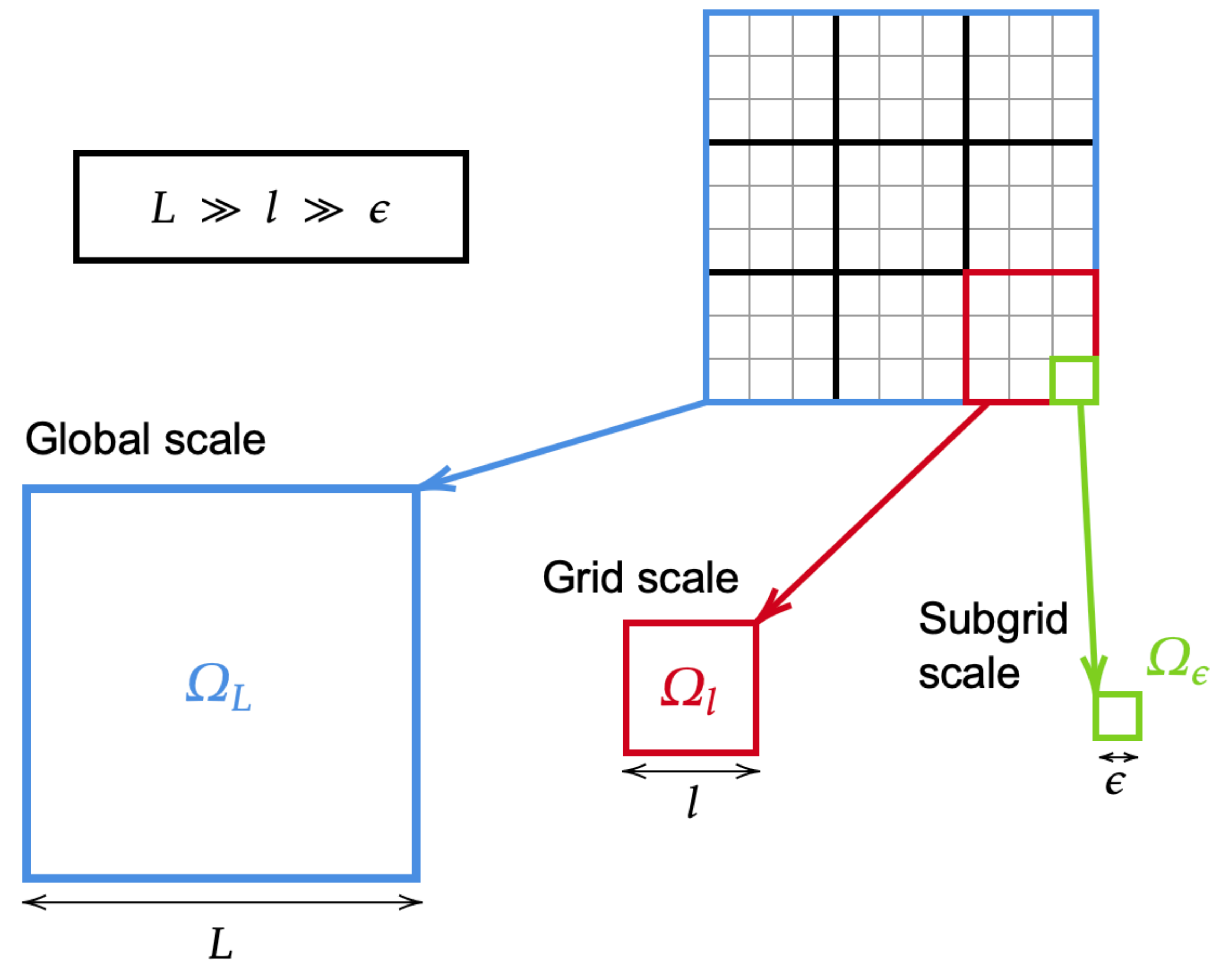

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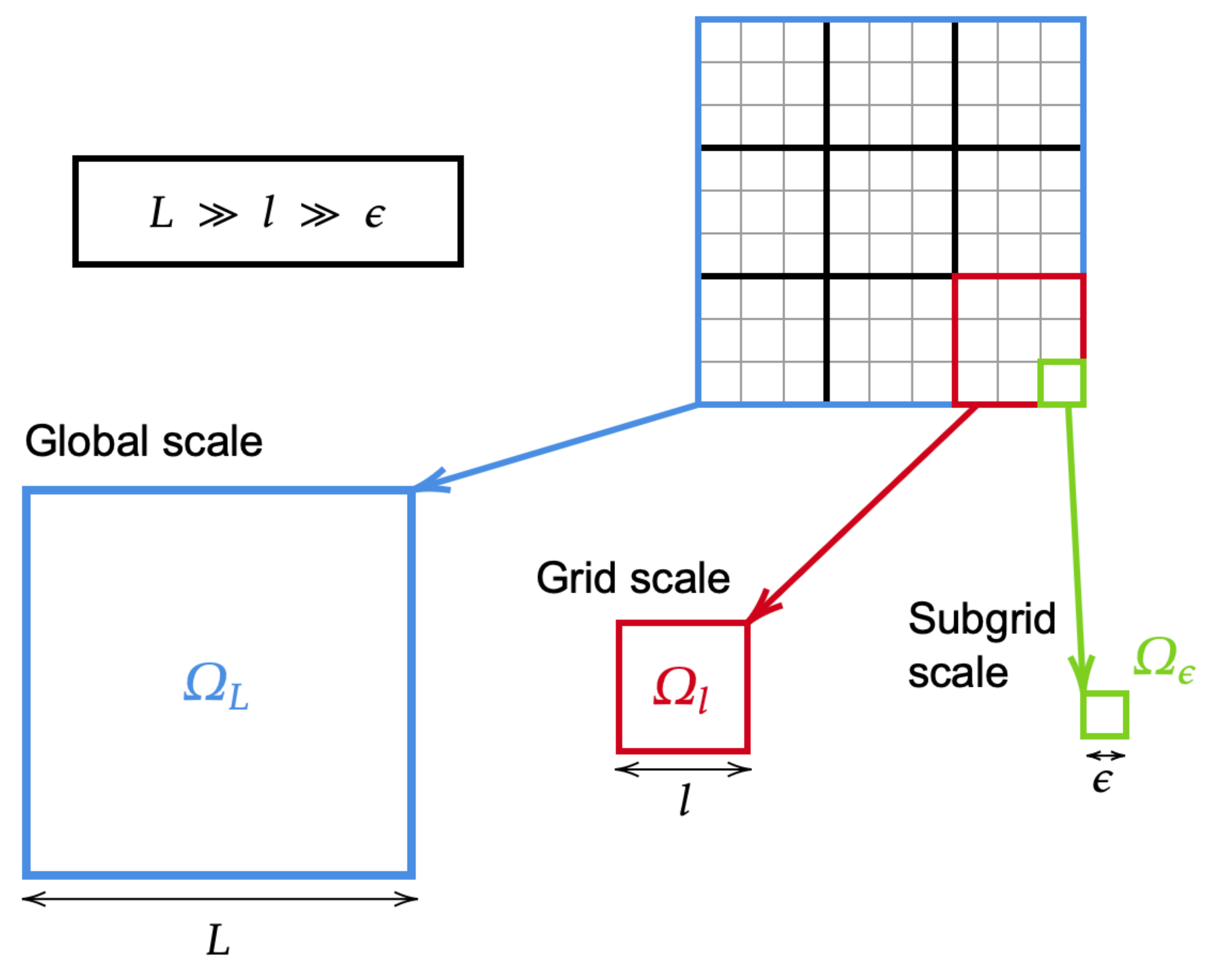

How to upscale permeability data from Ω_ϵ to Ω_l?

Upscaling algorithm: periodic homogenization

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- SGS flow approximated by Darcy's law

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Linear elliptic PDE for porous media flow.

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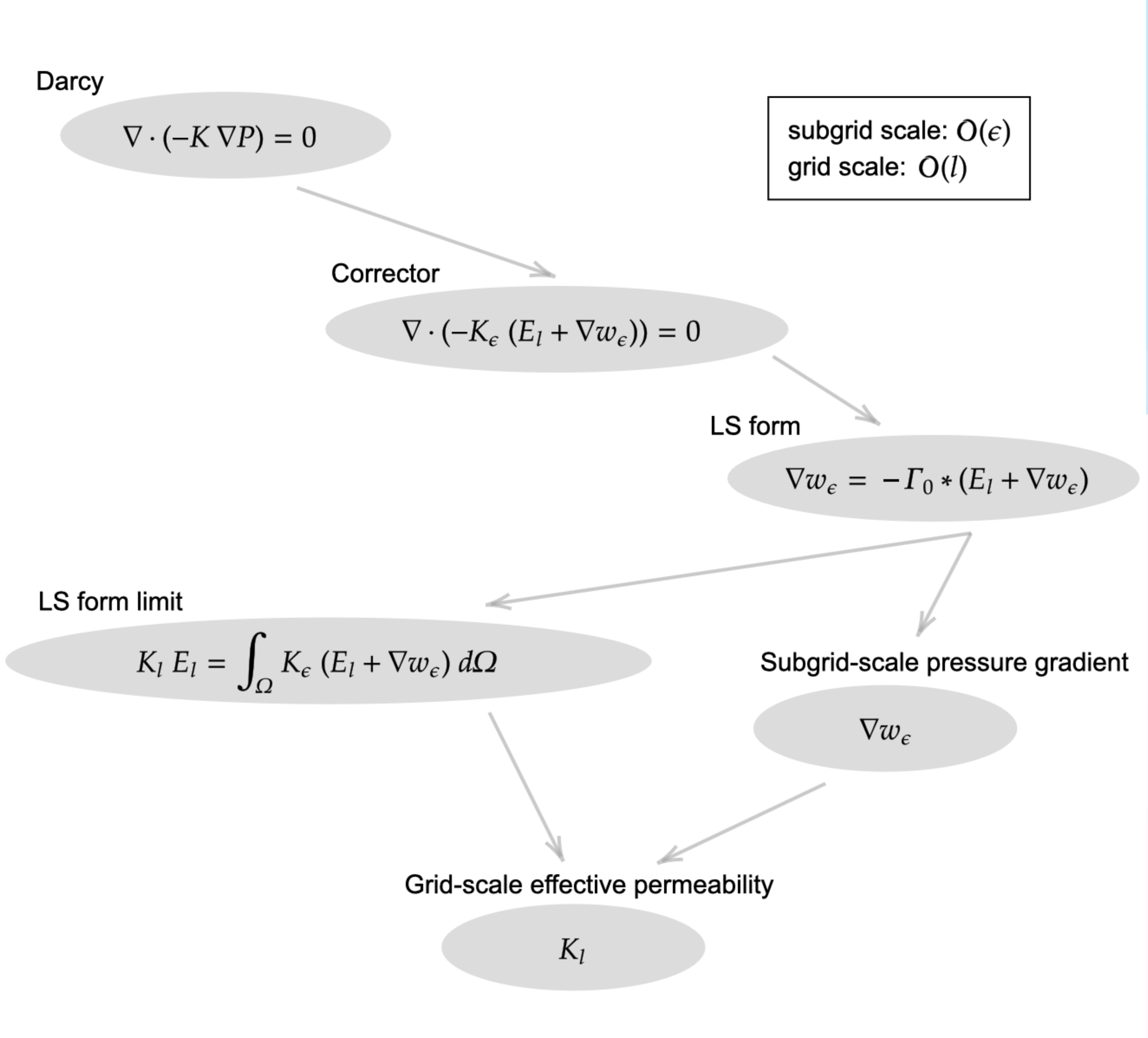

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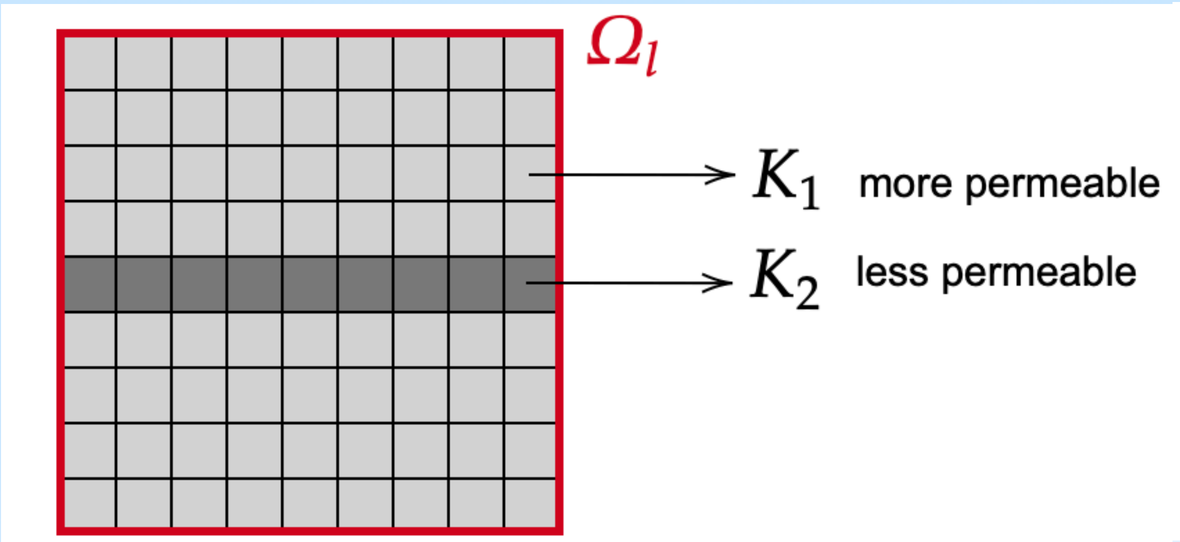
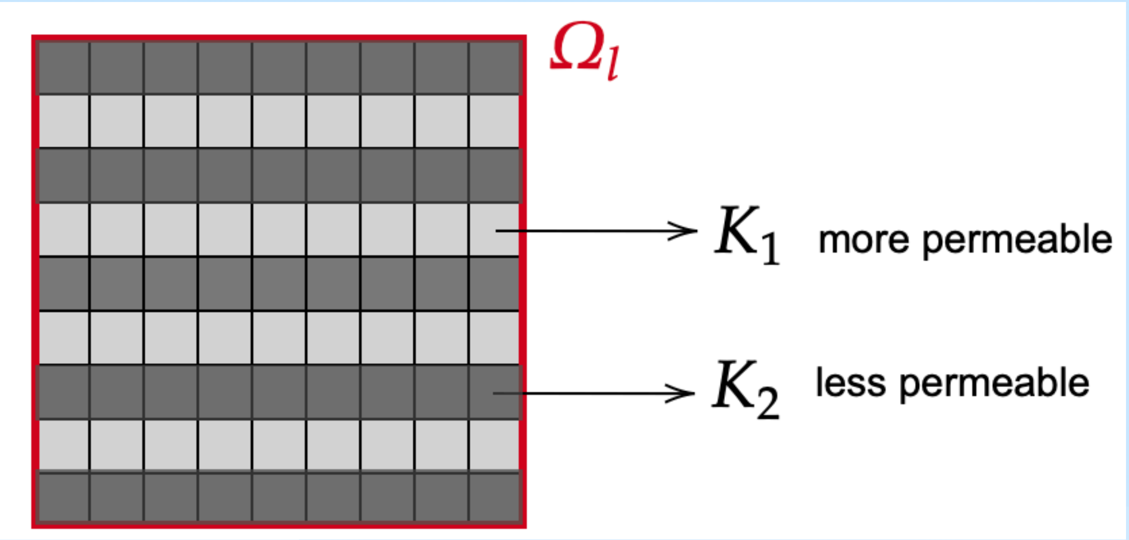
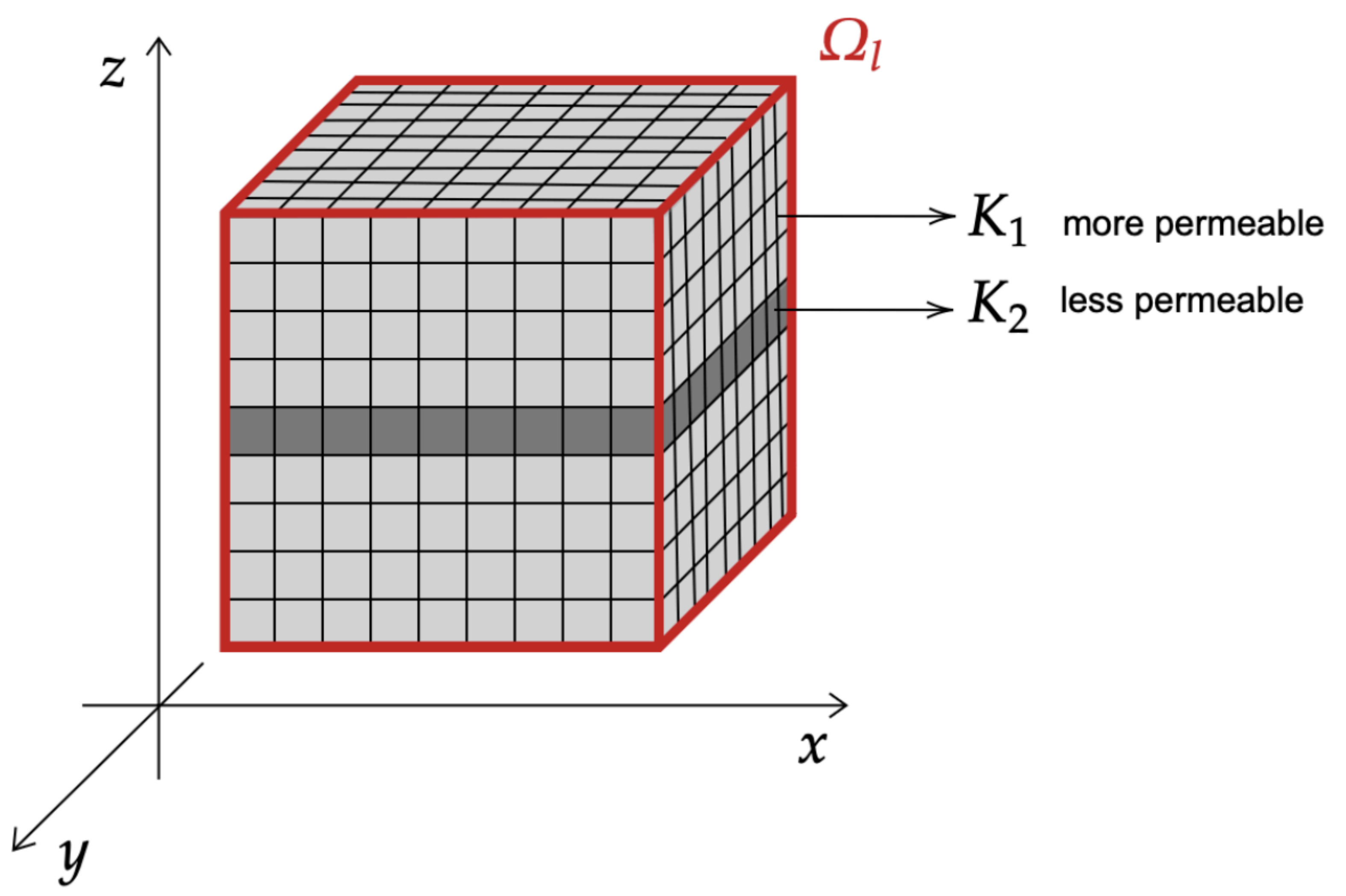


Blockage



Laminated



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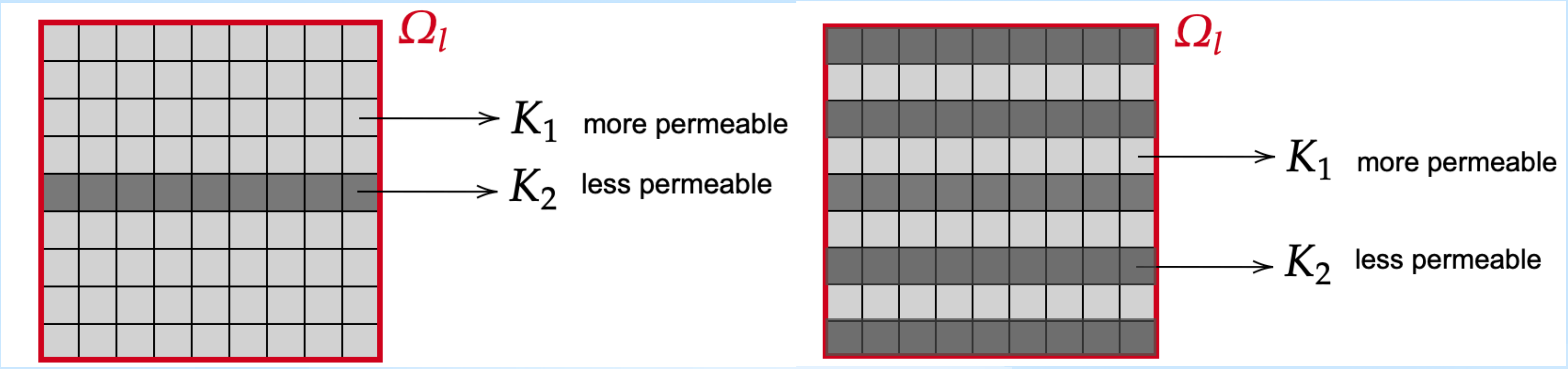
- Test cases with analytical upscaled permeability: layered structures, homogeneous material.

- Compare periodic homogenization (PH) with volume averaging (VA)
  (Renard & Ababou & 2022)



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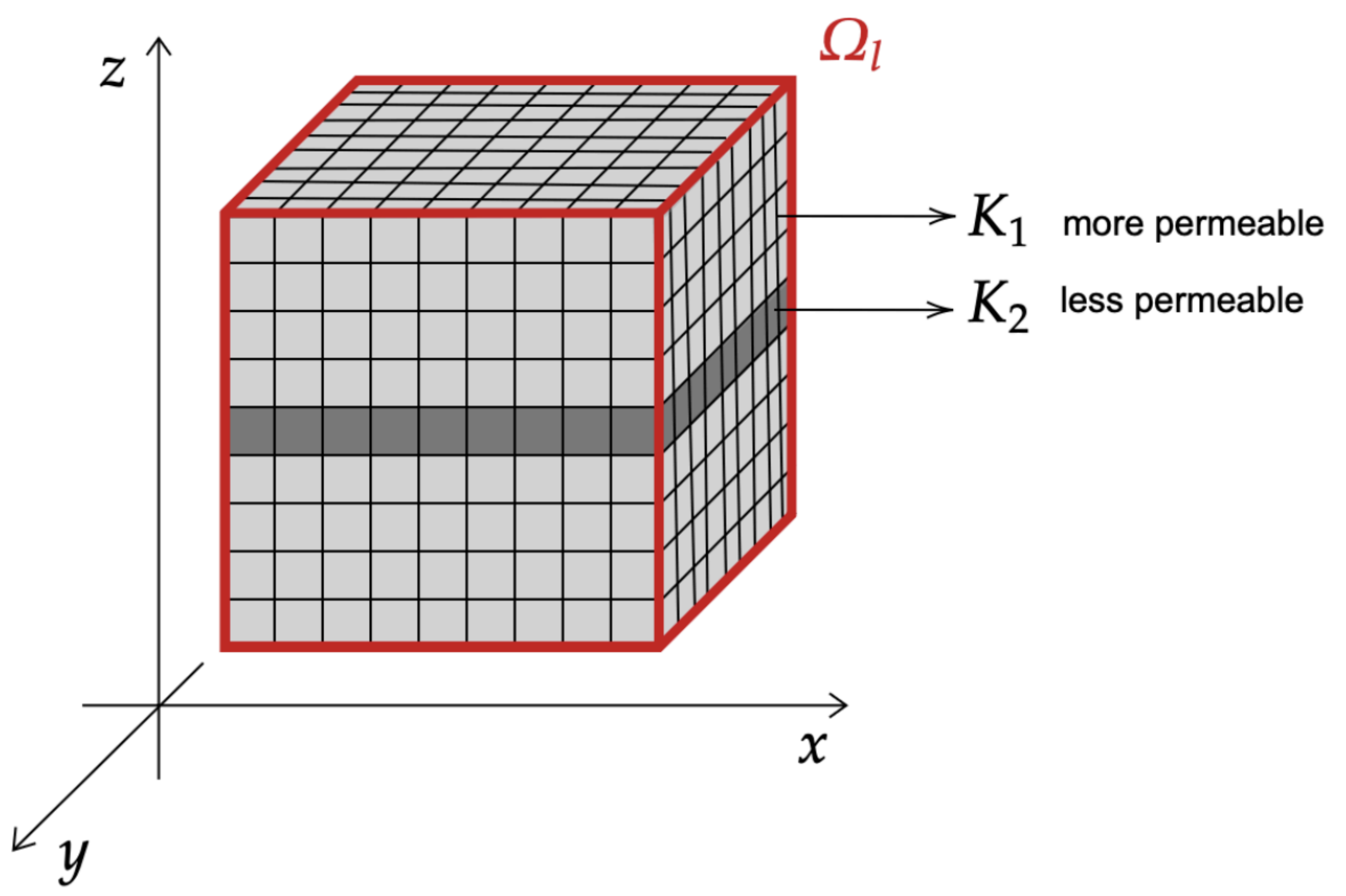

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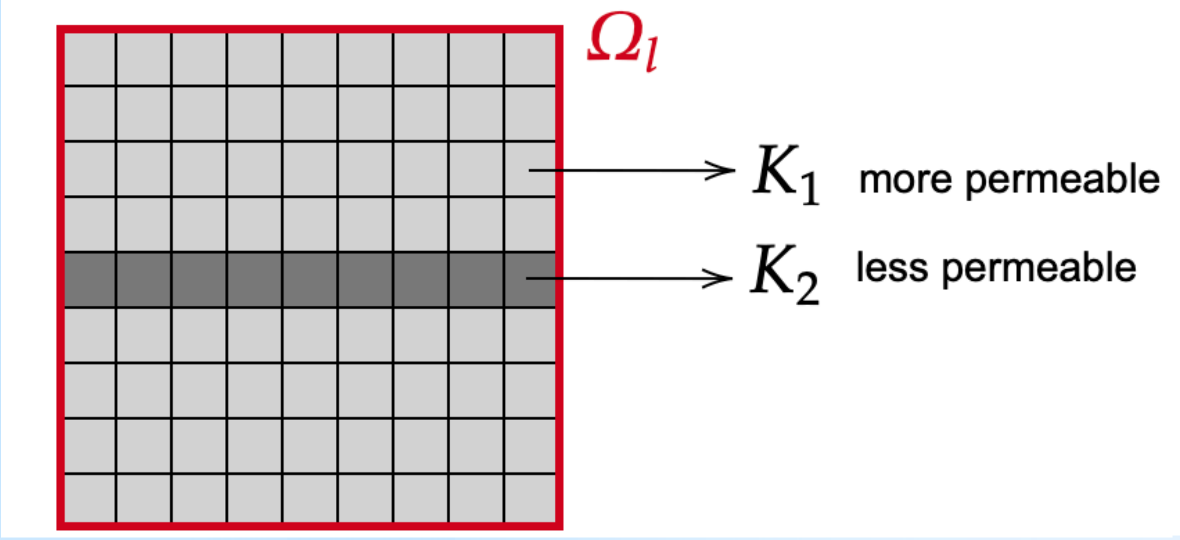

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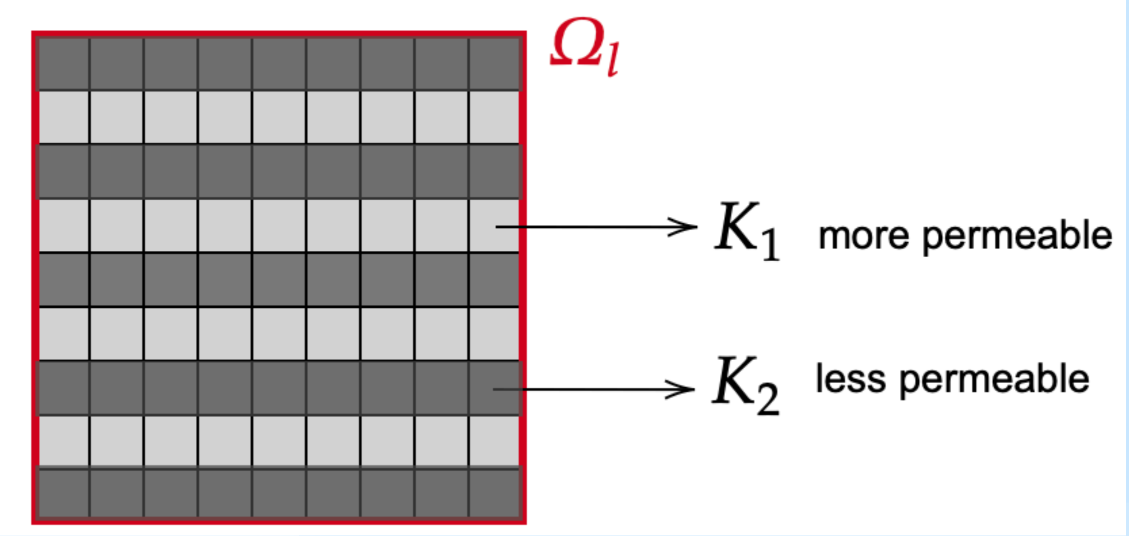

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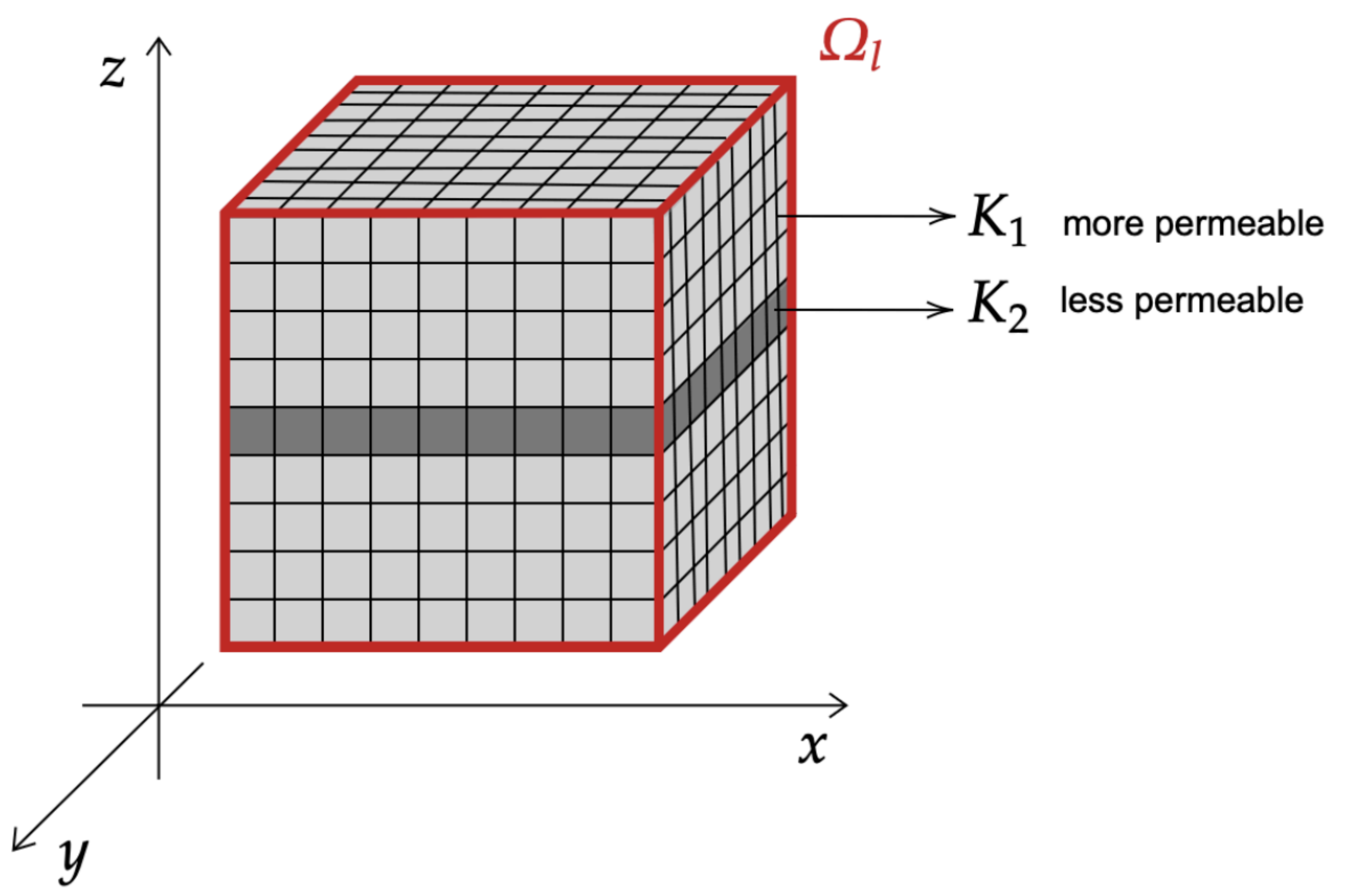

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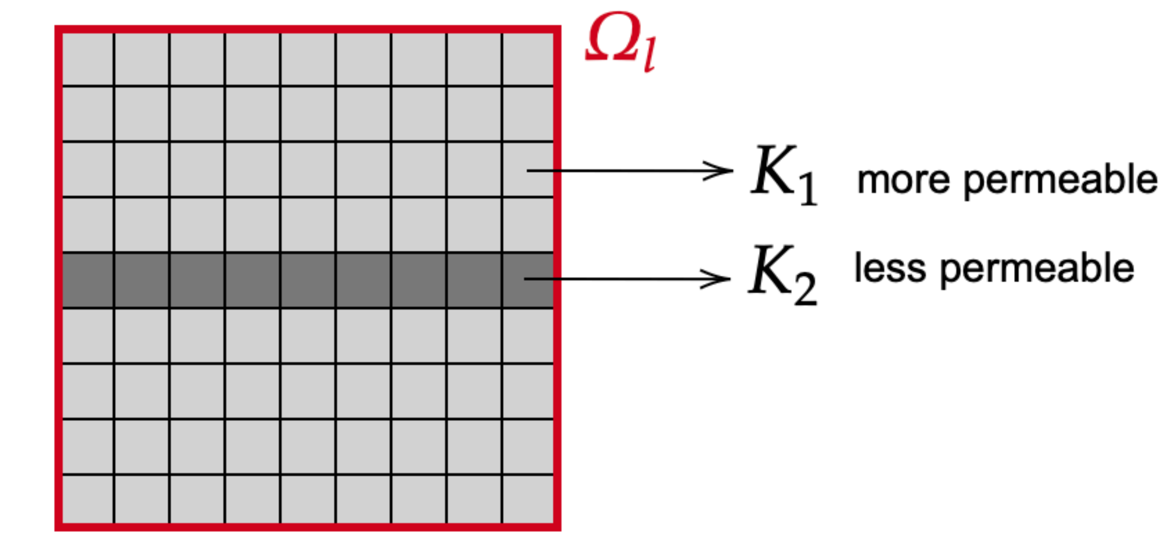

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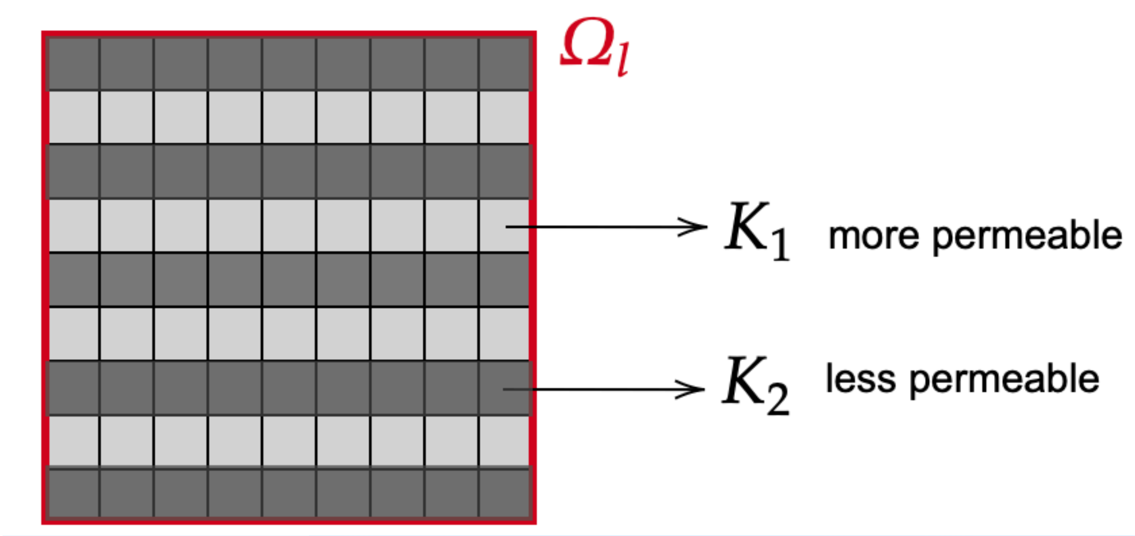

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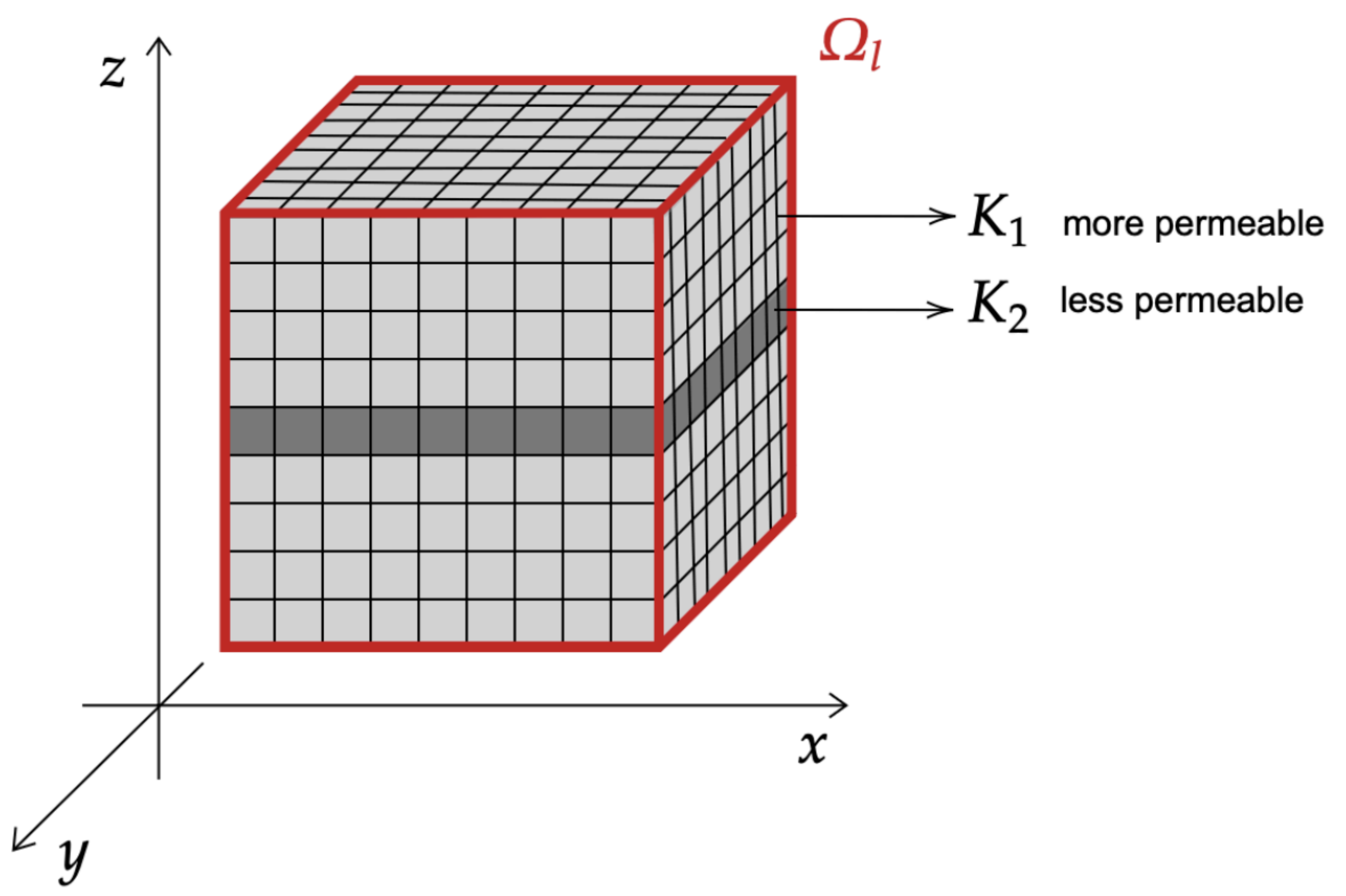

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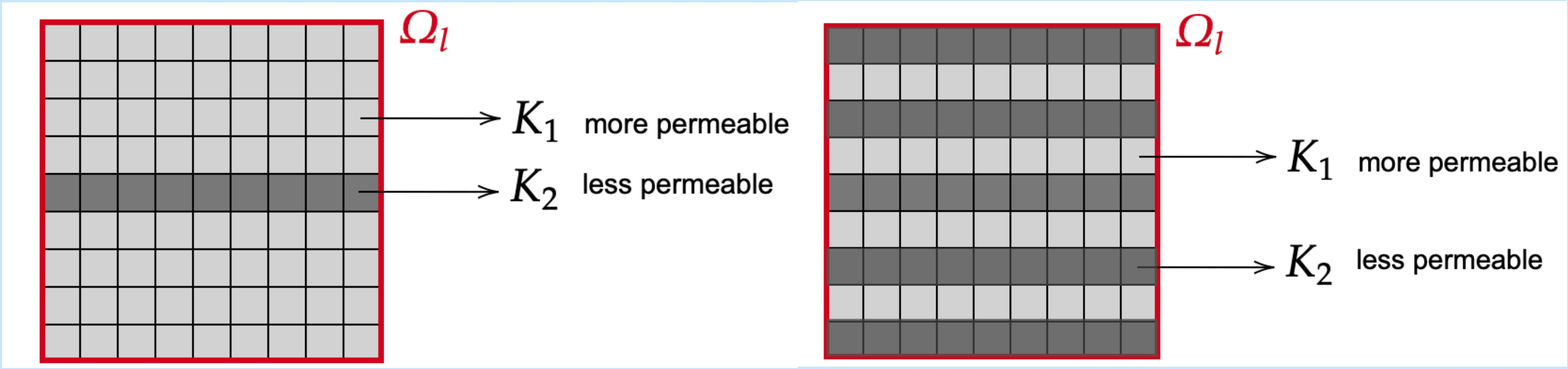

Blockage   Laminated

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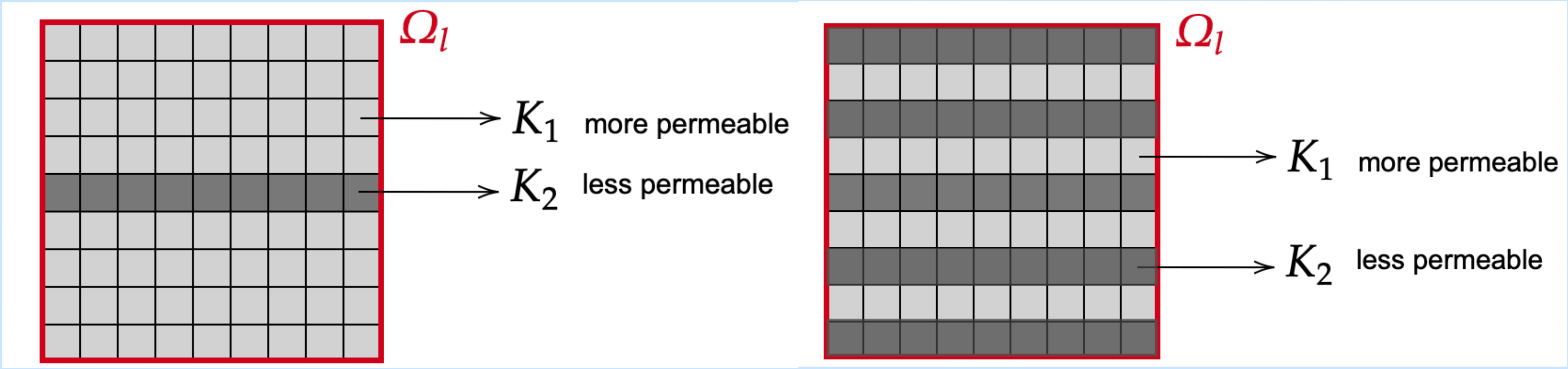

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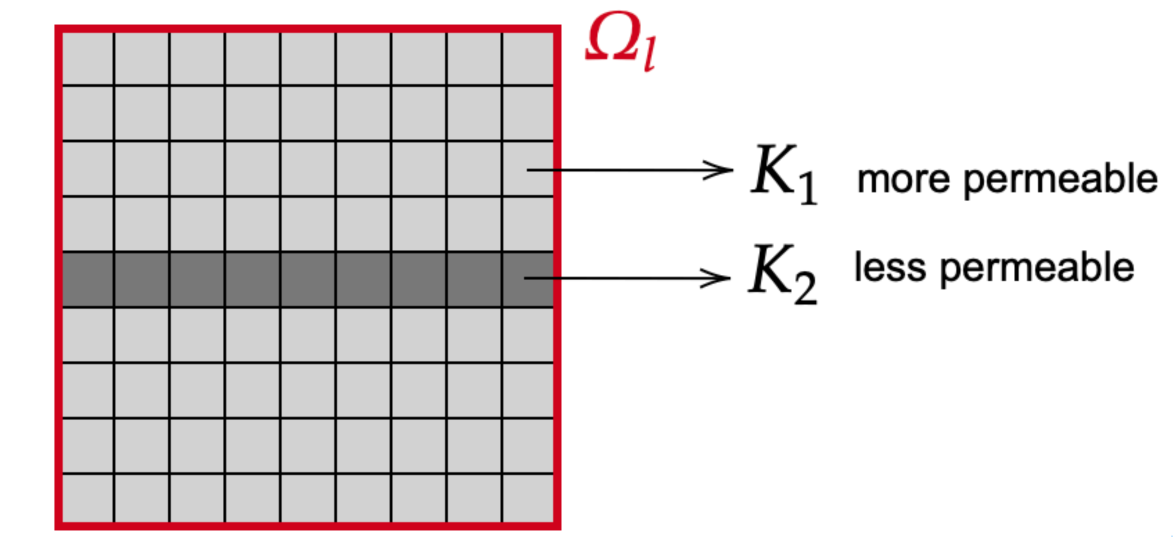

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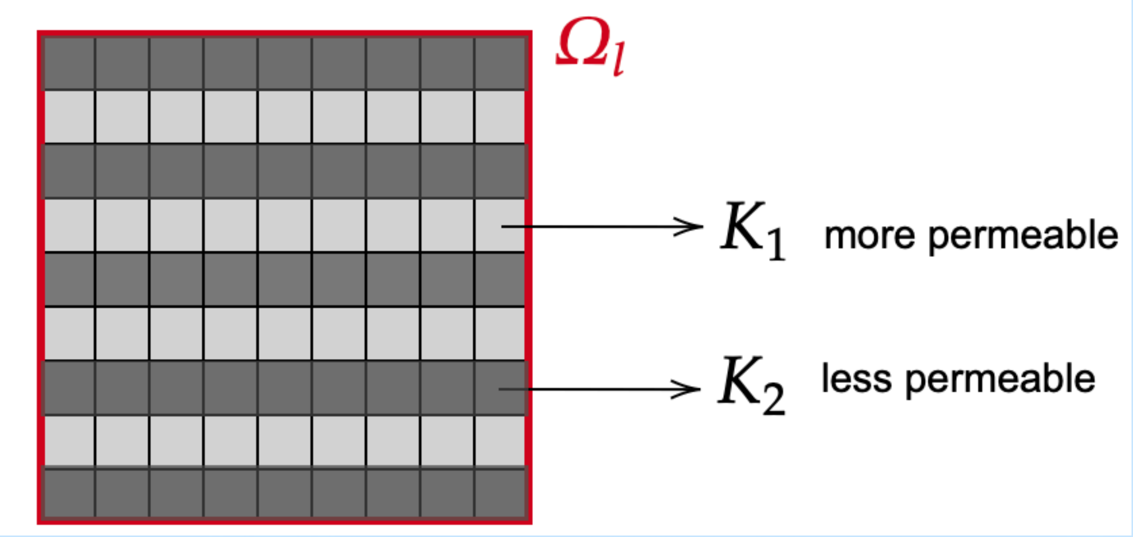

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  Reproduces fine-scale permeability.

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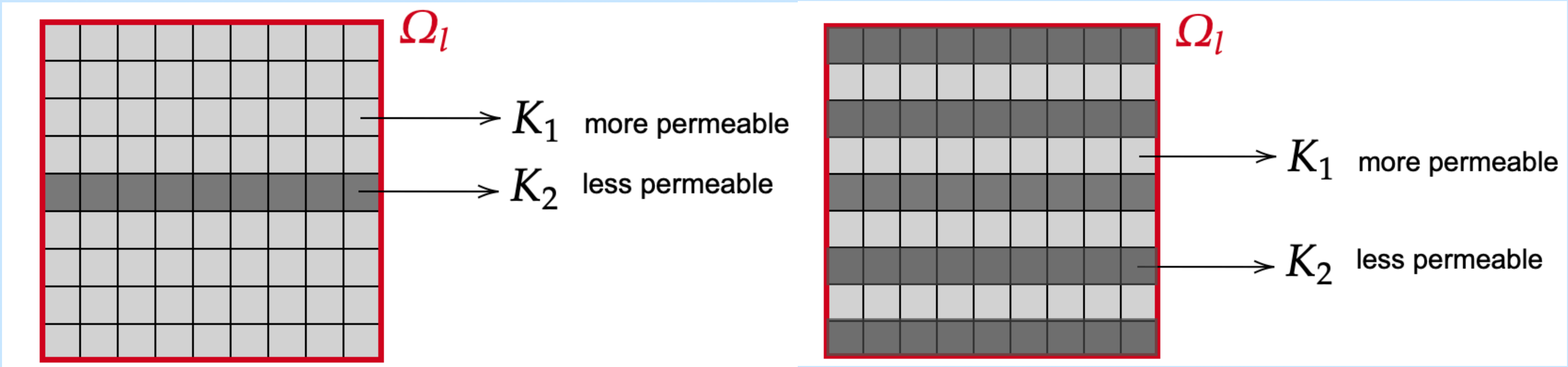

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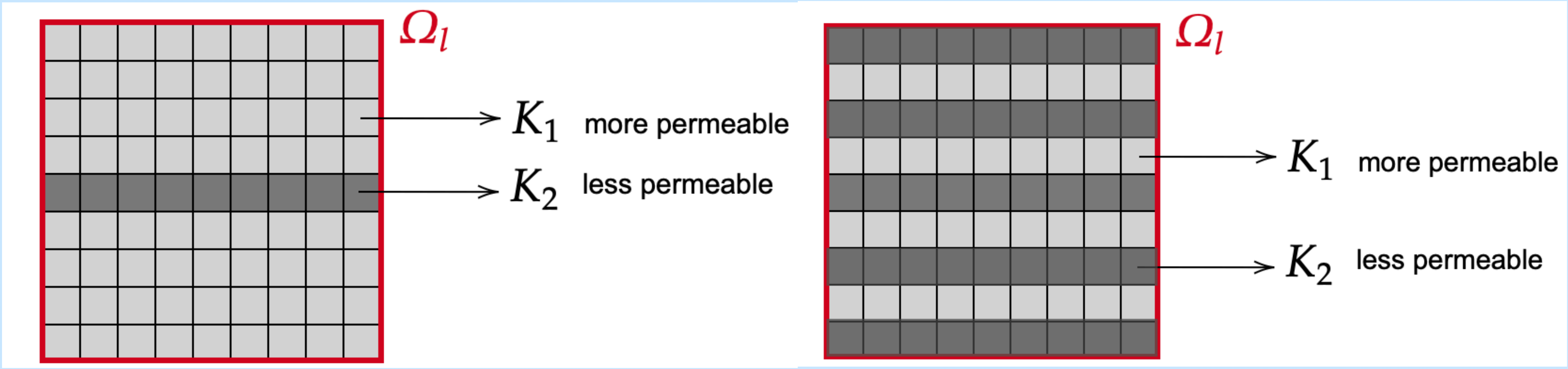

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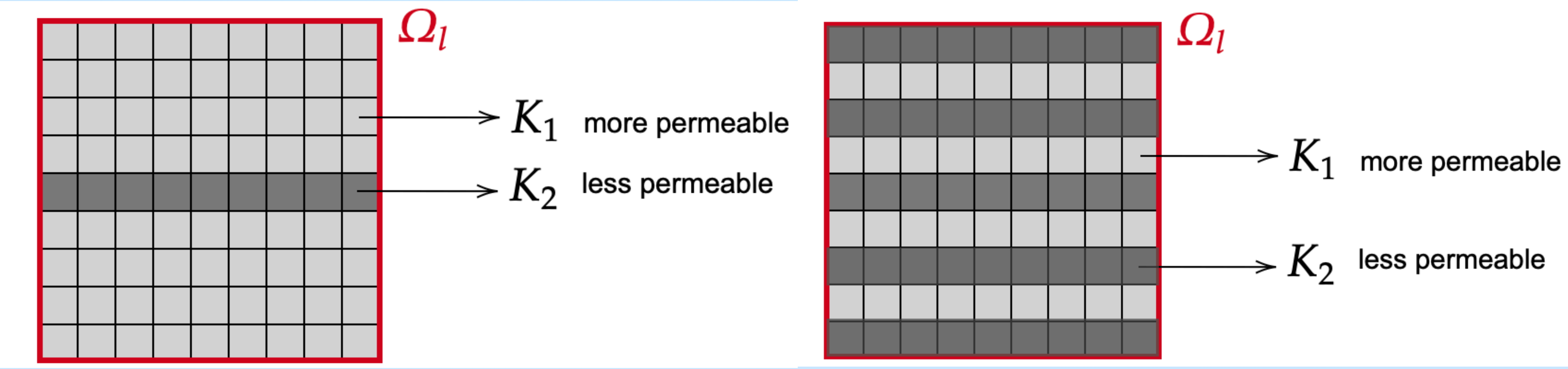

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Method	Run Time	K_1	K_2	K_l	
		11×11 Blockage			
PH	0.00212 s	$\begin{bmatrix} 0.7226 & 0 \\ 0 & 0.2667 \end{bmatrix}$	$\begin{bmatrix} 0.1473 & 0 \\ 0 & 0.4958 \end{bmatrix}$	$\begin{bmatrix} 0.6703 & 0 \\ 0 & 0.2784 \end{bmatrix}$	☺
VA	1.4901 s	"	"	$\begin{bmatrix} 0.5943 & 0.0758 \\ 0.0758 & 0.2038 \end{bmatrix}$	😐
PH	0.00145 s	$\begin{bmatrix} 0.7226 & 0.4338 \\ 0.4338 & 0.2667 \end{bmatrix}$	$\begin{bmatrix} 0.1473 & 0.1253 \\ 0.1253 & 0.4958 \end{bmatrix}$	$\begin{bmatrix} 0.6537 & 0.4181 \\ 0.4181 & 0.2784 \end{bmatrix}$	☺
VA	2.0687 s	"	"	$\begin{bmatrix} 2.6477 & 2.7776 \\ 2.7776 & 2.9138 \end{bmatrix}$	☹

Blockage permeability: PH reproduces exact results.

31x31 Laminated

PH	0.00935 s	$\begin{bmatrix} 0.7226 & 0 \\ 0 & 0.2667 \end{bmatrix}$	$\begin{bmatrix} 0.1473 & 0 \\ 0 & 0.4958 \end{bmatrix}$	$\begin{bmatrix} 0.4256 & 0 \\ 0 & 0.3502 \end{bmatrix}$	☺
VA	1.440 s	"	"	$\begin{bmatrix} 0.3984 & 0.0635 \\ 0.0635 & 0.2908 \end{bmatrix}$	😐
PH	0.00681 s	$\begin{bmatrix} 0.7226 & 0.4338 \\ 0.4338 & 0.2667 \end{bmatrix}$	$\begin{bmatrix} 0.1473 & 0.1253 \\ 0.1253 & 0.4958 \end{bmatrix}$	$\begin{bmatrix} 0.3627 & 0.3212 \\ 0.3213 & 0.3502 \end{bmatrix}$	☺
VA	7.469 s	"	"	$\begin{bmatrix} 5.6020 & 7.2797 \\ 7.2797 & 17.312 \end{bmatrix}$	☹

Laminated permeability: PH reproduces exact results.

31x31 Homogenous

PH	0.00707 s	$\begin{bmatrix} 0.7226 & 0 \\ 0 & 0.2667 \end{bmatrix}$	N/A	$\begin{bmatrix} 0.7226 & 0 \\ 0 & 0.2667 \end{bmatrix}$	☺
VA	1.5966 s	"	"	$\begin{bmatrix} 0.6532 & 0.0694 \\ 0.0694 & 0.1974 \end{bmatrix}$	☹
PH	0.00798 s	$\begin{bmatrix} 0.7226 & 0.4338 \\ 0.4338 & 0.2667 \end{bmatrix}$	N/A	$\begin{bmatrix} 0.7226 & 0.4338 \\ 0.4338 & 0.2667 \end{bmatrix}$	☺
VA	7.637 s	"	"	$\begin{bmatrix} 15.615 & 10.065 \\ 10.065 & 6.4875 \end{bmatrix}$	☹

Homogeneous case: upscaled permeability is same as fine scale permeability!

Application to 2D shallow water equations

$$\frac{\partial \tilde{h}}{\partial t} + \nabla \cdot \tilde{\boldsymbol{m}} = 0$$

$$\frac{\partial \tilde{\boldsymbol{m}}}{\partial t} + \nabla \cdot (\tilde{\boldsymbol{m}} \otimes \boldsymbol{u}) + \phi \nabla \left(\frac{1}{2} g h^2 \right) = \nu_{\mathrm{e}} \nabla \left(\tilde{h} \cdot \nabla \boldsymbol{u} \right) - \mathbb{1}(\boldsymbol{x})\, \nu \mathbf{k}^{-1} \tilde{\boldsymbol{m}} + \tilde{h} \boldsymbol{F}$$

Porous shallow water equations *(pSWE)* **in conservation form**

Dropped constant density factor ρ_0.

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- Assume $\nu_e = \nu$ and absorb viscosity into permeability, $\mathbf{K} := \nu \mathbf{k}$

$$\mathbb{1}(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \phi(x) < 1, \\ 0, & \text{otherwise.} \end{cases}$$ is mask defining fluid/non-fluid regions.

Periodic homogenization for pSWE

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  No adjustable parameters!

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  Upscaled permeability depends only on ratio of fine scale permeabilities.

Periodic homogenization for pSWE

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- Set solid porosity to ϕ_{min}= 0.05.

Numerical model

$$\frac{\partial \tilde{h}}{\partial t} + \nabla \cdot \tilde{\boldsymbol{m}} = 0$$

$$\frac{\partial \tilde{\boldsymbol{m}}}{\partial t} + \nabla \cdot (\tilde{\boldsymbol{m}} \otimes \boldsymbol{u}) + \phi \nabla \left(\frac{1}{2} g h^2 \right) = \nu \nabla \left(\tilde{h} \cdot \nabla \boldsymbol{u} \right) - \mathbb{1}(\boldsymbol{x}) \, \mathbf{K}^{-1} \tilde{\boldsymbol{m}} + \tilde{h} \boldsymbol{F}$$

Porous shallow water equations *(pSWE)* **in conservation form**

Numerical model

$$\frac{\partial \tilde{h}}{\partial t} + \nabla \cdot \tilde{\boldsymbol{m}} = 0$$

$$\frac{\partial \tilde{\boldsymbol{m}}}{\partial t} + \nabla \cdot (\tilde{\boldsymbol{m}} \otimes \boldsymbol{u}) + \phi \nabla \left(\frac{1}{2} g h^2 \right) = \nu \nabla \left(\tilde{h} \cdot \nabla \boldsymbol{u} \right) - \mathbb{1}(\boldsymbol{x})\, \mathbf{K}^{-1} \tilde{\boldsymbol{m}} + \tilde{h} \boldsymbol{F}$$

Porous shallow water equations *(pSWE)* **in conservation form**

- Second order finite volume/finite difference method on Arakawa C-grid *(staggered grid)*.

Numerical model

$$\frac{\partial \tilde{h}}{\partial t} + \nabla \cdot \tilde{\boldsymbol{m}} = 0$$

$$\frac{\partial \tilde{\boldsymbol{m}}}{\partial t} + \nabla \cdot (\tilde{\boldsymbol{m}} \otimes \boldsymbol{u}) + \phi \nabla \left(\frac{1}{2} g h^2\right) = \nu \nabla \left(\tilde{h} \cdot \nabla \boldsymbol{u}\right) - \mathbb{1}(\boldsymbol{x})\, \mathbf{K}^{-1} \tilde{\boldsymbol{m}} + \tilde{h} \boldsymbol{F}$$

Porous shallow water equations *(pSWE)* **in conservation form**

- Second order finite volume/finite difference method on Arakawa C-grid *(staggered grid)*.
- Periodic inflow/outflow boundary conditions.

Numerical model

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Porous shallow water equations *(pSWE)* **in conservation form**

- Second order finite volume/finite difference method on Arakawa C-grid *(staggered grid)*.
- Periodic inflow/outflow boundary conditions.
- Free-slip lateral boundary conditions.

Numerical model

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- Second order finite volume/finite difference method on Arakawa C-grid *(staggered grid)*.
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- RK4 in time.

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- Masking for solid no-slip boundary conditions for non-porous SWE *(reference fine scale Ω_ϵ simulation)*.

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- Cell-edge permeability is the harmonic mean of porosities of adjacent cells.

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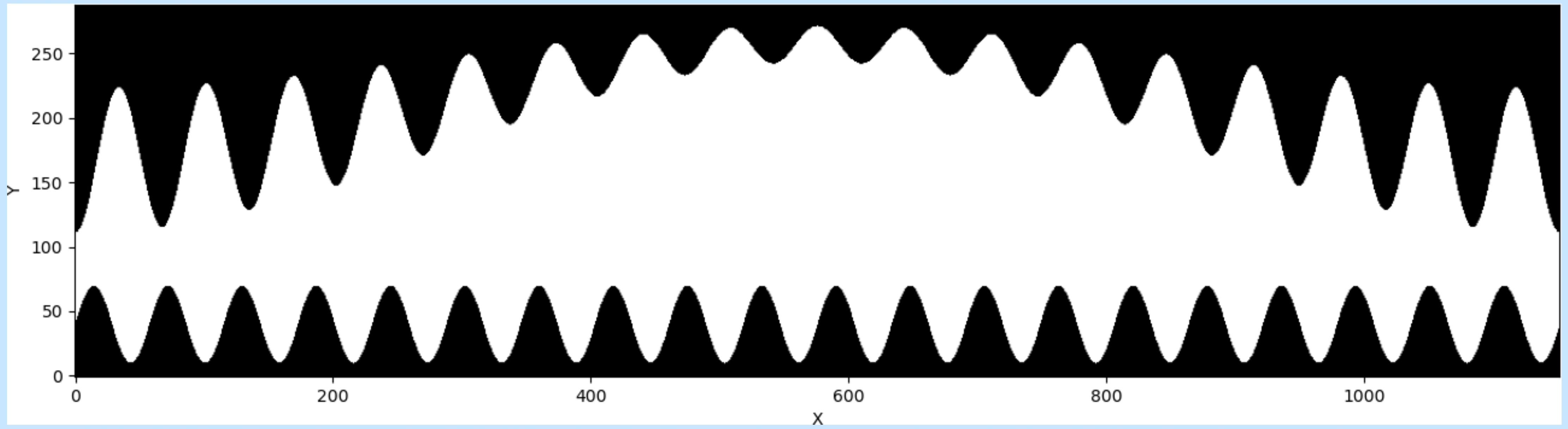
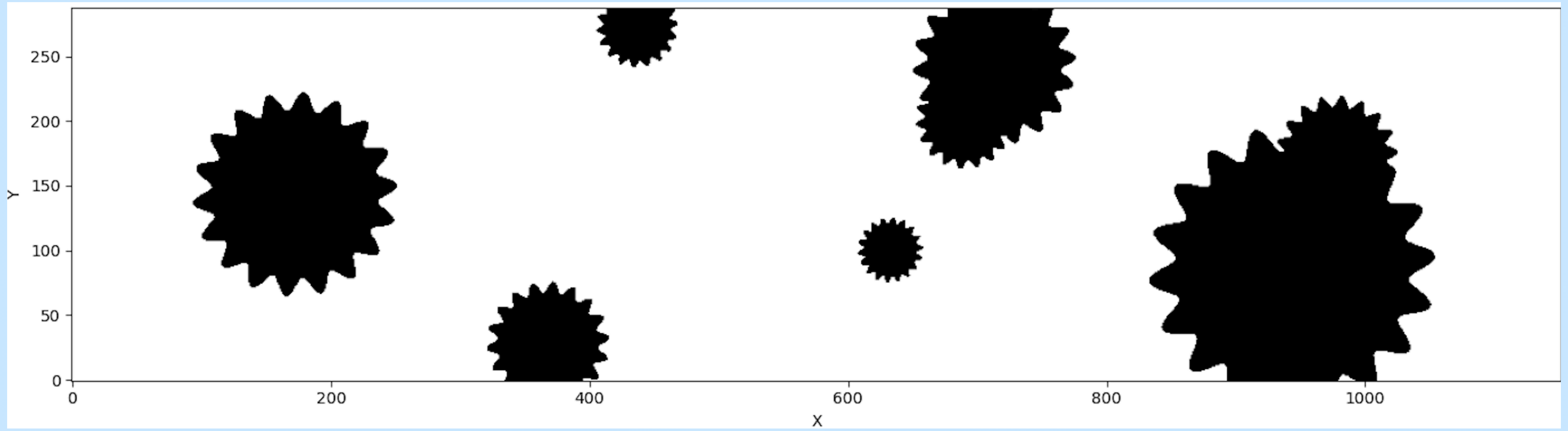
- Cell-edge permeability is the harmonic mean of porosities of adjacent cells.
- Cell-edge friction *(inverse permeability matrix)* is arithmetic mean of adjacent cell resistivity matrices.
- Impose constant body force in $x-$direction *(simple model of large scale internal tides)*.

Topography test cases

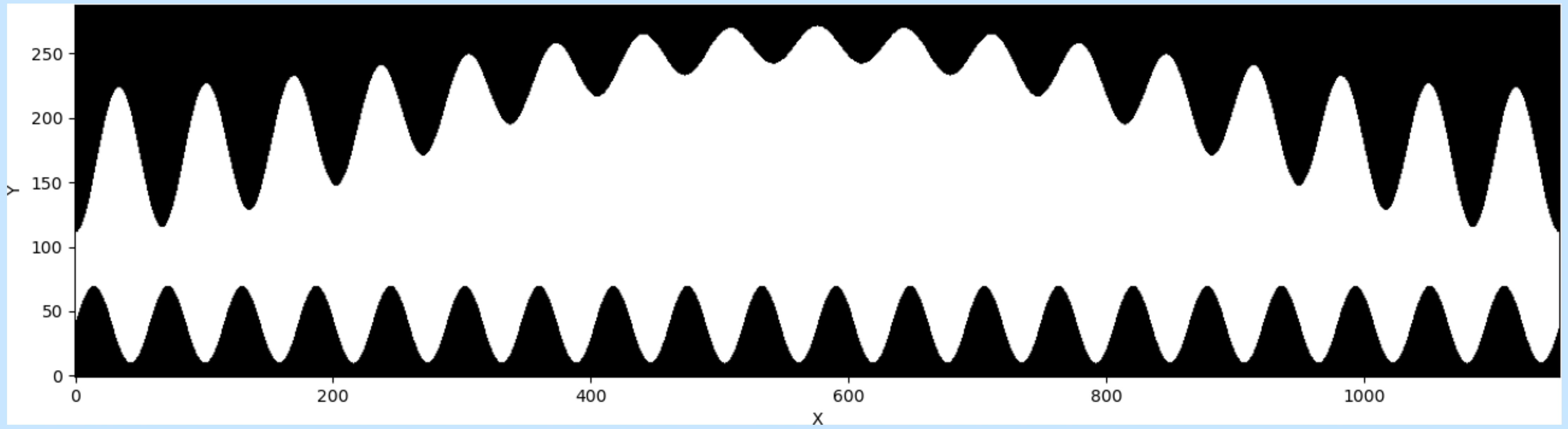


Asymmetric channel (CH)



Islands (IS)

Topography test cases



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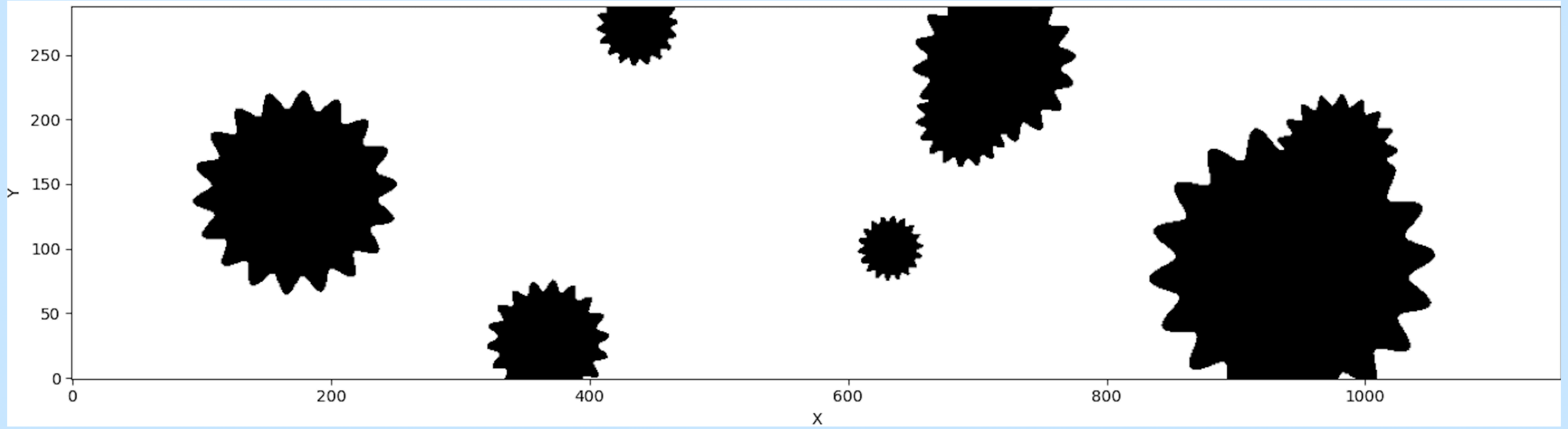

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- **Constant body force.** *Mean speed* $O(1)$.

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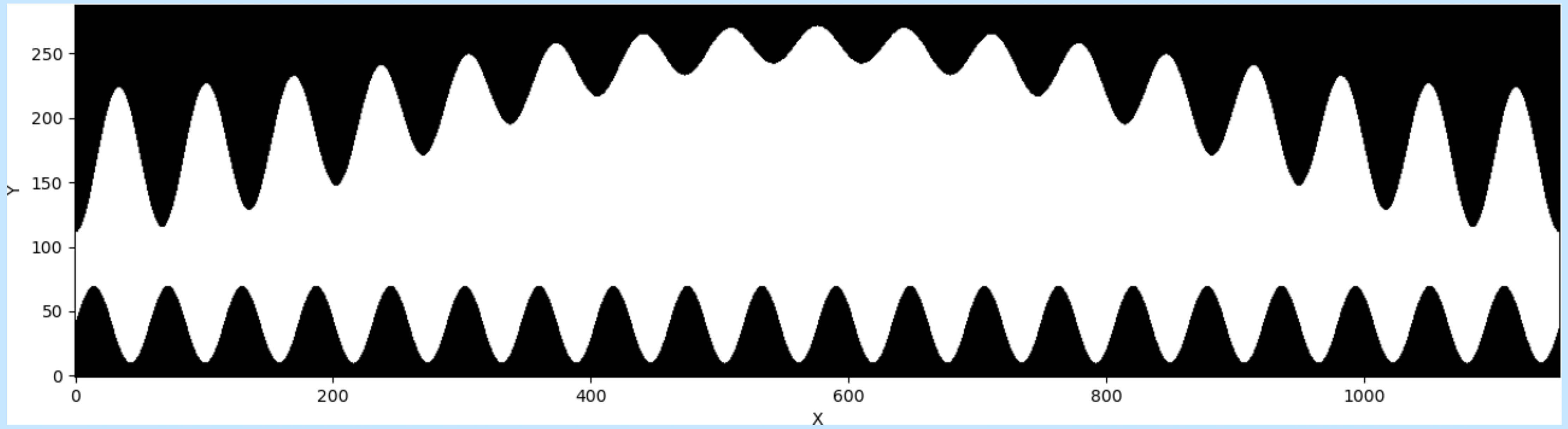

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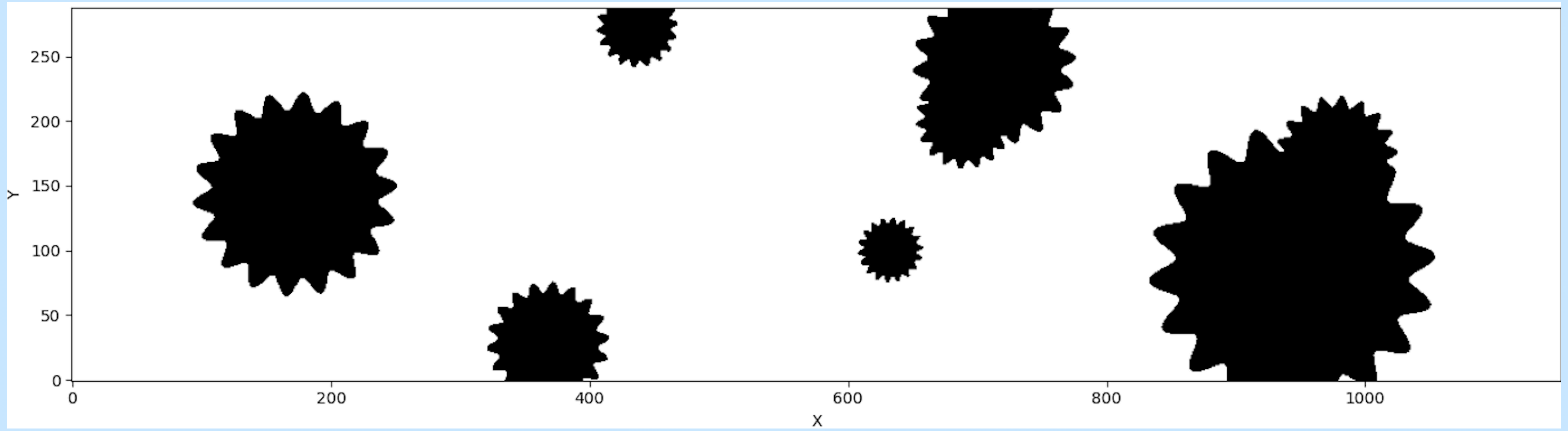

Islands (IS)

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  Clean test: separates topography upscaling errors from SGS turbulence errors.

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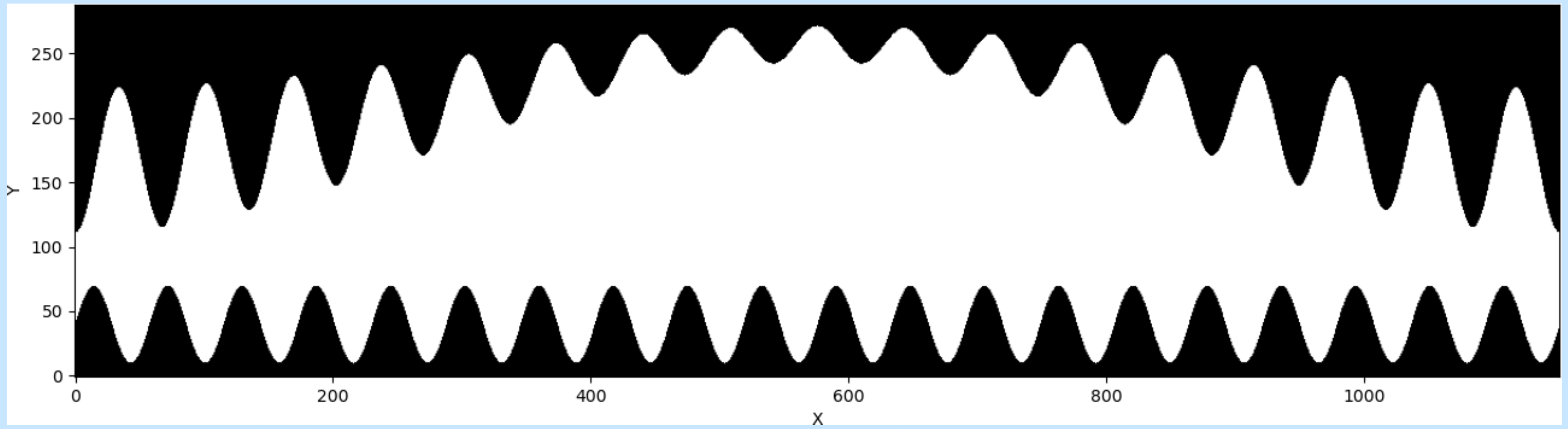

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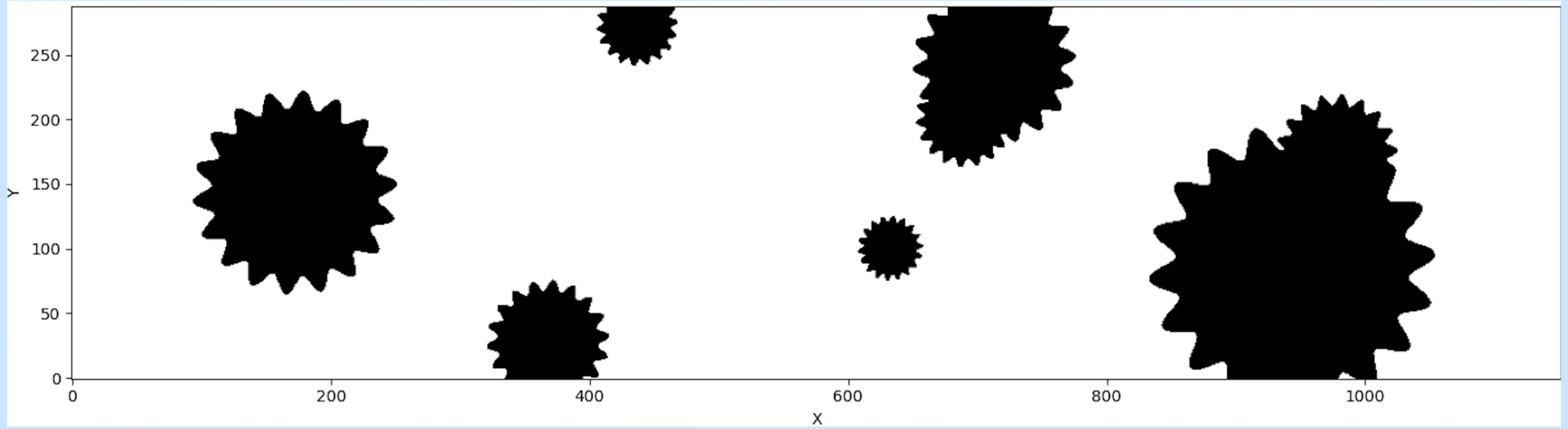

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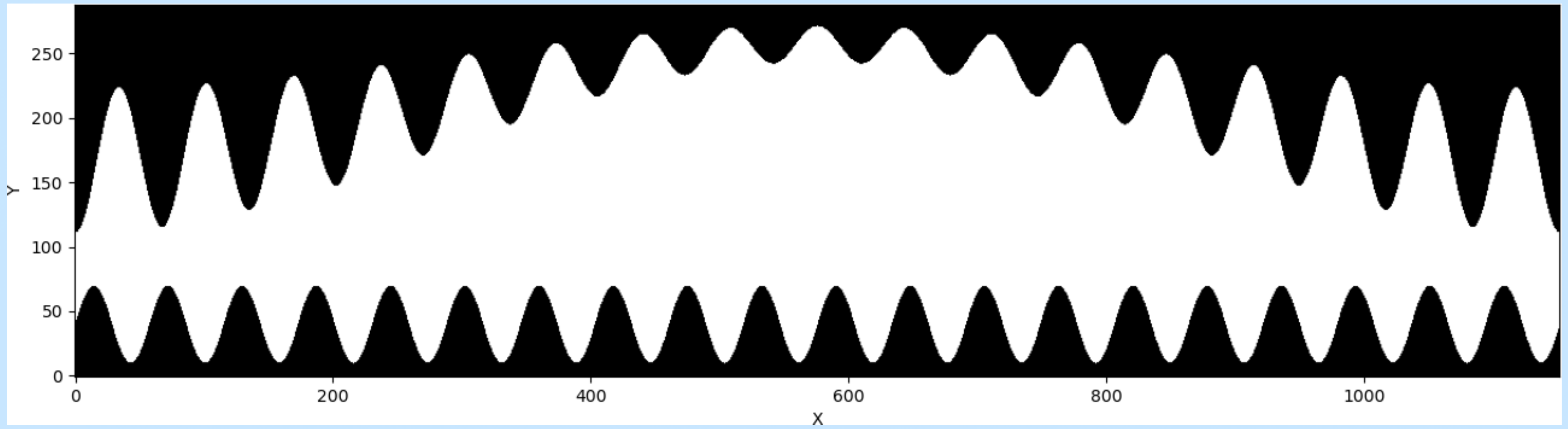

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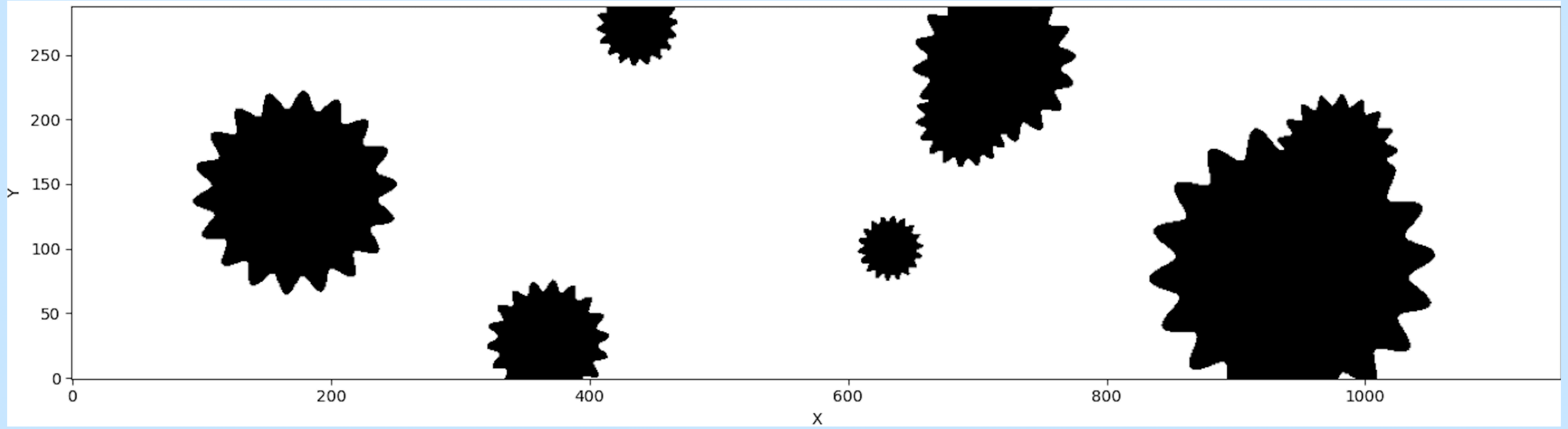

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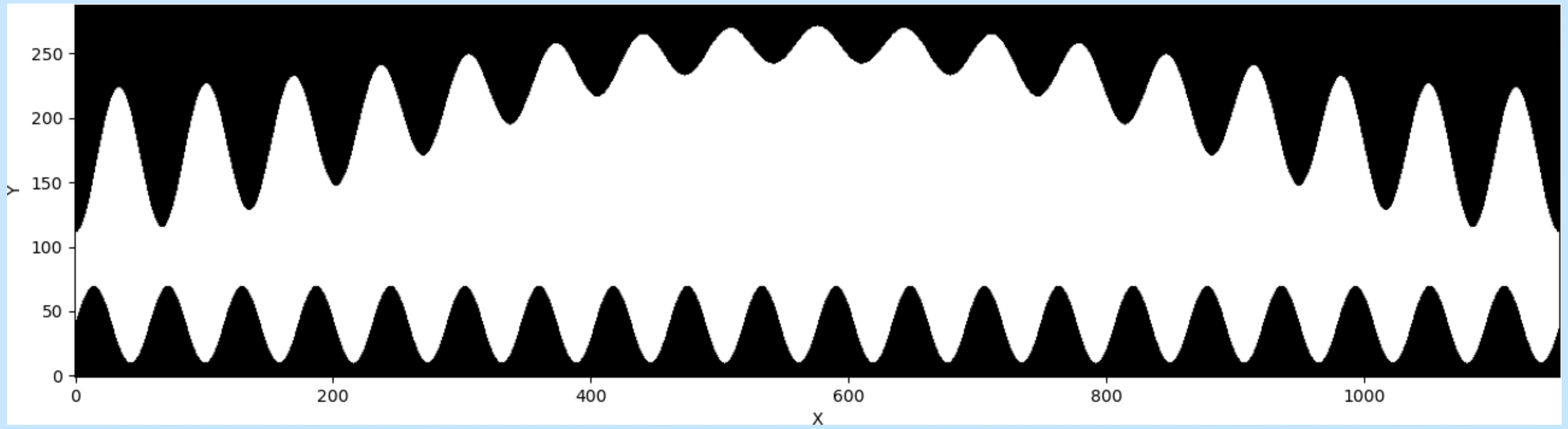

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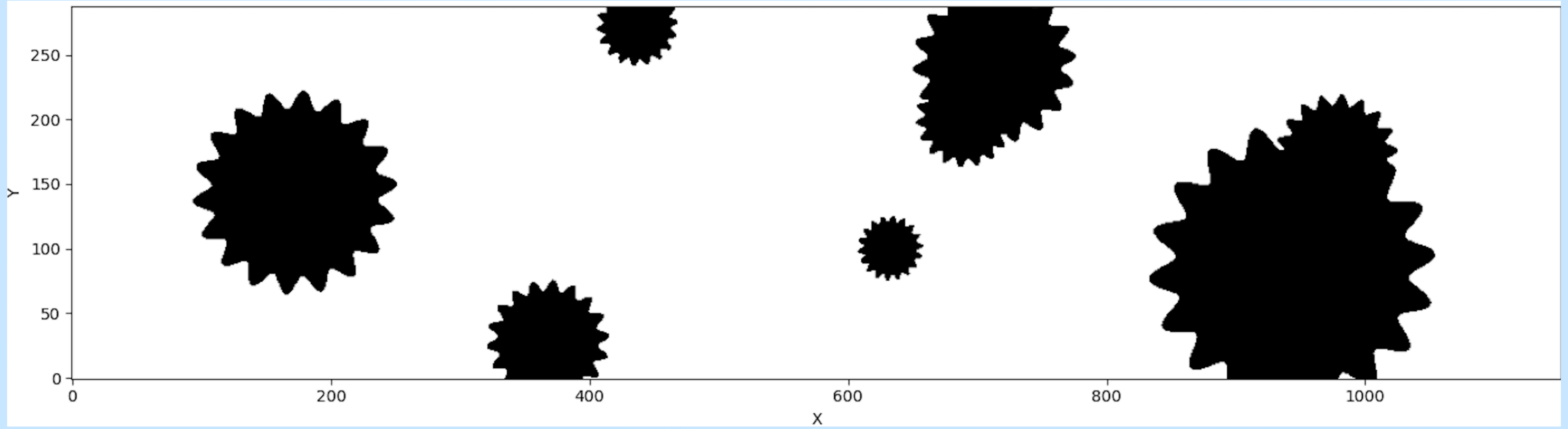

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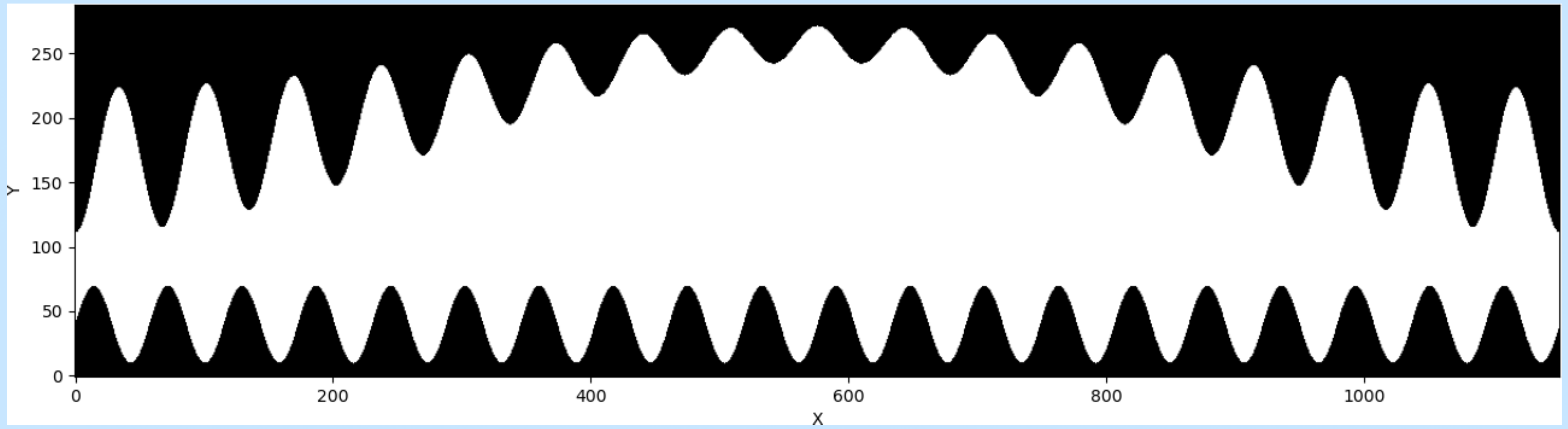

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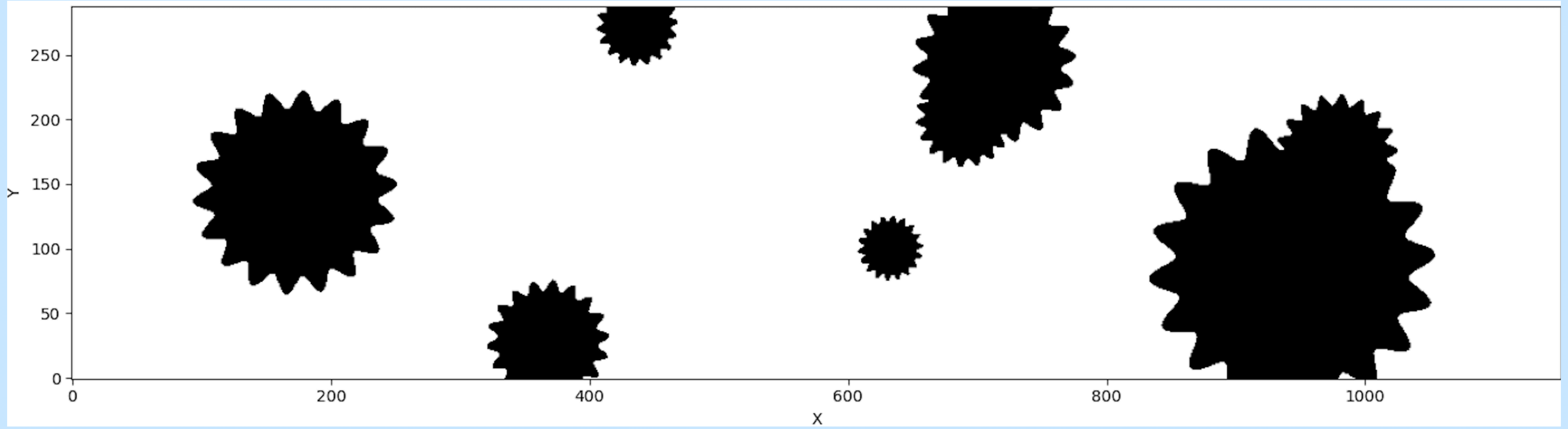

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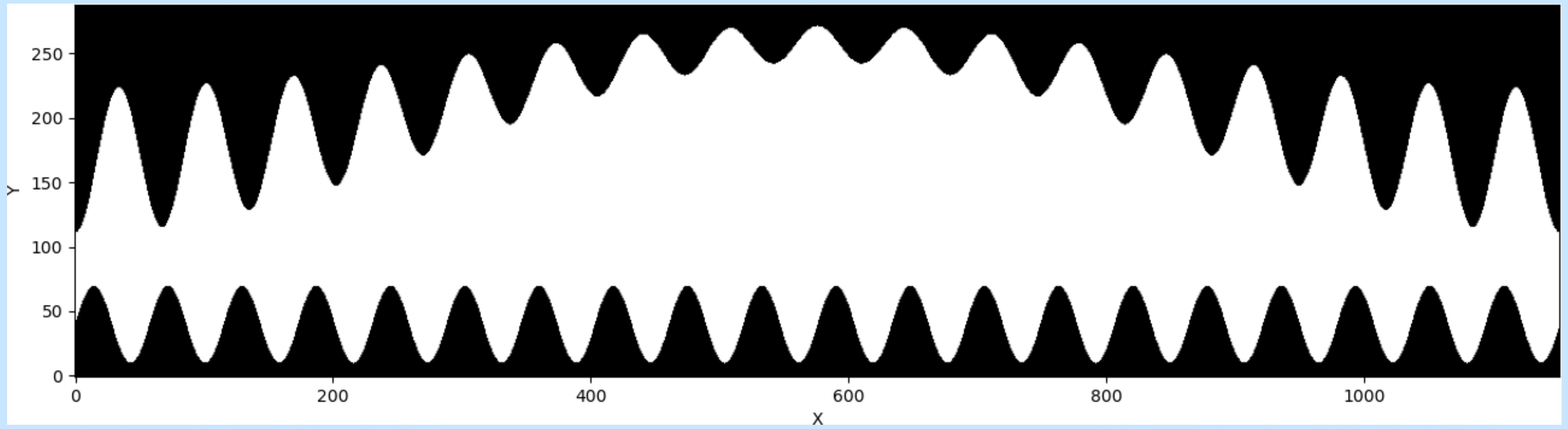
Topography test cases



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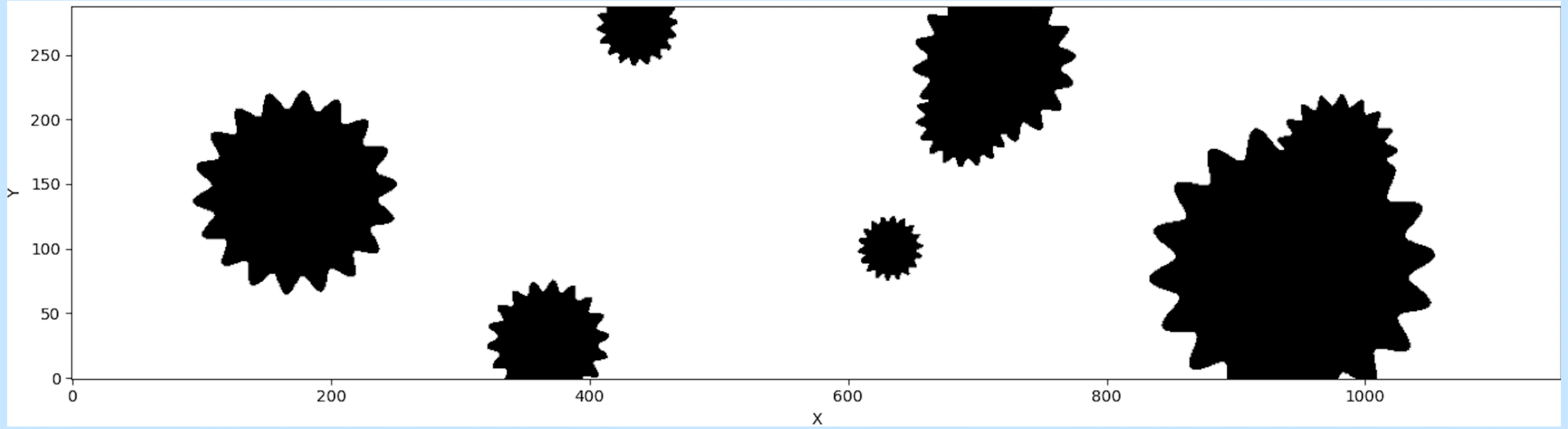

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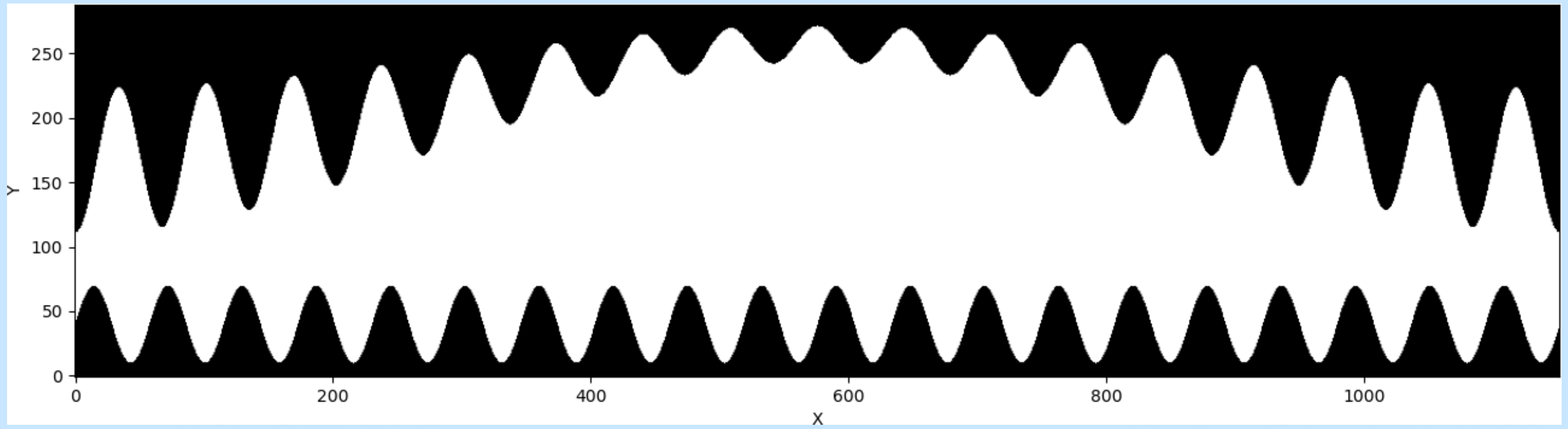
Diagnostics

Topography test cases



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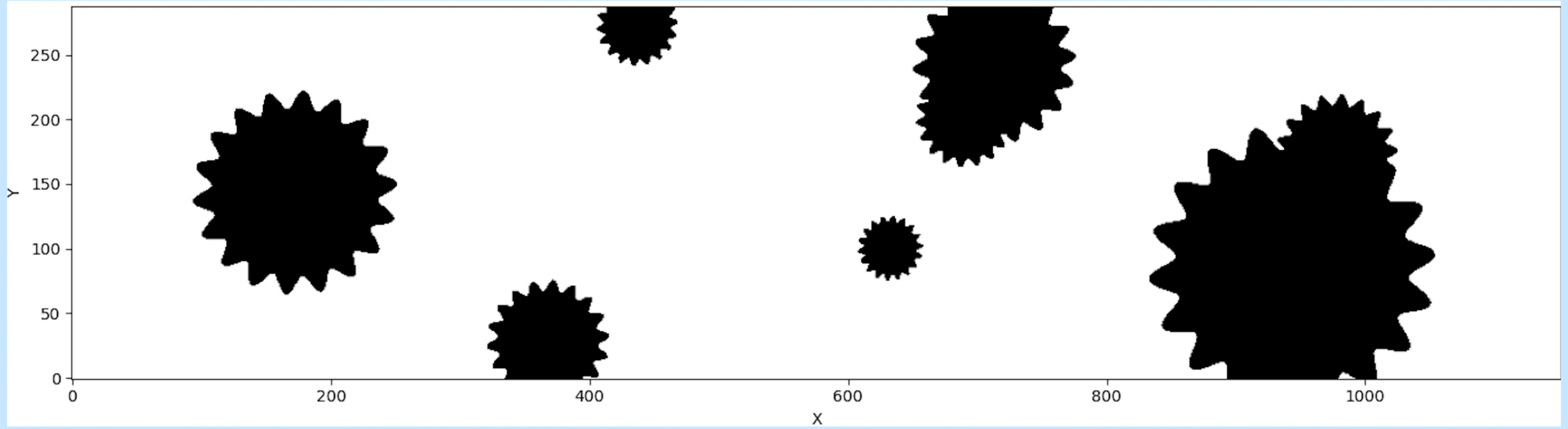

Islands (IS)

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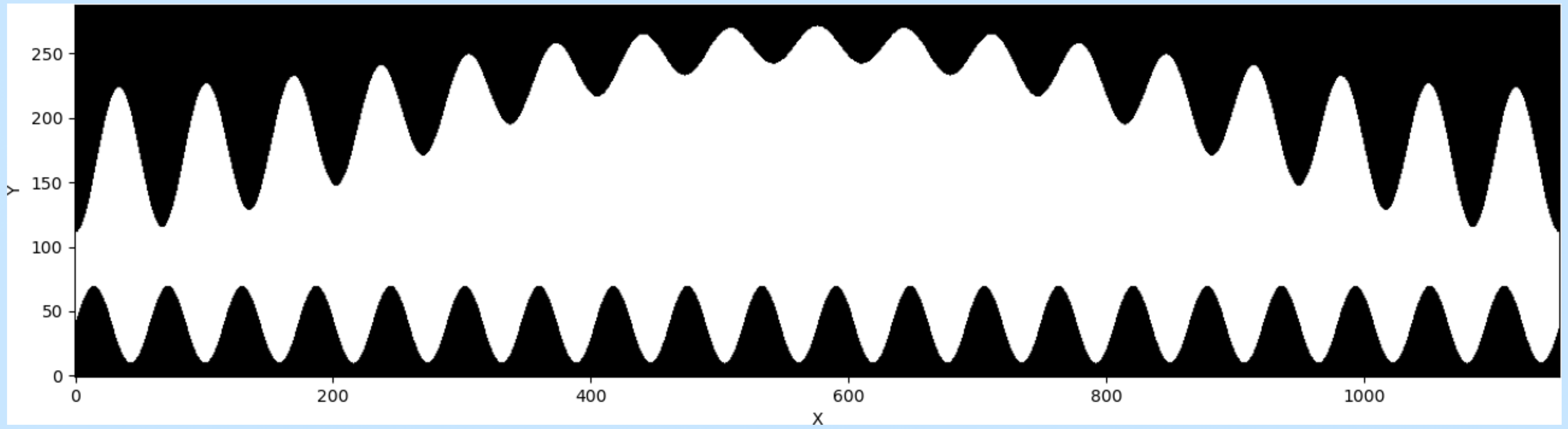
Diagnostics

- **Mean speed** of coarse simulations compared to mean speed of fine simulation averaged to coarse grid.

Topography test cases



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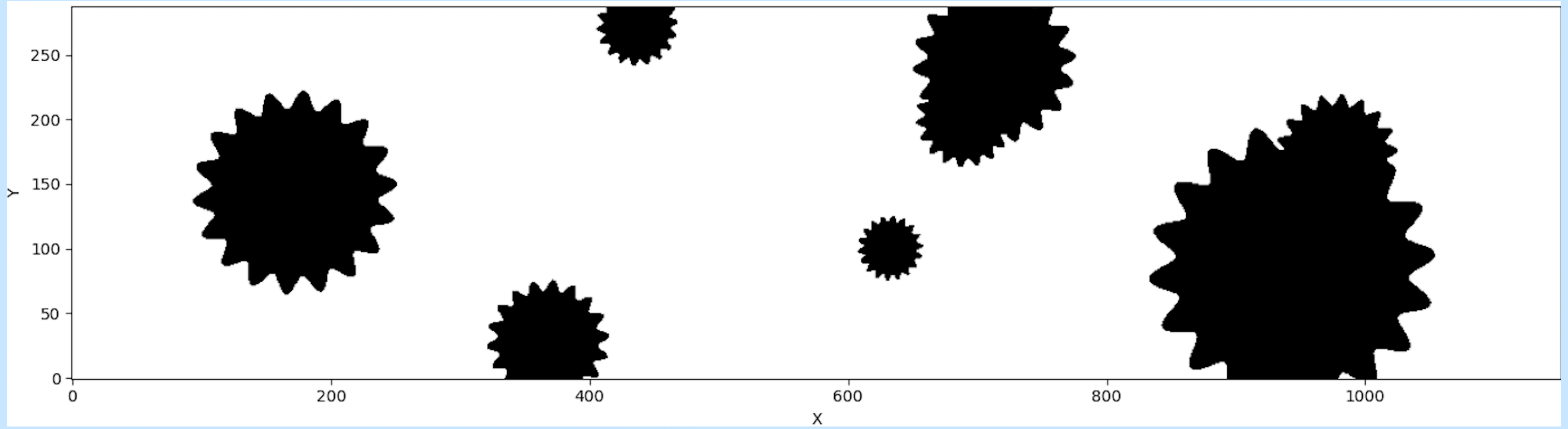

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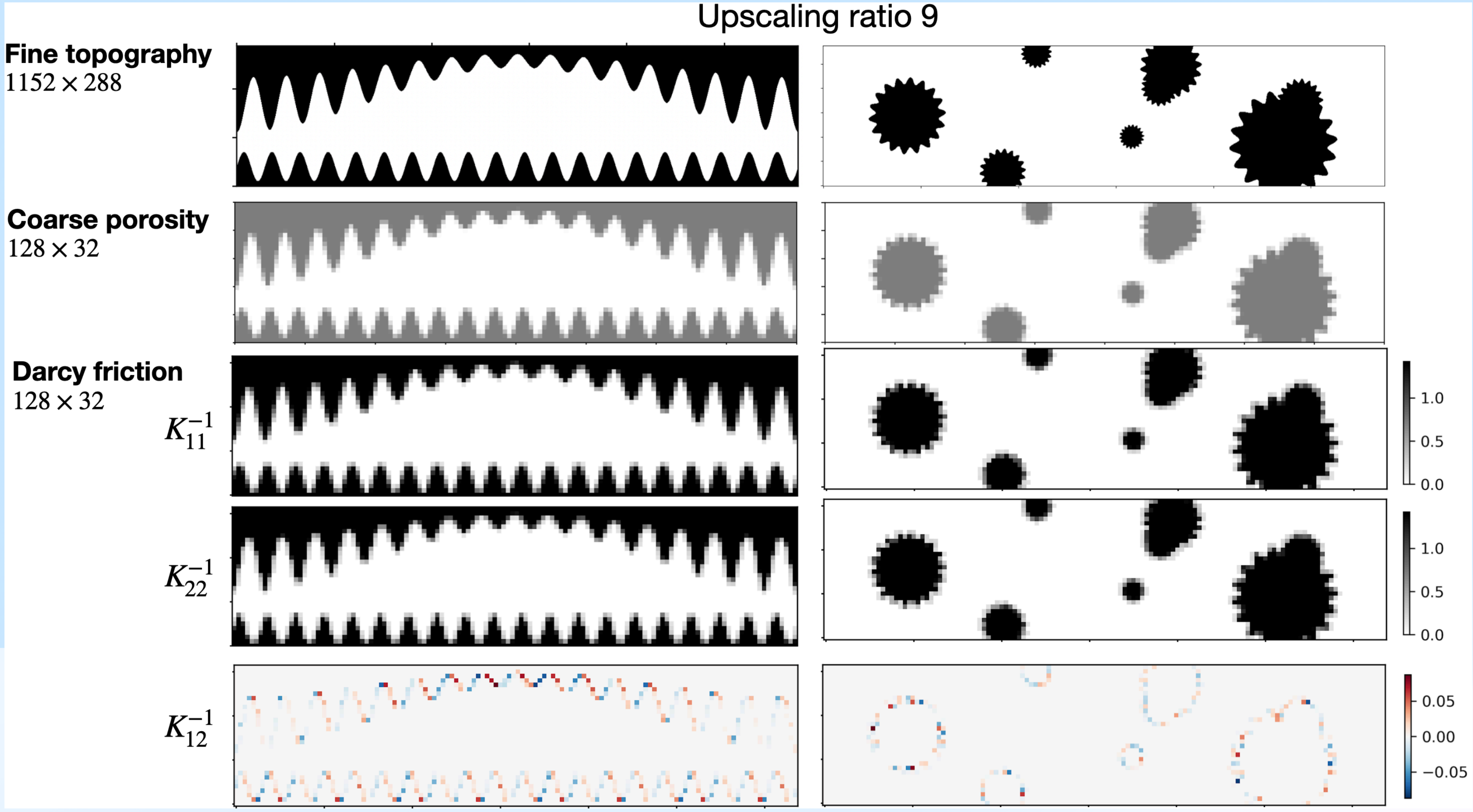
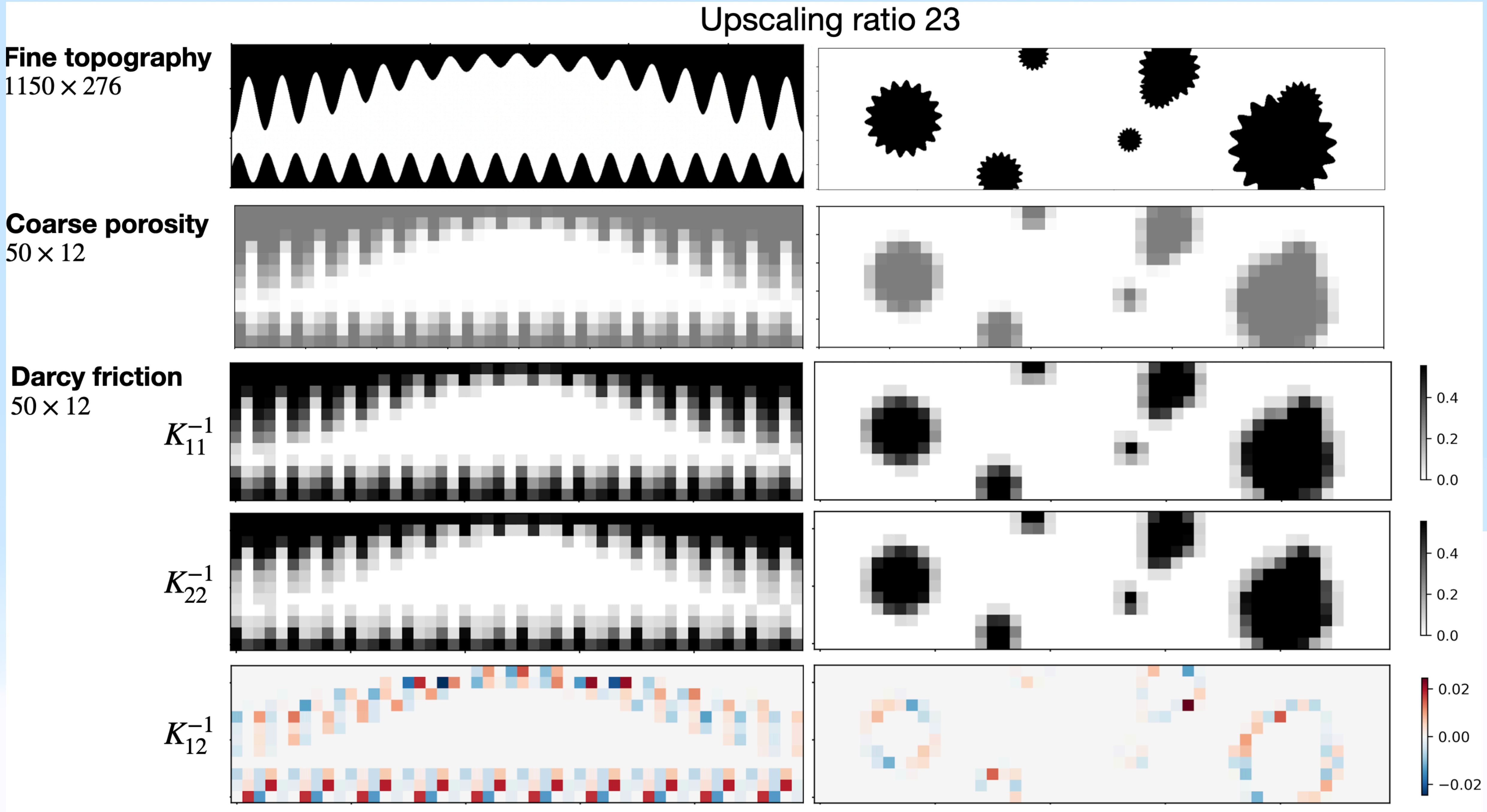
Diagnostics

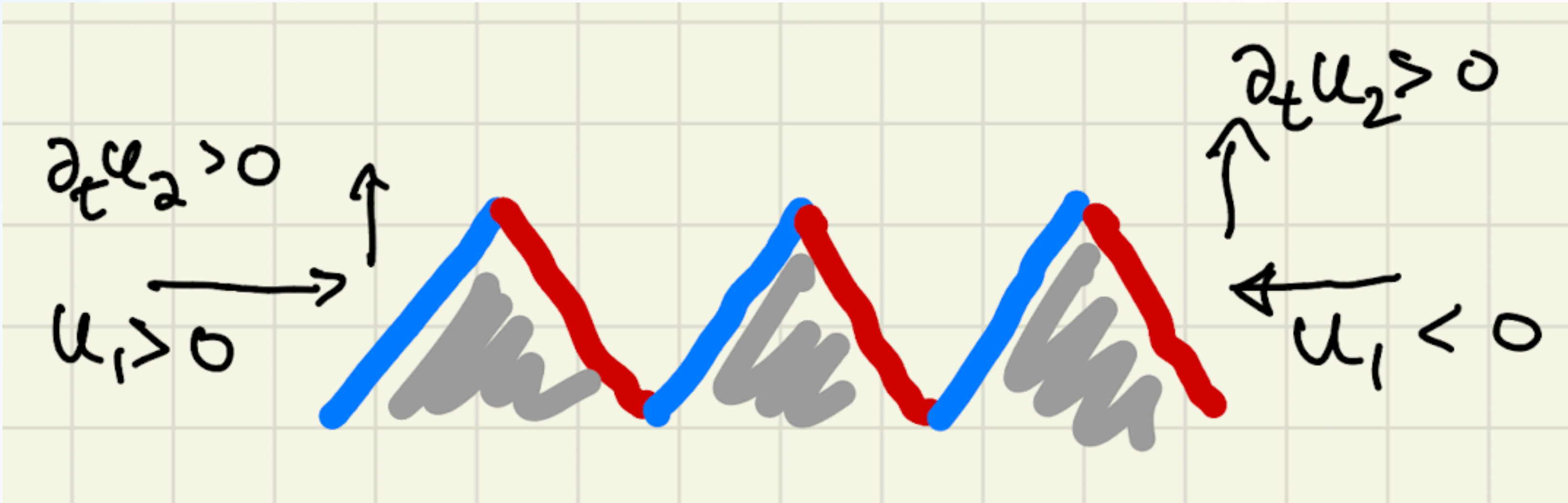
- **Mean speed** of coarse simulations compared to mean speed of fine simulation averaged to coarse grid.
- **Total dissipated power** *(TDP)* of coarse simulations compared to fine simulation averaged to coarse grid.

Results: upscaled topography



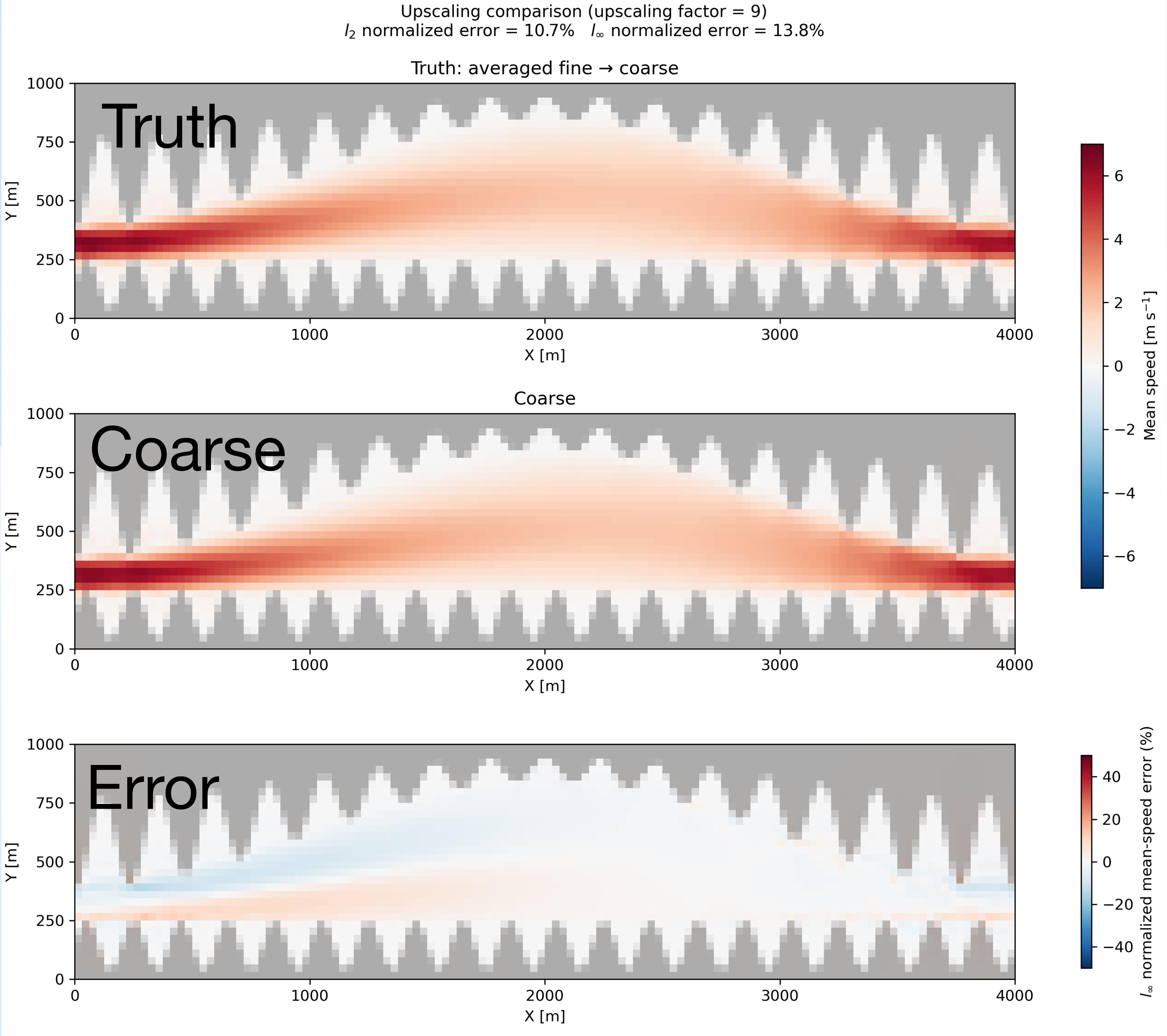


Off-diagonal Darcy friction matrix entries K_{12}^{-1} represent flow deflection by SGS topography:

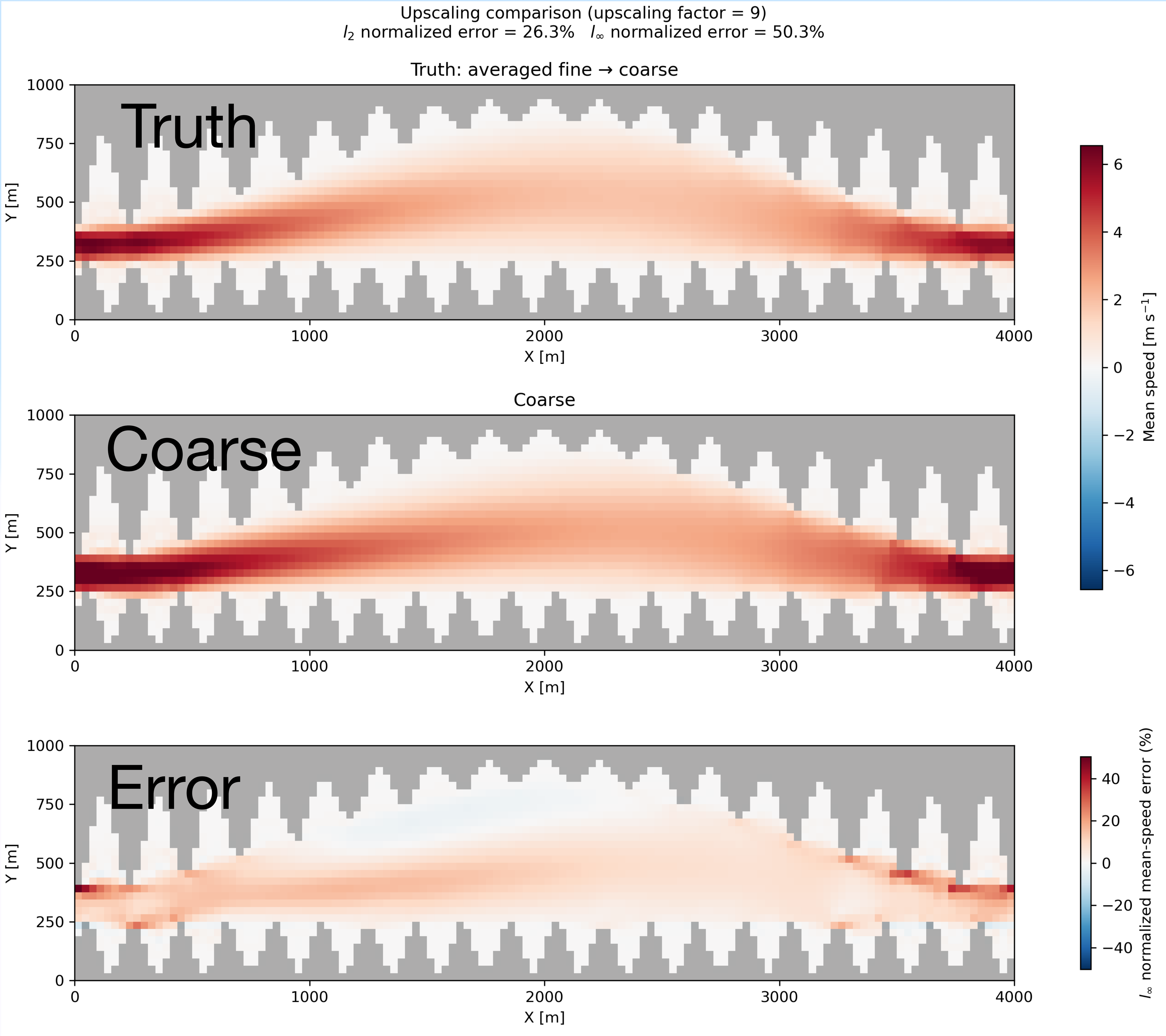


Upscale ratio 9: mean speed for $Re \approx 40$

Upscaled (UP)

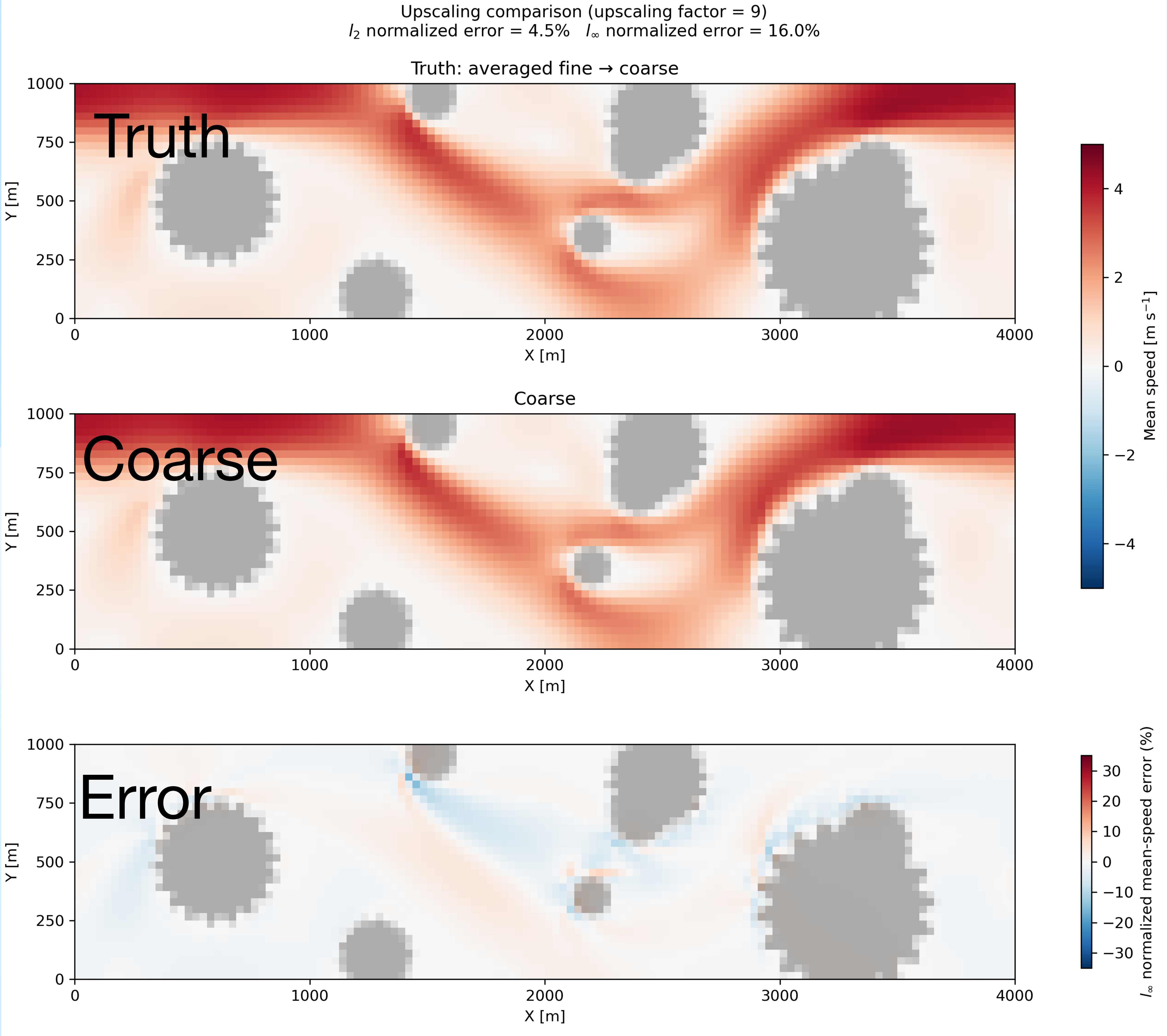


Averaged (AV)

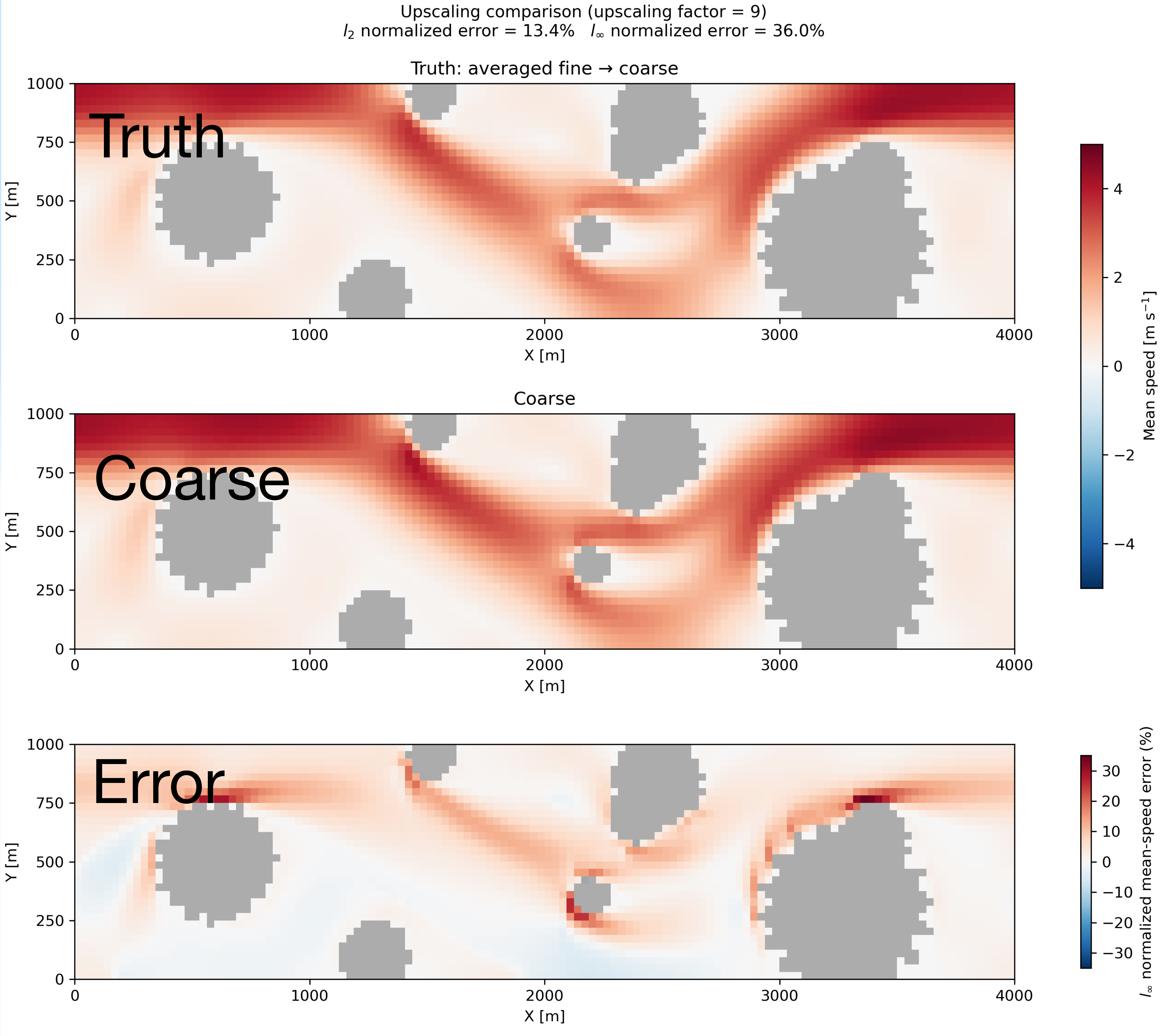


Upscale ratio 9: mean speed for $Re \approx 175$

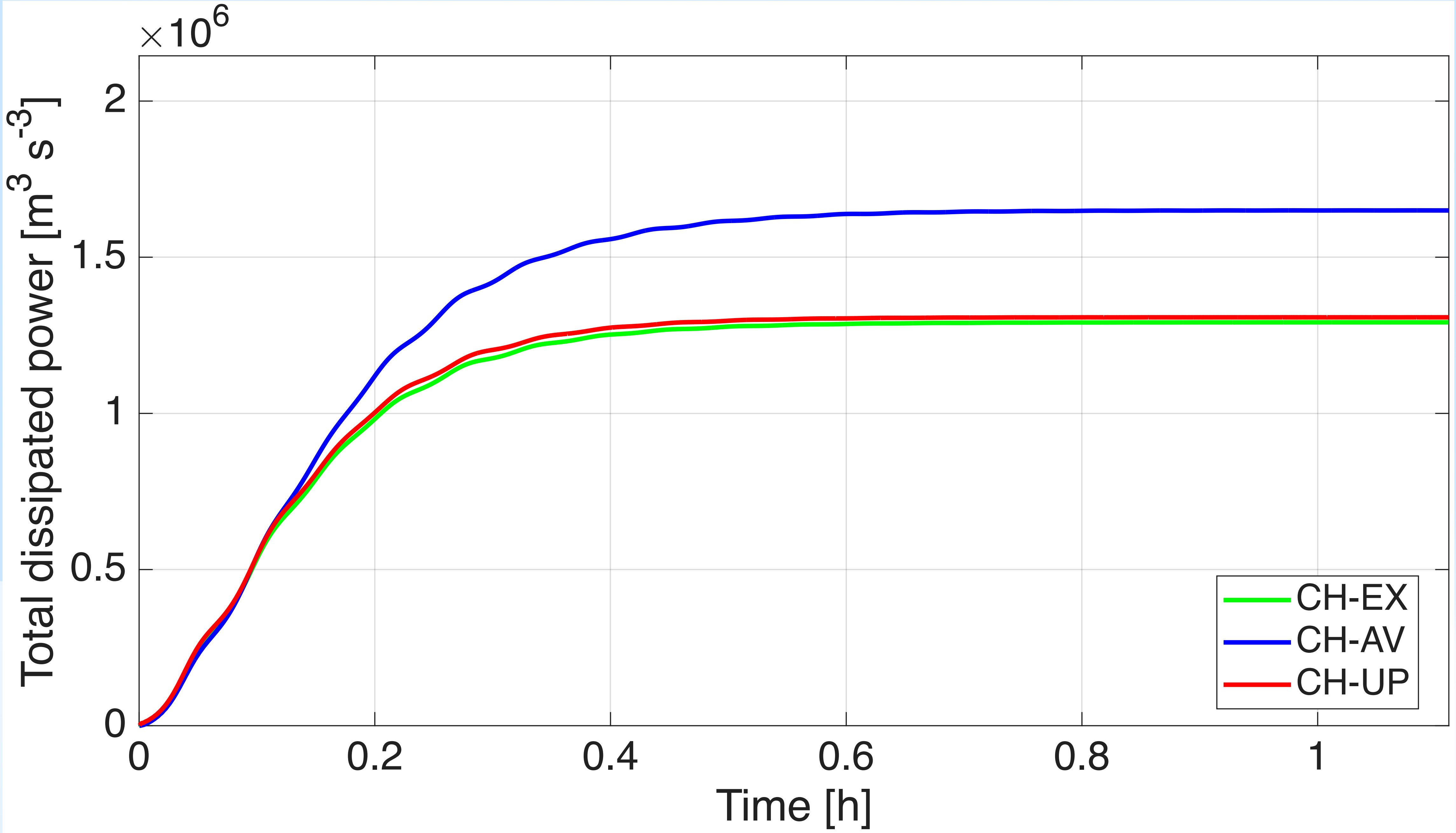
Upscaled (UP)



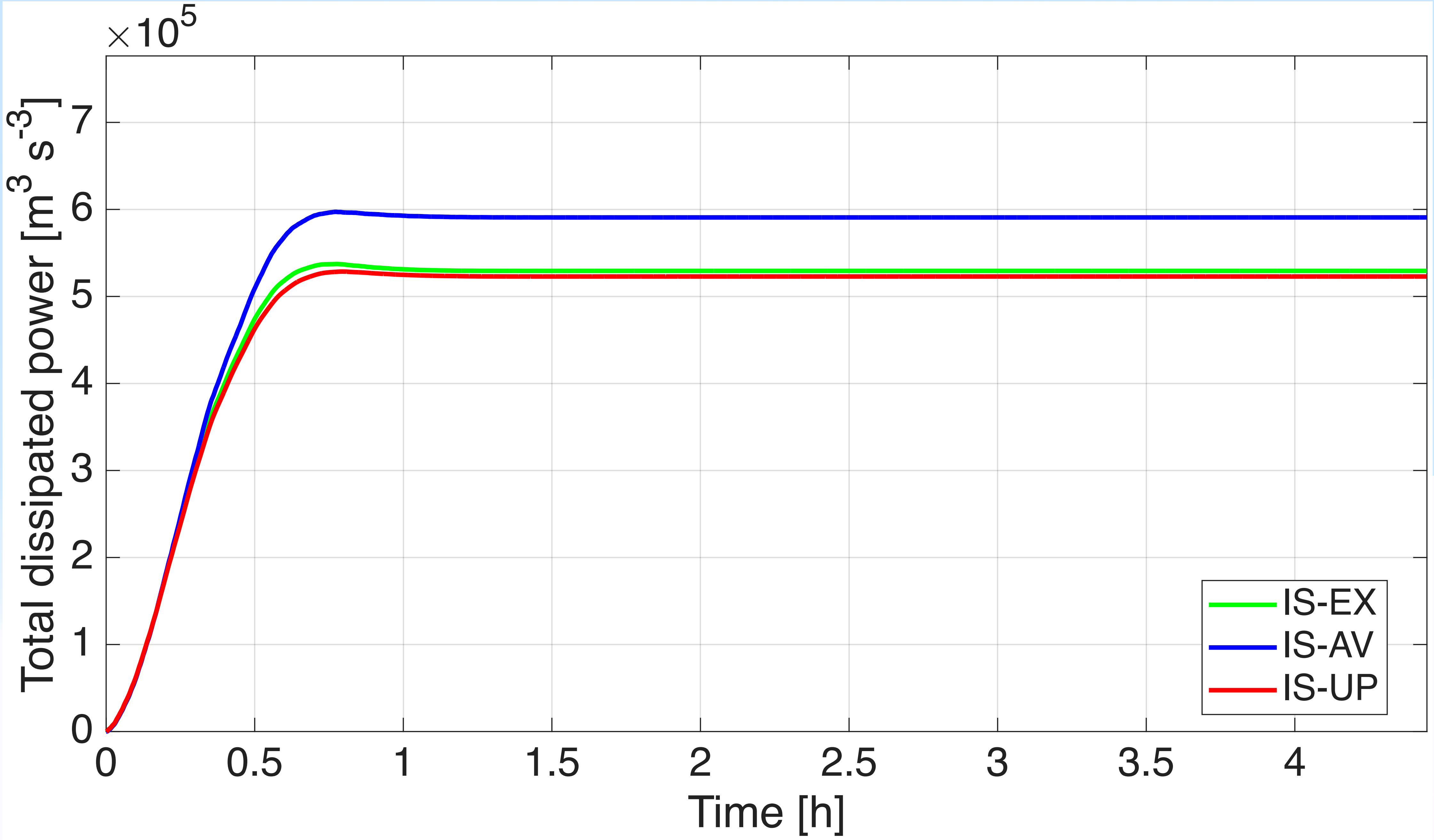
Averaged (AV)



Upscale ratio 9: total dissipated power



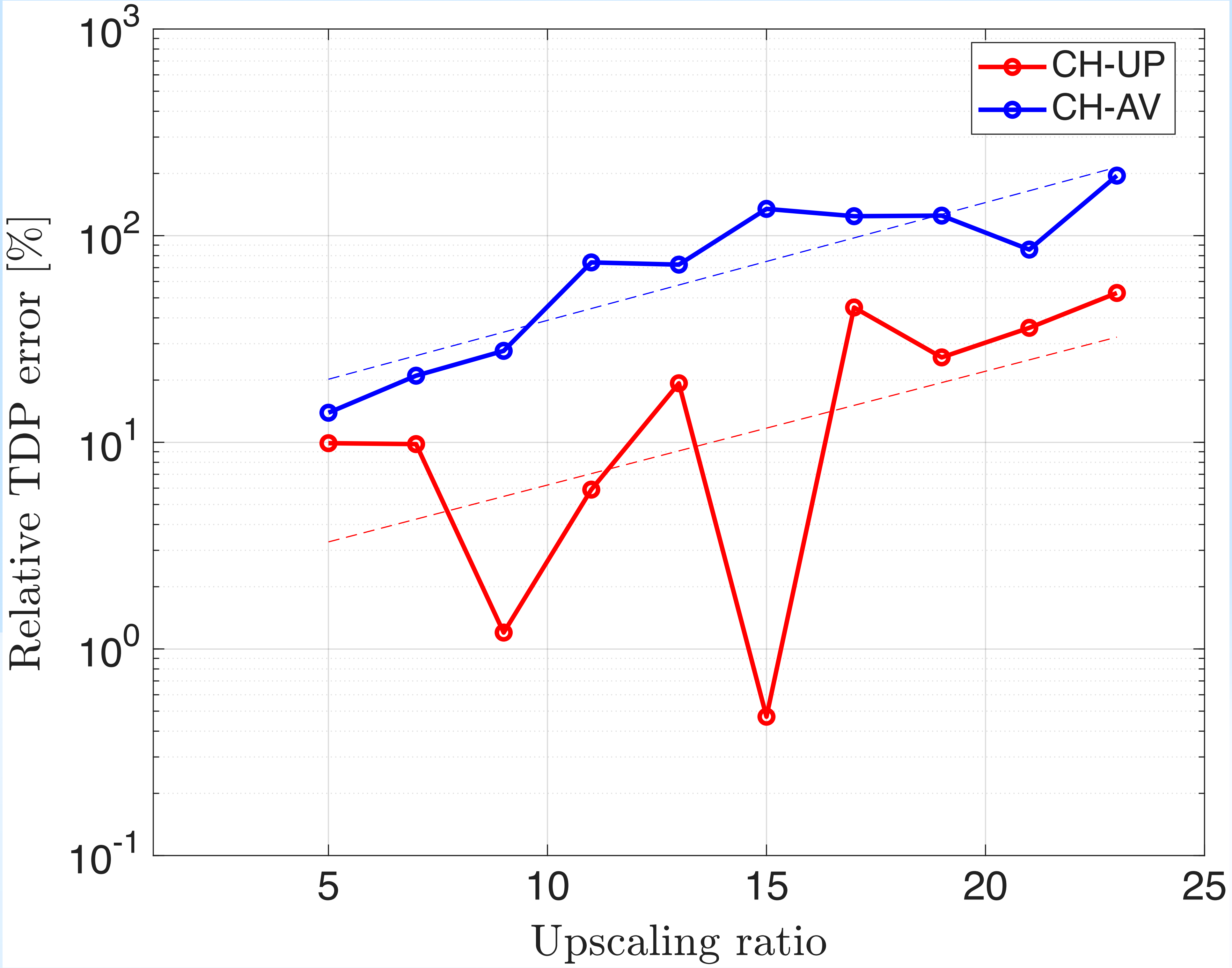
Channel configuration, $Re \approx 40$



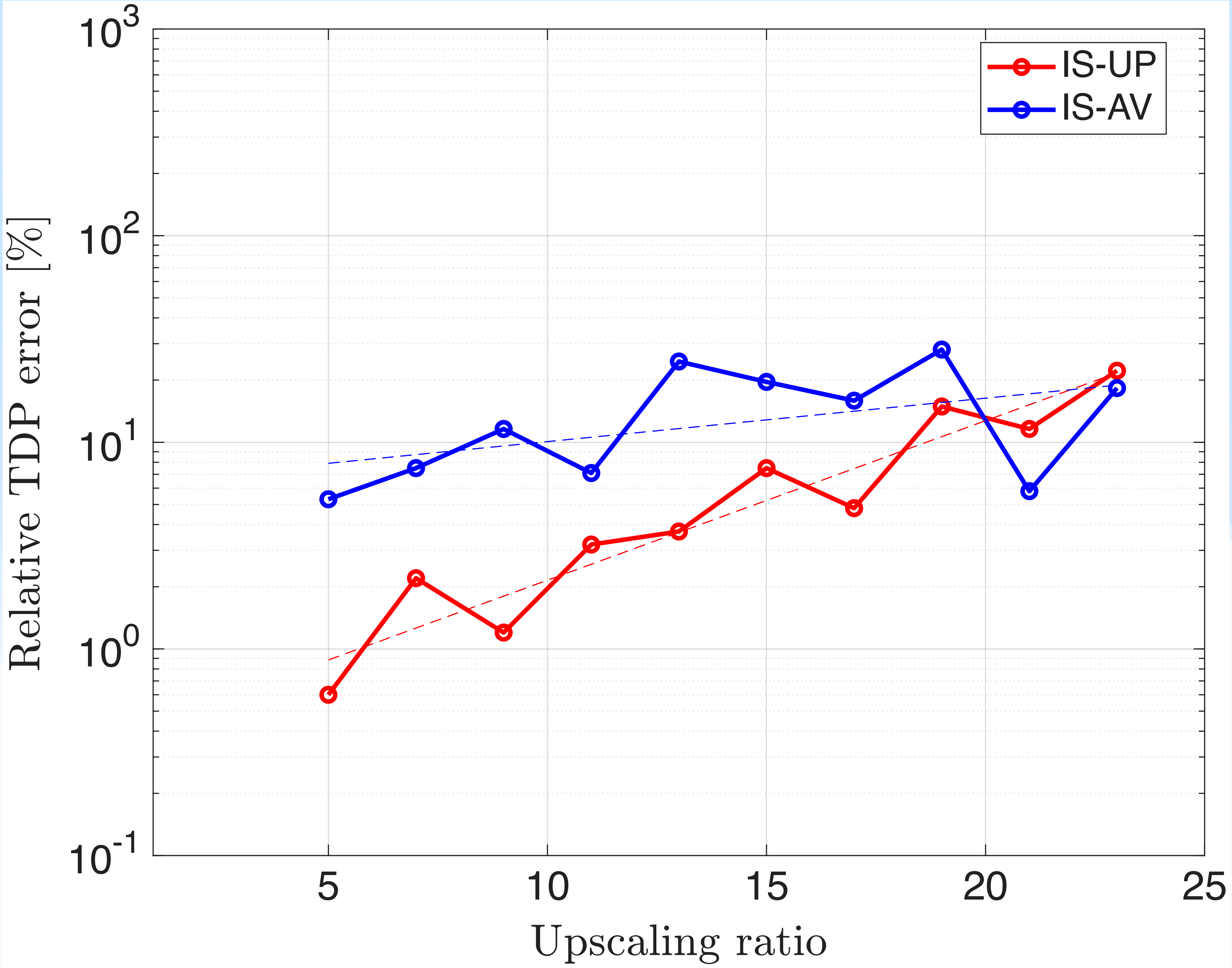
Islands configuration, $Re \approx 175$

EX = exact UP = periodic homogenization *(lower error)* AV = averaged

Scaling of TDP error with upscaling ratio



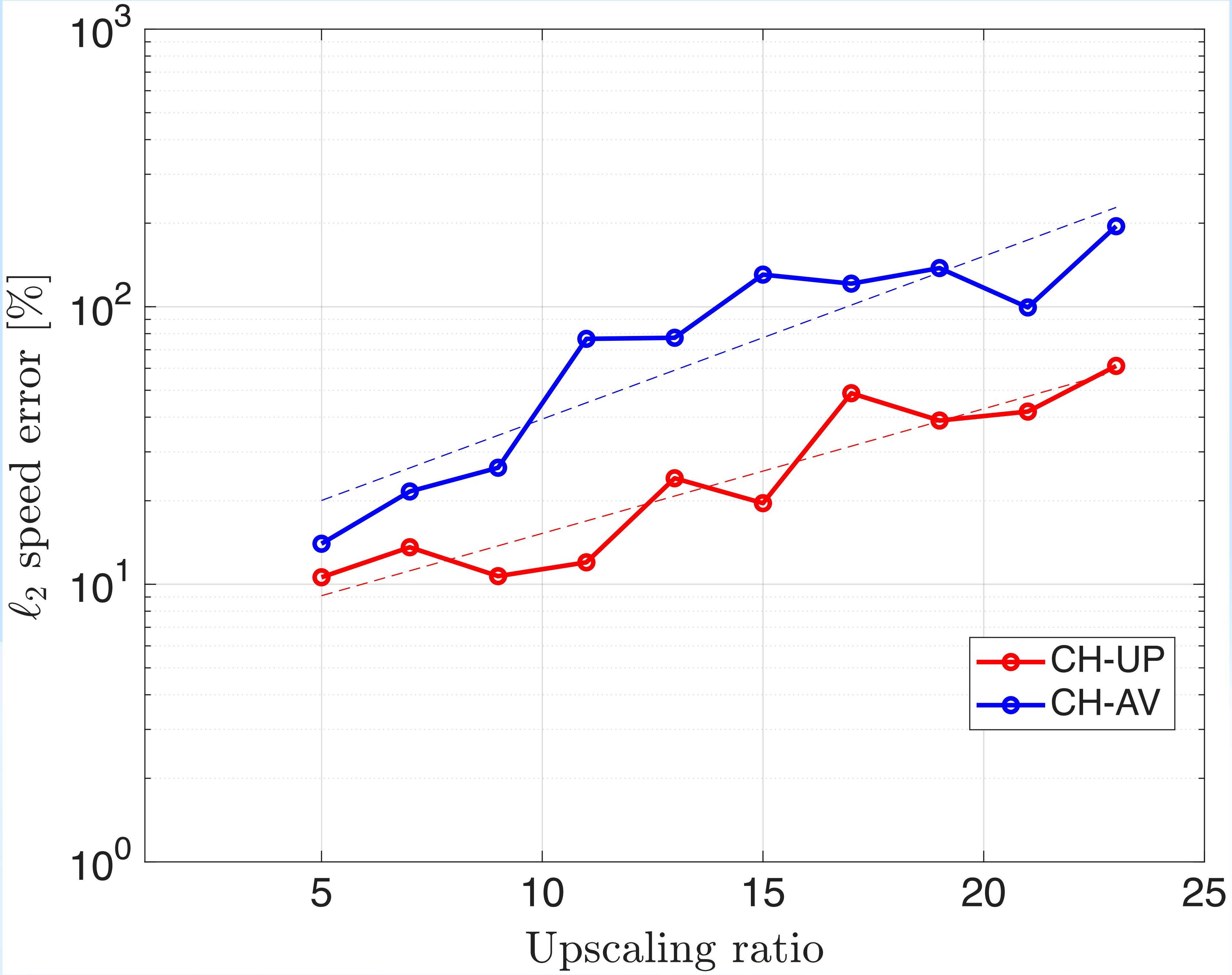
Channel configuration



Islands configuration

UP = periodic homogenization *(lower error)* AV = averaged *(higher error)*

Scaling of speed error with upscaling ratio



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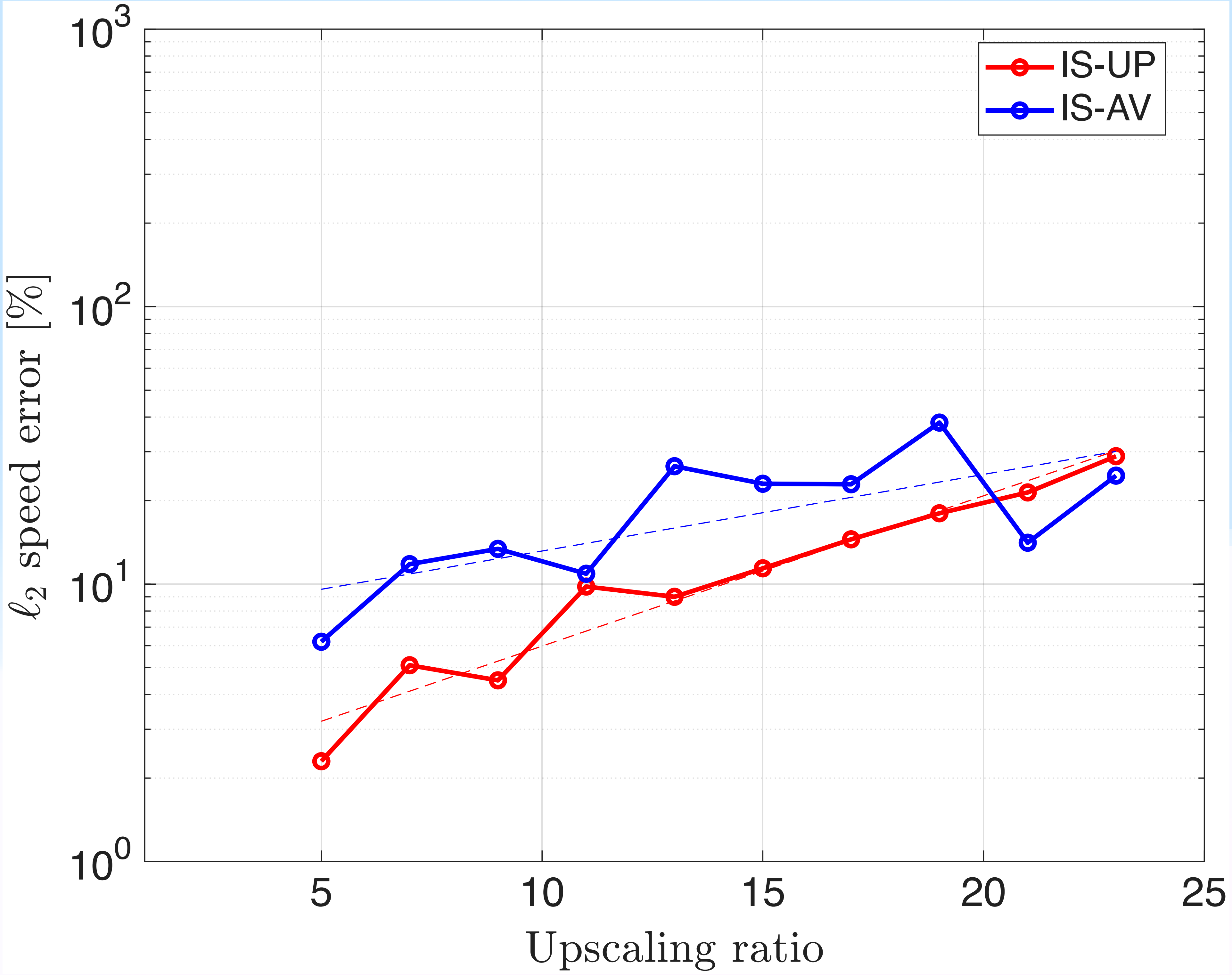

Islands configuration

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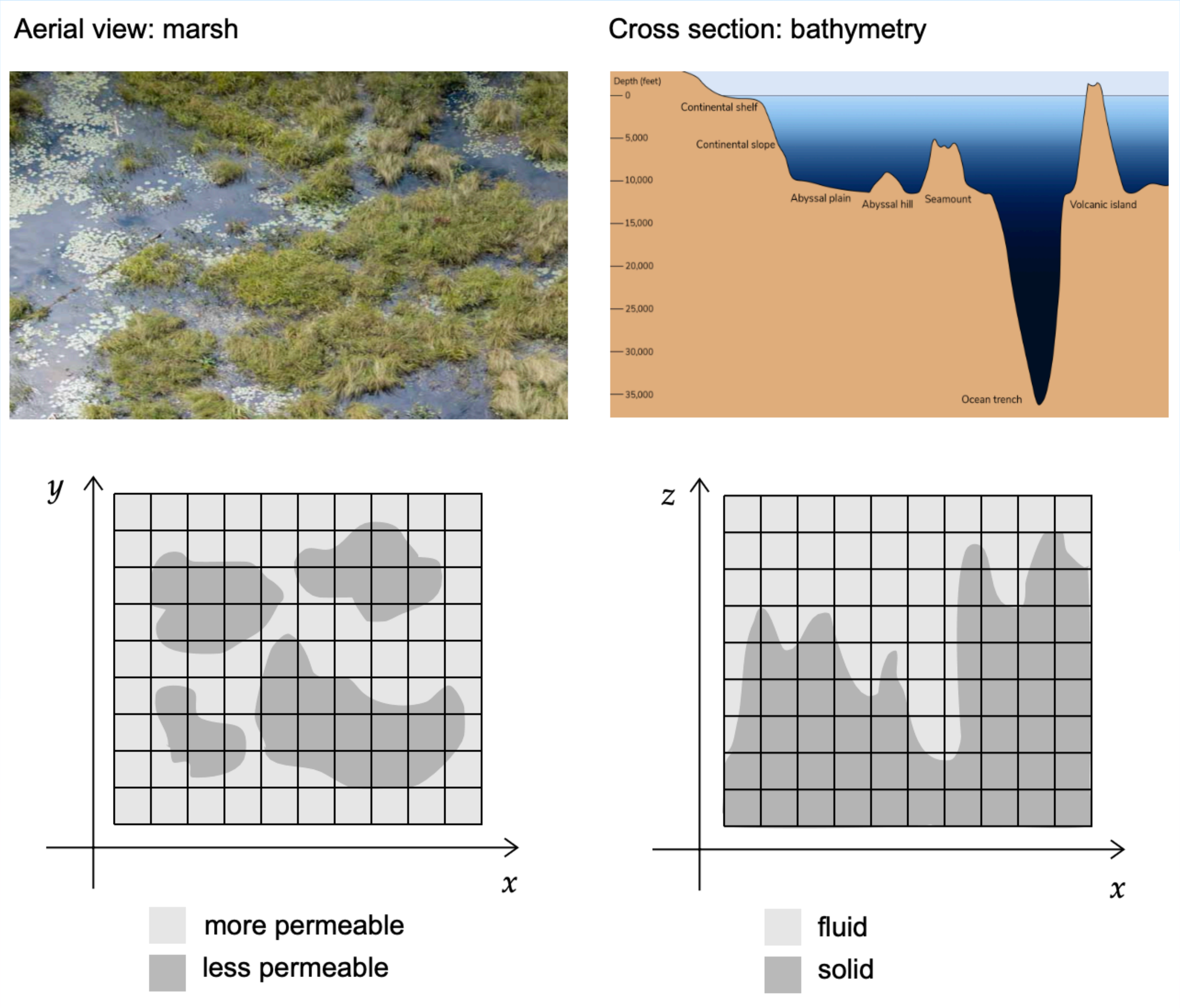
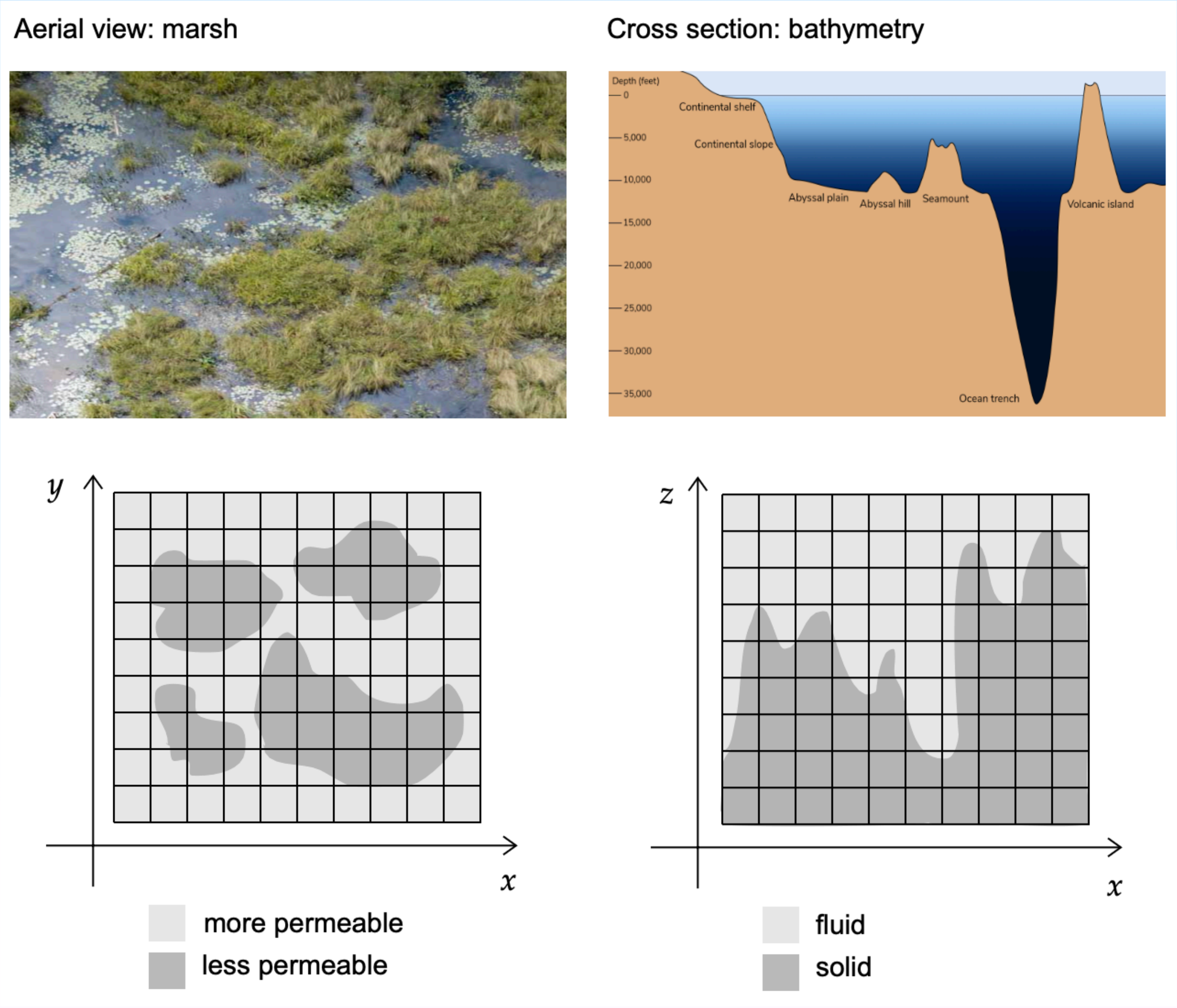
Summary and conclusions



Summary and conclusions

- New method for upscaling subgrid scale (SGS) topography based on porous medium flow model.
  Porous SWE upscaled topography.



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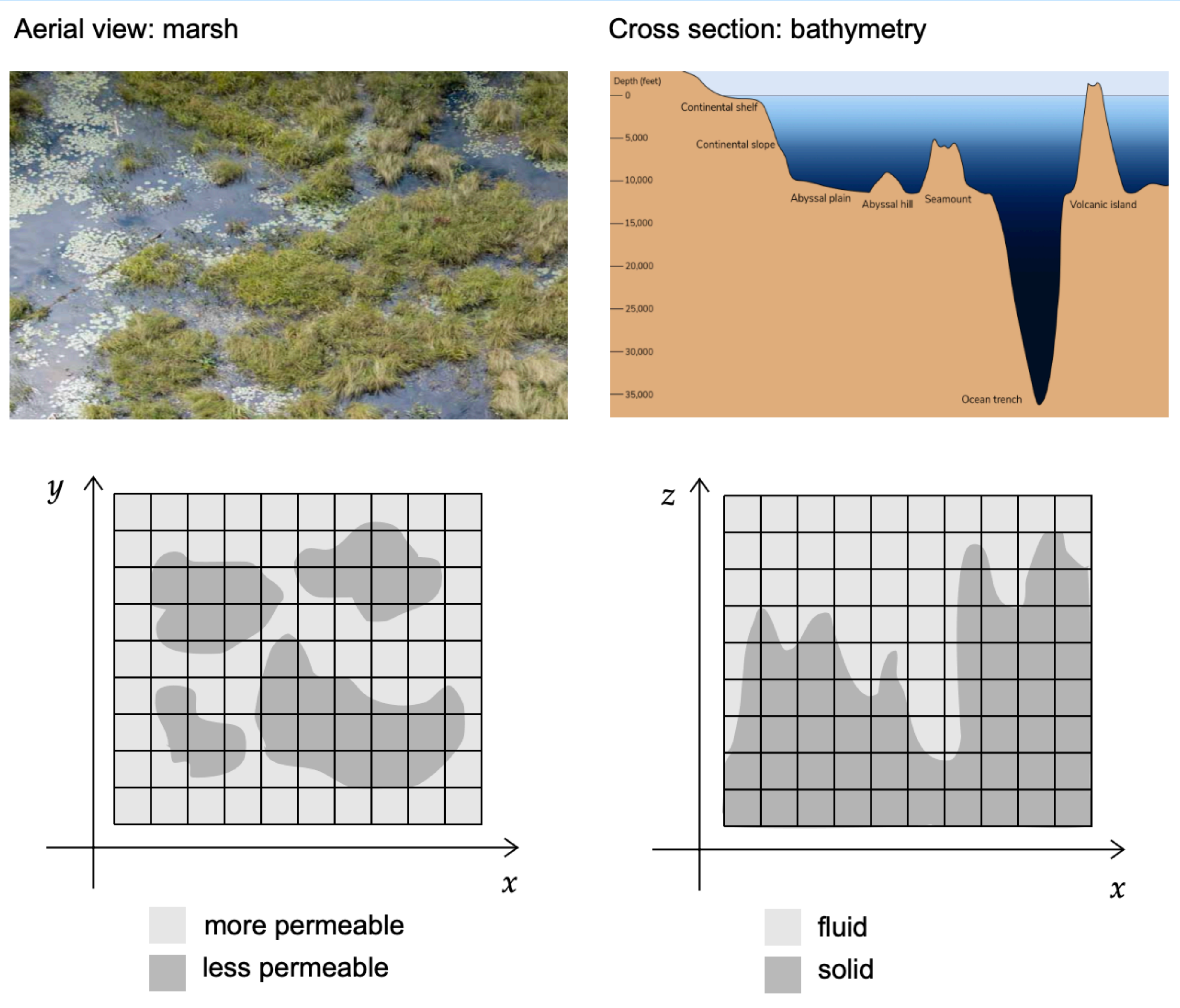

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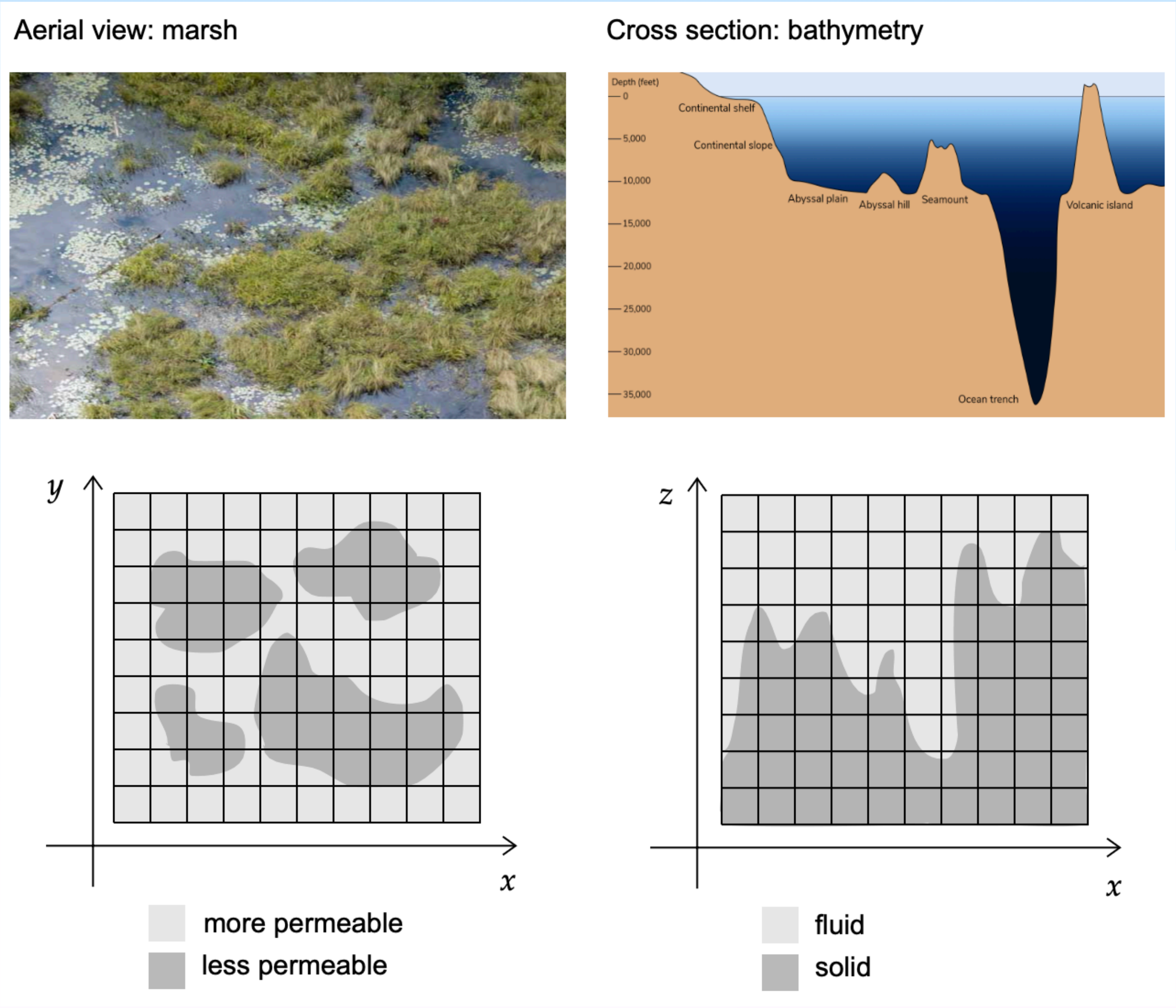

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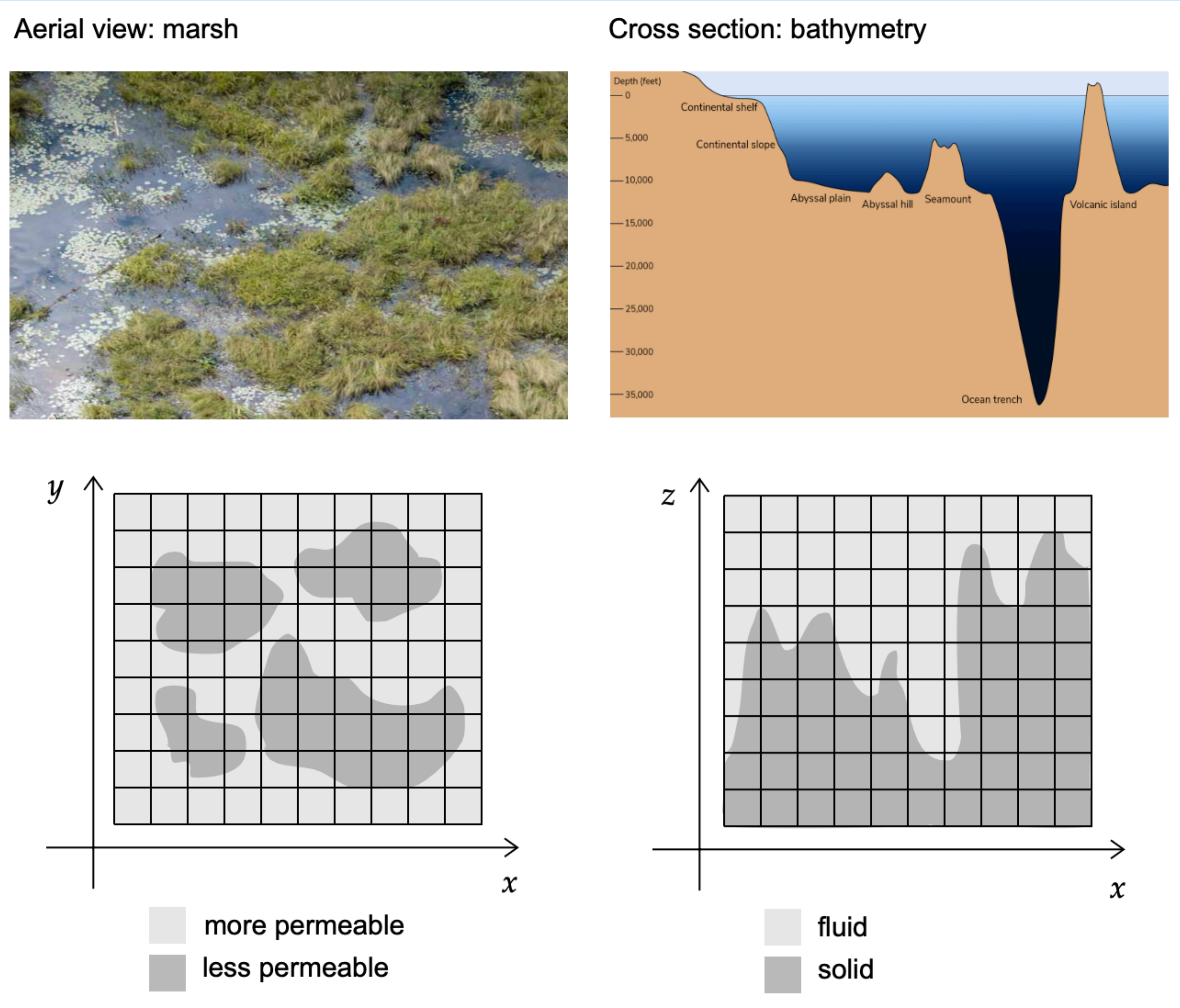
- Upscaled porosity is simply SGS porosity averaged over computational grid cells. *No adjustable parameters!*



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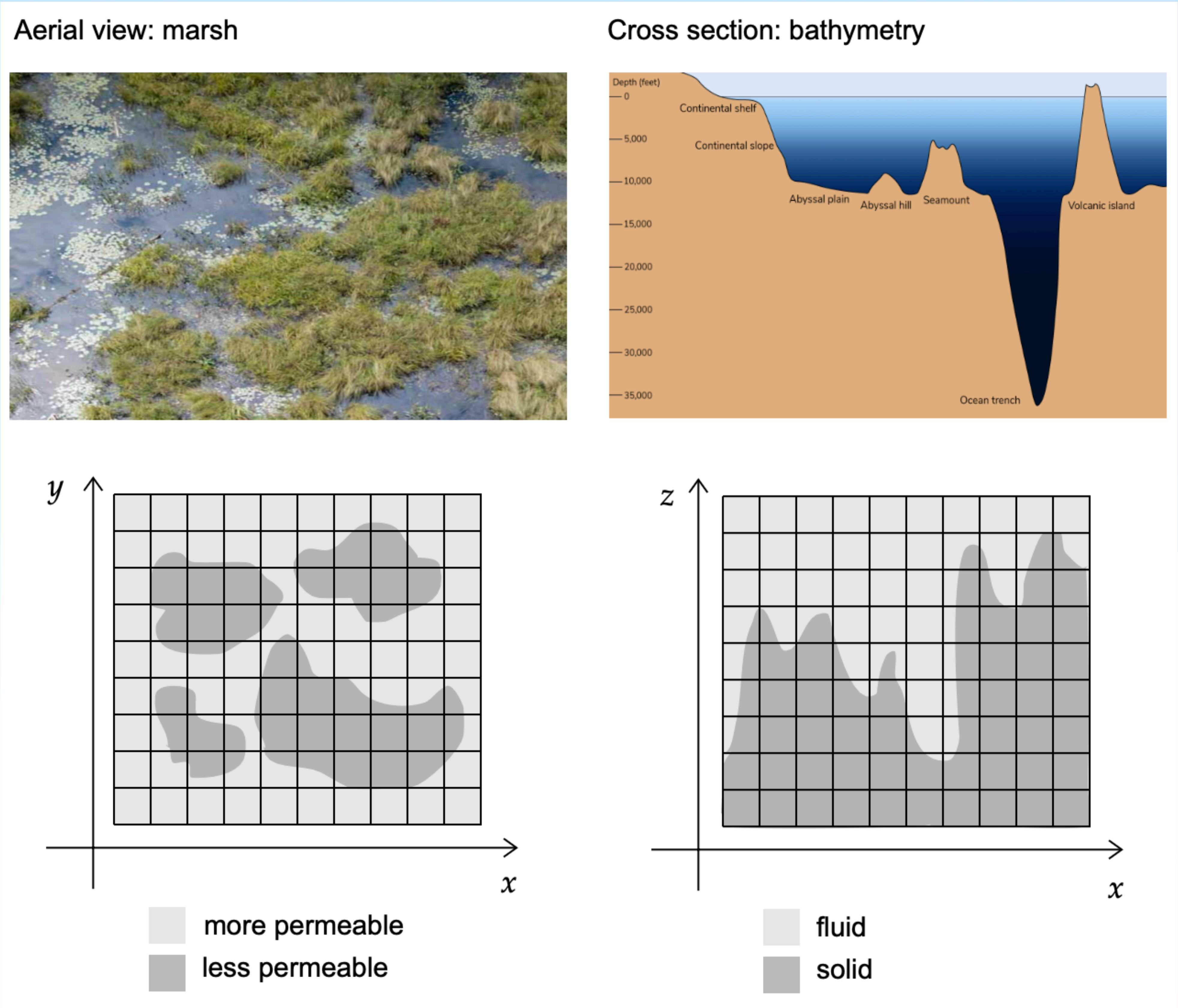

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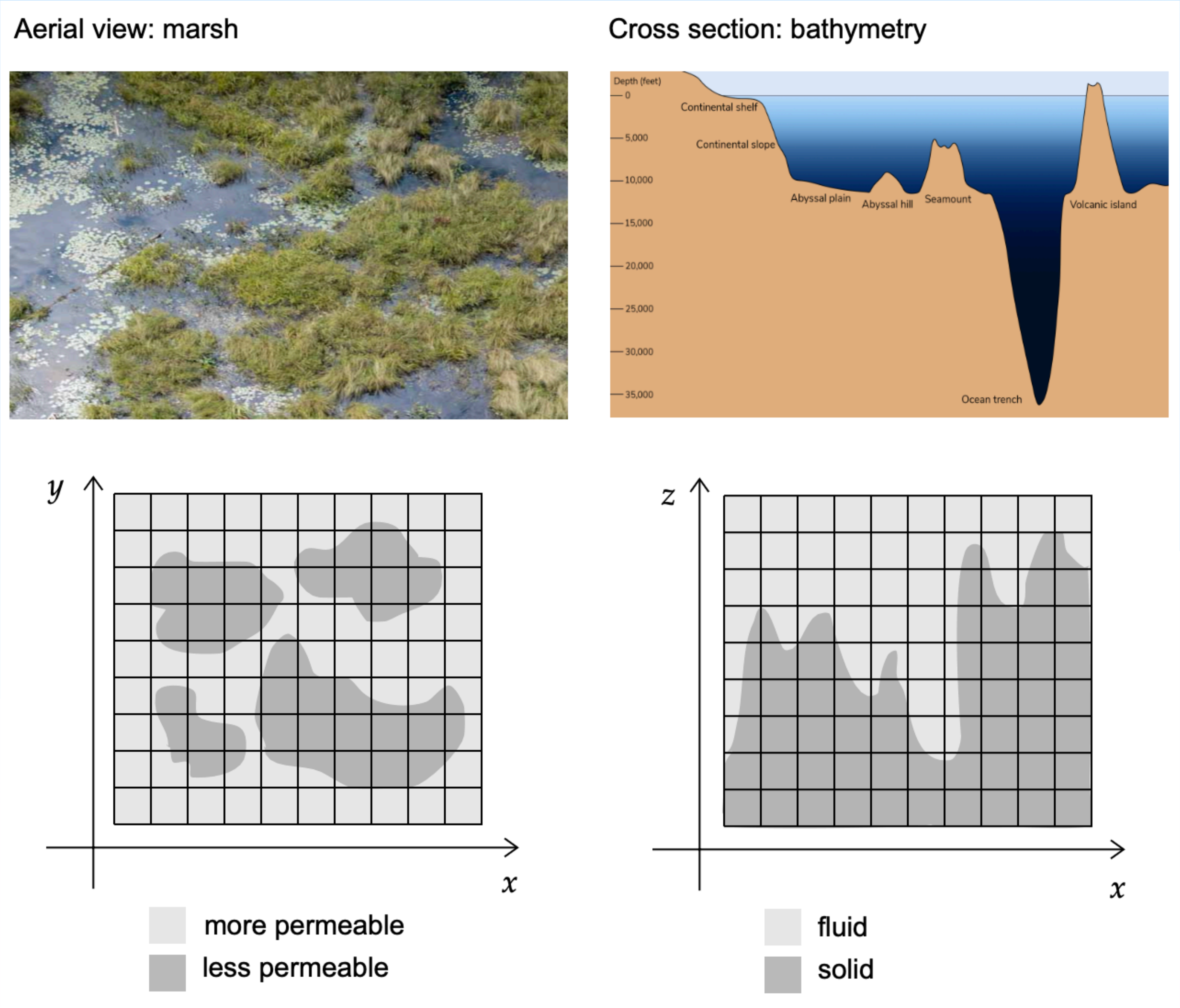

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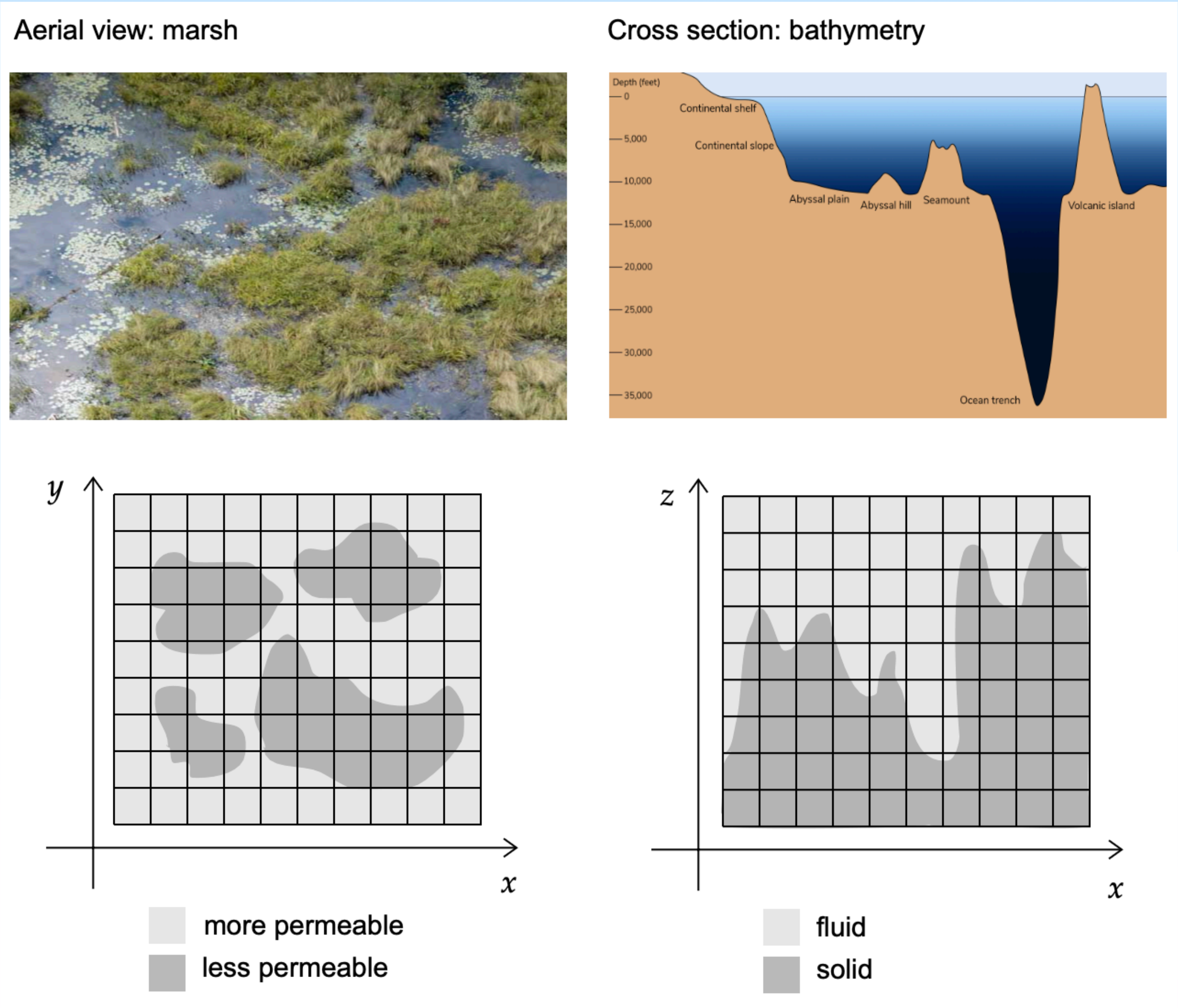

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