Thickness-weighted averaged eddy–mean decomposition of available potential energy

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Mesoscale parameterisations

Parameterisations represent "subgrid-scale" motion not resolved by the model.

Classic problem: mesoscale eddy parameterisations (Gent & McWilliams, 1990; Greatbatch & Lamb, 1990).

Several approaches to parametrising eddy flux diffusivity use *eddy energy* E_e (Eden & Greatbatch, 2008; Marshall, Maddison & Berloff, 2012), eg. in GEOMETRIC:

$$\langle b'\mathbf{v}'\rangle = -\kappa \nabla_\mathrm{h} \langle b \rangle , \qquad \text{with} \qquad \kappa = \alpha E_\mathrm{e} \frac{\sqrt{\mathrm{Ri}}}{f_0}.$$

Important to distinguish the energy conversions due to stirring (KE $\leftrightarrow$ APE) or mixing (APE $\rightarrow$ BPE) represented by mesoscale parameterisations.

APE is defined to be positive, whereas total energy is defined up to an arbitrary constant – important for constraining the possible values of κ?

Two ways to average

Large-scale represented by mean $\overline{(\cdot)}$, or thickness-weighted average (TWA) $\widehat{(\cdot)} = \overline{h(\cdot)}/\overline{h}$.

1. Transport by mean velocity $\overline{\mathbf{u}} = \overline{\mathbf{v}} + \overline{w}\mathbf{k}$ plus parameterised "bolus" velocity $\mathbf{u}_*$:

$$\frac{\partial \overline{C}}{\partial t} + (\overline{\mathbf{u}} + \mathbf{u}_*) \cdot \nabla \overline{C} = \nabla \cdot \mathbf{F}_{\text{diff}}$$

$$\frac{\partial \overline{\mathbf{v}}}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{v}} + \nabla p = b\mathbf{k} + \nabla \cdot \mathsf{R}$$

2. Transport by thickness-weighted averaged (TWA) velocity $\mathbf{u}^\sharp = \widehat{\mathbf{v}} + w^\sharp \mathbf{k}$:

$$\frac{\partial \widehat{C}}{\partial t} + \mathbf{u}^\sharp \cdot \nabla \widehat{C} = \nabla \cdot \mathbf{F}_{\text{diff}}$$

$$\frac{\partial \widehat{\mathbf{v}}}{\partial t} + \mathbf{u}^\sharp \cdot \nabla \widehat{\mathbf{v}} + \nabla p = b\mathbf{k} + \nabla \cdot \mathsf{R} - \nabla \cdot \mathsf{E}$$

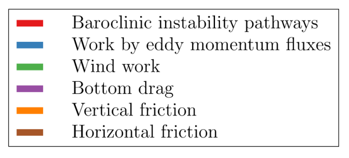
Parameterisation either in bolus velocity $\mathbf{u}_*$ or Eliassen–Palm flux $\nabla \cdot \mathsf{E}$.

Energy conversions in TWA vs. non-TWA

The two parameterisations change the resolved energy pathways:

- GM: MPE $\rightarrow$ EKE
- GL: MKE $\rightarrow$ EKE

Loose *et al.* (2022, 2023) show that the most “natural” choice for isopycnal models is a thickness-weighted energy budget with GL parameterisation.



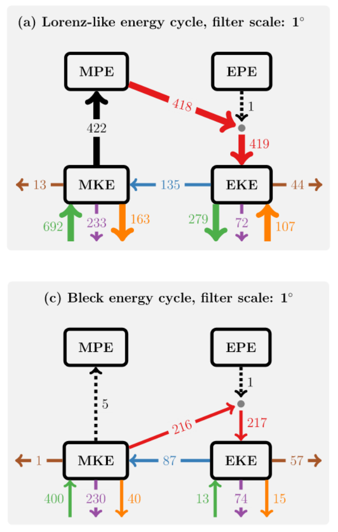


Figure: Energy conversions in non-TWA [Lorenz-like] and TWA [Bleck] energy cycles (Loose *et al.*, 2022, JPO)

Aims of this work

Understand how available potential energy is represented in TWA:

- Use TWA formulation of Young (2012), simple Boussinesq, zonal average, Lorenz reference height vertical coordinate.
- Establish link between terms in the Eliassen–Palm flux tensor and APE.

Close the energy budget:

- Start with an energetically consistent model.
- Write resolved and eddy KE budgets.
- Create proxies for resolved and eddy APE.
- Constrain closures based on total energy budget.

Reference state definition

Adiabatic and mass-conserving transformation into a minimum-energy reference state by flattening isopycnals (Lorenz, 1955).
See Saenz *et al.* (2015) for ocean implementation.

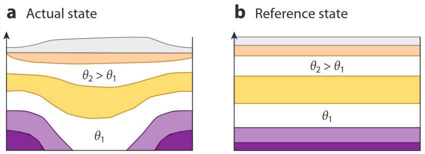


Figure: Illustration of Lorenz reference state (Tailleux, 2013, Annu. Rev. Fluid Mech.).

Reference height $Z_*(b,t)$: height to which parcel with buoyancy b is mapped.

Reference buoyancy profile: $b_*(\tilde{z},\tilde{t})$, w/

$$b_*(Z_*(b,t)) = b$$

and

$$\frac{\partial b_*}{\partial z} \geq 0.$$

With appropriate BCs:

$$\frac{\partial b_*}{\partial t}(\tilde{z},\tilde{t}) = \langle \varpi_b \rangle_* (\tilde{z},\tilde{t})$$

with $\langle (\cdot) \rangle_* (\tilde{z},t)$ average over isosurface $Z_* = \tilde{z}$ (Winters & D'Asaro, 1996), and where

$$\varpi_b = \frac{\mathrm{D}b}{\mathrm{D}t} = \sigma_b - \nabla \cdot \mathbf{j}_b.$$

APE definition

Local APE is defined by

$$A(b,z,t) = -\int_{Z_*(b,t)}^{z} \left(b - b_*(z',t)\right) \mathrm{d}z',$$

which is the work done (positive) by a parcel during the transformation to the reference state.

Note: in the QG limit, this can be approximated by the classical form

$$A(b,z,t) \approx \mathcal{A}[N_*^2, Z_*, z] = \frac{1}{2} N_*^2 \left(z - Z_*\right)^2,$$

where $N_*^2 = \partial b_* / \partial \tilde{z}$ is the background buoyancy frequency (Roullet & Klein, 2009).

Reference height coordinates

Assuming $\partial Z_*/\partial z > 0$ (stable stratification), we can use Z_* as a new vertical coordinate.

Denote the new (non-orthogonal) system of coordinates by $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$.

- Buoyancy $b_*(\tilde{z}, \tilde{t})$.
- Isopycnal height $\zeta(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$ such that $\tilde{z} = Z_*(b(x, y, \zeta(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}), t), t)$.
- Montgomery potential $m = p - b_*\zeta$.

Define the zonal average and thickness-weighted average in these coordinates:

$$\overline{f}(\tilde{y}, \tilde{z}, \tilde{t}) = \frac{1}{L_x} \int_0^{L_x} f(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) \, \mathrm{d}\tilde{x}, \qquad \widehat{f} = \frac{\overline{hf}}{\overline{h}}$$

with perturbations $f' = f - \overline{f}$ and $f'' = f - \widehat{f}$, where $h = \partial\tilde{z}/\partial z$ is the Jacobian, or "thickness".

TWA operators

Follow work of Young (2012), Smith (1999), de Szoeke & Bennett (1993).

Material derivative:

$$\frac{\mathrm{D}^\sharp}{\mathrm{D}t} = \frac{\partial}{\partial \tilde{t}} + \widehat{v}\frac{\partial}{\partial \tilde{y}} + \widehat{\varpi}_* \frac{\partial}{\partial \tilde{z}},$$

where

$$\varpi_* = \frac{\mathrm{D}Z_*}{\mathrm{D}t}.$$

In the averaged system the divergence-free transporting velocity is $\mathbf{u}^\sharp = \widehat{v}\mathbf{j} + w^\sharp \mathbf{k}$, where

$$w^\sharp = \frac{\mathrm{D}^\sharp \overline{\zeta}}{\mathrm{D}t}.$$

Note $w^\sharp \neq \widehat{w}$.

Zonally-averaged equations

$$\frac{\mathrm{D}^\sharp \widehat{u}}{\mathrm{D}t} + \frac{1}{\overline{\overline{h}}}\frac{\partial}{\partial \tilde{y}}\left(\overline{h}\widehat{u''v''}\right) + \frac{1}{\overline{\overline{h}}}\frac{\partial}{\partial \tilde{z}}\left(\overline{h}\widehat{u''\varpi''_*} + \overline{\zeta'\frac{\partial m'}{\partial \tilde{x}}}\right) = \widehat{F_u},$$

$$\frac{\mathrm{D}^\sharp \widehat{v}}{\mathrm{D}t} + \frac{\partial \overline{m}}{\partial \tilde{y}} + \frac{1}{\overline{\overline{h}}}\frac{\partial}{\partial \tilde{y}}\left(\overline{h}\widehat{v''^2} + \frac{1}{2}N_*^2\overline{\zeta'^2}\right) + \frac{1}{\overline{\overline{h}}}\frac{\partial}{\partial \tilde{z}}\left(\overline{h}\widehat{v''\varpi''_*} + \overline{\zeta'\frac{\partial m'}{\partial \tilde{y}}}\right) = \widehat{F_v},$$

$$\frac{\partial \overline{m}}{\partial \tilde{z}} + N_*^2\overline{\zeta} = 0,$$

$$\frac{\partial \overline{h}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{y}}\left(\overline{h}\widehat{v}\right) + \frac{\partial}{\partial \tilde{z}}\left(\overline{h}\widehat{\varpi}_*\right) = 0,$$

$$\frac{\mathrm{D}^\sharp b}{\mathrm{D}t} = \widehat{\varpi}_b = \widehat{\sigma}_b - \nabla\cdot\mathbf{j}_b^\sharp.$$

The following need closures:

$$\widehat{u''v''},\quad \widehat{v''^2},\quad \widehat{u''\varpi''_*},\qquad \text{(Reynolds stress)}$$

$$\frac{1}{2}N_*^2\overline{\zeta'^2},\quad \overline{\zeta'\partial_{\tilde{x}}m'},\quad \overline{\zeta'\partial_{\tilde{y}}m'},\qquad \text{(Eliassen-Palm flux)}$$

TWA parameterisation

For "form drag" terms, can use Greatbatch & Lamb (1990) parameterisation:

$$\overline{\zeta' \frac{\partial m'}{\partial \tilde{x}}} = -\mu \frac{\partial \hat{u}}{\partial \tilde{z}}, \quad \overline{\zeta' \frac{\partial m'}{\partial \tilde{y}}} = -\mu \frac{\partial \hat{v}}{\partial \tilde{z}}$$

The other terms are usually neglected (cf. Saenz *et al.*, 2015; Saenz & Ringler, 2020; Loose *et al.*, 2023).

However, other terms – especially $\frac{1}{2}N_*^2\overline{\zeta'^2}$ – contribute significantly large-scale forcing by eddies.

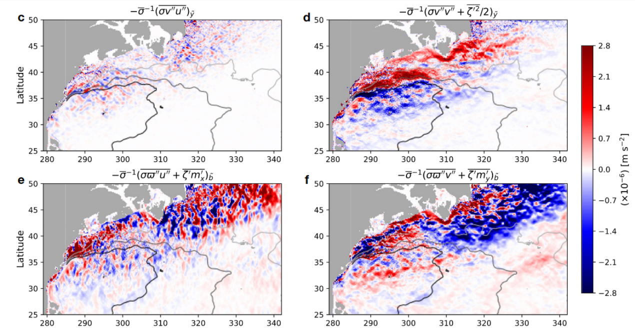


Figure: Snapshot of E–P flux tensor terms diagnosed from a 48 member ensemble at $(1/12)^\circ$ resolution (Uchida *et al.*, 2022, JAMES).

Eddy APE in TWA

$\frac{1}{2}N_*^2\overline{\zeta'^2}$ is an approximation to eddy APE.

- If we parameterise $\frac{1}{2}N_*^2\overline{\zeta'^2}$, how can we related it to eddy APE?
- How to even define eddy APE?

Kinetic energy is simple:

- MKE: $K_\mathrm{M} = \frac{1}{2}\left|\widehat{\mathbf{v}}\right|^2$
- EKE: $K_\mathrm{E} = \widehat{K} - K_\mathrm{M} = \frac{1}{2}\widehat{\left|\mathbf{v}''\right|^2}$

Mean–eddy decomposition: Eulerian vs. TWA

For an Eulerian average $\langle(\cdot)\rangle$, the APE split is also simple:

- MAPE: $A_{\mathsf{M}} = A(\langle b\rangle, z, t)$,
- EAPE: $A_{\mathsf{E}} = \langle A(b,z,t)\rangle - A_{\mathsf{M}}$.

APE is a convex function of b, so by Jensen's inequality,

$$\langle A(b,z,t)\rangle \geq A(\langle b\rangle, z, t).$$

Hence $A_{\mathsf{E}} \geq 0$ (Tailleux & Roullet, 2025; Wenegrat *et al.*, 2026).

However, in the TWA framework, all resolved fields are averaged with $\widehat{(\cdot)}$ except $\overline{\zeta}$ and $\overline{m}$.

Problem: $\widehat{A(b,\zeta,t)} - A(b,\overline{\zeta},t) \geq 0$ or ≤ 0?. What would a negative eddy APE even mean?

No nice inequality to split into positive eddy APE based on resolved height $\overline{\zeta}$.

Approximate APE closure: Prognosed quantities

Propose forms for resolved APE $A^\sharp$ and eddy APE a:

$$A^\sharp = A^\sharp(b, N^2, \overline{\zeta}), \quad a = a(b, N^2, q),$$

where $q = \frac{1}{2}\overline{\zeta'^2}$. Prognosis of these requires the following:

$$\frac{\mathrm{D}^\sharp b}{\mathrm{D}t} = \widehat{\varpi}_b, \qquad \frac{\mathrm{D}^\sharp N^2}{\mathrm{D}t} = \frac{\partial \left\langle \varpi_b \right\rangle_*}{\partial \tilde{z}} + \widehat{\varpi}_* \frac{\partial N^2}{\partial \tilde{z}},$$
$$\frac{\mathrm{D}^\sharp q}{\mathrm{D}t} = -\nabla \cdot \mathbf{J}_q + X_\zeta + P_q + T_q$$

where

$$\mathbf{J}_q = \overline{v'' q}\overline{\mathbf{e}}_2 + \overline{\varpi'' q}\overline{\mathbf{e}}_3, \quad X_\zeta = \overline{\zeta'(w - w^\sharp)},$$
$$P_q = \mathbf{J}_\zeta \cdot \nabla\overline{\zeta}, \quad \mathbf{J}_\zeta = \overline{\zeta' v''}\overline{\mathbf{e}}_2 + \overline{\zeta' \varpi''}\overline{\mathbf{e}}_3,$$
$$T_q = -\overline{h}\overline{\left(u''\frac{\partial q}{\partial \tilde{x}} + v''\frac{\partial q}{\partial \tilde{y}} + \varpi''_*\frac{\partial q}{\partial \tilde{z}}\right)}$$

Approximate APE closure: Energy budgets

Then the APE budgets are:

$$\frac{\mathrm{D}^\sharp A^\sharp}{\mathrm{D}t} = \frac{\partial A^\sharp}{\partial b}\widehat{\varpi}_b + \frac{\partial A^\sharp}{\partial N^2}\left(\frac{\partial \langle \varpi_b \rangle_*}{\partial \tilde{z}} + \widehat{\varpi}_* \frac{\partial N^2}{\partial \tilde{z}}\right) + \frac{\partial A^\sharp}{\partial \overline{\zeta}} w^\sharp$$

$$\frac{\mathrm{D}^\sharp a}{\mathrm{D}t} = \frac{\partial a}{\partial b}\widehat{\varpi}_b + \frac{\partial a}{\partial N^2}\left(\frac{\partial \langle \varpi_b \rangle_*}{\partial \tilde{z}} + \widehat{\varpi}_* \frac{\partial N^2}{\partial \tilde{z}}\right) + \frac{\partial a}{\partial q}\left(X_\zeta + P_q + T_q\right) - \nabla \cdot \left(\frac{\partial a}{\partial q}\mathbf{J}_q\right) + \mathbf{J}_q \cdot \nabla \frac{\partial a}{\partial q}$$

Resolved and eddy KE budget (Uchida *et al.* 2022):

$$\frac{\mathrm{D}^\sharp K^\sharp}{\mathrm{D}t} + \nabla \cdot \mathbf{J}_K = b w^\sharp - \varepsilon_{\mathrm{k}}^\sharp + \mathsf{E} : \nabla \widehat{\mathbf{v}}$$

$$\frac{\mathrm{D}^\sharp k}{\mathrm{D}t} + \nabla \cdot \mathbf{J}_k = b(\widehat{w} - w^\sharp) - (\widehat{\varepsilon}_{\mathrm{k}} - \varepsilon_{\mathrm{k}}^\sharp) - \mathsf{E} : \nabla \widehat{\mathbf{v}}$$

Closing the energy cycle

Use consistent Boussinesq model (Tailleux, Dubos & Hatton, 2026, GAFD, *accepted*) to close energy cycle.

Background potential energy (BPE):

$$\frac{\mathrm{D}^\sharp \widehat{B}}{\mathrm{D}t} + \nabla \cdot \mathbf{J}_B = \widehat{\varepsilon}_\mathrm{p} + \widehat{\varepsilon}_\mathrm{k} - \widehat{\Pi}_t.$$

with APE dissipation ε_p.

Next steps:

- Make correspondences between energy budget terms to constrain equation for $A^\sharp$ and a.
- Close prognostic equation for q.
- How to keep a positive?

Conclusion

Problems identified:

- Prognosis of height variance term in E–P flux tensor.
- Its relation to eddy APE.
- Conservation of energy, even with irreversible processes.

Future work:

- Identify appropriate energy conversions (EKE $\leftrightarrow$ EPE, EPE $\rightarrow$ BPE, etc.)
- Implement a simple eddy APE closure.
- Beyond linear EOS.
- Beyond simple Boussinesq.

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- Beyond linear EOS.
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Thank you!

Appendix 1: Diapycnal velocity diagnosis

The diapycnal velocity $\widehat{\varpi}_*$ is diagnosed with

$$\widehat{\varpi}_* = \frac{\widehat{\varpi}_b - \langle \varpi_b \rangle_*}{N_*^2} = \frac{\widehat{\varpi}_b - \frac{1}{L_y}\int_0^{L_y} \overline{h}\widehat{\varpi}_b \, \mathrm{d}\tilde{y}}{N_*^2}.$$

Appendix 2: Thermodynamically consistent Boussinesq

Equations:

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} + \nabla p = b\nabla z + \nabla \cdot \mathsf{T}, \tag{1}$$

$$\nabla \cdot \mathbf{u}, \tag{2}$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} = -\nabla \cdot \mathbf{j}_b + \sigma_b, \tag{3}$$

Energy conservation gives buoyancy production:

$$\sigma_b = \frac{\varepsilon + \mathbf{j}_b \cdot \nabla z}{c_p/(g\alpha) - z}. \tag{4}$$

Positive entropy production constrains $\mathbf{j}_b$.