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Interpreting filter based sequential spectra

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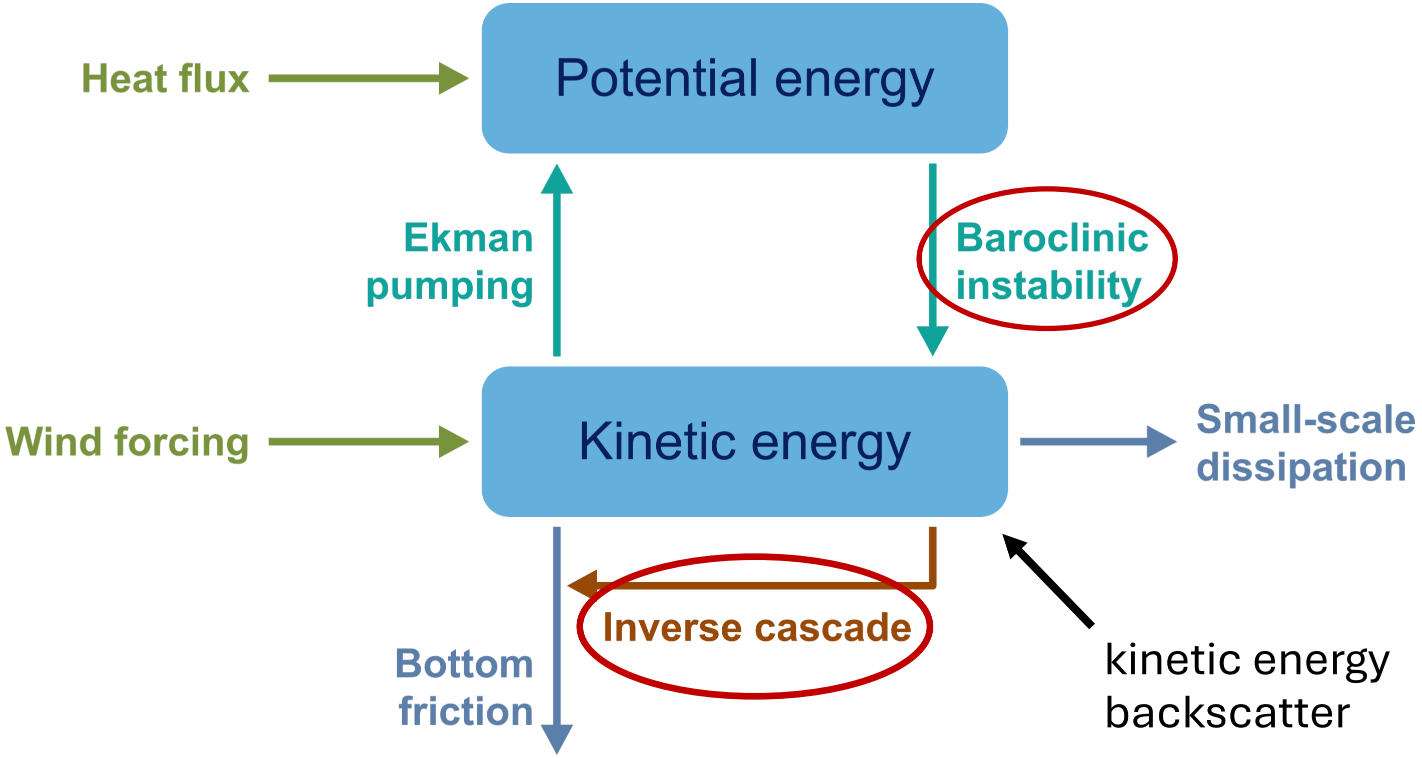
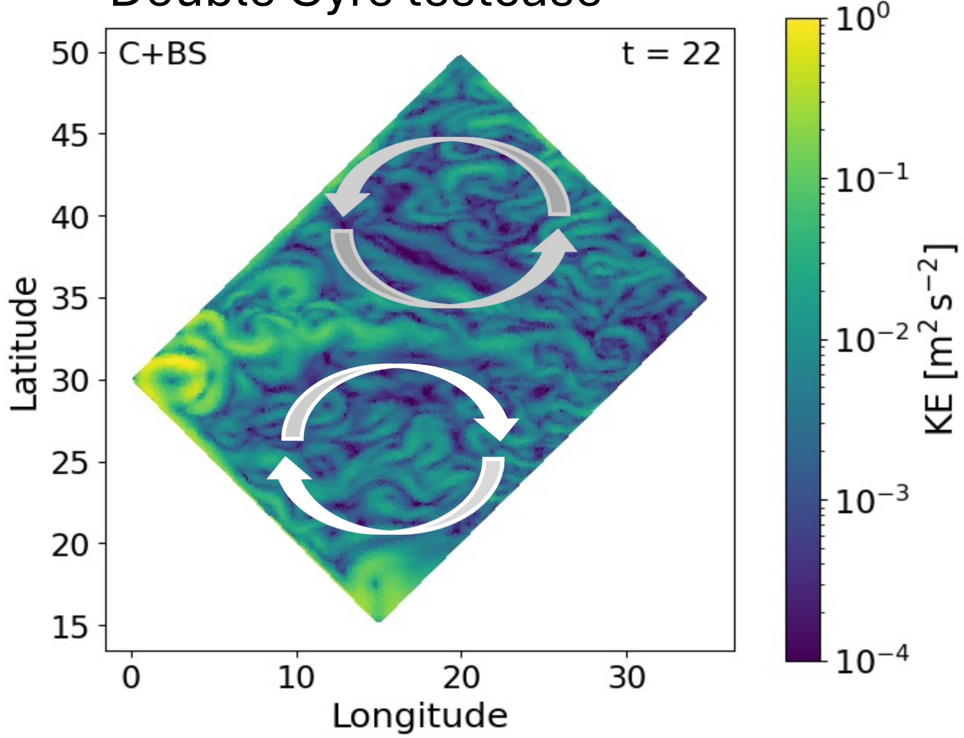
TRR181

Understanding the resolved and parameterised Ocean Energy Cycle

*How does **kinetic energy backscatter** transfer across scales and influence the Ocean energy cycle?*



Double Gyre testcase



Kinetic Energy Budget

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$$\underbrace{\frac{1}{2}\,\partial_t|\boldsymbol{u}_h|^2}_{\text{kinetic energy tendency}} = \underbrace{-\boldsymbol{u}_h\cdot(\boldsymbol{u}_h\cdot\nabla_h)\boldsymbol{u}_h}_{\text{non-linear transfer}} \underbrace{-\frac{1}{\rho_0}\,\boldsymbol{u}_h\cdot\nabla_h p}_{\to\text{buoyancy flux}} + \underbrace{\boldsymbol{u}_h\cdot V(\boldsymbol{u}_h)}_{\text{viscous dissipation}} + \underbrace{\boldsymbol{u}_h\cdot B(\boldsymbol{u},e)}_{\text{kinetic energy backscatter}} + \underbrace{\boldsymbol{u}_h\cdot\partial_z(\nu_v\,\partial_z\boldsymbol{u}_h)}_{\text{vertical viscosity}} + F$$

Domain integrated spectral analysis

Implicit Filter

github.com/FESOM/implicit_filter

biharmonic

Unresolved kinetic energy

harmonic

$$\partial_t e = -c_{\mathrm{dis}}\langle E_{\mathrm{dis}}\rangle - \langle E_{\mathrm{back}}\rangle + e_{\mathrm{dif}}$$

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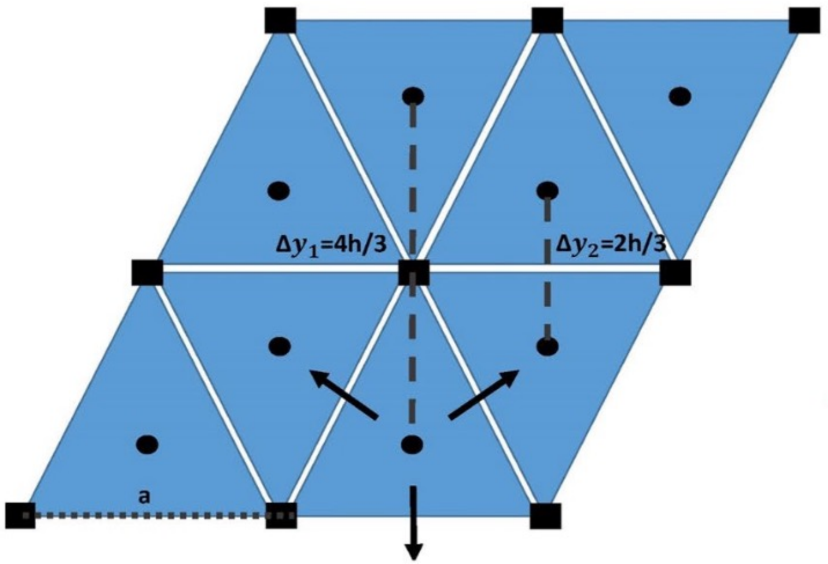
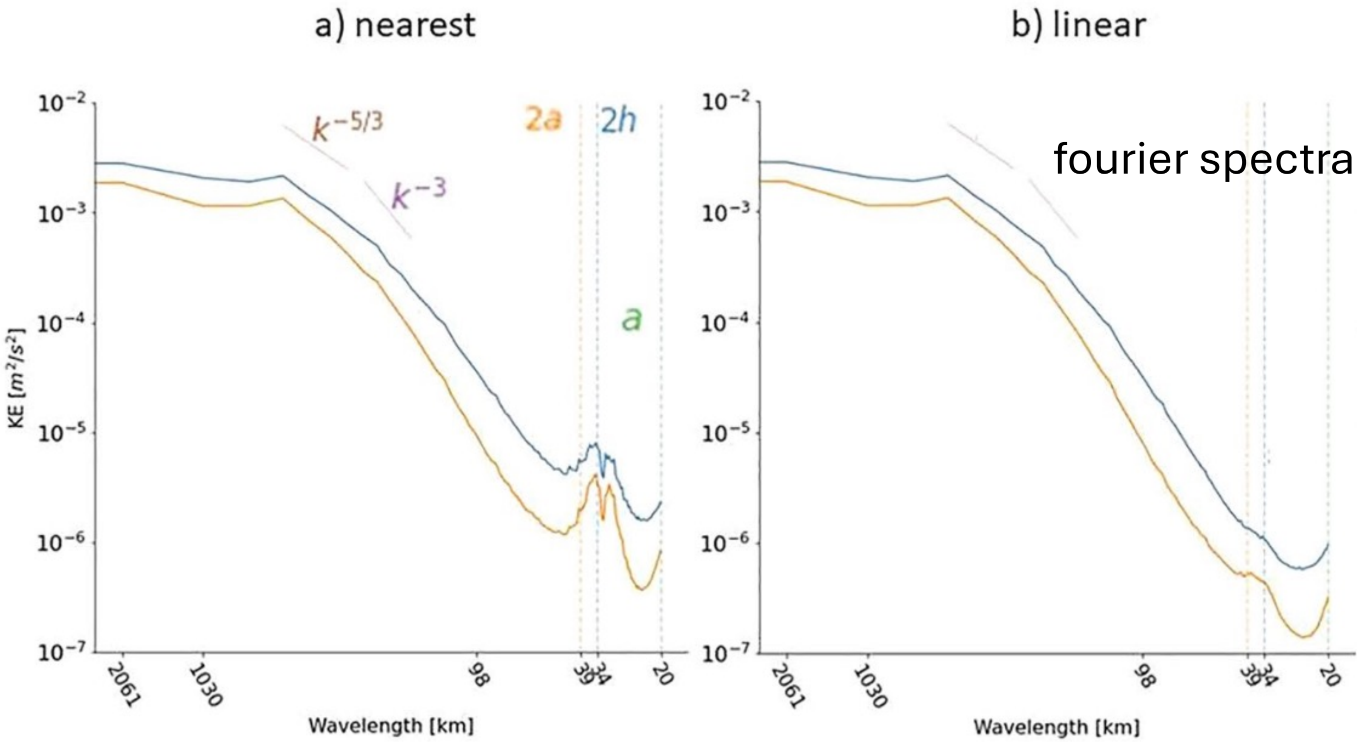
Why use the implicit filter?

Double Gyre testcase in FESOM2

- non-periodic bounded domain
- triangular grid



interpolation introduces uncertainties

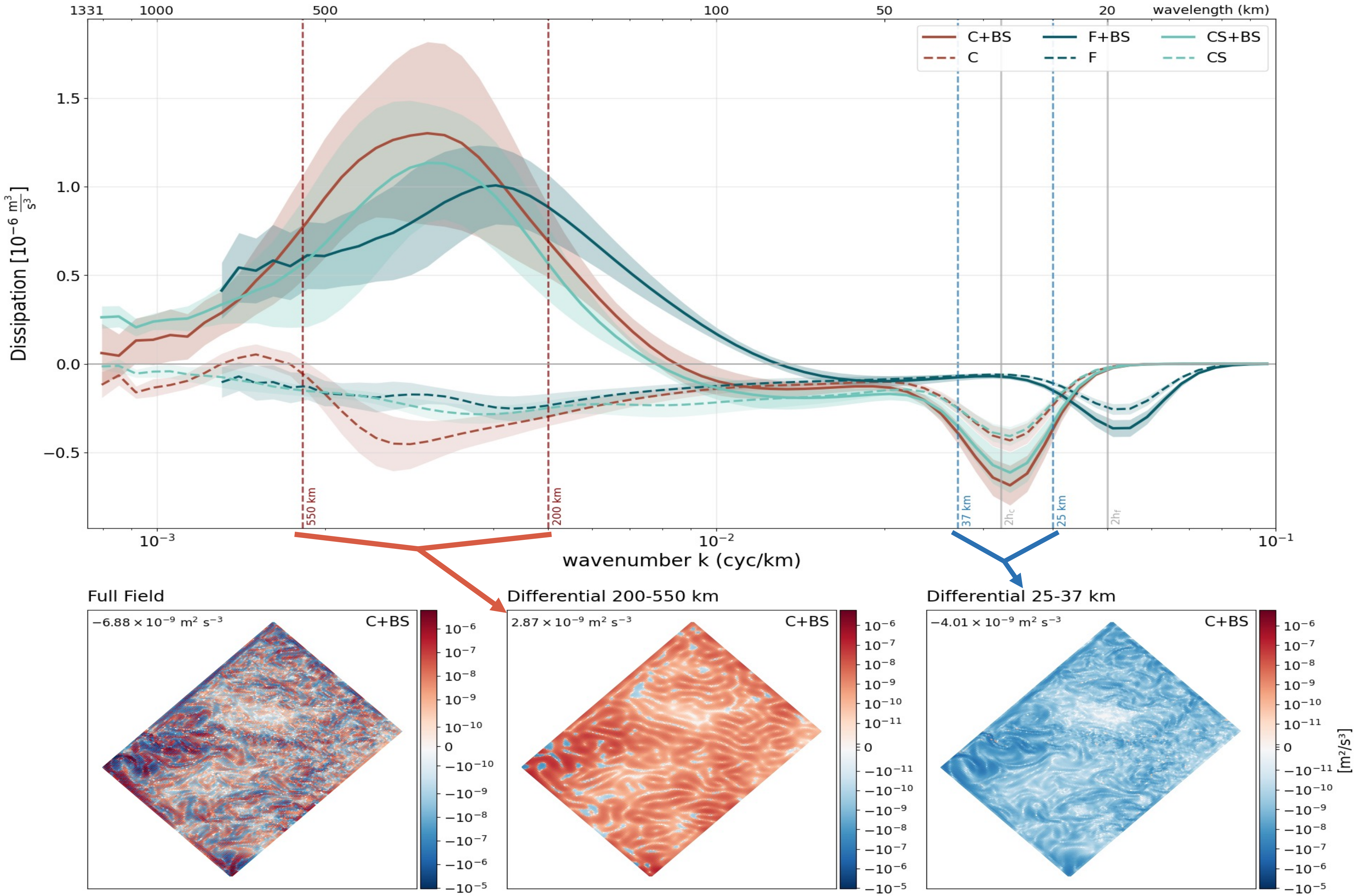


- implicit filter works on native grid
  - no subdomain
  - no windowing
  - no interpolation
- based on operators inherent in the model

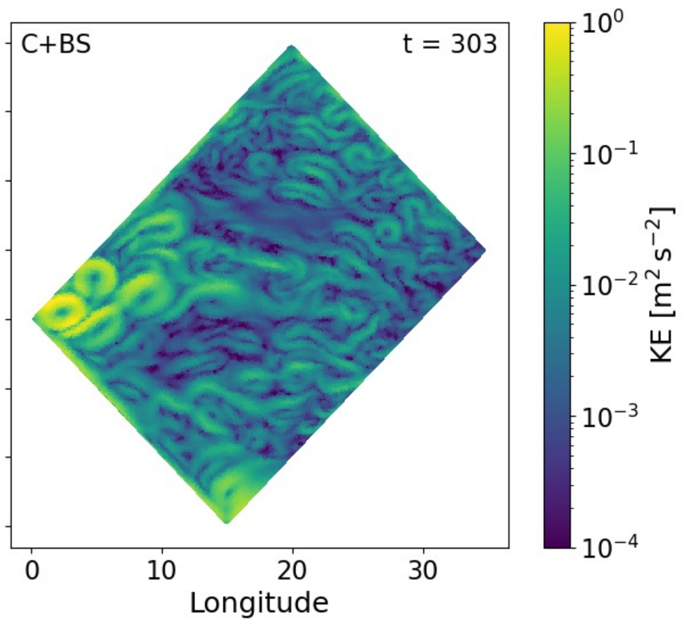
➡ **consistent diagnostic framework**

Figures from Juricke et al (2023)

Why use the implicit filter?



directly associate spectral features with 2D differential fields



Domain integrated spectral analysis

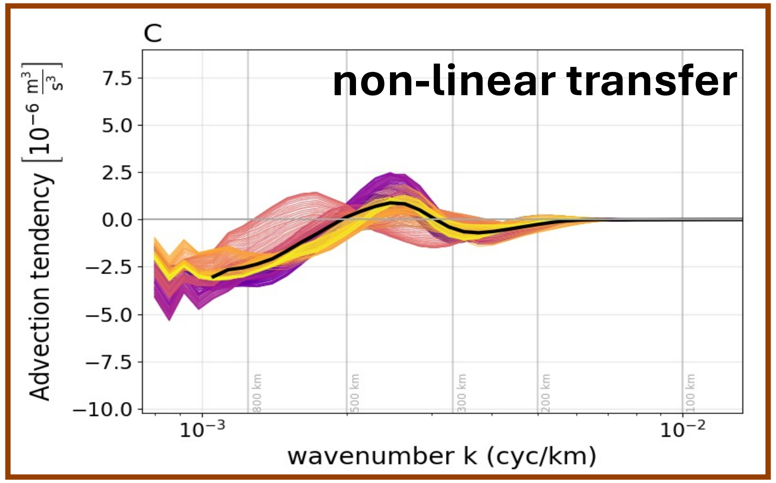
Implicit Filter

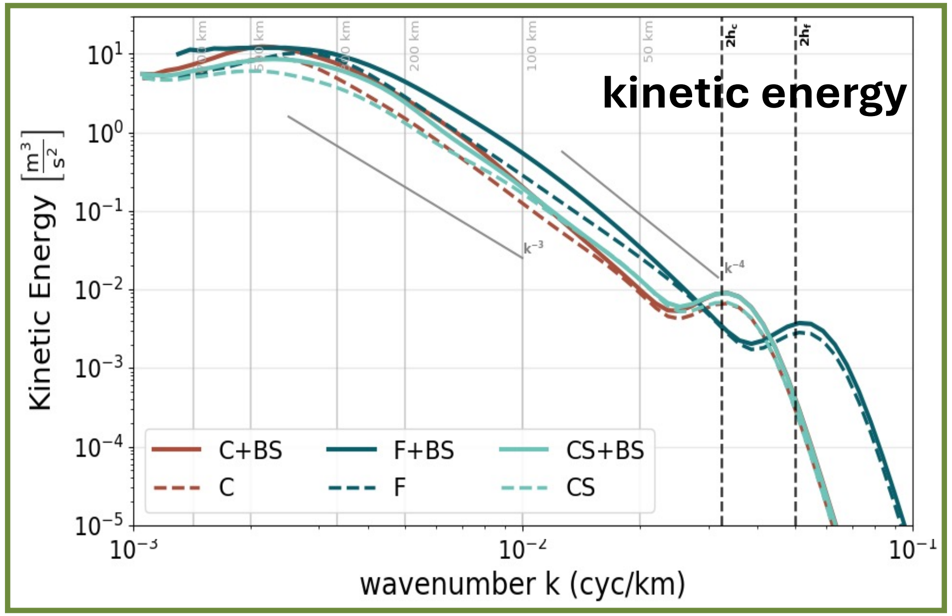
github.com/FESOM/implicit_filter

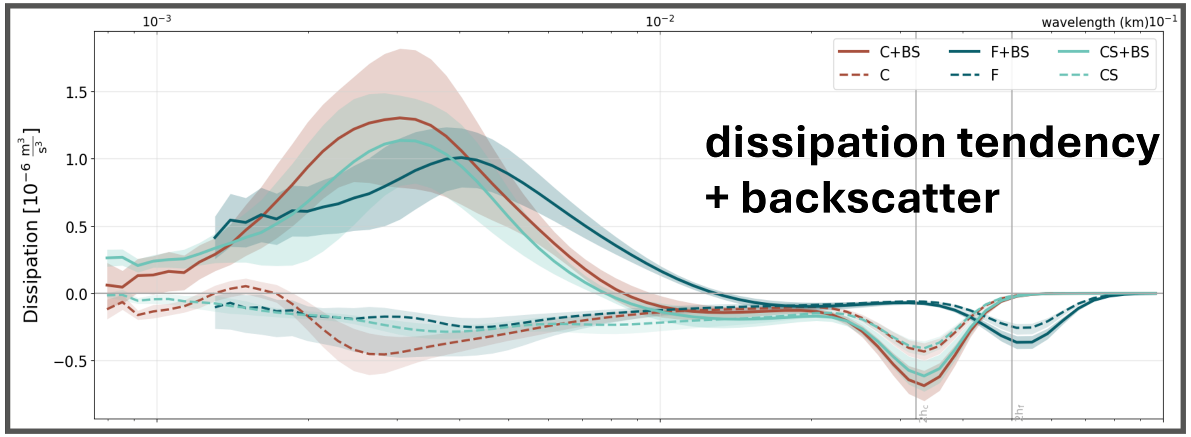
Kinetic Energy Budget

How do I interpret these spectra?

→ fourier spectra

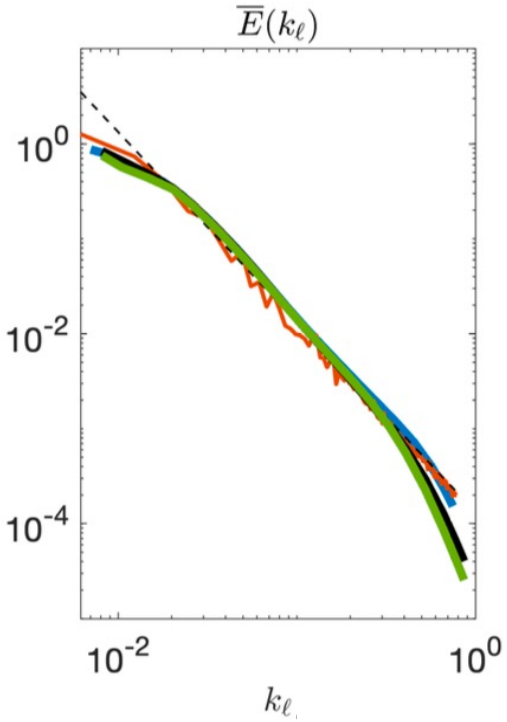




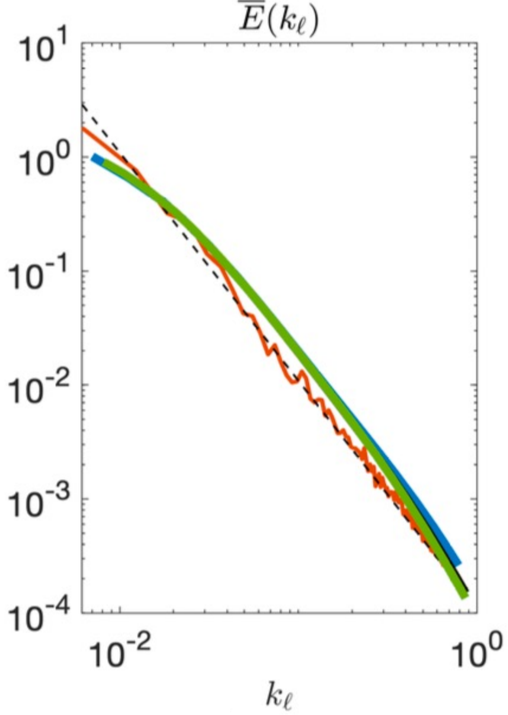


How can we interpret differential filter spectra?

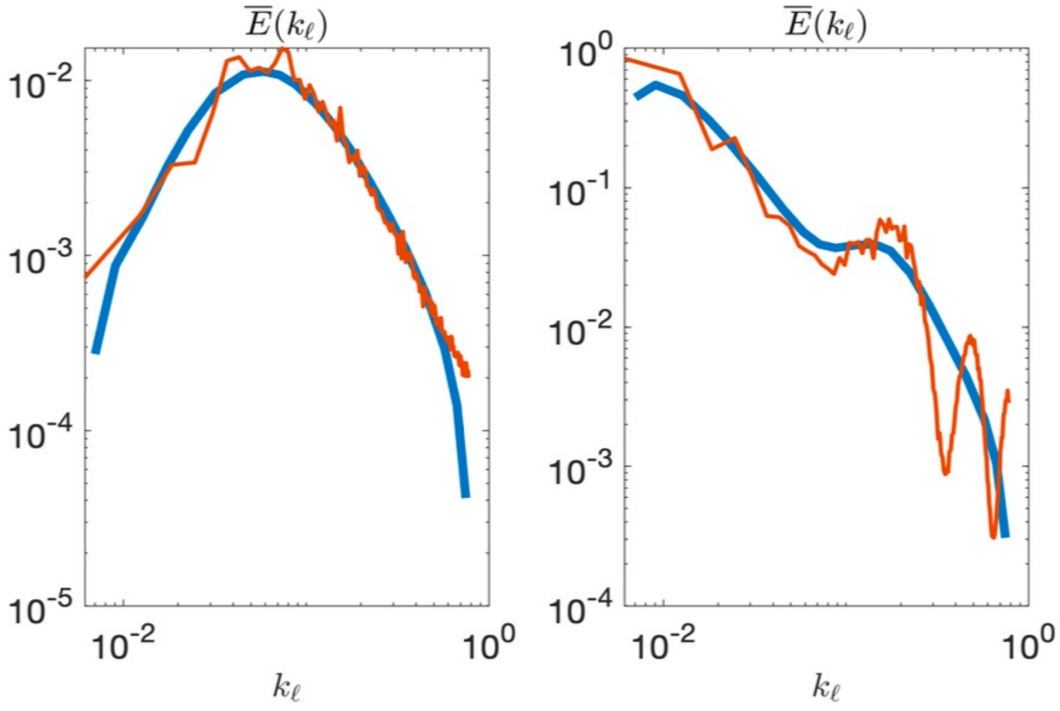
spectral slopes



spectral density



position and amplitude of spectral peaks



- ➢ theoretical understanding of mechanisms → 1D analytical analysis
- ➢ practical guideline on implementation and interpretation → parameter sweeps with synthetic spectra

Figures from Danilov et al (2024)

Sequential filtering as a convolution in fourier space

$$E_\ell(k_\ell) = \int E(k)\, F_\ell(k)\, \mathrm{d}k \qquad \rightarrow \qquad k_\ell E_\ell(k_\ell) = \int kE(k)\, W(s)\, \mathrm{d}s$$

$$z = \frac{k}{k_\ell}$$

$$= \int E(k)\, W(z)\, \mathrm{d}z$$

$$W(z) = k_\ell F_\ell$$

$$s = \ln z \; = \ln k - \ln k_\ell$$

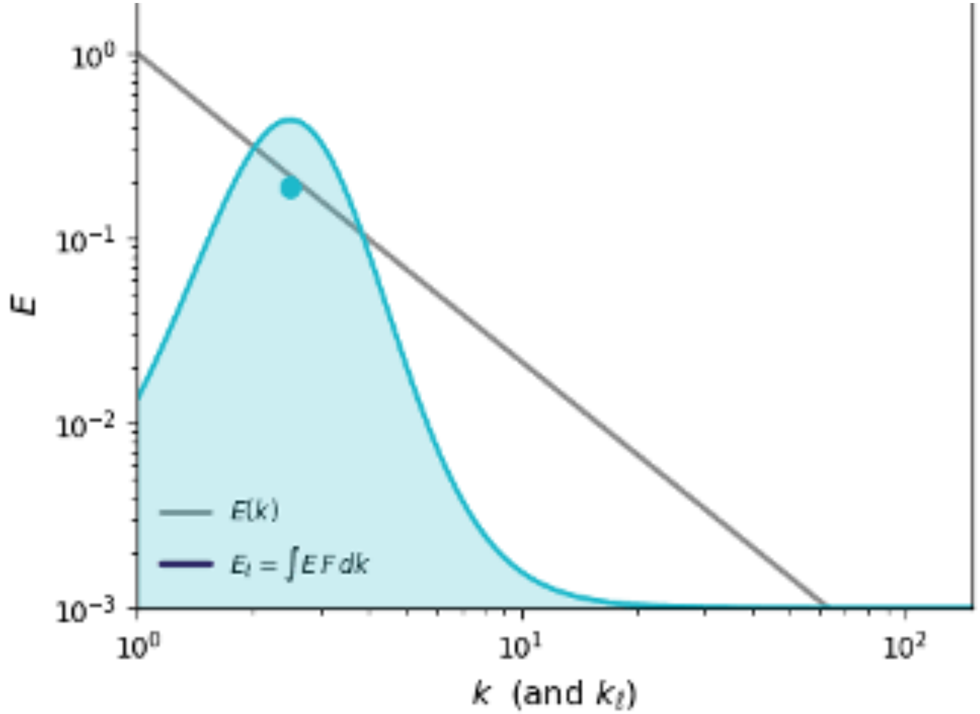
$$E\ \mathrm{d}k \; = \; kE\ \mathrm{d}\ln k$$

- form factor is normalised

$$\int_{-\infty}^{\infty} W(s)\, \mathrm{d}s = 1$$

- translation invariant

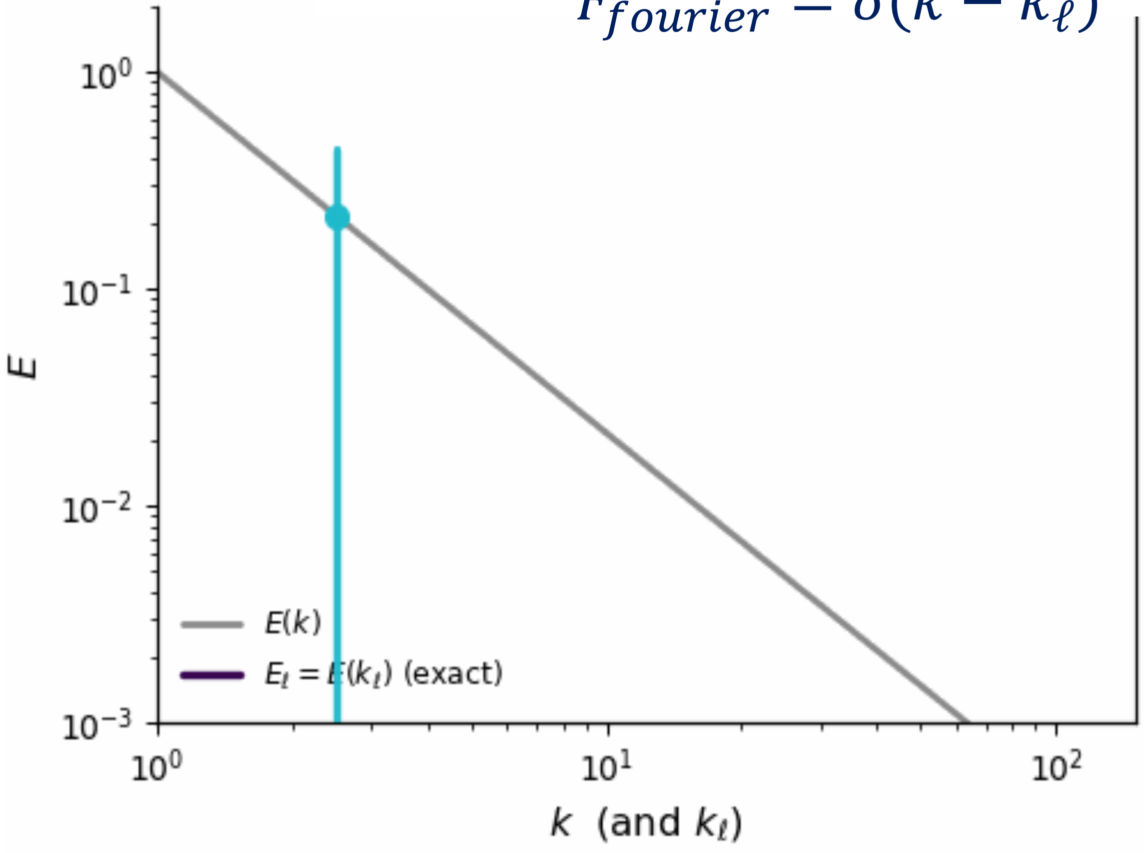
$$W \; = W(\ln k \; - \; \ln k_\ell)$$



Form factor: Fourier vs Coarse graining spectra

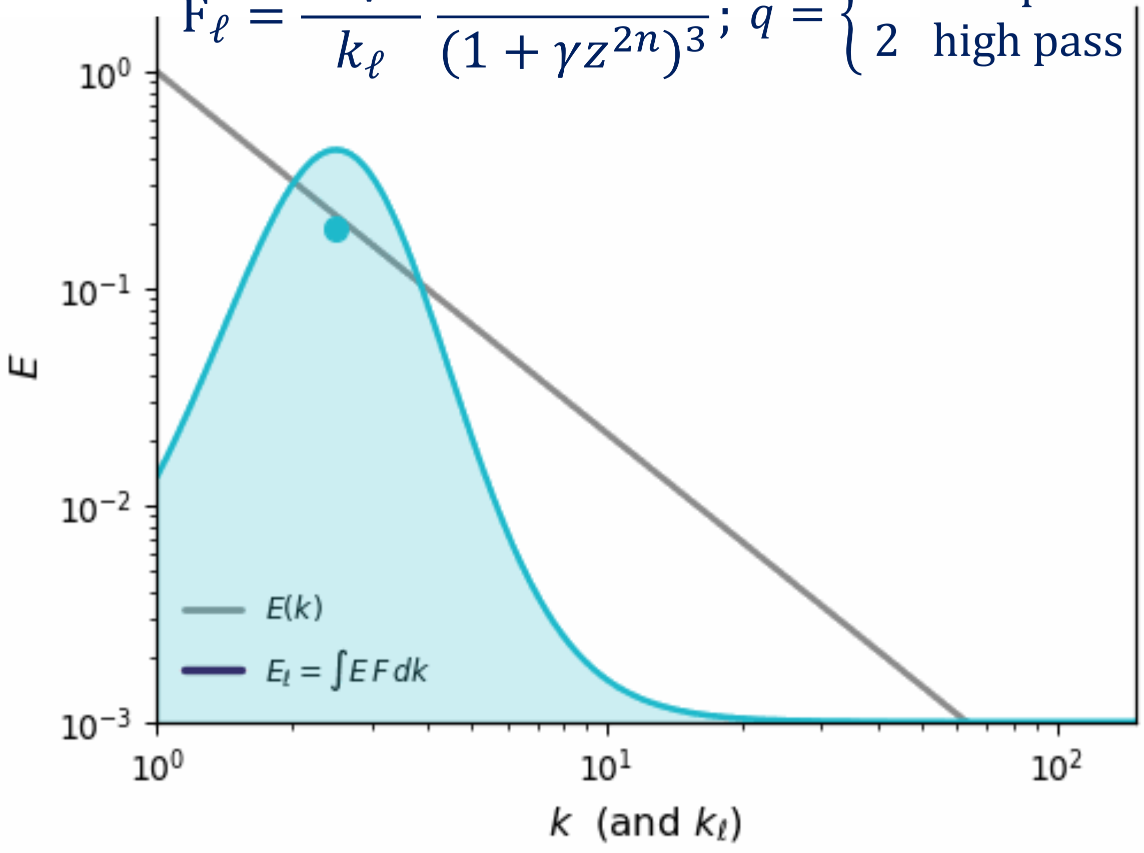
single wavenumber

$$F_{fourier} = \delta(k - k_\ell)$$

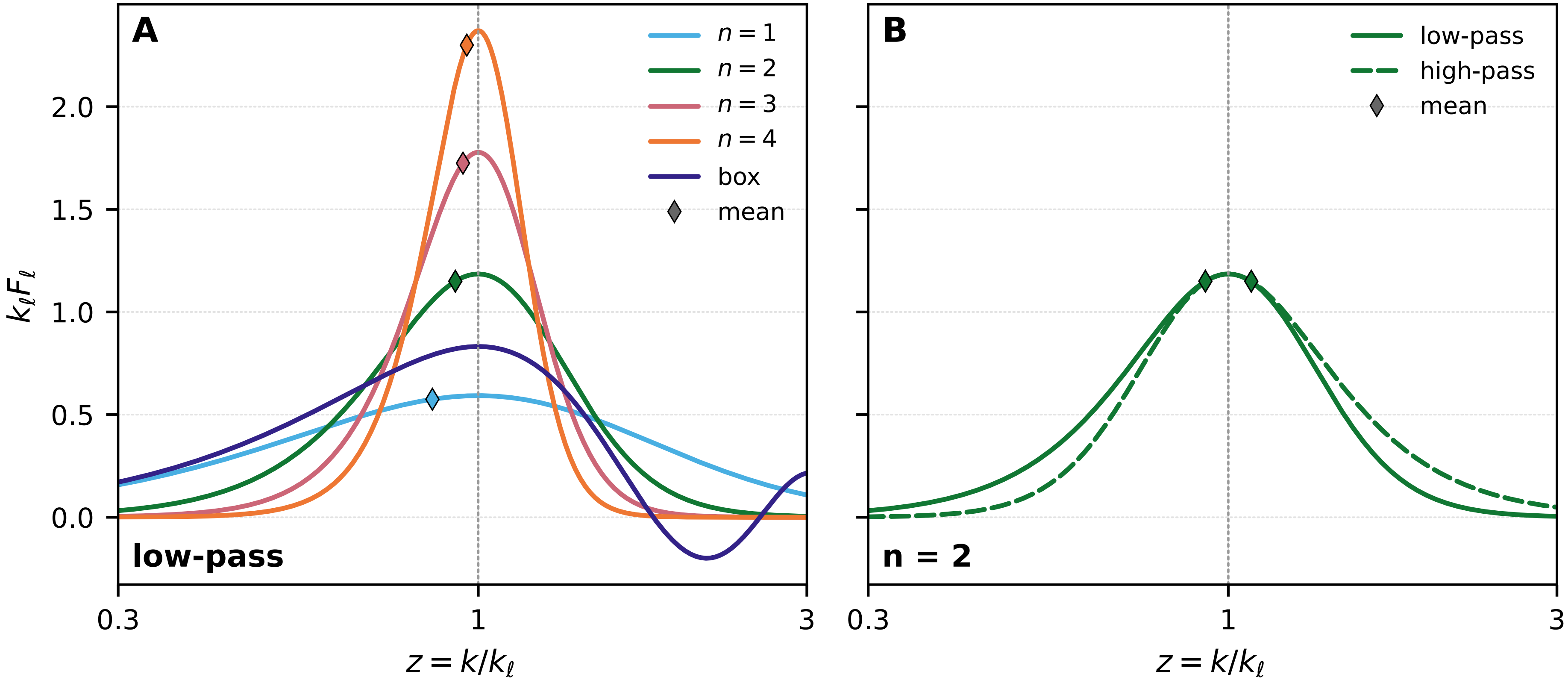


wavenumber band

$$F_\ell = \frac{4n\gamma^q}{k_\ell} \frac{z^{2qn}}{(1+\gamma z^{2n})^3}; \; q = \begin{cases} 1 & \text{low pass} \\ 2 & \text{high pass} \end{cases}$$



Form factor characterised by position and width



Position:

$$\langle s \rangle = \mp \frac{1 - \ln 2}{2n} = \mp \frac{0.22}{\mathrm{n}}$$

$$\gamma^{\mathrm{H}} = 2, \gamma^{\mathrm{L}} = \frac{1}{2}$$

Width:

$$\mathrm{FWHM}(n) = \frac{2.175}{n} \approx 2\sqrt{\mathrm{Var}(s)}$$

Power-law spectra $E = Ck^{-\alpha}$

$$k_\ell E_\ell(k_\ell) = \int kE(k)\,W(s)\,\mathrm{d}s = C{k_\ell}^{1-\alpha} \cdot \int e^{(1-\alpha)s}\,W(s)\,\mathrm{d}s$$

$$= C{k_\ell}^{1-\alpha} \cdot \mathrm{M_W}(1-\alpha)$$

converges for $0 < \mu < 6\mathrm{n}$ $\quad \mu = 2nq + 1 - \alpha$

$$\mathrm{M_W}(1-\alpha) = 4n\gamma^q \int \frac{z^{2nq+1-\alpha}}{(1+\gamma z^{2n})^3}\,\mathrm{d}s$$

moment generating function

for red spectra $\boxed{\alpha^L_{\mathrm{crit}} = 2n+1\,, \quad \alpha^H_{\mathrm{crit}} = 4n+1}$

Sadek and Aluie (2018), Zhao and Aluie (2025)

$\mu \leq 0$: lower limit of integral dominates

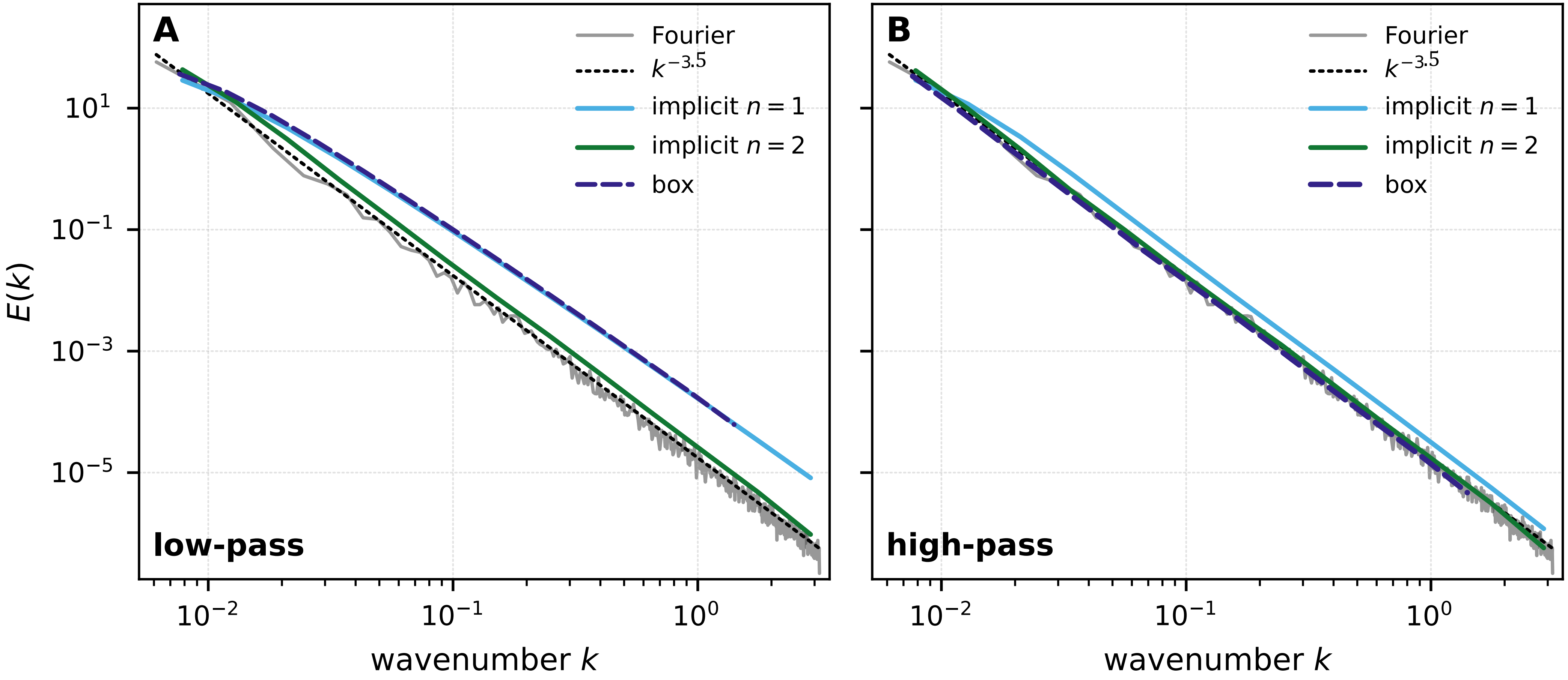
saturation at ${k_\ell}^{-\alpha_{\mathrm{crit}}}$

$$\mathrm{E}_\ell \approx 4n\gamma^q C{k_\ell}^{-\alpha} \int_{\ln\left(\frac{k_{\min}}{k_\ell}\right)} \frac{z^{2nq+1-\alpha}}{(1+\gamma z^{2n})^3}\,\mathrm{d}s \approx \underbrace{\frac{4n\gamma^q C{k_{\min}}^{\mu}}{|\mu|}}_{\text{const.}} k_\ell^{-(2nq+1)}$$

$\alpha_{\mathrm{crit}} \propto \mathrm{n}$

$\Rightarrow$ increasing order recovers steeper slopes

Power-law spectra $E = Ck^{-\alpha}$



saturation for slopes $\geq \alpha_{\text{crit}}$

$$\alpha^{L}_{\text{crit}} = 2n + 1\,, \quad \alpha^{H}_{\text{crit}} = 4n + 1$$

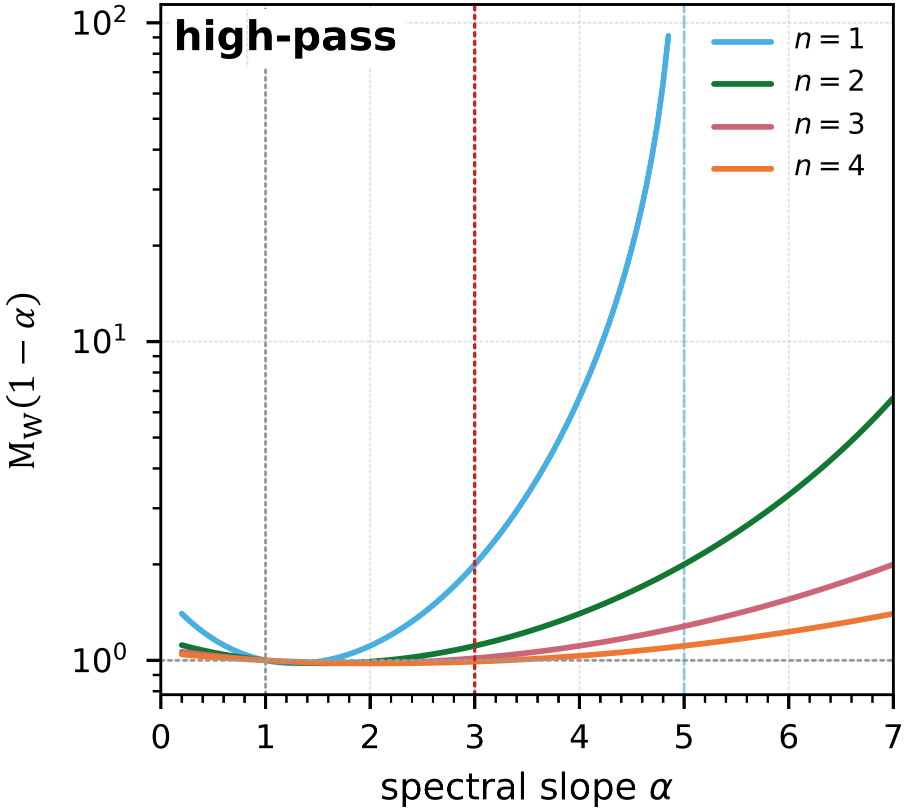
Power-law spectra $E = Ck^{-\alpha}$

$$k_\ell E_\ell(k_\ell) = \int kE(k)\,W(s)\,\mathrm{d}s = C{k_\ell}^{1-\alpha} \cdot \int e^{(1-\alpha)s}\,W(s)\,\mathrm{d}s$$

$$= C{k_\ell}^{1-\alpha} \cdot \underbrace{\mathrm{M_W}(1-\alpha)}_{\substack{\text{const.} \neq 1 \\ \text{(in general)}}}$$

➢ power law multiplied with constant factor

⇔ constant vertical shift in logarithmic coordinates



cumulant generating function:

$$\ln \mathrm{M_W}(1-\alpha) = \underbrace{\langle s\rangle(1-\alpha)}_{\langle s\rangle \propto \frac{1}{n}} + \underbrace{\frac{1}{2}\mathrm{Var}(s)(1-\alpha)^2}_{\mathrm{Var}(s) \propto \frac{1}{n^2}} + \cdots$$

➢ density shift grows towards α_{crit} where the spectrum saturates

➢ increasing filter order reduces density shift

Spectral peaks $kE = P(\ln k)$

position and *amplitude* ?

$\downarrow$ $\langle x \rangle$

$\downarrow$ $\mathrm{Var}(x)$

energy in peak is conserved $\Rightarrow \dfrac{h_{P_\ell}}{h_P} \approx \sqrt{\dfrac{\mathrm{Var}(x_P)}{\mathrm{Var}(x_{P_\ell})}}$

if peak shape is recovered

P: peak in $\ln k$

$M_P(t)$: mgf of peak

x_P: coordinate of peak in ln k

$$M_{P_\ell} = \cdots = M_P(t) \cdot M_W(-t) \quad \Rightarrow \ln\left(M_{P_\ell}\right) = \ln\left(M_P(t)\right) + \ln\left(M_W(-t)\right)$$

only filter order !

$$\Rightarrow \left\langle x_{P_\ell} \right\rangle = \langle x_P \rangle - \langle s \rangle$$

filtered peak shifted by $-\langle s \rangle$ to
HP: larger scales
LP: smaller scales

$$\Rightarrow Var\left(x_{P_\ell}\right) = Var(x_P) + Var(s)$$

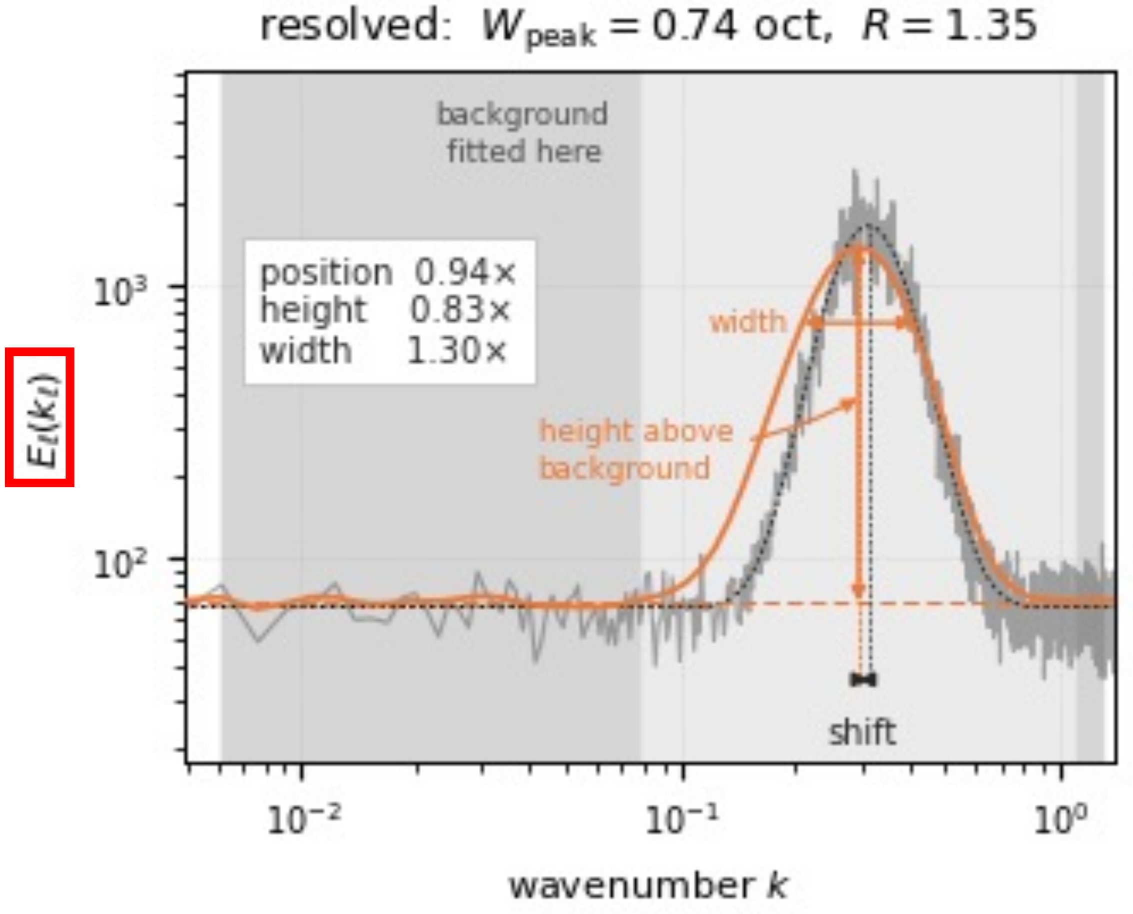
variances add

change in mean and variance **independent** of original peak (in log–coordinates)

$\langle s \rangle \propto \dfrac{1}{n}$ $\quad \mathrm{Var}(s) \propto \dfrac{1}{n^2}$

damping and relative broadening depend on ratio $\sqrt{\dfrac{\mathrm{Var}(s)}{\mathrm{Var}(x_{P_\ell})}}$ $\Rightarrow$ stronger for narrower peaks and lower filter order !

Artificial peak experiments with log-gauss peaks



ratio $$R := \frac{\mathrm{FWHM}_{\text{peak}}}{\mathrm{FWHM}(n)}$$

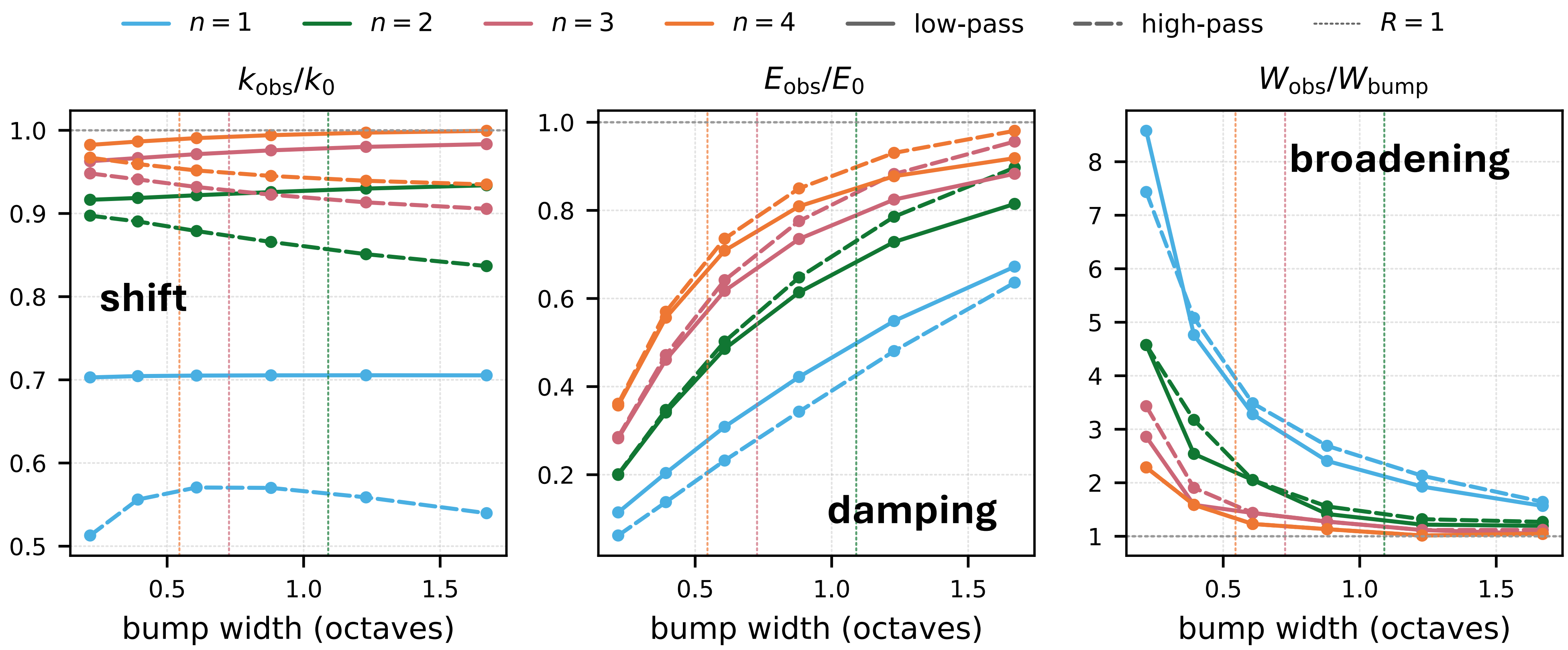
$$\mathrm{FWHM}(n) = \frac{2.175}{n}\,\text{oct} \approx 2\sqrt{\mathrm{Var}(s)}$$

broadening $$\frac{\mathrm{FWHM}_{\text{obs}}}{\mathrm{FWHM}_{\text{peak}}} \approx \sqrt{1 + \frac{1.4}{R^2}}$$

damping $$\frac{h_{\text{obs}}}{h_{\text{peak}}} \approx \frac{1}{\sqrt{1 + 1.40/R^2}}$$

$R \lesssim 1$ → shape of peak is not resolved ⇒ shape of filtered peak ≈ shape of form factor

Synthetic peak experiments with log gauss peaks



$$\mathrm{FWHM}(n) = \frac{2.175}{n}\,\mathrm{oct}$$

ratio

$$R \coloneqq \frac{\mathrm{FWHM}_{\mathrm{peak}}}{\mathrm{FWHM}(n)}$$

broadening $$\frac{\mathrm{FWHM}_{\mathrm{obs}}}{\mathrm{FWHM}_{\mathrm{peak}}} \approx \sqrt{1 + \frac{1.4}{R^2}}$$

damping $$\frac{h_{\mathrm{obs}}}{h_{\mathrm{peak}}} \approx \frac{1}{\sqrt{1 + 1.40/R^2}}$$

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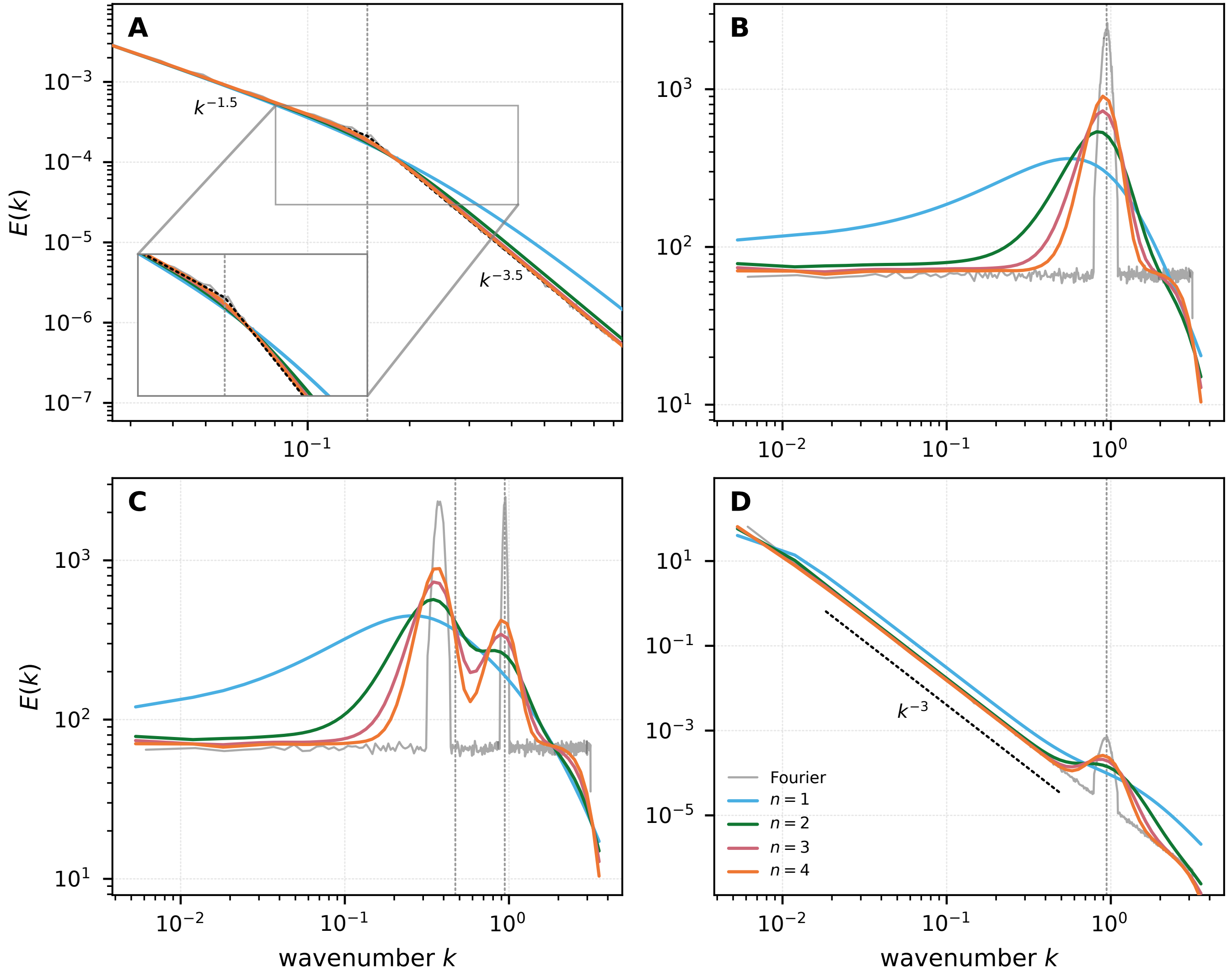
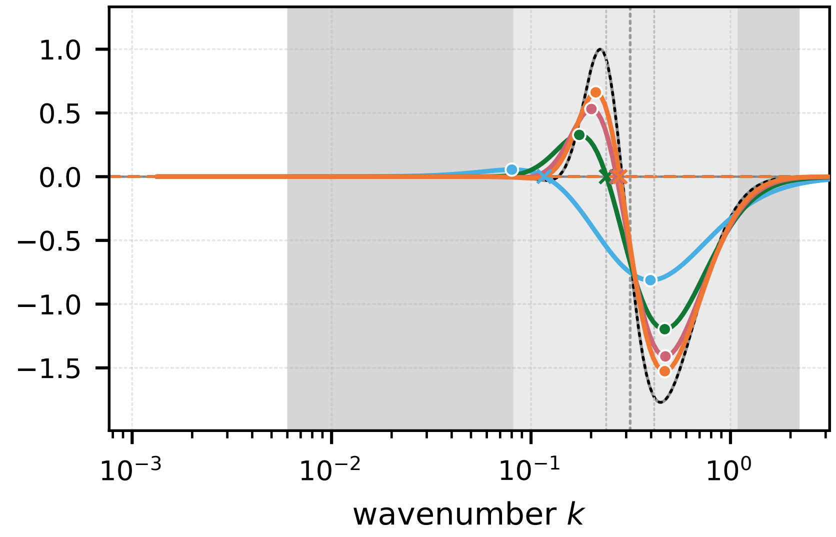
Examples of synthetic spectra experiments



- slope break
- multiple peaks
- peaks above background slope
- dipole structure



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Domain and grid scales

truncation effect *domain and grid scales*

- field carries energy only between $k_{\min}$ and $k_{\max}$
- close to integral limits form factor flanks would reach beyond $k_{\min}$ or $k_{\max}$

$$k_\ell E_\ell(k_\ell) = \int_{\ln\left(\frac{k_{\min}}{k_\ell}\right)}^{\ln\left(\frac{k_{\max}}{k_\ell}\right)} kE(k)\, W(s)\, \mathrm{d}s$$

- cut off at integral limits

power law spectra $\rightarrow$ dip in spectral density

peaks $\rightarrow$ displaced away from $k_{\min}$ or $k_{\max}$

- higher order decreases margin of truncation effect

discretisation effect *only grid scales*

- discrete Laplacian: $\lambda(k) = \frac{2}{h} \sin \frac{kh}{2}$

$$kh \ll 1 \ \rightarrow \lambda \approx k$$

- increasing compression of axis $\frac{\mathrm{d}\lambda}{\mathrm{d}k} \rightarrow 0$

  towards $\lambda\left(\frac{\pi}{h}\right) = \frac{2}{h}$

- energy conservation $\rightarrow$ enhanced spectral density
- cannot be changed by filter order

How do I interpret spectra?

$n = 4$, high-pass
$\mathrm{FWHM}(n = 4) = 0.54\ \mathrm{oct}$

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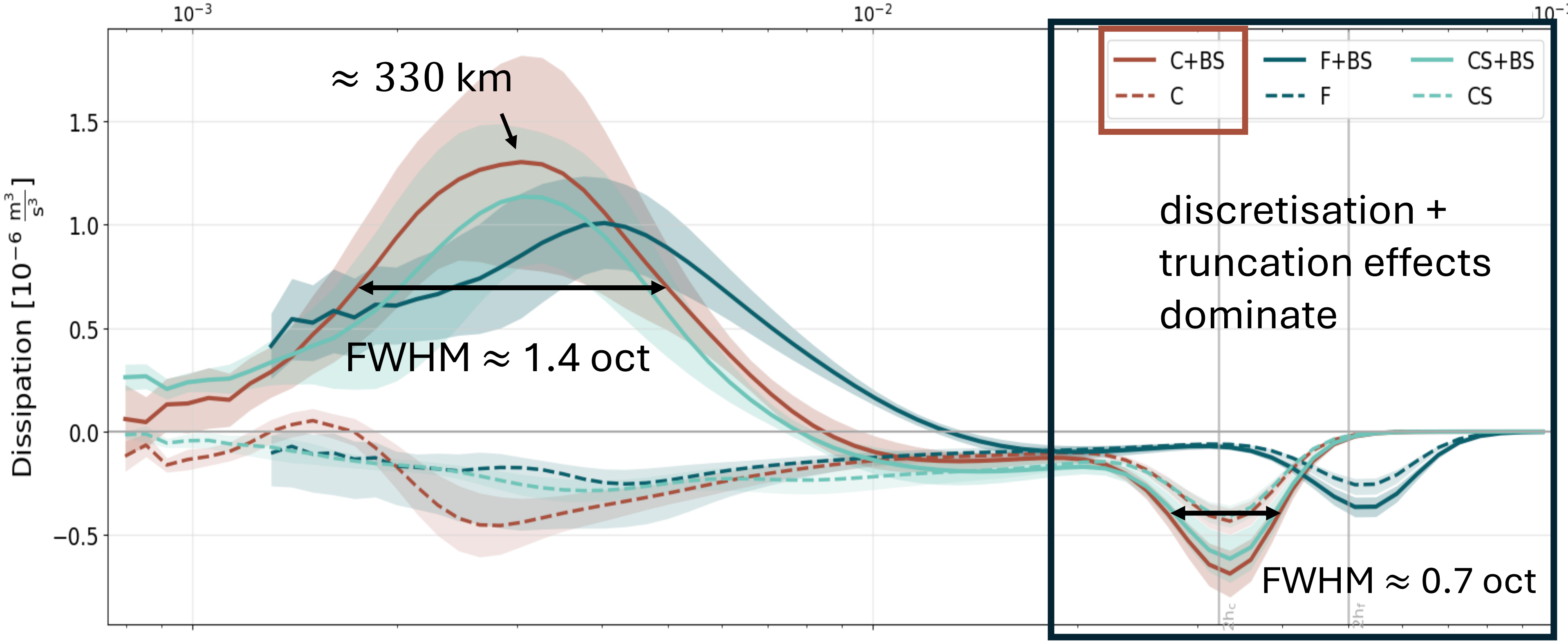

$$\widetilde{\mathrm{R}} = \frac{\mathrm{FWHM_{obs}}}{\mathrm{FWHM}(n)}$$

$$\mathrm{R} = \frac{\mathrm{FWHM_{peak}}}{\mathrm{FWHM}(n)}$$

damping

$$\frac{h_{\mathrm{obs}}}{h_{\mathrm{peak}}} \approx \frac{1}{\sqrt{1 + 1.40/R^2}}$$

broadening

$$\frac{\mathrm{FWHM_{obs}}}{\mathrm{FWHM_{peak}}} \approx \sqrt{1 + \frac{1.4}{R^2}}$$

<u>injection</u>

$\widetilde{\mathrm{R}} \approx 2.2 \Rightarrow \mathrm{R} \approx 1.85 \Rightarrow h_{\mathrm{fourier}} \approx 1.2\, h_{\mathrm{obs}}$

maximum $\Rightarrow$ at ≈ 300 km

<u>grid scale dissipation</u>

- damping & shift due to truncation effect
- amplification of peak due to discretisation effect

no reliable estimate of damping and shift possible, only area preserved

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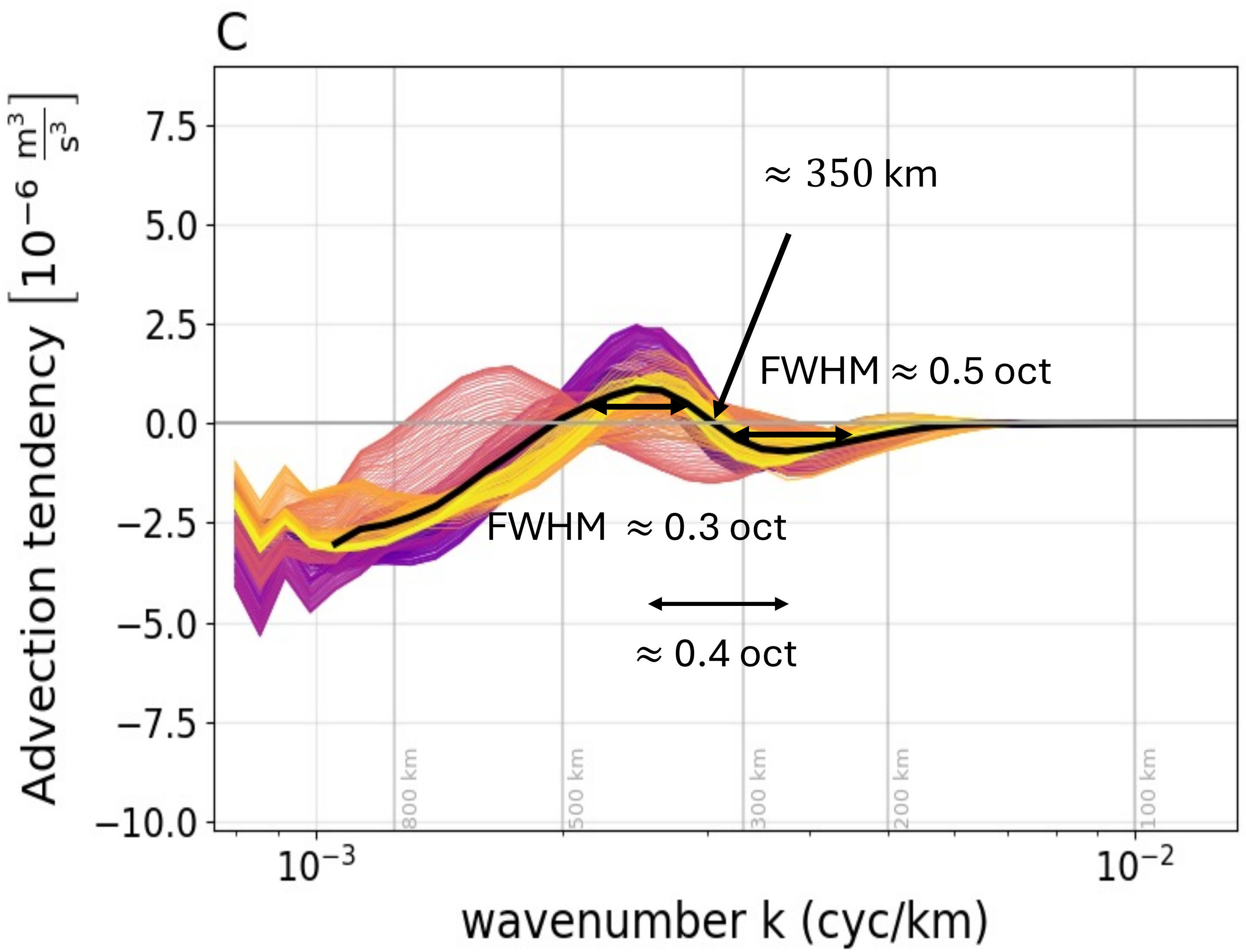
How do I interpret my spectra?

$$\widetilde{R} = \frac{\mathrm{FWHM_{obs}}}{\mathrm{FWHM}(n)}$$

n = 4, high-pass

$\mathrm{FWHM}(n = 4) = 0.54$ oct



positive lobe

$\widetilde{R} \approx 0.55$ ⇒ width set by formfactor

⇒ estimate of lower bound for fourier peak $h_{\mathrm{fourier}} \gtrsim 3h_{\mathrm{obs}}$

negative lobe

$\widetilde{R} \approx 0.9$ ⇒ $h_{\mathrm{fourier}} \approx 2h_{\mathrm{obs}}$

root

shifted by ≈ 0.09 oct ⇒ at ≈ 340 km

Take Away…

- implicit filter
  - provides a <u>consistent</u> framework for analyses on unstructured grids
  - allows for direct association of spectra with 2D structures
  - able to recover relevant features for energy cycle analysis

- coarse graining spectra ≠ fourier spectra
  - differences due to finite form factor
  - decrease with increasing filter order

→ new vcycle- preconditioner allows using orders $\gg 4$

Kacper Nowak
Implicit Filter
github.com/FESOM/implicit_filter

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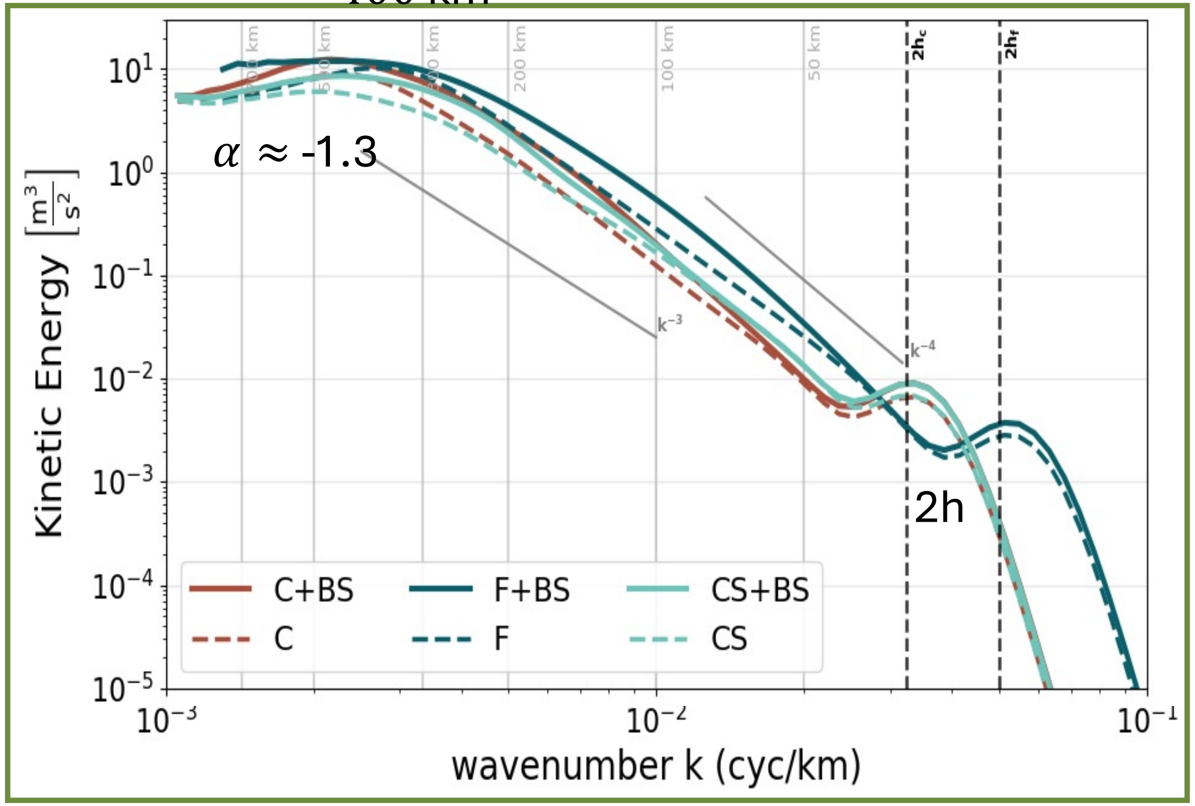
How do I interpret my spectra? through ‘Fourier Eyes’

≈ 400 km



largescale dip

$\gtrsim 1.5$ oct above $k_{\min}$ ⇒ no truncation effect

α ≈ -1.3 ⇒ density bias ≈ 20%

solver instability at largest scale

powerlaw slope

$\alpha \approx 3 - 4 \ll 17 = \alpha^{H}{}_{\mathrm{crit}}(n = 4)$

⇒ slopes are recovered

⇒ density bias negligible

gridscale bump

discretisation and form factor truncation dominate ⇒ location and amplitude not reliable, only total energy

n = 4, high-pass

$\alpha^{H}_{\mathrm{crit}} = 4n + 1 = 17$