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QUANTUM DYNAMICS and CONTROL of MESOSCOPIC SYSTEMS

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QUANTUM DYNAMICS and CONTROL of MESOSCOPIC SYSTEMS

Lecture 1 (Wednesday, Sept. 30):

Quantum dynamics and control:

Landau-Zener-Stückelberg-Majorana transitions
and interference

Lecture 2 (Thursday, Oct. 1):

Superconductivity: from foundations to qubits

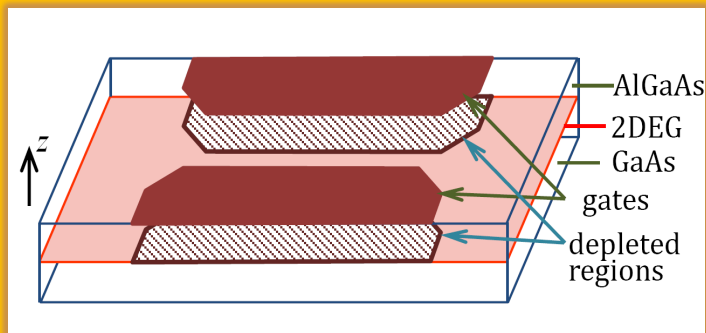
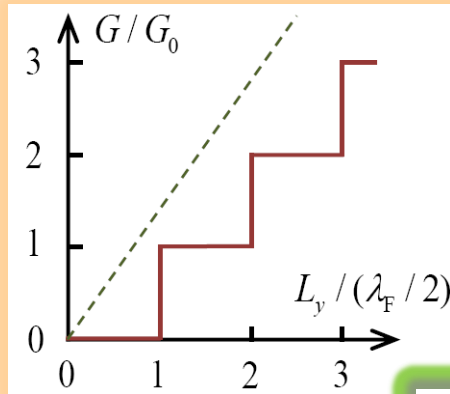
+Part II: Reflectometry – or how to probe such systems

Lecture 3 (Thursday, Oct. 1):

Normal quantum circuits: from 2DEG to qubits

Outline of Lecture 3

- ❖ Low-dimensional structures
- ❖ Conductance quantization
- ❖ Single-electron transistor / charge qubit

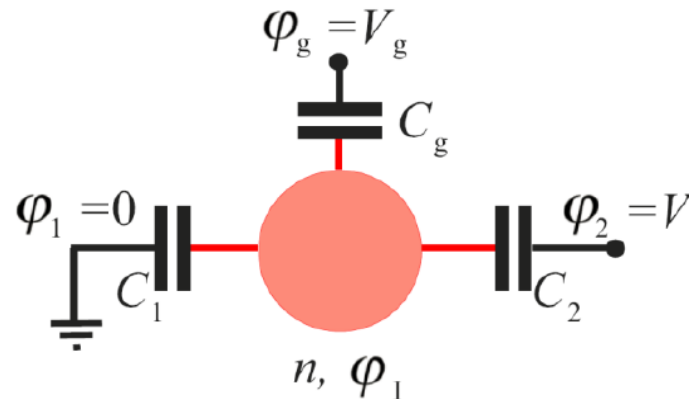


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FUNDAMENTALS OF MESOCOPIC PHYSICS

Study guide



Low-dimensional structures

Many quantum properties, inherent to superconducting circuits, can be realized in “normal-metal” coherent systems.

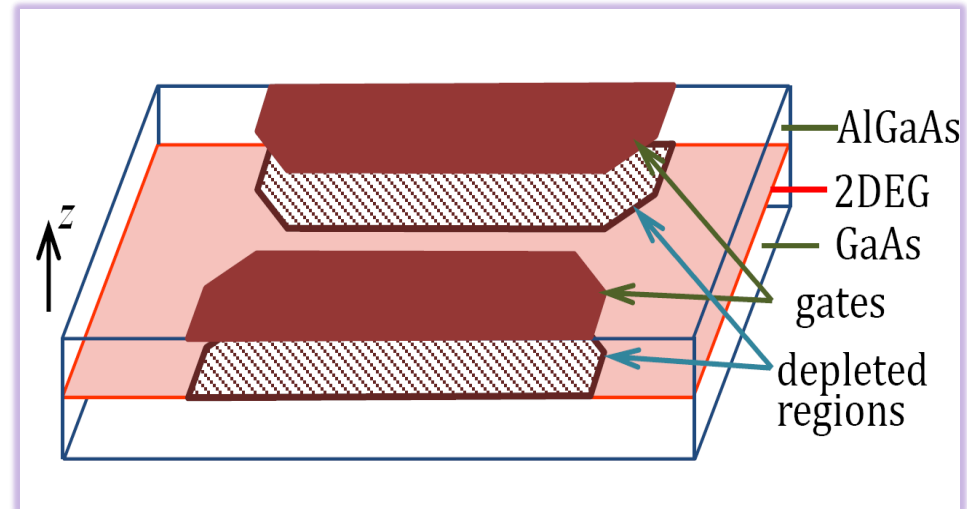
This relates to systems of the size much less than the coherence length L_ϕ . The **phase coherence** means that the particles can be described by a wave function.

Here we will describe these systems, making accent on such specific properties, as **conductance quantization** and **Coulomb blockade**.

Such systems are realized on the base of a *two-dimensional electron gas* (2DEG), in which electrons are restricted to move in one direction and freely move in the plane, perpendicular to that direction.

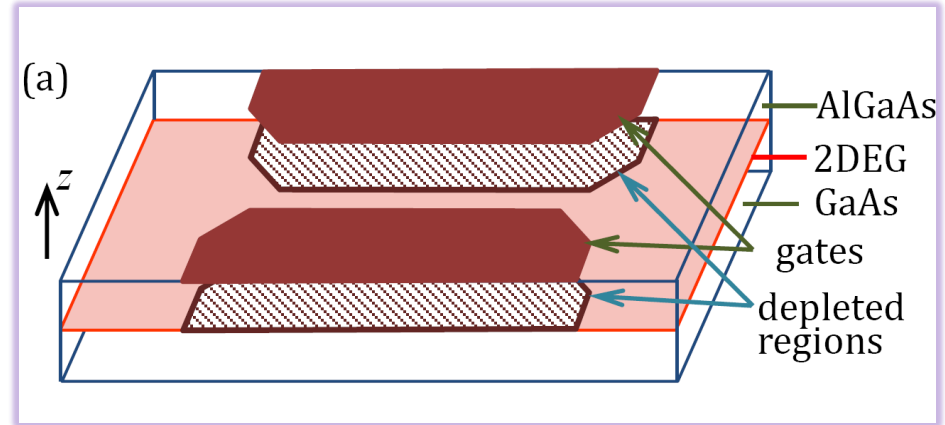
Such situation appears in a **heterojunction of two semiconductors**, for example, gallium arsenide and aluminum-doped gallium arsenide, $\text{GaAs}/\text{Al}_x\text{Ga}_{1-x}\text{As}$.

Additional electrodes (gates) allow to “press out” electrons from the area of their **geometrical shadows**, by means of the negative gate voltage, $V_g < 0$, which are not accessible to the electrons of the 2DEG. In this way the low-dimensional structures are formed (**quantum wires** and **dots**).



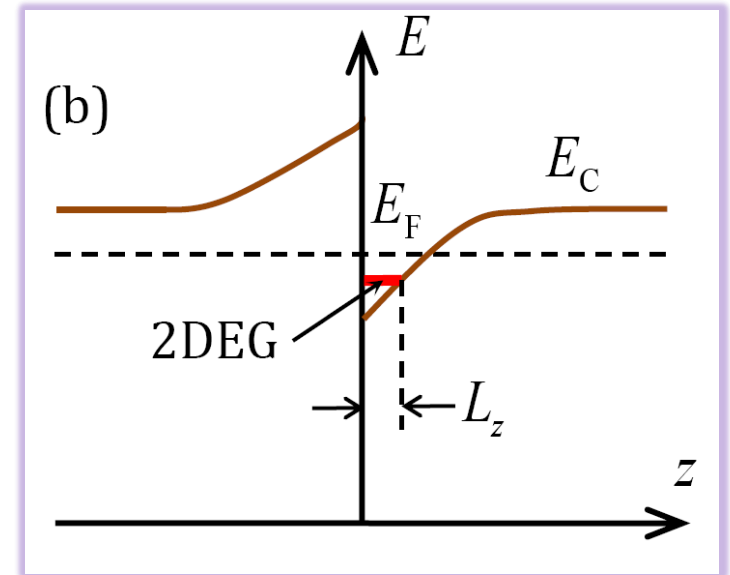
2DEG

In such heterojunction, due to the incommensurability of the lattices, the curvature of the conduction bands appears. As a result, in the vicinity of the junction, the potential well arises around the boundary.



It is important that

- the quantization in the transverse direction results in that in the potential well there is **only one energy level**;
- the electrons possess **high mobility in the longitudinal direction**;
- the collective of electrons can be described as **quasiparticles** with the electron charge e and mass m , different, from the free electron mass m_e .



Characteristic parameters for a 2DEG are the following: coherence length $L_\phi \sim 1 \mu\text{m}$, Fermi wavelength $\lambda_F \sim 50 \text{ nm}$ (for conventional metals $\lambda_F \sim 0.1 \text{ nm}$), effective mass of the quasiparticles $m \simeq 0.07m_e$ ($m_e \approx 10^{-30} \text{ kg}$), density $n \sim 10^{12} \text{ cm}^{-2}$.

Spectrum quantization in a 3D quantum box

To describe low-dimensional structures, consider the *spectrum quantization* in the simplest model, when electrons are locked in a three-dimensional well of the size $L_x \times L_y \times L_z$.

Then, the assumption of the *phase coherence* allows to describe the system in a **single-particle approximation** with the Schrödinger equation for the conduction electrons (quasiparticles) with the effective mass m :

$$-\frac{\hbar^2}{2m} \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} \psi = E\psi, \quad (x_1 = x, \quad x_2 = y, \quad x_3 = z).$$

We apply the *zero boundary conditions on the wave function* at $x_i = 0, L_i$.

We obtain the solution by separating the variables and normalizing the wave function by unity:

$$\psi(x, y, z) = \psi_1(x)\psi_2(y)\psi_3(z) \quad \psi_i(x_i) = \sqrt{\frac{2}{L_i}} \sin k_i x_i, \quad k_i = \frac{\pi n_i}{L_i}, \quad n_i = 1, 2, \dots;$$

$$E = \sum_{i=1}^3 E_i, \quad E_i = \frac{\hbar^2 k_i^2}{2m} = \frac{\hbar^2 \pi^2}{2m} \frac{n_i^2}{L_i^2}.$$

Homework:
Derive and check this solution
of the Schrodinger equation

Hence, the state of the electrons in the three-dimensional potential well is defined by the set of three numbers n_i . Respective basis functions and energies are the following:

$$\psi = \prod_{i=1}^3 \psi_i^{(n_i)}(x_i) = \psi_1^{(n_1)}(x)\psi_2^{(n_2)}(y)\psi_3^{(n_3)}(z) \quad \text{and} \quad E_{n_1, n_2, n_3} = \sum_i E_i^{(n_i)}.$$

Quantum wires and dots

The maximally filled number of the levels $n_i^{max} = N_i$ is defined by the chemical potential μ , which in the absence of fields equals to the **Fermi energy** ε_F .

Then the maximal wave number equals to $k_F = \frac{p_F}{\hbar} = \frac{\sqrt{2m\varepsilon_F}}{\hbar}$; for the maximal number of the levels we get

$$N_i = \left[\frac{L_i k_F}{\pi} \right] = \left[\frac{L_i}{\lambda_F/2} \right],$$

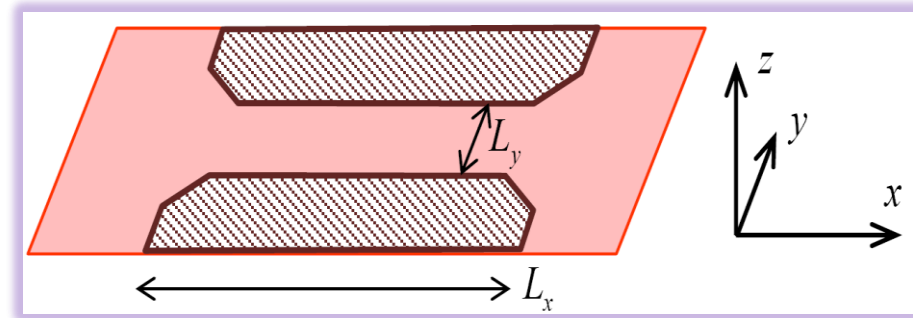
where the **Fermi wavelength** is $\lambda_F = 2\pi/k_F$. This result means that, if $L_i < \lambda_F/2$, then none of the energy levels is filled and the current cannot flow through such sample, at low voltages.

For the transverse quantization in heterojunctions, which we discussed above, the potential is such that its characteristic width $L_z \geq \lambda_F/2$.

So, the conductivity of the 2DEG is defined by **single subband** with $n_z = 1$; the energy of 2DEG is then counted from $E_z = \hbar^2 \pi^2 / 2m L_z^2$.

If other dimensions are also of the order of the Fermi wave-length, then such system becomes one- or zero-dimensional.

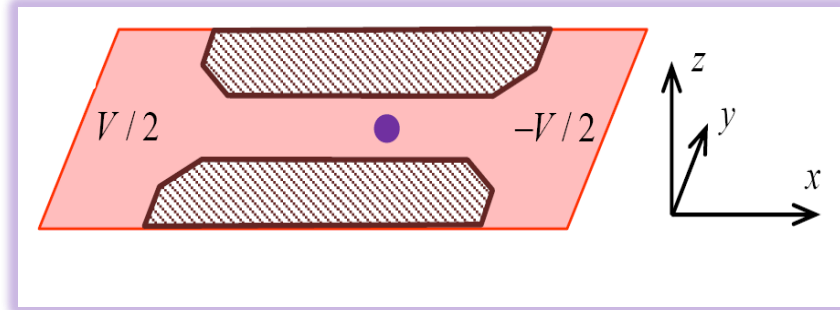
Such conductors are called **quantum wires** and **quantum dots**, respectively. Note that if, say, in the x direction the system has large size, $L_x \gg \lambda_F$, then $N_x \gg 1$ and the distance between the energy levels becomes small, $\Delta E_x \sim \varepsilon_F / N_x \rightarrow 0$, therefore the spectrum can be considered as **continuous**: $E_x = p^2 / 2m$.



Conductance quantization

Consider the conductivity of a quantum wire – a one-dimensional conductor – which is connected to bulk conductors.

These conductors play the role of the reservoirs, characterized by the temperature T and the chemical potential μ .



We assume that to the banks of the contact the voltage is applied $V > 0$, and respectively the chemical potentials are shifted by the value of the electric potential, $\pm eV/2$.

The wire is assumed to be *ballistic*, which means that the dephasing length is larger than its length, $L_\varphi \gg L$. We further assume that there is a scatterer (an impurity) within the wire.

The solution of the Schrödinger equation, $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2}$, is

$$\Psi_p(x, t) = \frac{1}{\sqrt{L}} e^{-i\frac{\varepsilon}{\hbar}t} e^{i\frac{p}{\hbar}x}, \quad \varepsilon = \frac{p^2}{2m}.$$

Where m , p , and ε are the mass, momentum, and kinetic energy of an electron, and $L \equiv L_x$. The wave function of an electron is normalized by the unity, $\int_0^L dx |\Psi|^2 = 1$.

The current, created by the electron motion with the momentum p , can be calculated as the following

$$I_p = e \frac{i\hbar}{2m} \left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x} \right) = \frac{e\hbar}{m} \text{Im} \left(\Psi^* \frac{\partial \Psi}{\partial x} \right) = \frac{ep}{mL} = \frac{ev}{L}$$

Conductance quantization

To obtain the total current, we have to sum over all the states, accounting their occupations by multiplying by the distribution function $f(p)$.

Assuming that the conductor is long ($L \gg \lambda_F$), we can change the summation by the integration:

indeed, for $p = p_n = \hbar \frac{\pi n}{L}$, we have $\Delta p = p_{n+1} - p_n = \pi \hbar / L \rightarrow 0$.

Therefore, we can use that $1 = \frac{L}{\hbar \pi} \Delta p$ and make the replacement $\Delta p \rightarrow dp$.

We obtain : $\sum_n \dots = \frac{L}{\pi \hbar} \sum_n \Delta p \dots \rightarrow \frac{L}{\pi \hbar} \int_0^{p_F} dp \dots \approx \frac{L}{2\pi \hbar} \int_{-\infty}^{\infty} dp \dots$

Then for the total current, accounting the two-fold degeneracy in spin, we have

$$I = \sum I_p f(p) \approx 2 \frac{L}{h} \int_{-\infty}^{\infty} dp I_p f(p) = \frac{2e}{mh} \int_{-\infty}^{\infty} dp p f(p) = \frac{2e}{mh} \int_0^{\infty} dp p (f(p) - f(-p)).$$

Let us turn the integration to the one over energy $\varepsilon = p^2/2m$; then $d\varepsilon = p dp/m$ and we obtain $I = \frac{2e}{h} \int_0^{\infty} d\varepsilon (f_{>}(\varepsilon) - f_{<}(\varepsilon))$,

where $f_{>}(\varepsilon)$ and $f_{<}(\varepsilon)$ correspond to the electrons with $v > 0$ and $v < 0$, respectively. Let us discuss these distribution functions.

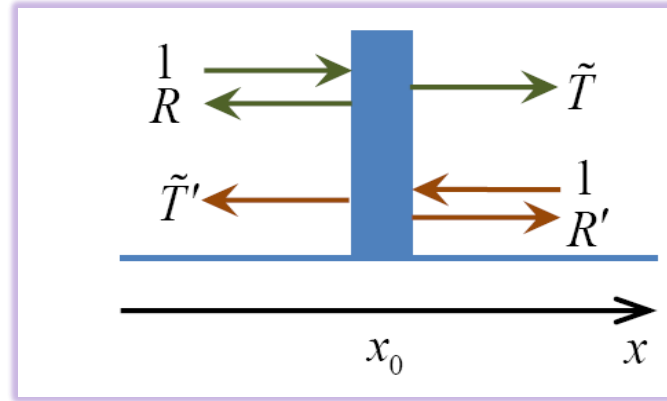
Conductance quantization

Electrons incident from the left and right banks are described by the Fermi distribution functions

$$f_{1,2} = f_F(\varepsilon - \mu_{1,2}) = \left[1 + \exp\left(\frac{\varepsilon - \mu_{1,2}}{k_B T}\right) \right]^{-1}$$

with respective chemical potentials $\mu_{1,2} = \mu \pm eV/2$.

The electrons from the left bank with the probability $\tilde{T}$ pass by the scatterer and with the probability R are reflected from it; the electrons from the right bank with the probability $\tilde{T}'$ are transmitted and with the probability R' are reflected



The probabilities of the states occupations are characterized by the distribution functions; and to the right from the scatterer, we have:

$$f_{<}(\varepsilon) = f_2 \text{ and } f_{>}(\varepsilon) = \tilde{T}f_1 + R'f_2.$$

Then, accounting $1 - R' = \tilde{T}'$, we obtain

$$I = \frac{2e}{h} \int_0^\infty d\varepsilon (f_{>}(\varepsilon) - f_{<}(\varepsilon)) = \frac{2e}{h} \int_0^\infty d\varepsilon (\tilde{T}f_1 - \tilde{T}'f_2).$$

Homework:
Recall about f_F and
check derivations here.

Next, we assume that the potential barrier is **symmetric**; then $\tilde{T}' = \tilde{T}$.

Consider in what follows the **response linear in the voltage**, which means that we assume $|eV| \ll \mu$. Then we can expand into series the distribution functions:

$$f_F\left(\varepsilon - \mu \mp \frac{eV}{2}\right) \approx f_F(\varepsilon - \mu) \mp \frac{eV}{2} f'_F(\varepsilon - \mu), \quad f'_F = -\frac{1}{4k_B T} \cosh^{-2} \frac{\varepsilon - \mu}{k_B T} \xrightarrow{T \rightarrow 0} -\delta(\varepsilon - \mu)$$

Conductance quantization

Then, for the *conductance* of the quantum wire, we obtain

$$G = \frac{I}{V} = G_0 \int_0^\infty d\varepsilon \tilde{T}(\varepsilon) \left(-\frac{\partial f_F}{\partial \varepsilon} \right),$$

which is the so-called *Landauer formula*. We introduced the *conductance quantum* $G_0 = \frac{2e^2}{h}$.

At zero temperature (*i.e.* at $T \ll \mu$), for the conductance we get:

$$G|_{T=0} = G_0 \tilde{T}(\mu).$$

Even in the absence of a scatterer, when the transmission probability $\tilde{T} = 1$, the conductance of a quantum wire is not infinite – such a one-dimensional conductor creates the resistance $R_0 = 1/G_0 \approx 13 \text{ kOhm}$. This means that $G_0 \approx 7.8 \cdot 10^{-5} \text{ siemens}$ (1 siemens = 1 Ohm⁻¹).

Let us discuss the situation, when the quantum wire is **not a strictly one-dimensional conductor**, but rather a conductor of finite width L_y – a two-dimensional conducting channel.

Then the number of the levels, which are filled, is $N_y = \lceil 2L_y/\lambda_F \rceil$, and these levels correspond to the transverse quantization.

We have to sum over them, and then the Landauer formula takes the form:

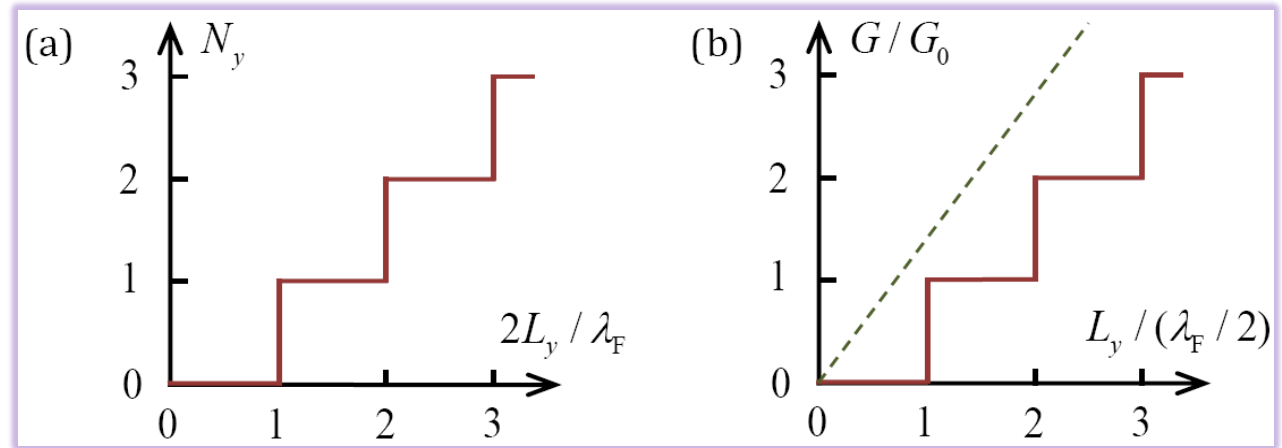
$$G|_{T=0} = G_0 \sum_{n=1}^{N_y} \tilde{T}_n(\mu).$$

Conductance quantization

The number N_y depends on the channel width L_y , defined by the gate voltage V_g .

So, the Landauer formula describes the stepwise dependence of the conductance on the voltage.

$$G\Big|_{T=0} = G_0 \sum_{n=1}^{N_y} \tilde{T}_n(\mu)$$



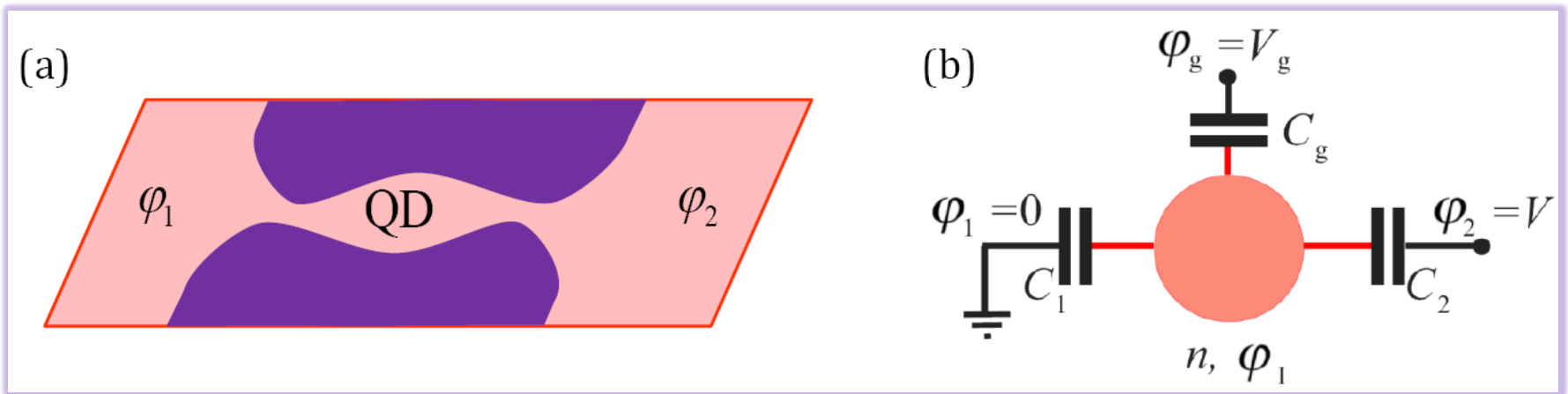
If $\tilde{T}_n = 1$, then the height of the steps on the width dependence of the conductance equals to G_0 .

If $\tilde{T}_n < 1$, then the conductance decreases correspondingly, and the resistance of the conductor increases.

Note that for a macroscopic conductor, the conductivity is $G = \sigma \frac{S}{L}$, that is **linearly dependent** on the conductor width, while for a mesoscopic conductor this dependence is **stepwise**.

Quantum dot = single-electron transistor (SET)

Consider now the quantum transport through a quantum dot. A *quantum dot* (or, an island, a grain) is formed by several capacitances C_j , to which the electric potentials φ_j are applied.



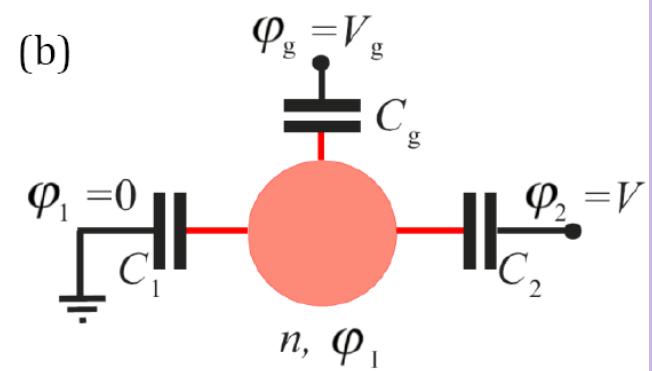
The gate voltage V_g is used to control the quantum-dot state and, correspondingly, the transport current, created by the bias voltage V . Note that the superconducting analogue of a quantum dot is the Cooper-pair box.

Energy levels in a quantum dot

Let us calculate the energy, related to adding n electrons to initially electro-neutral island.

The added charge equals to the sum of the charges on the inner plates of the capacitors, to which the potential difference $\Delta\varphi_j = \varphi_I - \varphi_j = \varphi_I - V_j$ is applied (φ_I stands for the island potential):

$$ne = \sum C_j \Delta\varphi_j = \varphi_I \sum C_j - \sum C_j V_j \equiv C_\Sigma \varphi_I + en_g,$$



$$C_\Sigma = \sum C_j, \quad n_g = -\frac{1}{e} \sum C_j V_j.$$

Expression for the island potential follows: $\varphi_I = e(n - n_g)/C_\Sigma$.

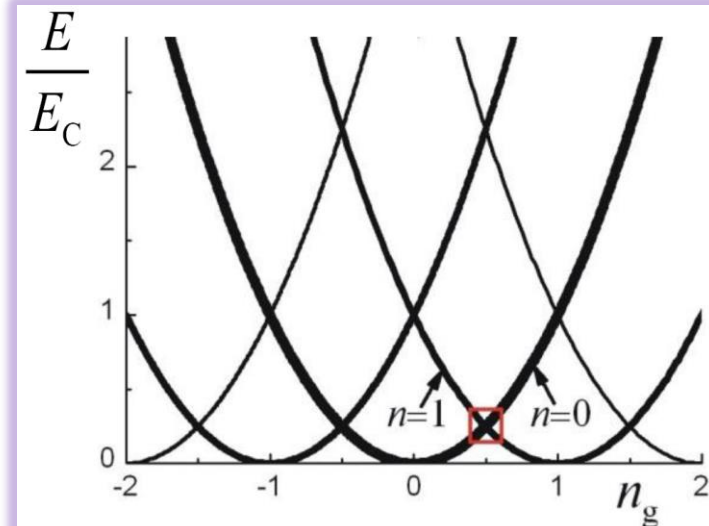
Then for the electrostatic energy of the island we get

$$E = \sum \frac{C_j \varphi_I^2}{2} = \frac{e^2 (n - n_g)^2}{2C_\Sigma^2} \sum C_j = E_C (n - n_g)^2,$$

where $E_C = e^2/2C_\Sigma$ is the characteristic charging energy of the island.

If we minimize the energy, then we obtain $n_0 = n_g$.

Since the number of electrons is integer, this corresponds to the number in the interval $n_0 - \frac{1}{2} \leq n \leq n_0 + \frac{1}{2}$.

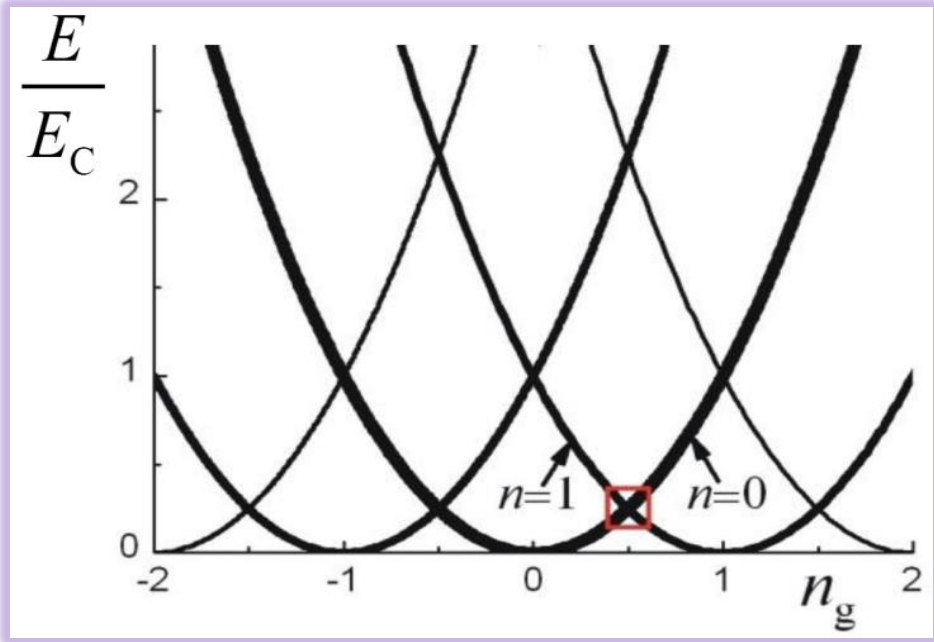


Coulomb blockade. Charge qubit

Consider the system energy levels at $V = 0$, versus $n_g = C_g V_g / |e|$.

At the integer values of n_g , the energy levels, are maximally separated from one another.

In this case, in order to **change the island charge by the charge of one electron**, we need to **change the system energy by $E = E_C$** .



So, at integer n_g , the application of a small bias voltage cannot change the island charge, and the current through it would not pass: the **Coulomb blockade**.

So, at $|V| < E_C/|e|$, the current through the island equals to zero; at $|V| > E_C/|e|$ the **current, linear in voltage, appears**, which is related to the resonant tunneling of single electrons.

If n_g is a half-integer, then **changing the island charge does not require energy**: the states with $n = 0$ and $n = 1$ have equal energies – and the island is **deblockaded**.

At vanishingly small bias voltage, the **current will flow through the quantum dot**. Such a device is called a **single-electron transistor (SET)**.

The two-state situation ($n = 0$ and 1) provides the platform for a **charge qubit**.

Conclusions

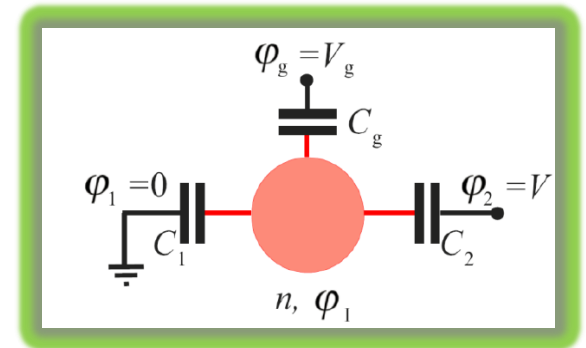
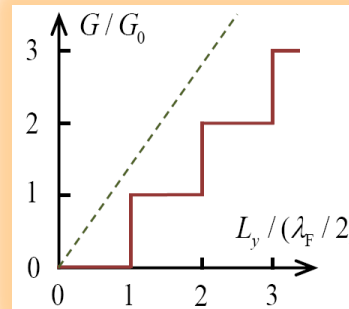
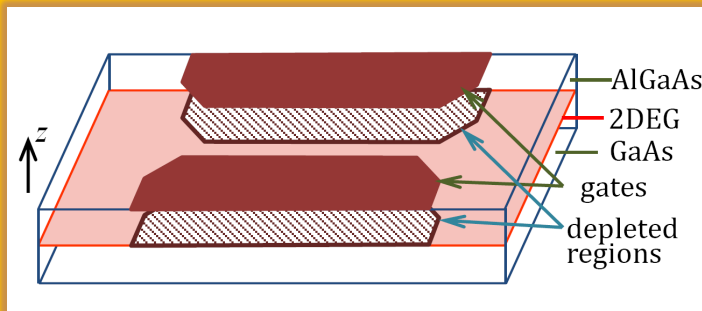
Mesoscopic-size conductors differ from macroscopic conductors. For them important are the **charge and spectrum quantization**.

The **conductance** of a mesoscopic conductor has a **step-wise dependence on its width**, which is distinct from the linear dependence for a macroscopic conductor.

The height of the steps is defined by the conductance quantum, $G_0 = 2e^2/h$.

If the conductor has **one conducting channel** (*i.e.* its width is of the order of the Fermi wavelength), then, even in the absence of impurities, **its resistance has finite value**, $R_0 = 1/G_0 \approx 13 \text{ k}\Omega$.

The conductance of a mesoscopic-size island is described by the **Coulomb-blockade regime**: the electrons tunnel one by one and the gate voltage can control the current through such a single-electron transistor. This provides the basis for **charge qubits**.



Acknowledgements: thanks to Omar Di Stefano and Oleksandr Kitsenko for the slides; financial support of the National Research Foundation of Ukraine (grant 2025.07/0044)