

Tutorial - Fractional charge tunneling in the FQHE

Christophe MORA

We consider a quantum point contact (QPC) connecting two edge channels of a fractional quantum Hall (FQH) system at filling factor $\nu = 1/3$. In the weak-tunneling regime, quasiparticles of fractional charge $e^* = \nu e$ can tunnel between the two edge channels. The tunneling process is described by the Hamiltonian

$$H_T = \mathcal{T} + \mathcal{T}^\dagger, \quad (1)$$

which gives rise to the tunneling current operator

$$\hat{I}_T = \frac{ie^*}{\hbar} (\mathcal{T}^\dagger - \mathcal{T}). \quad (2)$$

Here, the tunneling operator $\mathcal{T} = \hbar\gamma\psi_2^\dagger(0)\psi_1(0)$ describes the transfer of an anyon of charge e^* across the QPC, located at $x = 0$, from the upper to the lower channel. The operators $\psi_1^\dagger$ and $\psi_2^\dagger$ create a quasiparticle on the upper and lower edge channels, respectively, which are labeled by 1 and 2.

We use the subscript 0 to denote averages evaluated in the absence of tunneling. The anyon correlators at $x = 0$ and zero temperature have the form

$$\langle \psi_2^\dagger(t)\psi_2(t') \rangle_0 = \langle \psi_1^\dagger(t)\psi_1(t') \rangle_0 = \left(\frac{1}{\tau_c + i(t-t')} \right)^\nu \quad (3)$$

where τ_c is a short-time cutoff ($\tau_c \rightarrow 0$). In addition, it is straightforward to show, using time-translation invariance, that $\langle \psi_1(t)\psi_1^\dagger(t') \rangle_0 = \langle \psi_1^\dagger(t)\psi_1(t') \rangle_0$.

- 1) Using the expression of the unitary operator in the interaction representation, derive the perturbative Kubo formula for the average tunnel current

$$\langle \hat{I}_T(t) \rangle = \langle \hat{I}_T \rangle_0 + \frac{i}{\hbar} \int_{-\infty}^t dt' \langle [H_T(t'), \hat{I}_T(t)] \rangle_0 \quad (4)$$

- 2) Using again time-translation invariance, show that this expression simplifies as

$$\langle \hat{I}_T \rangle = \frac{e^*}{\hbar^2} \int_{-\infty}^{+\infty} dt' \langle [\mathcal{T}^\dagger(t'), \mathcal{T}(t)] \rangle_0 = \frac{e^*}{\hbar^2} \int_{-\infty}^{+\infty} dt \langle [\mathcal{T}^\dagger(t), \mathcal{T}(0)] \rangle_0 \quad (5)$$

- 3) In the presence of a finite gate voltage V applied on the upper edge, the field $\psi_1(t)$ is simply dressed by the phase factor $e^{-ie^*Vt/\hbar}$. Compute the following correlators under this polarisation

$$\langle \mathcal{T}^\dagger(t)\mathcal{T}(0) \rangle_0, \quad \langle \mathcal{T}(0)\mathcal{T}^\dagger(t) \rangle_0 \quad (6)$$

- 4) Compute the average tunnel current to obtain

$$\langle \hat{I}_T \rangle = \frac{2\pi e^* \gamma^2}{\Gamma(2\nu)} \left(\frac{e^* V}{\hbar} \right)^{2\nu-1} \quad (7)$$

You can use the integral

$$\int_{-\infty}^{+\infty} dt \frac{e^{i\alpha t}}{(0^+ + it)^{2\nu}} = \frac{2\pi}{\Gamma(2\nu)} \alpha^{2\nu-1} \theta(\alpha) \quad (8)$$

where $\theta(x)$ denotes the heaviside function.

- 5) Compute the noise $S = \int dt \langle \delta \hat{I}_T(t) \delta \hat{I}_T(0) \rangle$ and show that it reveals the fractional charge $e^* = \nu e$ carried by the tunneling anyons,

$$\frac{S}{I_T} = e^* \quad (9)$$

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1) Perturbative Kubo formula

In the interaction representation, the evolution operator starting from $t = -\infty$ (where tunneling is adiabatically switched off) reads

$$U(t, -\infty) = \mathcal{T} \exp \left(-\frac{i}{\hbar} \int_{-\infty}^t dt' H_T(t') \right) \approx 1 - \frac{i}{\hbar} \int_{-\infty}^t dt' H_T(t')$$

to first order in γ (i.e. to first order in H_T). The physical (full Heisenberg) average of an operator $\hat{I}_T$ taken in the interaction picture is obtained as

$$\langle \hat{I}_T(t) \rangle = \left\langle U^\dagger(t, -\infty) \hat{I}_T(t) U(t, -\infty) \right\rangle_0.$$

Expanding to first order,

$$\langle \hat{I}_T(t) \rangle \approx \langle \hat{I}_T(t) \rangle_0 + \frac{i}{\hbar} \int_{-\infty}^t dt' \langle H_T(t') \hat{I}_T(t) \rangle_0 - \frac{i}{\hbar} \int_{-\infty}^t dt' \langle \hat{I}_T(t) H_T(t') \rangle_0,$$

so that

$$\langle \hat{I}_T(t) \rangle = \langle \hat{I}_T \rangle_0 + \frac{i}{\hbar} \int_{-\infty}^t dt' \langle [H_T(t'), \hat{I}_T(t)] \rangle_0, \quad (4)$$

which is exactly the announced Kubo formula (4).

2) Simplification using time-translation invariance

Expanding the commutator with $H_T = T + T^\dagger$ and $\hat{I}_T = \frac{ie^*}{\hbar} (T^\dagger - T)$ produces four types of terms. The “anomalous” correlators $\langle T(t')T(t) \rangle_0$ and $\langle T^\dagger(t')T^\dagger(t) \rangle_0$ vanish identically: since $T \sim \psi_2^\dagger \psi_1$ changes the quasiparticle number on each edge separately, these terms would connect states with different particle number on channel 1 or channel 2.

Only the cross terms $T(t')T^\dagger(t)$ and $T^\dagger(t')T(t)$ survive. Moreover $\langle \hat{I}_T \rangle_0 = 0$ since $\langle T \rangle_0 = \langle T^\dagger \rangle_0 = 0$. Using time-translation invariance, the correlators depend only on $t' - t$, which allows extending the integration domain from $\int_{-\infty}^t dt'$ to $\int_{-\infty}^{+\infty} dt'$ (the change of variable $s = t' - t \rightarrow -s$ maps the missing half-line onto its complement and reconstructs the full integral once the four terms are regrouped in pairs). One obtains

$$\langle \hat{I}_T \rangle = \frac{e^*}{\hbar^2} \int_{-\infty}^{+\infty} dt' \langle [T^\dagger(t'), T(t)] \rangle_0 = \frac{e^*}{\hbar^2} \int_{-\infty}^{+\infty} dt \langle [T^\dagger(t), T(0)] \rangle_0, \quad (5)$$

which is exactly (5), the last equality being a simple relabeling of the dummy time variable once the time origin is fixed at 0 using time-translation invariance.

3) Correlators in the presence of a gate voltage V

With $\psi_1(t) \rightarrow \psi_1(t) e^{-ie^* V t / \hbar}$, we have $T(t) = \hbar \gamma \psi_2^\dagger(t) \psi_1(t) e^{-ie^* V t / \hbar}$. Since channels 1 and 2 are independent, the correlators factorize:

$$\langle T^\dagger(t) T(0) \rangle_0 = \hbar^2 \gamma^2 e^{ie^* V t / \hbar} \langle \psi_1^\dagger(t) \psi_1(0) \rangle_0 \langle \psi_2(t) \psi_2^\dagger(0) \rangle_0.$$

Using (3) together with the identity $\langle \psi_i(t) \psi_i^\dagger(t') \rangle_0 = \langle \psi_i^\dagger(t) \psi_i(t') \rangle_0$ (valid for $i = 1, 2$ by symmetry of the two edges), each factor equals $(1/(\tau_c + it))^\nu$, so that

$$\langle T^\dagger(t) T(0) \rangle_0 = \hbar^2 \gamma^2 \frac{e^{ie^* V t / \hbar}}{(\tau_c + it)^{2\nu}} \quad (6a)$$

A similar calculation (each factor now equals $(1/(\tau_c - it))^\nu$) gives

$$\langle T(0) T^\dagger(t) \rangle_0 = \hbar^2 \gamma^2 \frac{e^{ie^* V t / \hbar}}{(\tau_c - it)^{2\nu}} \quad (6b)$$

4) Average tunneling current

Inserting (6a)–(6b) into (5),

$$\langle \hat{I}_T \rangle = e^* \gamma^2 \int_{-\infty}^{+\infty} dt e^{ie^* V t / \hbar} \left[\frac{1}{(\tau_c + it)^{2\nu}} - \frac{1}{(\tau_c - it)^{2\nu}} \right].$$

Setting $\alpha \equiv e^* V / \hbar$, the first term is given directly by (8): $\frac{2\pi}{\Gamma(2\nu)} \alpha^{2\nu-1} \theta(\alpha)$. For the second term, the substitution $t \rightarrow -t$ maps it onto the same integral with $\alpha \rightarrow -\alpha$, i.e. $\frac{2\pi}{\Gamma(2\nu)} (-\alpha)^{2\nu-1} \theta(-\alpha)$, which vanishes for $V > 0$ since $\theta(-\alpha) = 0$. For $V > 0$ we are left with

$$\langle \hat{I}_T \rangle = \frac{2\pi e^* \gamma^2}{\Gamma(2\nu)} \left(\frac{e^* V}{\hbar} \right)^{2\nu-1}, \quad (7)$$

which reproduces exactly the announced result (7): a nonlinear I – V law with exponent $2\nu - 1$ (for $\nu = 1/3$, exponent $-1/3$), characteristic of tunneling into a chiral Luttinger liquid.

5) Noise and the fractional charge

At the same leading order in γ^2 , one computes $\langle \hat{I}_T(t) \hat{I}_T(0) \rangle_0$ using the unperturbed correlators. The anomalous terms vanish as in question 2, leaving

$$\langle \hat{I}_T(t) \hat{I}_T(0) \rangle_0 = \frac{e^{*2}}{\hbar^2} \left[\langle T^\dagger(t) T(0) \rangle_0 + \langle T(t) T^\dagger(0) \rangle_0 \right].$$

The second term is computed as in question 3 and equals $\hbar^2 \gamma^2 e^{-ie^* V t / \hbar} / (\tau_c + it)^{2\nu}$, so

$$\langle \hat{I}_T(t) \hat{I}_T(0) \rangle_0 = 2e^{*2} \gamma^2 \frac{\cos(e^* V t / \hbar)}{(\tau_c + it)^{2\nu}}.$$

The constant piece $\langle \hat{I}_T \rangle^2$, which must be subtracted to isolate the fluctuating part $\delta \hat{I}_T$, is of order γ^4 and thus negligible at this order. Writing $\cos(\alpha t) = \frac{1}{2} (e^{i\alpha t} + e^{-i\alpha t})$ and using (8), only the $e^{i\alpha t}$ term contributes for $V > 0$, giving

$$S = e^{*2} \gamma^2 \frac{2\pi}{\Gamma(2\nu)} \left(\frac{e^* V}{\hbar} \right)^{2\nu-1}.$$

Comparing with the result of question 4,

$$\frac{S}{\langle \hat{I}_T \rangle} = \frac{e^{*2} \gamma^2 \frac{2\pi}{\Gamma(2\nu)} (e^* V / \hbar)^{2\nu-1}}{e^* \gamma^2 \frac{2\pi}{\Gamma(2\nu)} (e^* V / \hbar)^{2\nu-1}} = e^*, \quad (9)$$

which is exactly (9). This is the central physical result: in the weak-tunneling regime, the ratio of shot noise to current directly measures the elementary charge of the tunneling quasiparticles – here $e^* = e/3$ – the historic experimental signature of fractional charge (Saminadayar–Glattli, de Picciotto *et al.*, 1997).