Research Overview

# The Application of the Sine-Gordon Equation in Spin Systems

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# Outline

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   - Breathers
   - Applications

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# What is the Sine-Gordon Model?

## Definition

The Sine—Gordon (SG) model is a foundational (1+1)—dimensional nonlinear relativistic wave equation. It is a key example of an integrable system, meaning it can be solved exactly.

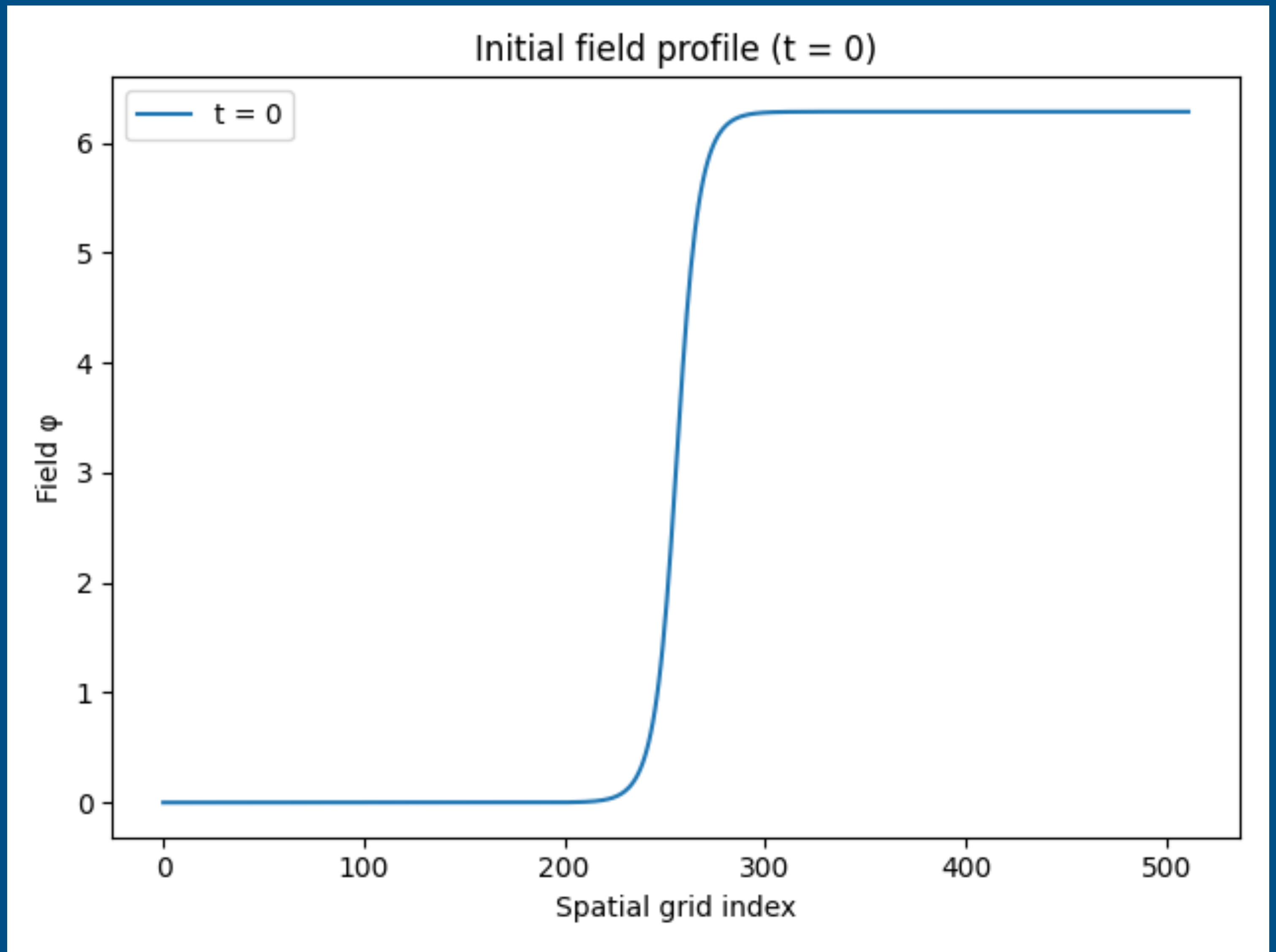
It describes a wide range of physical phenomena, from particle physics (as a quantum field theory) to condensed matter physics (like crystal dislocations and spin chains).

## The Equation

$$u_{tt} - u_{xx} + \sin(u) = 0$$

The field variable is $\boldsymbol{u(x, t)}$. The $\boldsymbol{\sin(u)}$ term provides the crucial non-linear, periodic restoring force required for solitons.

# Classical Solution: The Soliton (Kink)




$$u(x,t) = 4\arctan(e^{\gamma(x-x_0-vt)})$$

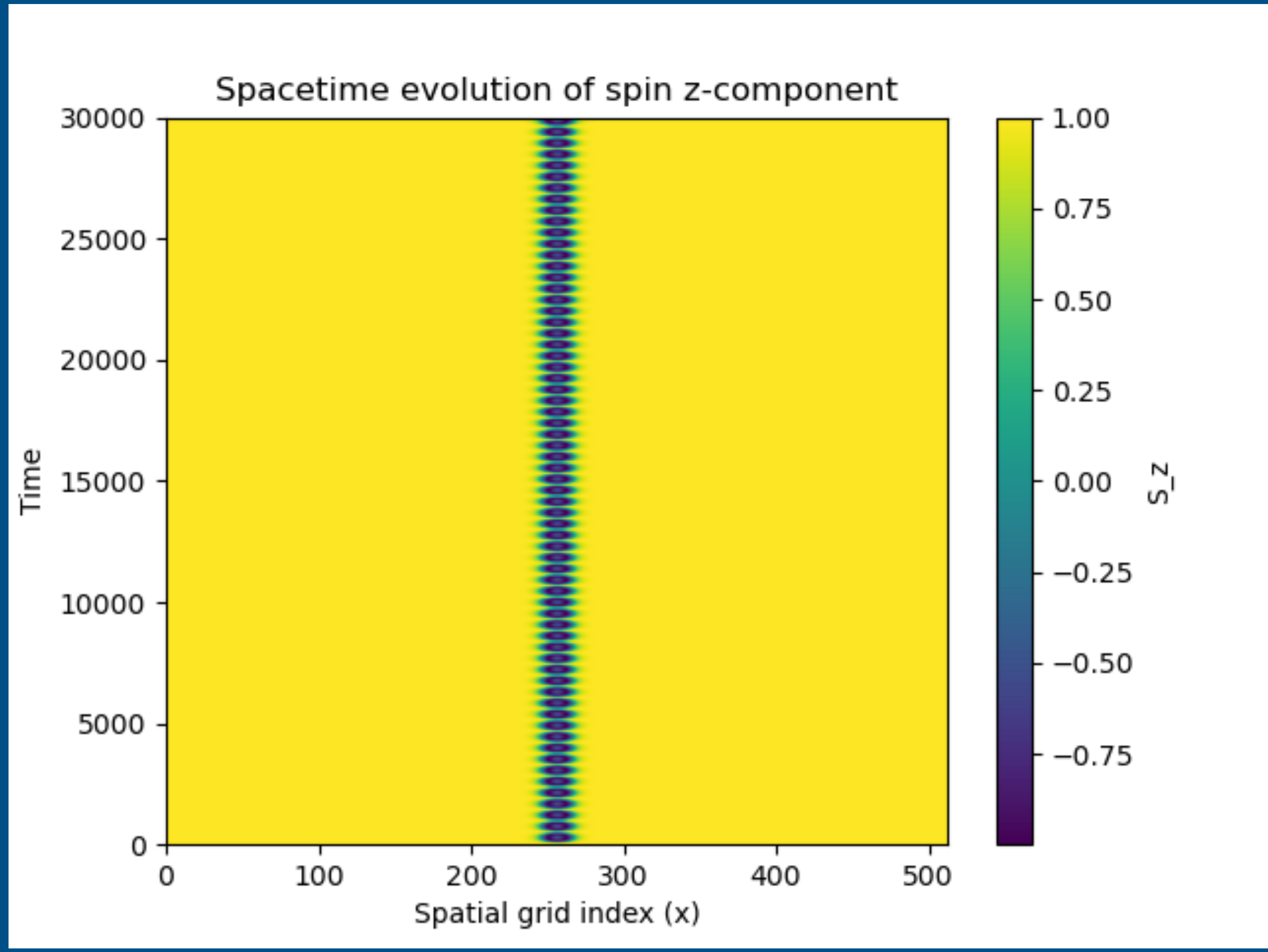
## What is a Soliton?

A soliton (or "kink") is a stable, particle—like, traveling wave solution. It is a topological excitation.

It represents a "twist" in the field that connects one vacuum state to an adjacent one. For example, the field $\phi$ transitions from $0$ to $2\pi$ as it moves through space.

An anti-soliton (or "anti—kink") does the opposite, twisting from $2\pi$ down to $0$. These solutions are incredibly stable and can pass through each other without being destroyed.

# Classical Solution: The Breather




$$u(x,t) = 4\arctan\left(\frac{\sqrt{1-\omega^2}}{\omega}\sin(\omega(t-t_0))\mathrm{sech}(\sqrt{1-\omega^2}(x-x_0))\right)$$

What is a Breather?

A breather is a non-topological, localized solution. It is a bound state of a soliton and an anti—soliton.

Unlike the kink, it doesn't travel indefinitely. Instead, it remains localized in space and oscillates in time—it “breathes." It is a dynamic, pulsating particle—like excitation.

# Applications of the Sine-Gordon Model

### Josephson Junctions

Describes the propagation of magnetic flux quanta (known as **fluxons**) in long superconducting junctions. The fluxon is a perfect physical realization of the sG soliton.

### 1D Spin Chains

Acts as the low-energy effective field theory for 1D antiferromagnets (e.g., XXZ spin chain). Solitons (**spinons**) and breathers (**magnons**) map to the chain's quantum excitations.

### Quantum Field Theory

Provides a foundational "toy model" for particle physics. It has a famous duality with the **Massive Thirring Model**, connecting classical solitons to quantum fermions.

# Excitations in a Spin Chain

n certain quantum spin-½ chains, especially those with staggered magnetic fields or Dzyaloshinskii–Moriya interactions—the low-energy physics is effectively described by the Sine-Gordon field theory. In this mapping, the physical excitations of the spin chain correspond to the soliton, antisoliton, and breather modes of the Sine-Gordon model. Experimental techniques such as neutron scattering have observed these excitations directly.

# Excitations in a Spin Chain

## Soliton = Domain Wall (Spinon)

In the spin chain, a soliton (kink) corresponds to a domain wall. This is the boundary where the magnetic ordering of the spins flips (e.g., from mostly "up" to mostly "down").

These domain walls are the fundamental spin—1/2 excitations of the chain, often called spinons.

## Breather = Bound State (Magnon)

A breather corresponds to a bound pair of these domain walls (a spinon—antispinon pair).

This collective excitation propagates through the chain as a single, neutral (spin—0) particle. This is the quantum equivalent of a magnon bound state (a quantized spin wave).

# Excitations in a Spin Chain

The goal of our project is to explore the possibility of generating breather excitations in such spin systems in a way that allows them to be used for quantum computation and the storage of quantum information. The reason we focus on breathers is that these wave packets are non-dispersive and non-dissipative due to the system's nonlinearity, which makes them potentially robust carriers of information.

As an initial step, we attempted to semi-classically simulate configurations with different winding numbers in a one-dimensional system.

# General Research Workflow – Illustrated Through the Sine–Gordon Numerical Study

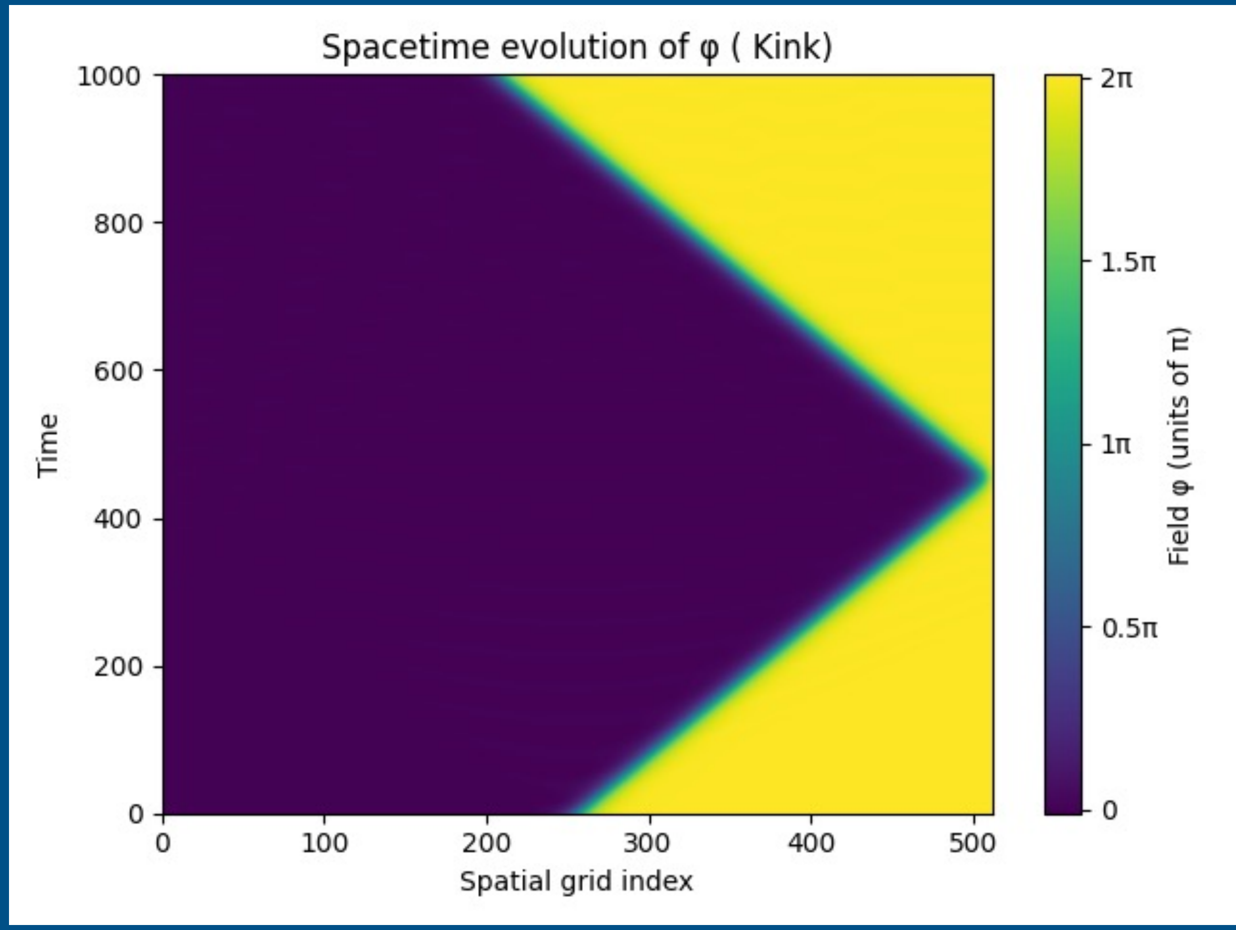
**Step 1 — Model Construction & Numerical Implementation**
Formulating the Sine–Gordon equation and implementing its numerical solution in Python.
Key components include:
• Definition of the spatial Laplacian operator
• Choice of time-evolution scheme (e.g., leap-frog / Symplicit propagator)
• Specification of initial and boundary conditions
• Visualization through space-time and phase-space plots

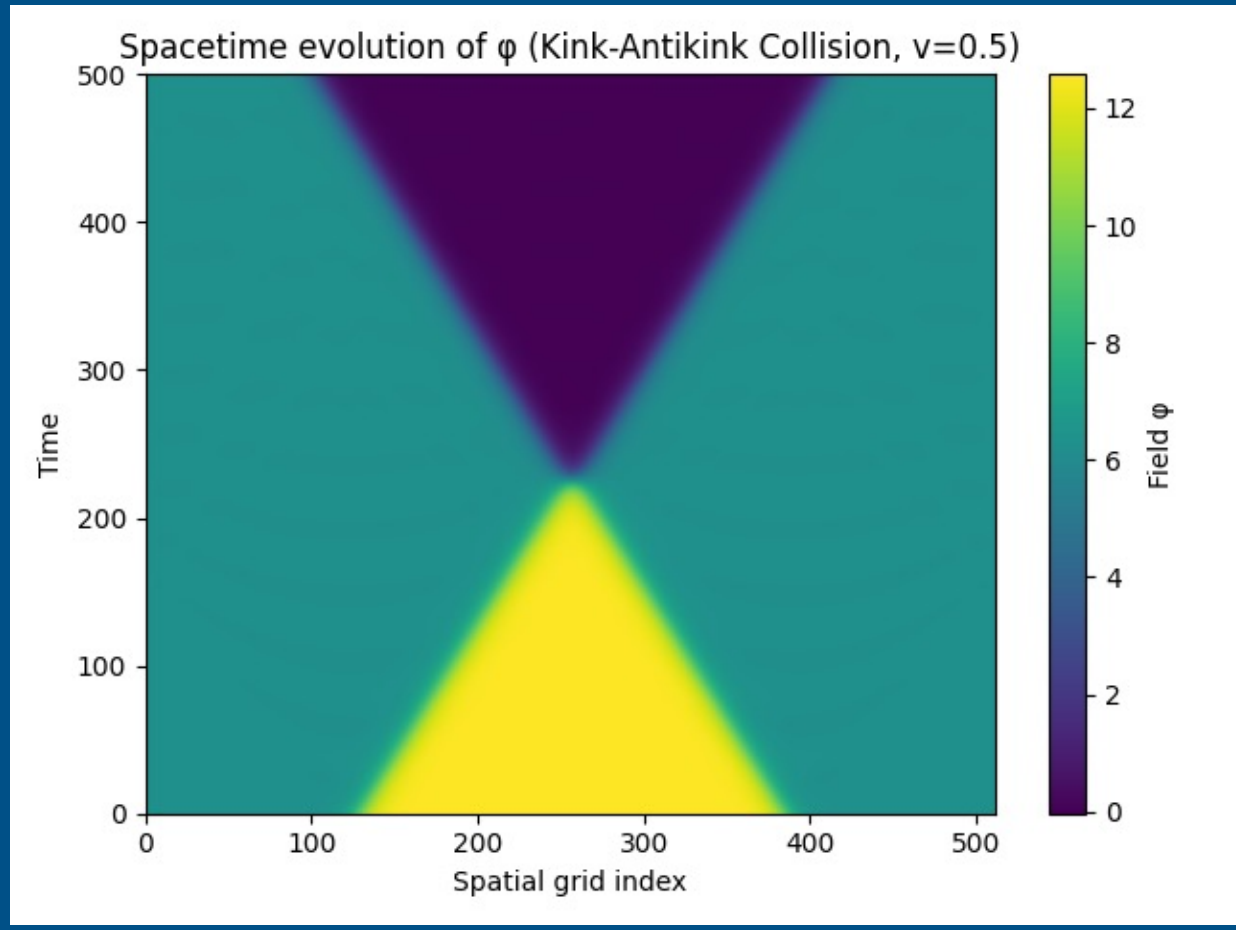
**Step 2 — Iterative Refinement (Trial-and-Error Methodology)**
Systematically exploring parameter ranges, stability conditions, and numerical discretization's to identify physically meaningful behaviours (e.g., soliton formation, breather stability).

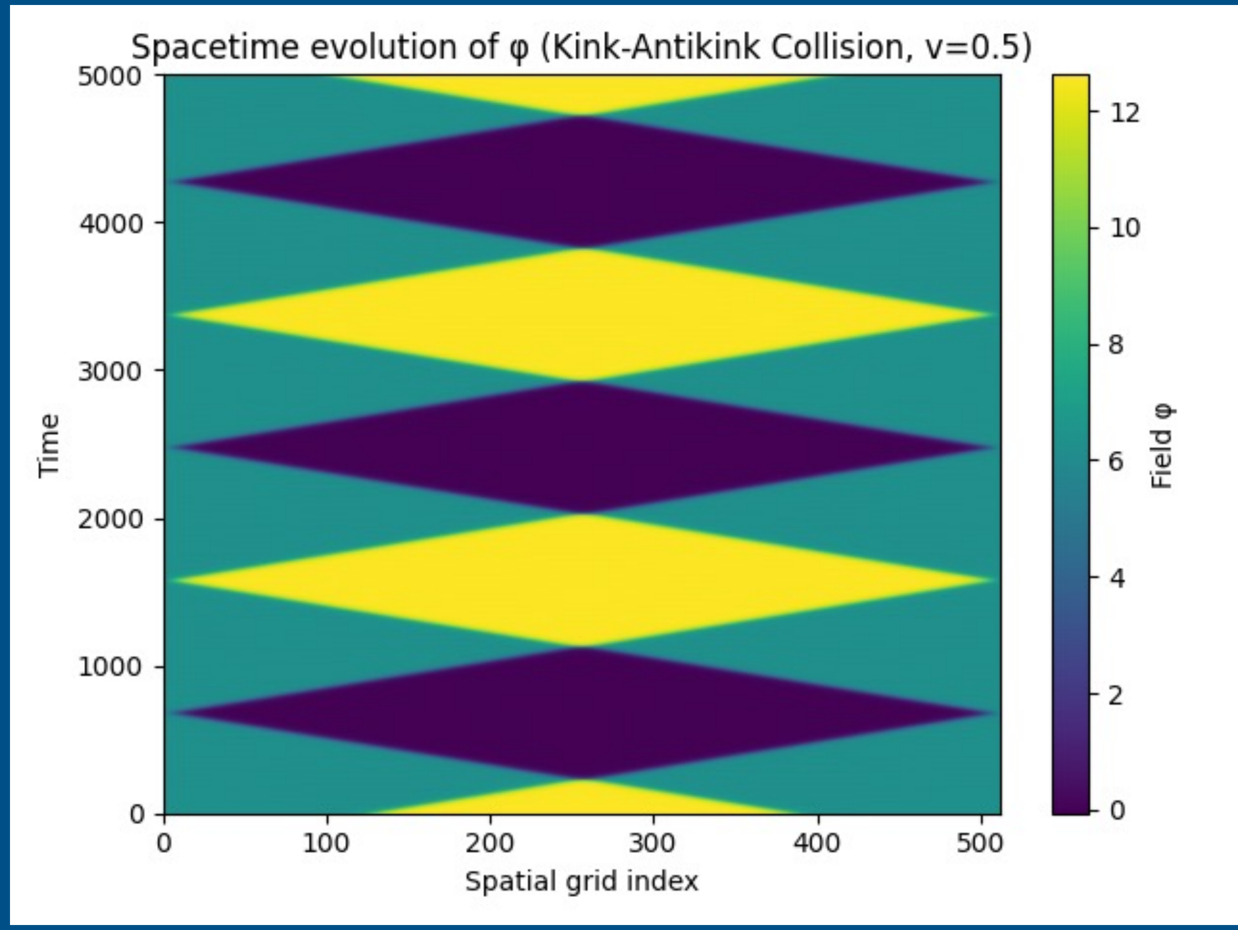
**Step 3 — Focusing on a Well-Defined Research Direction**
Narrowing the investigation to a specific phenomenon or parameter regime—for example, stabilizing breather excitations or analysing winding-number sectors.

**Step 4 — Analysis, Interpretation, and Dissemination**
Critically evaluating the numerical results, comparing them with analytical expectations, and preparing the findings for presentation or publication.

# fix-boundary condition



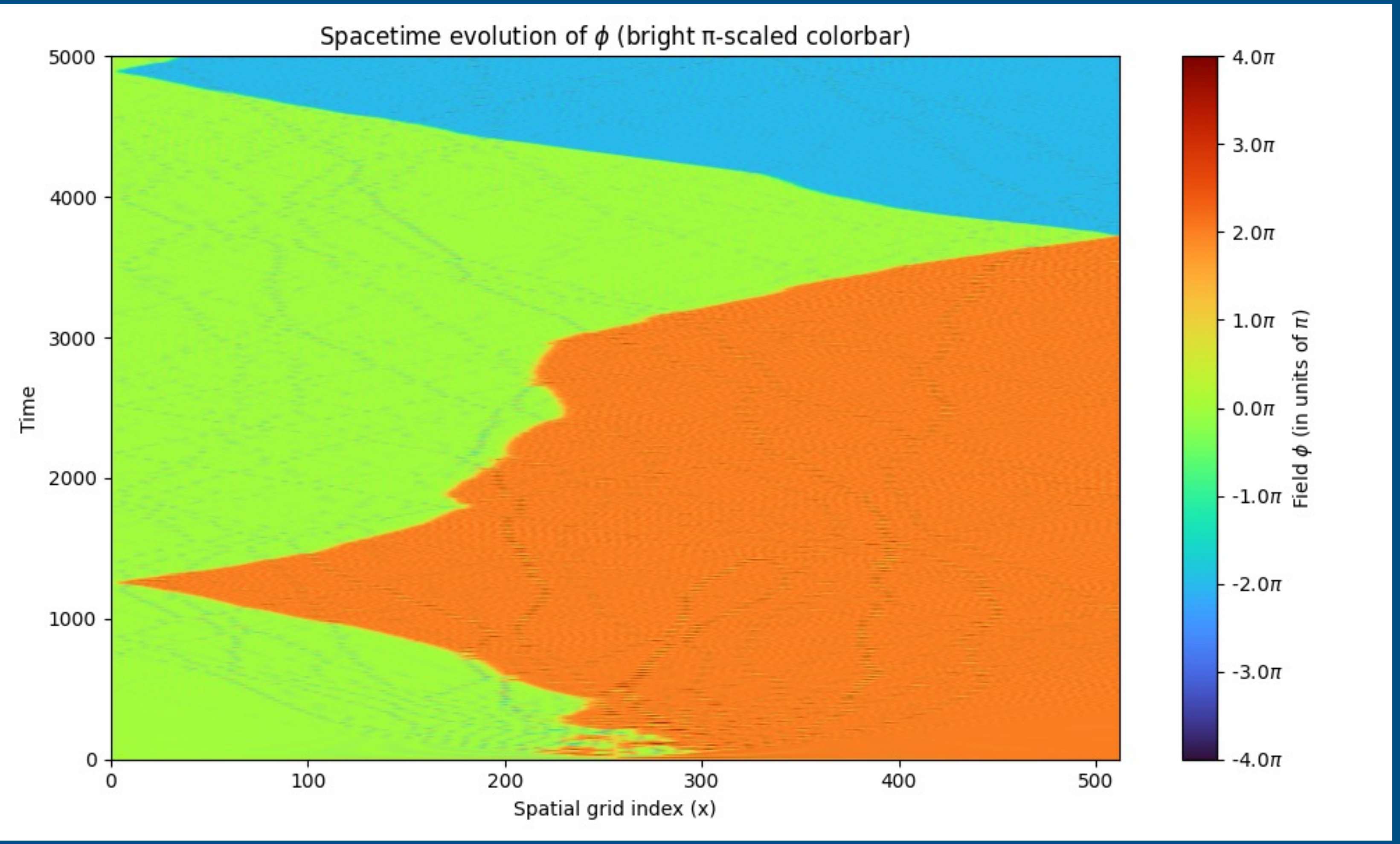

(a)

(b)

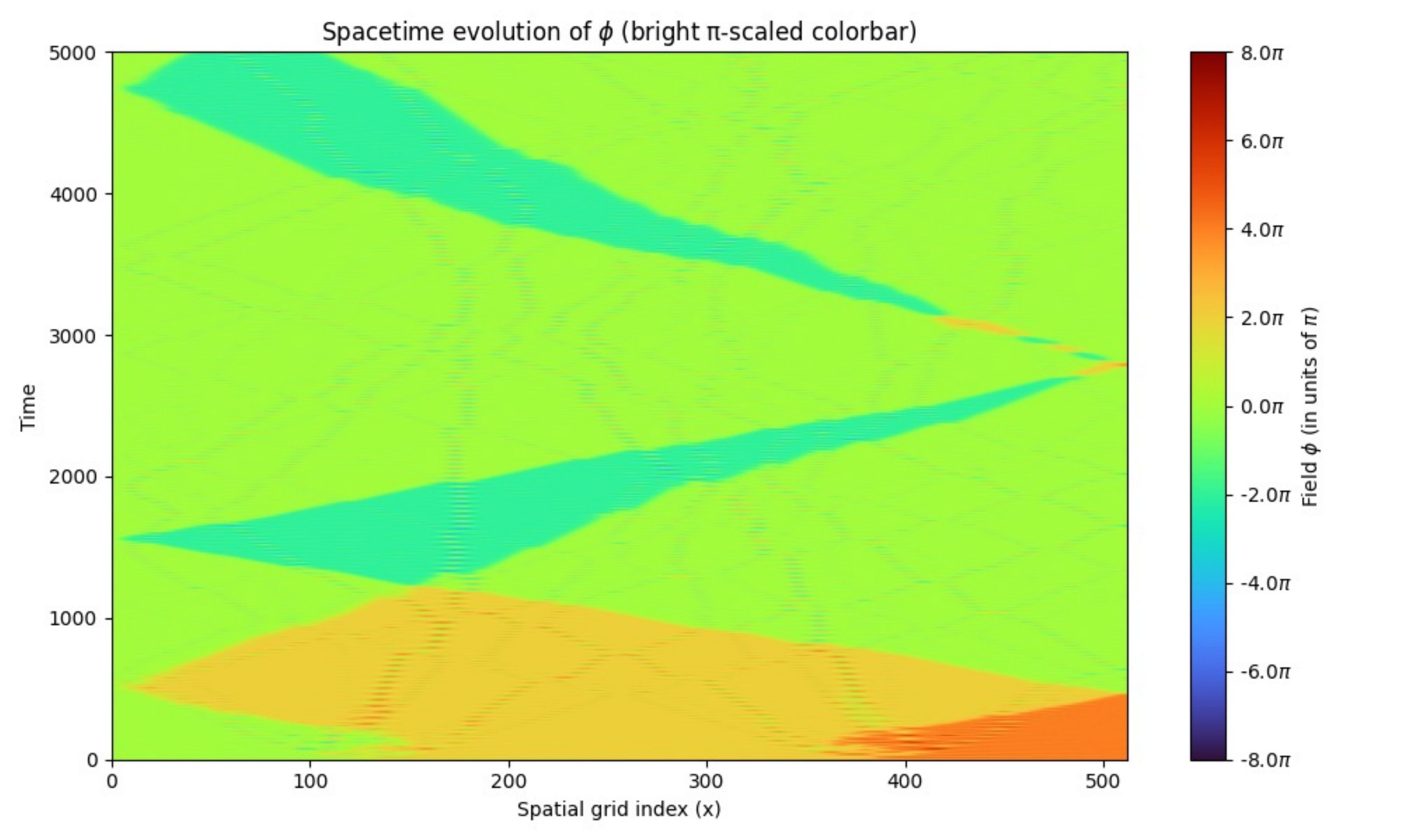
(c)

(a) Single kink, (b) kink and antikink, (c) kink–antikink pair at later time

# Sawtooth initial codnidtion




(a)

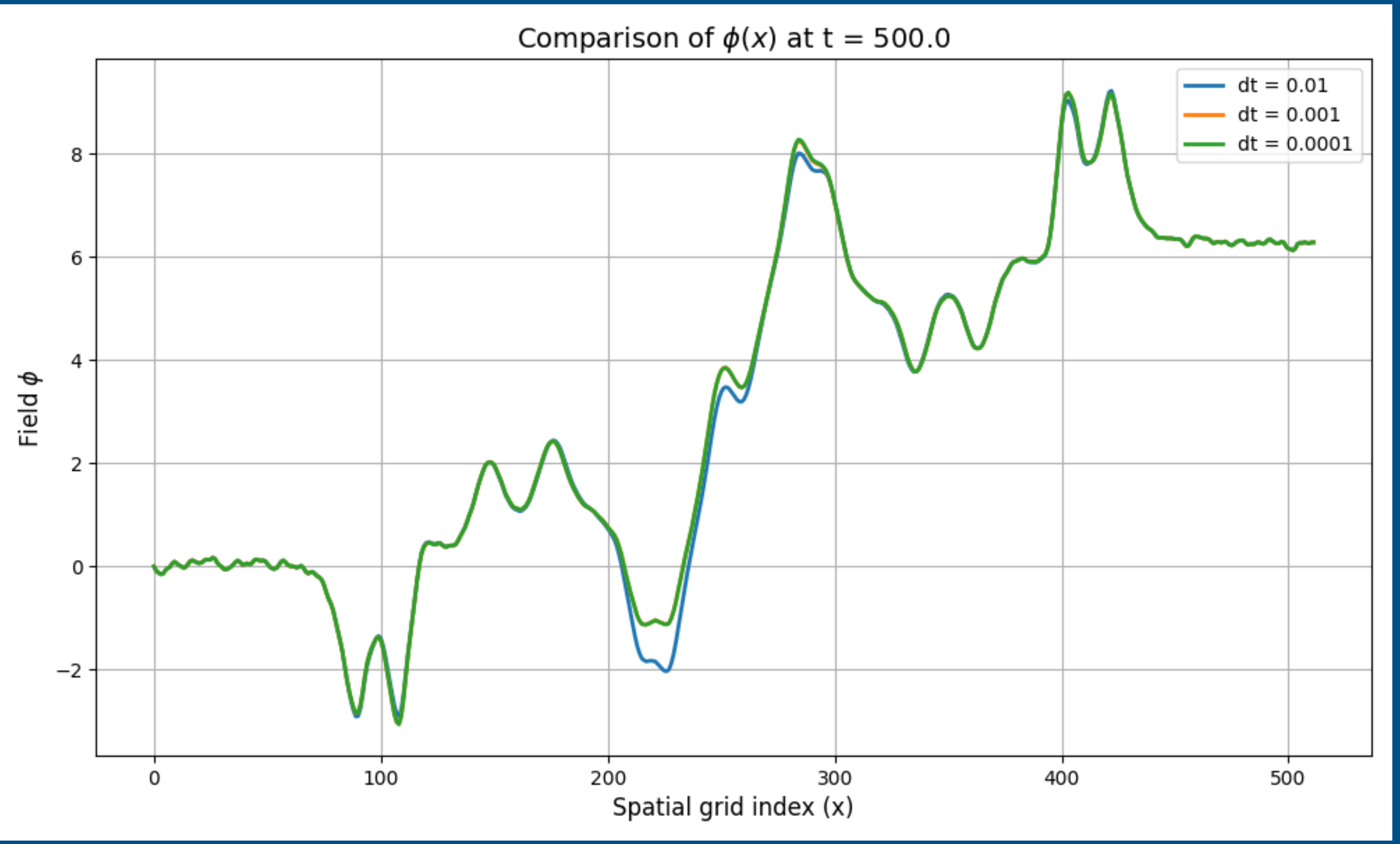



(b)

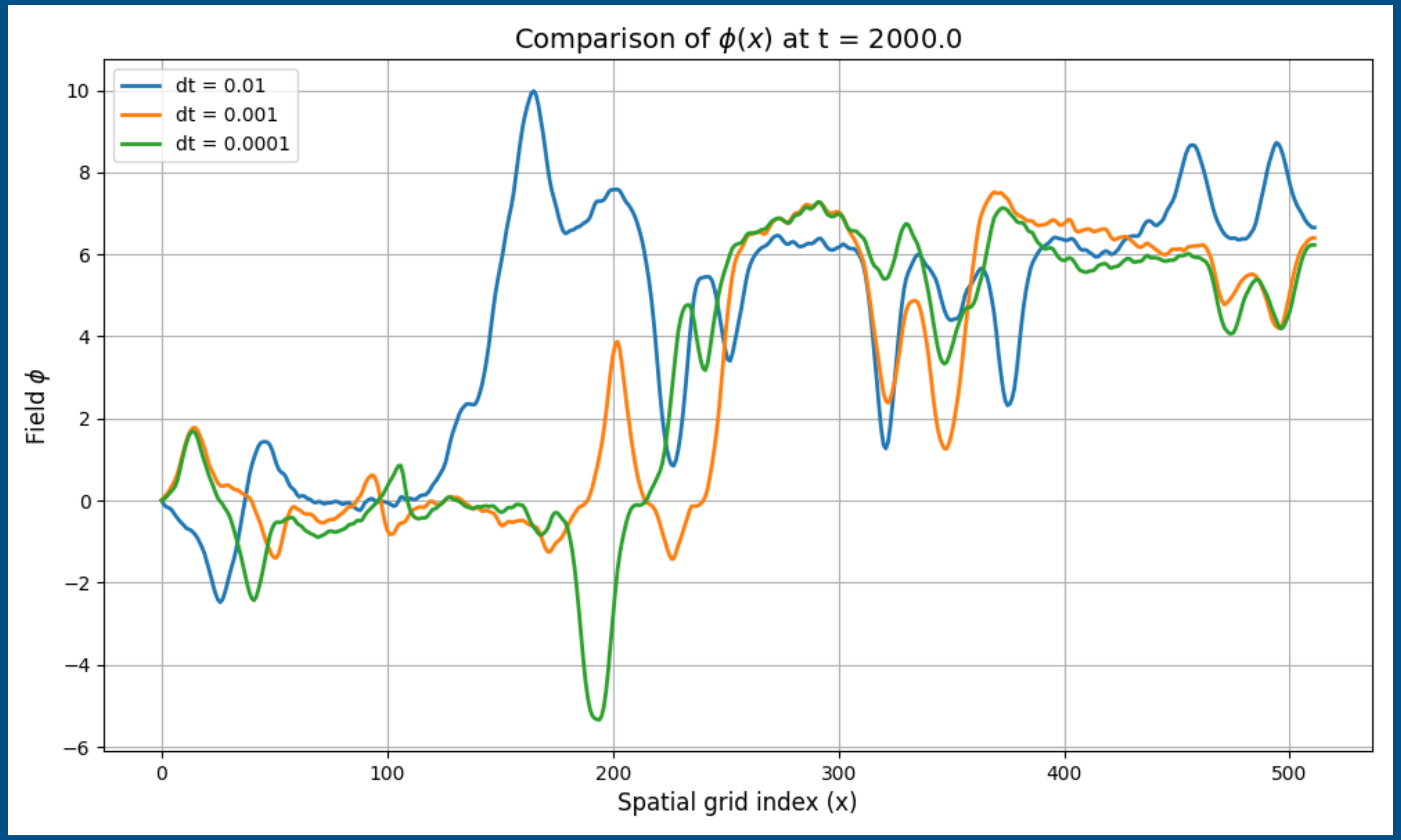
Spacetime evolution of a spin chain under a smoothly increasing field, with fixed left and open right boundary conditions, for (a) winding number 1 and (b) winding number 2

# Results

**The system is likely nonlinear and chaotic: tiny changes in initial conditions grow significantly over time.**




(a)

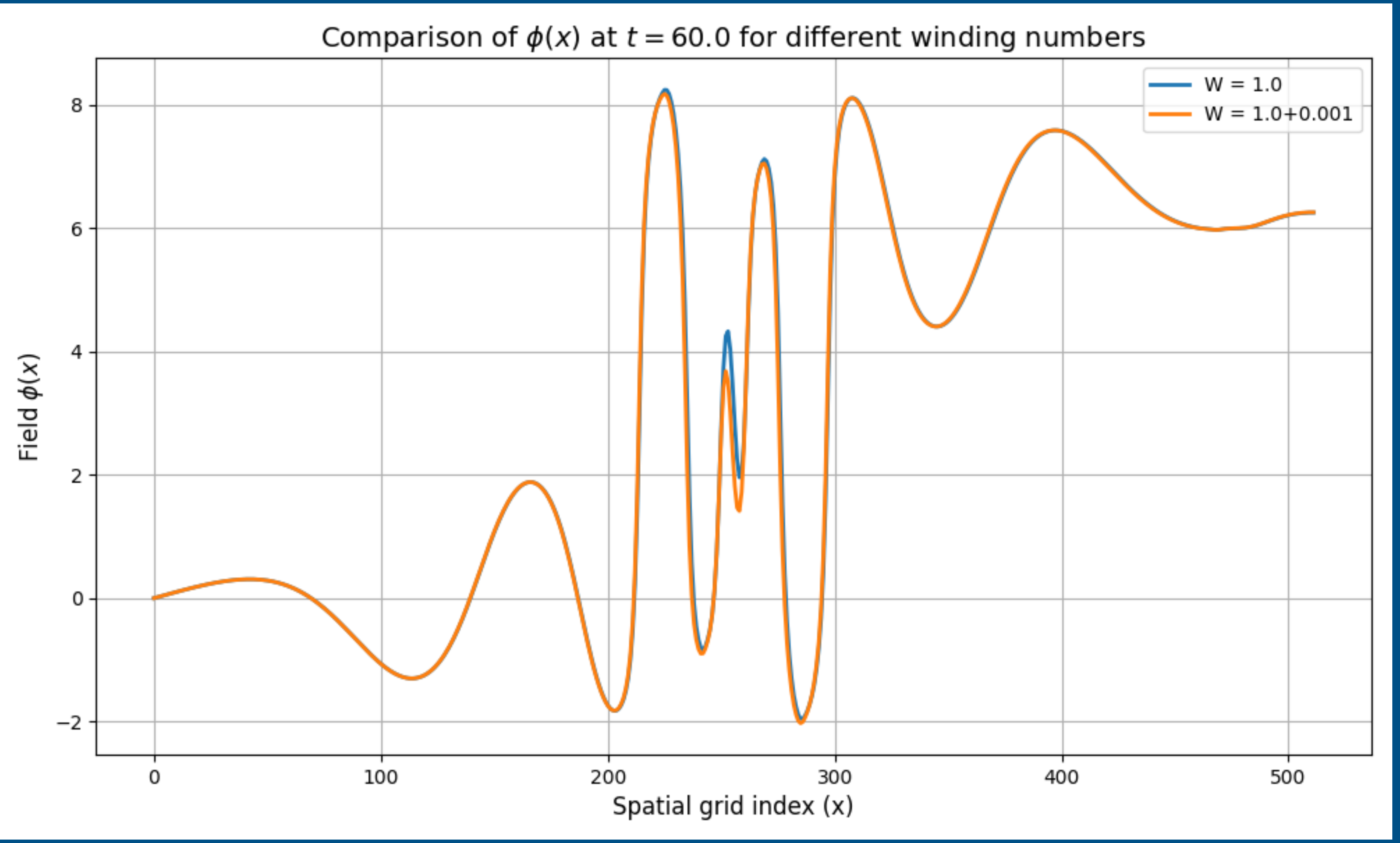



(b)

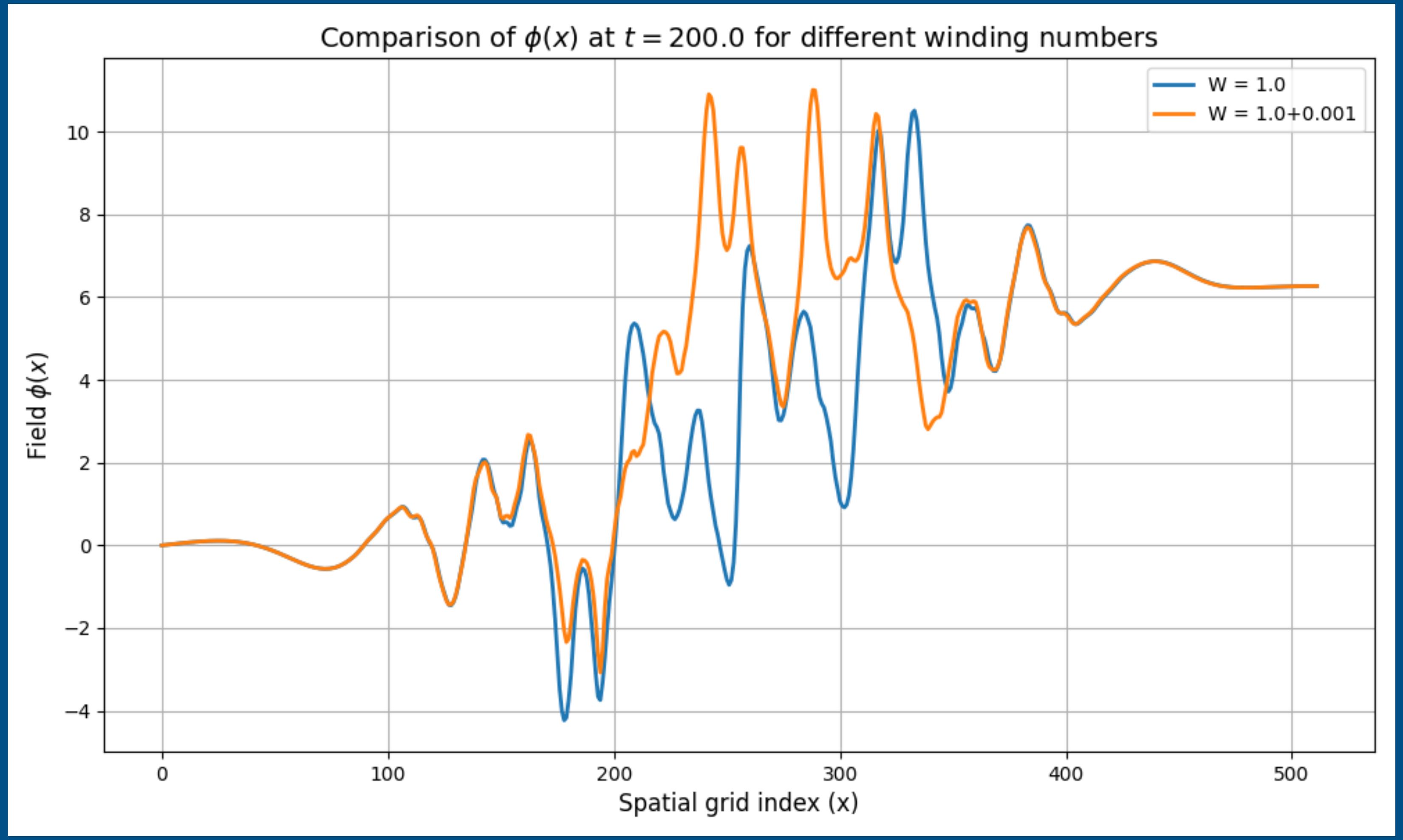
**Results for a specific winding number at different times.**
(a) Up to t = 500, results are nearly identical.
(b) At t = 2000, results diverge significantly.

# Results

**The images below depict the system's behavior, providing an explanation for why different time steps result in distinct outcomes.**



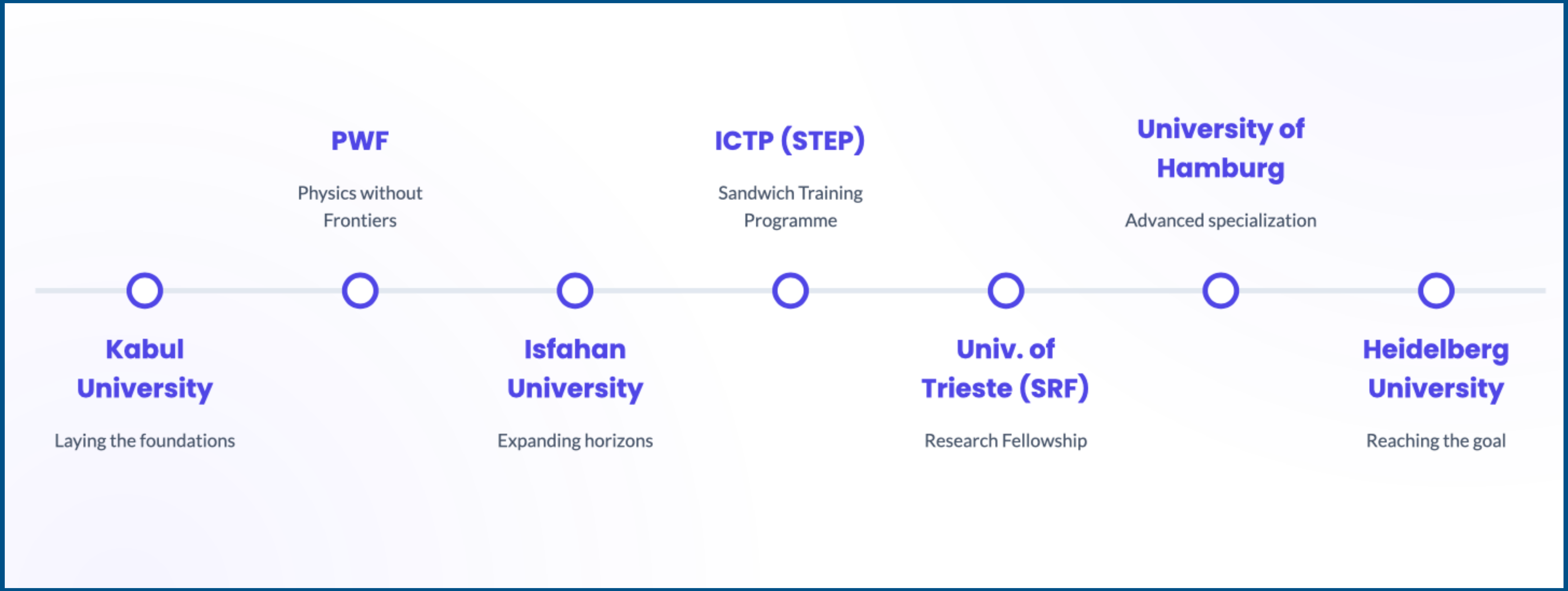

(a)




(b)

(a) shows that a small perturbation in the initial condition does not cause noticeable changes at early time t = 60, while (b) shows that at later time t = 200, the same small perturbation leads to significant differences.

# My Path to Heidelberg University



**Dr. Thore Posske**
*University of Hamburg*

- Topological phases & spin chains
- Majorana fermions, Kondo effect
- Robust quantum information

**Prof. Thomas Gasenzer**
*Heidelberg University*

- Far-from-equilibrium quantum dynamics
- Ultracold gases & turbulence
- Neural networks for quantum systems

# Thank You!

## Special Thanks

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