LAWS OF QUANTUM COMMUNICATION

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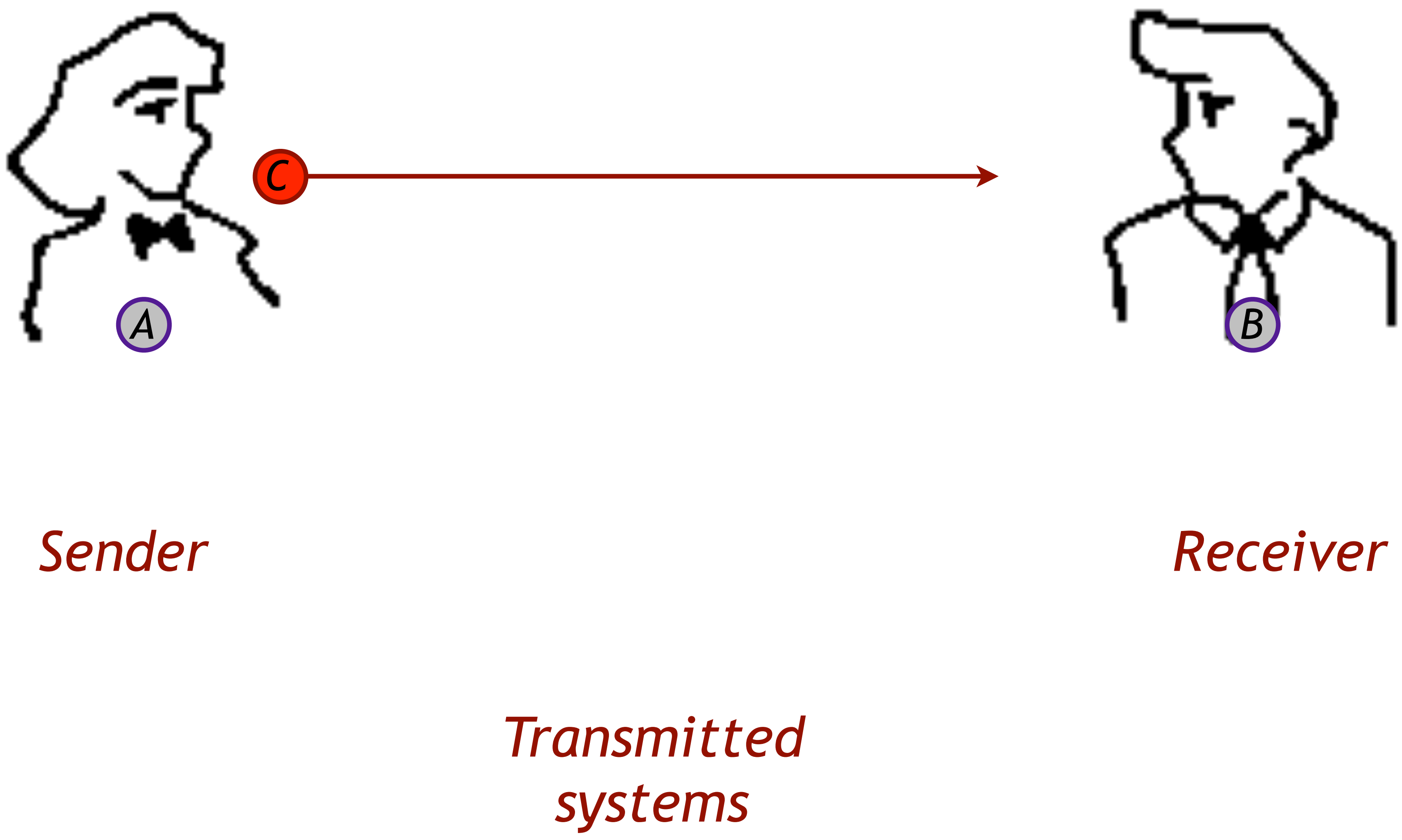
1st LAW: INFORMATION GAIN

2nd LAW: INFORMATION CAUSALITY

3rd LAW: ENTANGLEMENT GAIN

COMMUNICATION SCENARIO



PRIMARY AIM: ESTABLISH CORRELATIONS

CORRELATIONS: BY LOOKING AT ONE PART YOU CAN SAY SOMETHING ABOUT THE OTHER PART

$$I_{X:Y} = S_X + S_Y - S_{XY}$$
$$I_{X:Y} = S_X - S_{X|Y}$$

$$S_X = -\sum_x p_x \log p_x$$
$$S_X = -\mathrm{Tr}(\rho_X \log \rho_X)$$

FIND MUTUAL INFORMATION BETWEEN PARTICLES IN THE BELL SINGLET STATE:

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

ANSWER: 2 BITS

Relative entropy and its properties:

$$S(\rho||\tau) = -\mathrm{Tr}(\rho \log \tau) - S(\rho)$$

$$S(\rho||\tau) \geq 0$$

$$S(\rho||\tau) = 0 \iff \rho = \tau$$

Mutual information is the relative entropy distance to the closest product state.

$$\min_{\pi_X \otimes \pi_Y} S(\rho_{XY} || \pi_X \otimes \pi_Y) = S(\rho_{XY} || \rho_X \otimes \rho_Y) = I_{X:Y}$$

Proof. $S(\rho||\rho_A \otimes \rho_B) - S(\rho||\pi_A \otimes \pi_B)$

$$= -\mathrm{Tr}(\rho \log(\rho_A \otimes \rho_B)) - S(\rho) + \mathrm{Tr}(\rho \log(\pi_A \otimes \pi_B)) + S(\rho)$$

Use $\log(\alpha \otimes \beta) = \log(\alpha) \otimes \mathbb{1} + \mathbb{1} \otimes \log(\beta)$

$$= S(\rho_A) + S(\rho_B) + \mathrm{Tr}(\rho_A \log \pi_A) + \mathrm{Tr}(\rho_B \log \pi_B)$$

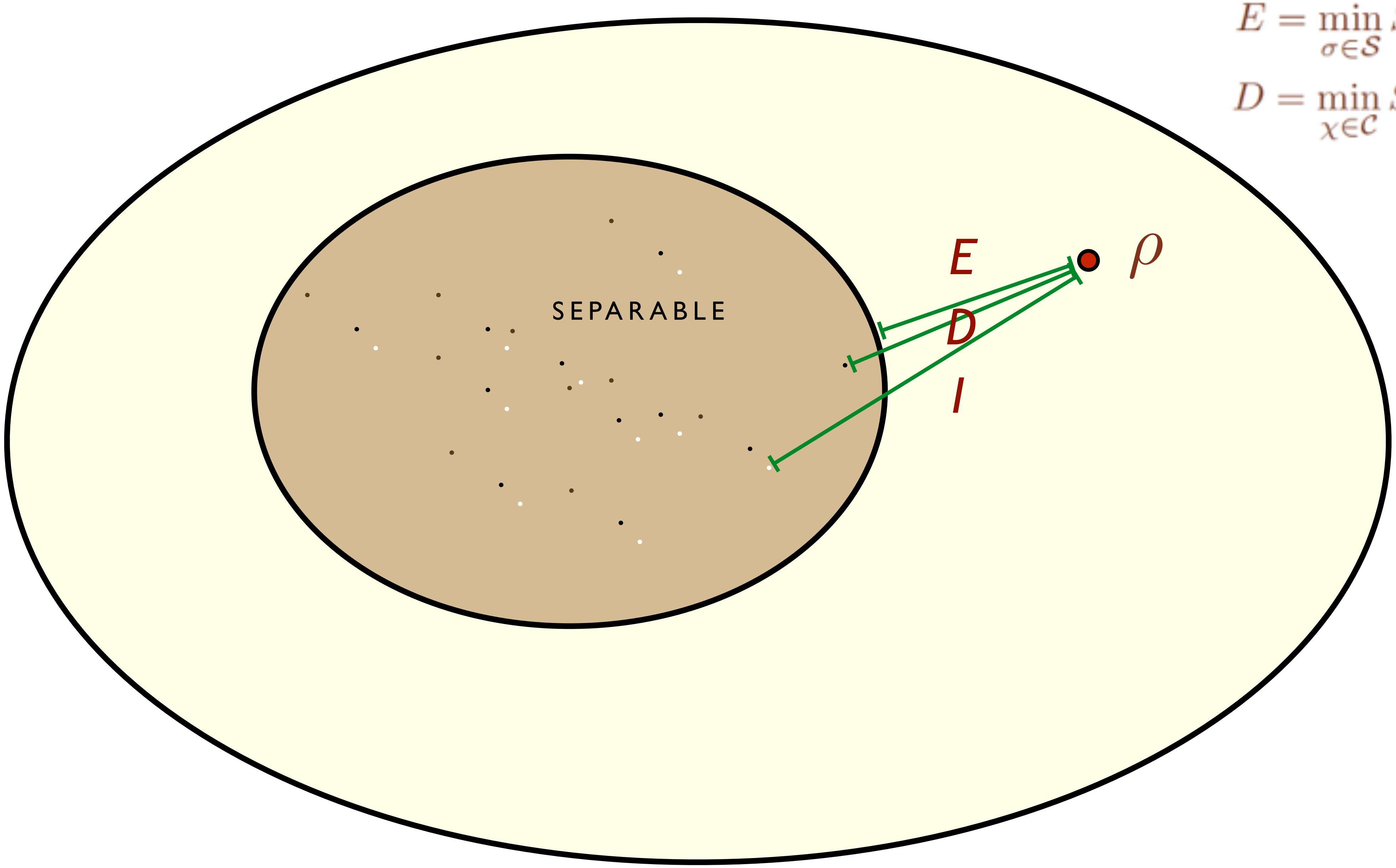
$$= -S(\rho_A||\pi_A) - S(\rho_B||\pi_B) \leq 0.$$

UNIFIED VIEW OF CORRELATIONS

$$I = \min_{\pi \in \mathcal{P}} S(\rho || \pi)$$

$$E = \min_{\sigma \in \mathcal{S}} S(\rho || \sigma)$$

$$D = \min_{\chi \in \mathcal{C}} S(\rho || \chi)$$



COMMUNICATION SCENARIO

Information gain caused by communication:

$$\Delta I = I_{A:CB} - I_{AC:B}$$

Entanglement gain caused by communication:

$$\Delta E = E_{A:CB} - E_{AC:B}$$

WARM UP: NO COMMUNICATION

$$\Delta I \leq ?$$

Unrelated communication $\rho_{AB} \otimes \rho_C$ *also gives no information gain*

$$I_{A:CB} - I_{AC:B} = I_{A:B} - I_{A:B} = 0$$

$$I_{A:CB} - I_{AC:B} \leq ?$$

$$I_{A:CB} - I_{AC:B} \leq I_{AB:C}$$

Information gain is bounded by the communicated information.

PROOF

Symmetry of conditional mutual information

$I_{A:B|C} = I_{AC:B} - I_{B:C}$ $\qquad$ $I_{A:B|C} = I_{A:CB} - I_{A:C}$

$$I_{A:CB} - I_{AC:B} = I_{A:C} - I_{B:C} \leq I_{A:C} \leq I_{AB:C}$$

Positivity of mutual information

Data processing inequality

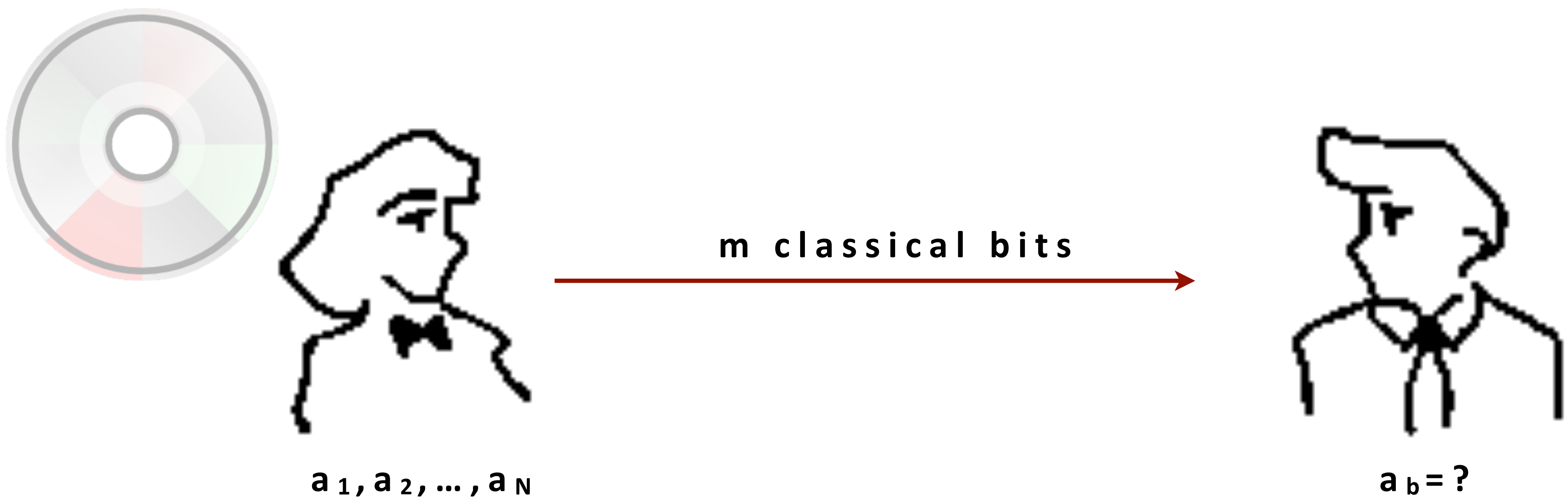
The first law is likely to be present even in future physical theories.

2nd LAW: INFORMATION CAUSALITY

1. BILLIE JEAN
2. BILLIE JEAN - RADIO
3. BILLIE JEAN - MIX

When Alice has sent 10 MB, Bob cannot choose which version he would like to listen.
She decides this when sending the data!

MATHEMATICAL STATEMENT



$$I(a_1{:}G\,|\,b=1) + \ldots + I(a_N{:}G\,|\,b=N) \leq m$$

PROOF

By data processing inequality:

$$I(a_k : G|b = k) \leq I(a_k : C, B)$$

classical message

whatever else in Bob's lab

$$\sum_{k=1}^{N} I(a_k : G|b = k) \leq \sum_{k=1}^{N} I(a_k : C, B) \leq I(a_1 \dots a_N : C, B)$$

Recall the conditional mutual information:

$$I(X : Y|Z) = I(XZ : Y) - I(Y : Z) = I(X : YZ) - I(X : Z)$$

Take $X = a_1, Z = a2 \dots a_N, Y = C, B$ *and start from the right-hand side:*

$$I(a_1 \dots a_N : C, B) = I(a_1 : a_2 \dots a_N, C, B) + I(a_2 \dots a_N : C, B) - I(a_1 : a_2 \dots a_N)$$

$$\geq I(a_1 : C, B) + I(a_2 \dots a_N : C, B)$$

from data processing

assumed independent

PROOF

$$\sum_{k=1}^{N} I(a_k : G|b = k) \leq \sum_{k=1}^{N} I(a_k : C, B) \leq I(a_1 \ldots a_N : C, B)$$

$$I(a_1 \ldots a_N : C, B) = I(a_1 \ldots a_N, B : C) + I(a_1 \ldots a_N : B) - I(B : C) \leq m$$

Independence of the input

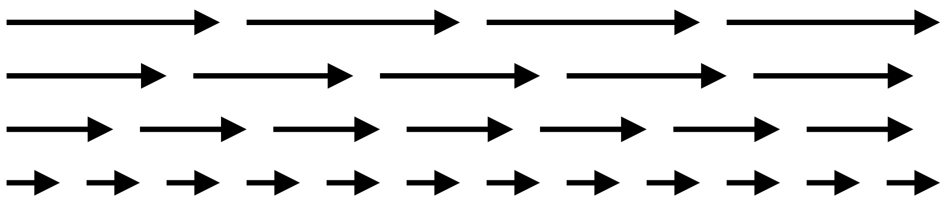
"Classical limit": mutual information between m classical bits and something is upper-bounded by m

Positivity of mutual information

The second law is likely to be present even in future physical theories.

SO WHAT?

$$\vec{E} = y\hat{x}$$



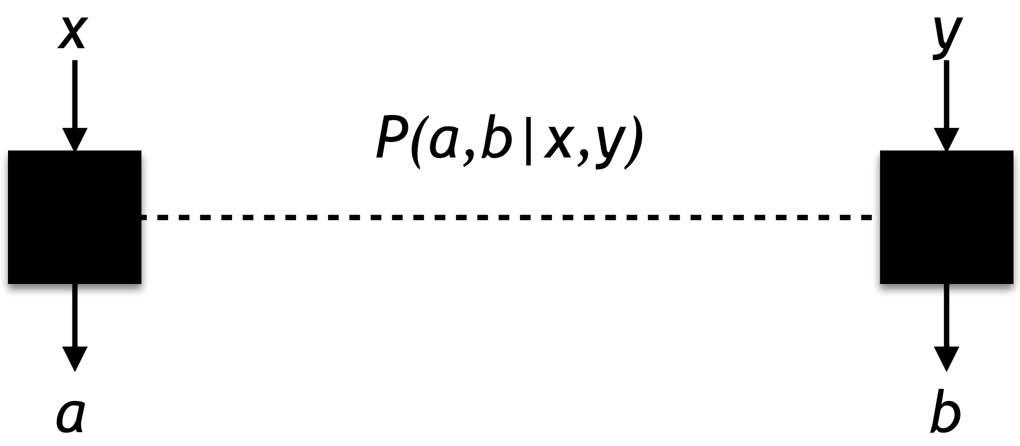
This is not a physical field.
No set of charges would ever produce it.

What's wrong with it?
It's against Maxwell's equation:

$$\vec{\nabla} \times \vec{E} = \vec{0}$$

Some correlations are also not physical.
Though they look quite innocent at first sight.

What's wrong with them?
They violate information causality.



PROBABILITY TABLE

$P(a,b|x,y)$

$P(0,0\|0,0)$	$P(0,1\|0,0)$	$P(0,0\|0,1)$	$P(0,1\|0,1)$
$P(1,0\|0,0)$	$P(1,1\|0,0)$	$P(1,0\|0,1)$	$P(1,1\|0,1)$
$P(0,0\|1,0)$	$P(0,1\|1,0)$	$P(0,0\|1,1)$	$P(0,1\|1,1)$
$P(1,0\|1,0)$	$P(1,1\|1,0)$	$P(1,0\|1,1)$	$P(1,1\|1,1)$

EXAMPLE: DETERMINISTIC BOXES

Output 1 independently of the input.

0	*0*	*?*	*?*
0	*1*	*?*	*?*
?	*?*	*?*	*?*
?	*?*	*?*	*?*

EXAMPLE: DETERMINISTIC BOXES

Output 1 independently of the input.

0	*0*	*0*	*0*
0	*1*	*0*	*1*
0	*0*	*0*	*0*
0	*1*	*0*	*1*

EXAMPLE: COMPLEMENTARY MEASUREMENTS

Pair of spin up particles is measured either along z axis (input 0) or along x axis (input 1)

0	*0*	*?*	*?*
0	*1*	*?*	*?*
?	*?*	*?*	*?*
?	*?*	*?*	*?*

EXAMPLE: COMPLEMENTARY MEASUREMENTS

Pair of spin up particles is measured either along z axis (x,y=0) or along x axis (x,y=1)

0	0	0	0
0	1	1/2	1/2
0	1/2	1/4	1/4
0	1/2	1/4	1/4

EXAMPLE: WHAT'S WRONG WITH THIS BOX?

It allows for faster than light communication!

1	0	1/2	0
0	0	0	1/2
1/2	0	0	0
0	1/2	0	1

NO SIGNALLING

$$P(a|x,y) = P(a|x)$$
$$P(b|x,y) = P(b|y)$$

$$\begin{array}{ccc|ccc}
P(0,0|0,0) & \oplus & P(0,1|0,0) & = \; P(0,0|0,1) & \oplus & P(0,1|0,1) \\
\oplus & & \oplus & \oplus & & \oplus \\
P(1,0|0,0) & \oplus & P(1,1|0,0) & = \; P(1,0|0,1) & \oplus & P(1,1|0,1) \\
\hline
\parallel & & \parallel & \parallel & & \parallel \\
P(0,0|1,0) & \oplus & P(0,1|1,0) & = \; P(0,0|1,1) & \oplus & P(0,1|1,1) \\
\oplus & & \oplus & \oplus & & \oplus \\
P(1,0|1,0) & \oplus & P(1,1|1,0) & = \; P(1,0|1,1) & \oplus & P(1,1|1,1)
\end{array}$$

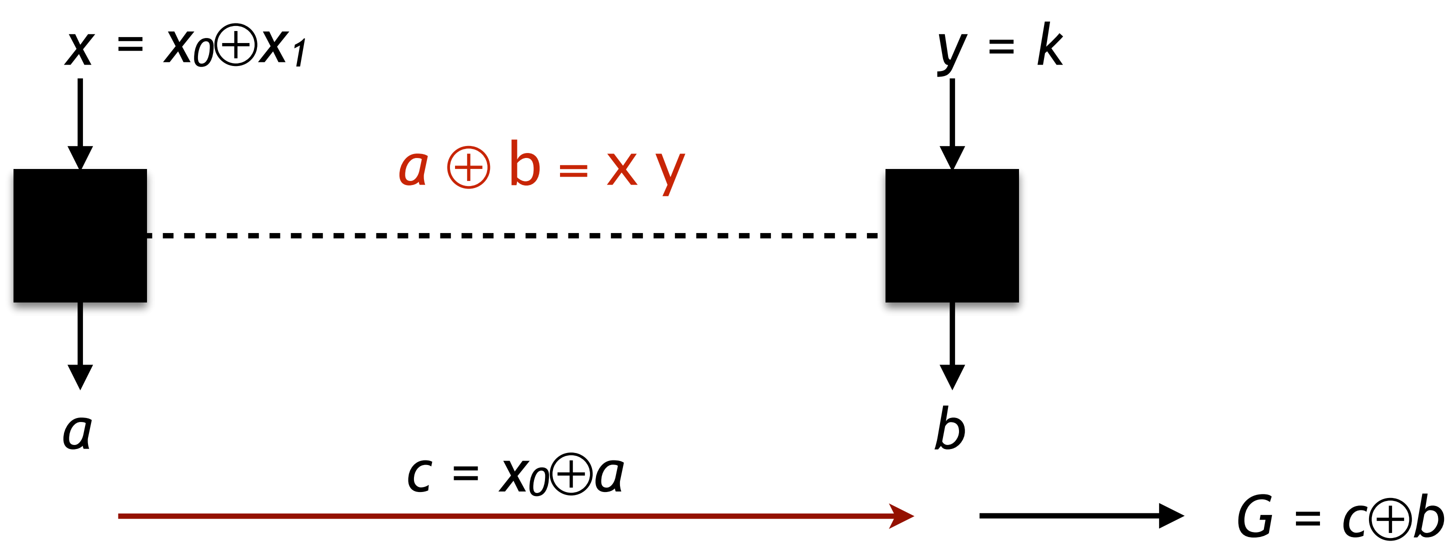
EXAMPLE: IS THIS BOX NO-SIGNALLING?

Yes. Is it realised in nature?

1/2	0	1/2	0
0	1/2	0	1/2
1/2	0	0	1/2
0	1/2	1/2	0

Problem. How to access any bit of Alice?

Solution. Use boxes from the last slide.



$$G = c \oplus b = x_0 \oplus a \oplus b = x_0 \oplus (x_0 \oplus x_1)k$$

SOME OTHER APPLICATIONS?

Information causality excludes no-signalling correlations which give access to too much of remote data.

3rd LAW: ENTANGLEMENT GAIN

$$\Delta E \leq ?$$

3rd LAW: ENTANGLEMENT GAIN

$$E_{A:CB} - E_{AC:B} \leq D_{AB|C}$$

QUANTUM DISCORD

Entangled states

$$\rho_{\text{ent}} \neq \sum_j p_j |a_j b_j \ldots\rangle\langle a_j b_j \ldots|$$

These can be any local states

Discorded states

$$\rho_{\text{dis}} \neq \sum_c p_c \rho_{AB|c} \otimes \Pi_c$$

These must be orthogonal states

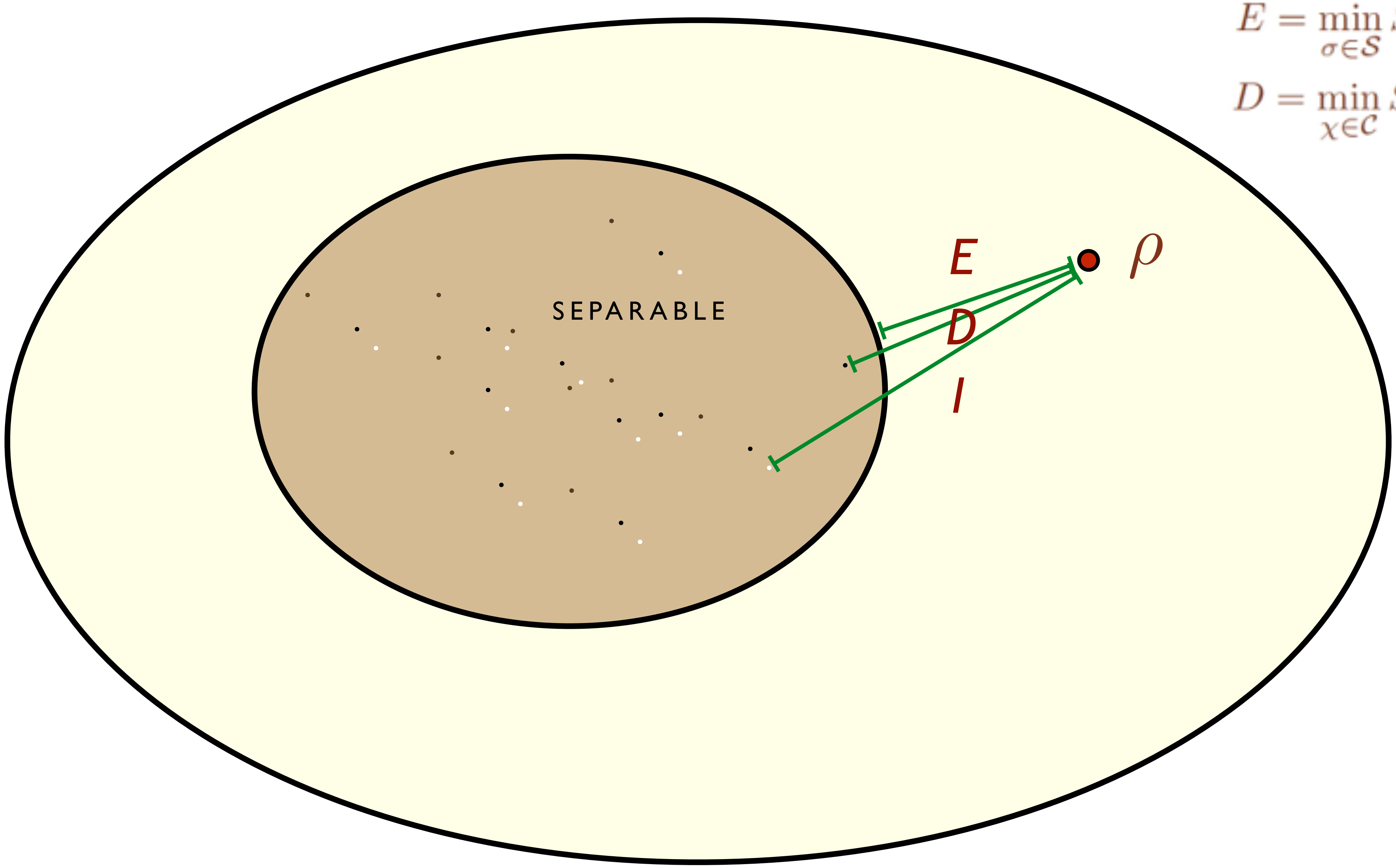
Quantum discord can be present in disentangled states!

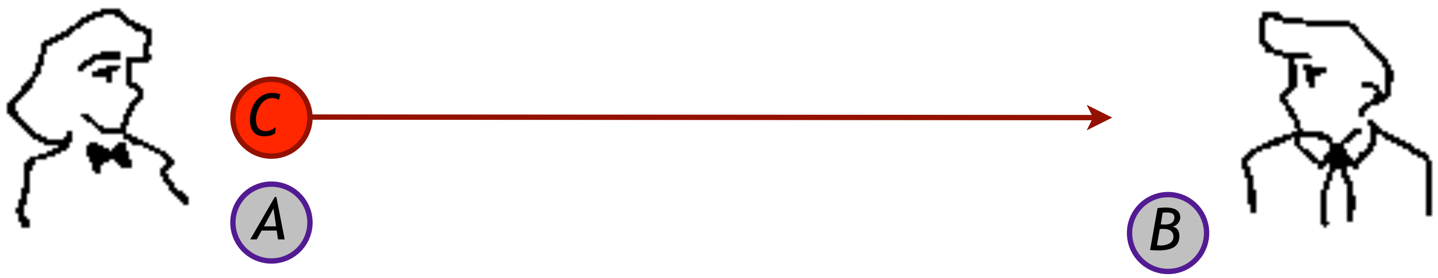
RELATIVE ENTROPY OF DISCORD

$$I = \min_{\pi \in \mathcal{P}} S(\rho || \pi)$$

$$E = \min_{\sigma \in \mathcal{S}} S(\rho || \sigma)$$

$$D = \min_{\chi \in \mathcal{C}} S(\rho || \chi)$$





$$\mathcal{E}_{AB:C}(\rho) = 0$$
$$\mathcal{E}_{AC:B}(\rho) = 0$$
$$\mathcal{E}_{A:CB}(\rho) > 0$$

1st LAW: INFORMATION GAIN

$$I_{A:CB} - I_{AC:B} \leq I_{AB:C}$$

2nd LAW: INFORMATION CAUSALITY

$$I(a_1:G|b=1) + \ldots + I(a_N:G|b=N) \leq m$$

3rd LAW: ENTANGLEMENT GAIN

$$E_{A:CB} - E_{AC:B} \leq D_{AB|C}$$

LITERATURE

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