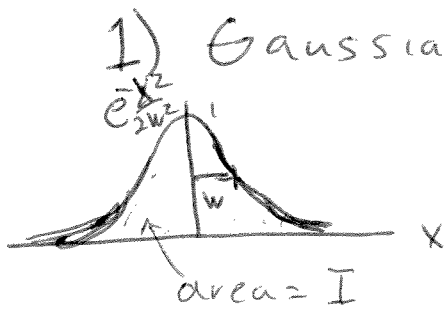


Brief Review of Fourier Transforms

Before defining the Fourier transform, let us go through two derivations that will be very useful.

1) Gaussian integral



$$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2w^2}} dx$$

To solve, consider instead

$$I^2 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2w^2}} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2w^2}} dy = \iint_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2w^2}} dx dy$$

$$I^2 = \int_0^{\infty} \int_0^{2\pi} e^{-\frac{r^2}{2w^2}} r d\phi dr = 2\pi \int_0^{\infty} e^{-\frac{r^2}{2w^2}} r dr$$

$$= 2\pi \int_0^{\infty} e^{-\frac{u}{w^2}} du = -2\pi w^2 e^{-\frac{u}{w^2}} \Big|_0^{\infty} = -2\pi w^2 (0 - 1)$$

assuming $\text{Re}[w^2] > 0$

$$I^2 = 2\pi w^2$$

$$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2w^2}} dx = \sqrt{2\pi} w$$

2) Dirac's delta as the integral of an exponential

consider $I(x) = \int_{-\infty}^{\infty} e^{ikxP} dp$, $K = \text{real constant}$

To force the integral to converge, insert "unity" as

$$1 = \lim_{q \rightarrow \infty} e^{-\frac{p^2}{2q^2}}$$

$$I(x) = \lim_{q \rightarrow \infty} \int_{-\infty}^{\infty} 1 e^{ikxP} dp = \lim_{q \rightarrow \infty} \int_{-\infty}^{\infty} e^{-\frac{p^2}{2q^2} + ikxP} dp$$

$$= \lim_{q \rightarrow \infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2q^2} [p^2 - 2pikxq^2]} dp = \lim_{q \rightarrow \infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2q^2} [p^2 - 2pikxq^2 + (ikxq^2)^2]} \frac{ikxq^2}{2q^2} dp$$

these two balance each other

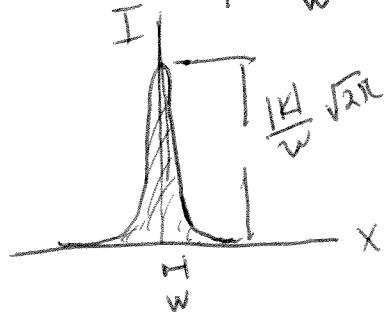
$$= \lim_{q \rightarrow \infty} \int_{-\infty}^{\infty} e^{-\frac{(p - ikxq^2)^2}{2q^2}} dp e^{-\frac{k^2 q^2 x^2}{2}}$$

$$= \lim_{q \rightarrow \infty} \int_{-\infty - ikxq^2}^{\infty - ikxq^2} e^{-\frac{\tau^2}{2q^2}} d\tau e^{-\frac{k^2 q^2 x^2}{2}} = \lim_{q \rightarrow \infty} \sqrt{2\pi} e^{-\frac{k^2 q^2 x^2}{2}} \frac{1}{q}$$

can use result from previous page with " $\tau \rightarrow x, q \rightarrow w$ "

The small imaginary parts of the limits of the integrals have no effect because the integrand $\rightarrow 0$

Let $K^2 q^2 = \frac{1}{w^2}$, then $I(x) = \lim_{w \rightarrow 0} \sqrt{2\pi} e^{-\frac{x^2}{2w^2}} \frac{w}{|K|}$



$$\text{Area} = \frac{\sqrt{2\pi} w}{|K|} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2w^2}} dx = \frac{2\pi}{|K|}$$

$$\text{so } I(x) = \int_{-\infty}^{\infty} e^{ikxP} dp = \frac{2\pi}{|K|} \delta(x)$$

We now define the Fourier transform as

$$\tilde{f}(p) = \hat{\mathcal{F}}_{x \rightarrow p} f(x) = \frac{|K|}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iKxp} dx$$

where K is a real constant chosen for convenience or convention. Common choices are:

$$K = \begin{cases} \pm 2\pi \\ k = \omega/c = \frac{2\pi}{\lambda} \text{ (wavenumber)} \\ \pm 1 \\ \hbar^{-1} = \frac{2\pi}{h} \text{ (in quantum mechanics)} \end{cases}$$

To find the inverse, consider

$$\begin{aligned} \int_{-\infty}^{\infty} \tilde{f}(p) e^{iKxp} dp &= \frac{|K|}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x') e^{-iKx'p} dx' e^{iKxp} dp \\ &= \frac{|K|}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') \underbrace{\int_{-\infty}^{\infty} e^{iK(x-x')p} dp}_{\frac{2\pi}{|K|} \delta(x-x')} dx' = \sqrt{\frac{2\pi}{|K|}} \underbrace{\int_{-\infty}^{\infty} f(x') \delta(x'-x) dx}_{f(x)} \end{aligned}$$

↑ substitute change order of \int .

therefore

$$f(x) = \hat{\mathcal{F}}_{p \rightarrow x}^{-1} \tilde{f}(p) = \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p) e^{iKxp} dp$$

Inverse Fourier transform

$$\left. \begin{array}{l} \hat{F}(p) = \hat{\int}_{x \rightarrow p} f(x) = \sqrt{\frac{|k|}{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikxp} dx \\ \text{F.T.} \end{array} \right| \begin{array}{l} f(x) = \hat{\int}_{p \rightarrow x}^{-1} \tilde{F}(p) = \sqrt{\frac{|k|}{2\pi}} \int_{-\infty}^{\infty} \tilde{F}(p) e^{ikxp} dp \\ \text{I.F.T.} \end{array} \right\}$$

The Fourier transform satisfies many properties (see list at end of these notes)

Here, we will use these:

$$\begin{aligned} \hat{\int}_{x \rightarrow p} [x^n f(x)] &= \sqrt{\frac{|k|}{2\pi}} \int_{-\infty}^{\infty} x^n f(x) e^{-ikxp} dx = \sqrt{\frac{|k|}{2\pi}} \frac{d^n}{dp^n} \int_{-\infty}^{\infty} f(x) e^{-ikxp} dx \\ &= \left(\frac{i}{k}\right)^n \frac{d^n}{dp^n} \sqrt{\frac{|k|}{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikxp} dx = \left(\frac{i}{k}\right)^n \tilde{F}^{(n)}(p) \end{aligned}$$

Conversely

$$\hat{\int}_{p \rightarrow x}^{-1} [p^n \tilde{F}(p)] = \left(\frac{-i}{k}\right)^n f^{(n)}(x), \text{ so}$$

$$\hat{\int}_{x \rightarrow p} f^{(n)}(x) = (ik)^n p^n \tilde{F}(p) /$$

$$\begin{aligned} \bullet \text{ Scaling } \leftarrow \text{real, positive} \\ \hat{\int}_{x \rightarrow p} f(ax) &= \sqrt{\frac{|k|}{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{-ikxp} dx = \sqrt{\frac{|k|}{2\pi}} \int_{-\infty}^{\infty} f(x') e^{-\frac{ikx'p}{a}} \frac{dx'}{a} \\ &\quad x' = ax, dx' = a dx \end{aligned}$$

$$= \frac{1}{a} \tilde{F}\left(\frac{p}{a}\right) /$$

$$\begin{aligned} \bullet \text{ Parseval's theorem} \\ \int_{-\infty}^{\infty} |f(x)|^2 dx &= \int_{-\infty}^{\infty} f^*(x) f(x) dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{|k|}{2\pi}} \tilde{F}(p) e^{ikxp} \right]^* \left[\sqrt{\frac{|k|}{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikxp} dx \right] dp \\ &= \int_{-\infty}^{\infty} \tilde{F}^*(p) \tilde{F}(p) dp = \int_{-\infty}^{\infty} |\tilde{F}(p)|^2 dp / \end{aligned}$$

• Uncertainty relation

First, derive Cauchy-Schwarz-Bunyakovski inequality:

$$\iint_{-\infty}^{\infty} |g(x)h(y) - g(y)h(x)|^2 dx dy \geq 0$$

$$\iint_{-\infty}^{\infty} [g^*(x)h^*(y) - g^*(y)h^*(x)] [g(x)h(y) - g(y)h(x)] dx dy \geq 0$$

"=" only if $h(x) = a g(x)$
constant

multiply:

$$\int_{-\infty}^{\infty} |g(x)|^2 dx \int_{-\infty}^{\infty} |h(y)|^2 dy - \int_{-\infty}^{\infty} g^*(x)h(x) dx \int_{-\infty}^{\infty} h^*(y)g(y) dy$$

$$- \int_{-\infty}^{\infty} g^*(y)h(y) dy \int_{-\infty}^{\infty} h^*(x)g(x) dx + \int_{-\infty}^{\infty} |g(y)|^2 dy \int_{-\infty}^{\infty} |h(x)|^2 dx \geq 0$$

x & y are dummy variables of integration, so we can change their names. Then, this reduces to

$$2 \left[\int_{-\infty}^{\infty} |g(x)|^2 dx \right] \left[\int_{-\infty}^{\infty} |h(x)|^2 dx \right] - 2 \left[\int_{-\infty}^{\infty} g^*(x)h(x) dx \right] \left[\int_{-\infty}^{\infty} h^*(x)g(x) dx \right] \geq 0$$

or

$$\left[\int_{-\infty}^{\infty} |g(x)|^2 dx \right] \left[\int_{-\infty}^{\infty} |h(x)|^2 dx \right] \geq \left| \int_{-\infty}^{\infty} g^*(x)h(x) dx \right|^2$$

Cauchy-Schwarz-Bunyakovski inequality.

Now, let us define:

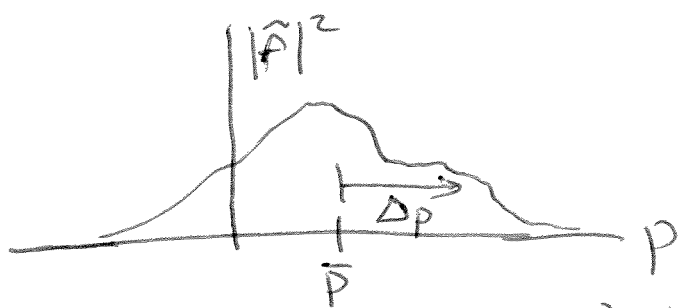
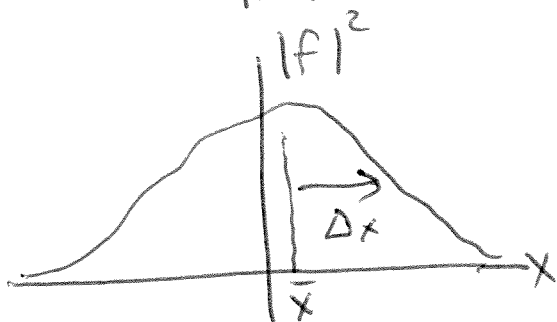
• squared norm of f: $\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(p)|^2 dp$

• average or centroid: $\bar{x} = \frac{\int_{-\infty}^{\infty} x |f(x)|^2 dx}{\|f\|^2}$

• standard deviation: $\Delta x = \left[\frac{\int_{-\infty}^{\infty} (x - \bar{x})^2 |f(x)|^2 dx}{\|f\|^2} \right]^{1/2}$

Similarly

$$\bar{p} = \frac{\int_{-\infty}^{\infty} p |\tilde{f}(p)|^2 dp}{\|f\|^2}, \quad \Delta p = \left[\frac{\int_{-\infty}^{\infty} (p - \bar{p})^2 |\tilde{f}(p)|^2 dp}{\|f\|^2} \right]^{1/2}$$



Δx & Δp estimate the widths of $|f(x)|^2$ & $|\tilde{f}(p)|^2$.

Let $g(x) = \frac{(x - \bar{x}) f(x)}{\|f\|}$, so $\int_{-\infty}^{\infty} |g(x)|^2 dx = \frac{\int_{-\infty}^{\infty} (x - \bar{x})^2 |f(x)|^2 dx}{\|f\|^2}$

$$= \frac{\Delta x^2}{\|f\|^2}$$

Similarly, let

$$\tilde{h}(p) = \frac{(p - \bar{p}) \tilde{f}(p)}{\|f\|}, \text{ so } \int_{-\infty}^{\infty} |\tilde{h}(p)|^2 dp = \frac{\int_{-\infty}^{\infty} (p - \bar{p})^2 |\tilde{f}(p)|^2 dp}{\|f\|^2}$$

$$= \frac{\Delta p^2}{\|f\|^2}$$

Note:

$$h(x) = \hat{F}_p^{-1} \tilde{h}(p) = \frac{1}{\|f\|} \left[\hat{F}_p^{-1} (p \tilde{f}(p)) - \bar{p} \hat{F}_p^{-1} (\tilde{f}(p)) \right]$$

$$= \frac{-i k f'(x) - \bar{p} f(x)}{\|f\|}$$

so $\int_{-\infty}^{\infty} g^*(x) h(x) dx = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} (x - \bar{x}) f^*(x) [-i k f'(x) - \bar{p} f(x)] dx$ (i)

$$= \frac{-i k}{\|f\|^2} \int_{-\infty}^{\infty} (x - \bar{x}) f^*(x) f'(x) dx - \frac{\bar{p}}{\|f\|^2} \int_{-\infty}^{\infty} (x - \bar{x}) f^*(x) f(x) dx$$

integrate by parts
 $u = (x - \bar{x}) f^*, dv = f' dx$

this term vanishes, but let's keep it anyway.

$$= \frac{-iK^{-1}}{\|f\|^2} \left[\int_{-\infty}^{\infty} (x-\bar{x}) f^*(x) f(x) dx - \int_{-\infty}^{\infty} [(x-\bar{x}) f^*(x)]' f(x) dx \right] - \frac{\bar{p}}{\|f\|^2} \int_{-\infty}^{\infty} (x-\bar{x}) f^*(x) f(x) dx$$

if we assume that $|f(x)|^2$ goes to zero at $\pm\infty$ faster than $\frac{1}{x}$, this goes to zero.

$$= \frac{+iK^{-1}}{\|f\|^2} \left[\int_{-\infty}^{\infty} f^*(x) f(x) dx + \int_{-\infty}^{\infty} (x-\bar{x}) f'^*(x) f(x) dx \right] - \frac{\bar{p}}{\|f\|^2} \int_{-\infty}^{\infty} (x-\bar{x}) f^*(x) f(x) dx$$

re group

$$= +iK^{-1} + \frac{1}{\|f\|^2} \left[\int_{-\infty}^{\infty} (x-\bar{x}) f^*(x) [-iK^{-1} f'(x) - \bar{p} f(x)] dx \right]^* \quad (ii)$$

So relations (i) & (ii) can be written as:

$$\int_{-\infty}^{\infty} g^*(x) h(x) dx = \begin{cases} T & (i) \\ +iK^{-1} + T^* & (ii) \end{cases}$$

where $T = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} (x-\bar{x}) f^*(x) [-iK^{-1} f'(x) - \bar{p} f(x)] dx$

averaging both options

$$\int_{-\infty}^{\infty} g^*(x) h(x) dx = \frac{(i) + (ii)}{2} = \underbrace{\text{Re}(T)} + \frac{iK^{-1}}{2}$$

this term is called Δ_{xp}

So $\int_{-\infty}^{\infty} |g(x)|^2 dx = \Delta_x^2$, $\int_{-\infty}^{\infty} |h(x)|^2 dx = \Delta_p^2$, $\int_{-\infty}^{\infty} g^*(x) h(x) dx = \Delta_{xp} + \frac{iK^{-1}}{2}$

Therefore, from Cauchy-Schwarz-Bunyakowski,

$$\Delta_x^2 \Delta_p^2 \geq \left| \Delta_{xp} + \frac{iK^{-1}}{2} \right|^2 = \Delta_{xp}^2 + \frac{1}{4K^2}$$

Schrödinger inequality

Since $\Delta x_p^2 \geq 0$,

$$\Delta x^2 \Delta p^2 \geq \Delta x_p^2 + \frac{1}{4k^2} \geq \frac{1}{4k^2}$$

$$\boxed{\Delta x \Delta p \geq \frac{1}{2|k|}}$$

Uncertainty relation.

That is, for a given Δx , $\Delta p \geq \frac{|k|}{2\Delta x|k|}$ and vice-versa.

Cannot have very small Δx & Δp simultaneously!!!

Note: $\Delta x \Delta p = \frac{1}{2|k|}$ Only if:

• $h(x) = ag(x)$ & • $\Delta x_p = 0$ ← constant

$h(x) = ag(x)$ means $-iK^+ f'(x) - \bar{p} f(x) = a(x-\bar{x}) f(x)$

Integrating: $\frac{f'(x)}{f(x)} = i\bar{p}K + iKa(x-\bar{x})$

$\ln f(x) + c = iK\bar{p}x + iKa \frac{(x-\bar{x})^2}{2}$ ← constant

$f(x) = \underbrace{e^{-c}}_{f_0 = \text{constant}} e^{iK\bar{p}x} e^{iKa \frac{(x-\bar{x})^2}{2}}$

$f'(x) = [iK\bar{p} + iKa(x-\bar{x})] f(x)$, so

$T = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} (x-\bar{x}) f^*(x) [\bar{p} + a(x-\bar{x}) - \bar{p}] f(x) dx = a \frac{\int_{-\infty}^{\infty} (x-\bar{x})^2 |f(x)|^2 dx}{\|f\|^2}$ (real)

so $\Delta x_p = \text{Re}(T) = 0$ only if a is imaginary, e.g. $a = \frac{i}{k\omega^2}$

so $\Delta x \Delta p = \frac{1}{2|k|}$ for $f(x) = \underbrace{f_0 e^{iK\bar{p}x} e^{-\frac{(x-\bar{x})^2}{2\omega^2}}}_{\text{Gaussian}}$

Exercises

- 1) Using $K=1$, find FT of $e^{-\frac{x^2}{2}}$
- 2) Using the previous result, find the F.T. of $e^{-\frac{x^2}{2\alpha^2}}$
- 3) Find Δx and Δp
- 4) Use your result from (1) to find the F.T. of: $x e^{-\frac{x^2}{2}}$, $x^2 e^{-\frac{x^2}{2}}$, $x^3 e^{-\frac{x^2}{2}}$
- 5) Based on your results find the values of the constants $\alpha, \beta, \gamma, \delta, a, b, c, d, h$, so that

$$\hat{f}_{x \rightarrow p} e^{-\frac{x^2}{2}} = \alpha e^{-\frac{p^2}{2}}$$

$$\hat{f}_{x \rightarrow p} [x e^{-\frac{x^2}{2}}] = \beta p e^{-\frac{p^2}{2}}$$

$$\hat{f}_{x \rightarrow p} ([x^2 + ax + b] e^{-\frac{x^2}{2}}) = \gamma [p^2 + ap + b] e^{-\frac{p^2}{2}}$$

$$\hat{f}_{x \rightarrow p} ([x^3 + cx^2 + dx + h] e^{-\frac{x^2}{2}}) = \delta [p^3 + cp^2 + dp + h] e^{-\frac{p^2}{2}}$$

2D Fourier transform

$$\hat{f}(p_x, p_y) = \int_{x \rightarrow p_x}^{\wedge} \int_{y \rightarrow p_y} f(x, y) = \frac{|K|}{2\pi} \iint_{-\infty}^{\infty} f(x, y) e^{-iK(xp_x + yp_y)} dx dy$$

$$f(x, y) = \int_{p_x \rightarrow x}^{\wedge} \int_{p_y \rightarrow y} \hat{f}(p_x, p_y) = \frac{|K|}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(p_x, p_y) e^{iK(xp_x + yp_y)} dp_x dp_y$$

Properties of Fourier transforms

$$\hat{f}(p) = \hat{\int}_{x \rightarrow p} f(x) = \sqrt{\frac{|q|}{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iqxp} dx$$

$$f(x) = \hat{\int}_{p \rightarrow x}^{-1} \hat{f}(p) = \sqrt{\frac{|q|}{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p) e^{iqxp} dp$$

• Shift

$$\begin{aligned} \hat{\int}_{x \rightarrow p} [f(x-a)] &= \sqrt{\frac{|q|}{2\pi}} \int_{-\infty}^{\infty} \underbrace{f(x-a)}_{x'} e^{-iqxp} dx \\ &= \sqrt{\frac{|q|}{2\pi}} \int_{-\infty}^{\infty} f(x') e^{-iqx'p} dx' e^{-iqap} \quad \begin{array}{l} x = x' + a \\ dx = dx' \end{array} \\ &= \hat{f}(p) e^{-iqap} \end{aligned}$$

• Phase

$$\begin{aligned} \hat{\int}_{x \rightarrow p} [f(x) e^{iqax}] &= \sqrt{\frac{|q|}{2\pi}} \int_{-\infty}^{\infty} [f(x) e^{iqax}] e^{-iqxp} dx \\ &= \sqrt{\frac{|q|}{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iqx(p-a)} dx = \hat{f}(p-a) \end{aligned}$$

• Scale

$$\begin{aligned} \hat{\int}_{x \rightarrow p} [f(ax)] &= \sqrt{\frac{|q|}{2\pi}} \int_{-\infty}^{\infty} \underbrace{f(ax)}_{x'} e^{-iqxp} dx \quad \begin{array}{l} x = x'/a \\ dx = dx'/a \end{array} \\ &= \sqrt{\frac{|q|}{2\pi}} \int_{-\infty \operatorname{sgn}(a)}^{\infty \operatorname{sgn}(a)} f(x') e^{-iq \frac{x'}{a} p} \frac{dx'}{a} = \frac{\operatorname{sgn}(a)}{a} \hat{f}\left(\frac{p}{a}\right) \\ &= \frac{1}{|a|} \hat{f}\left(\frac{p}{a}\right) \end{aligned}$$

• Product

$$\begin{aligned}
 \hat{\mathcal{F}}_{x \rightarrow p} [f(x)g(x)] &= \sqrt{\frac{|1\eta|}{2\pi}} \int_{-\infty}^{\infty} \underbrace{f(x)}_{\substack{\sqrt{\frac{|1\eta|}{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p') e^{i\eta x p'} dp' \\ \hat{f}(p')}} g(x) e^{-i\eta x p} dx \\
 &= \sqrt{\frac{|1\eta|}{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{|1\eta|}{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p') e^{i\eta x p'} dp' g(x) e^{-i\eta x p} dx \\
 &= \sqrt{\frac{|1\eta|}{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p') \sqrt{\frac{|1\eta|}{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\eta x (p-p')} dx dp' \\
 &= \sqrt{\frac{|1\eta|}{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p') \tilde{g}(p-p') dp' = \underline{\underline{\sqrt{\frac{|1\eta|}{2\pi}} \hat{f} * \tilde{g}(p)}}}
 \end{aligned}$$

• Convolution

$$\begin{aligned}
 \hat{\mathcal{F}}_{x \rightarrow p} [f * g(x)] &= \sqrt{\frac{|1\eta|}{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x') g(x-x') dx' e^{-i\eta x p} dx \\
 &= \int_{-\infty}^{\infty} \sqrt{\frac{|1\eta|}{2\pi}} \int_{-\infty}^{\infty} \underbrace{g(x-x')}_{x''} e^{-i\eta(x-x')p} dx f(x') e^{-i\eta x' p} dx' \\
 &\quad \substack{x'' = x-x', dx'' = dx} \\
 &= \int_{-\infty}^{\infty} \underbrace{\sqrt{\frac{|1\eta|}{2\pi}} \int_{-\infty}^{\infty} g(x'') e^{-i\eta x'' p} dx''}_{\tilde{g}(p)} f(x') e^{-i\eta x' p} dx' \\
 &= \tilde{g}(p) \int_{-\infty}^{\infty} f(x') e^{-i\eta x' p} dx' = \underline{\underline{\sqrt{\frac{2\pi}{|1\eta|}} \tilde{f}(p) \tilde{g}(p)}}}
 \end{aligned}$$

• Derivative

$$\begin{aligned} \hat{\mathcal{F}}_{x \rightarrow p} [f'(x)] &= \sqrt{\frac{|\eta|}{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-i\eta x p} dx \\ &= \sqrt{\frac{|\eta|}{2\pi}} \left[\cancel{f(x) e^{-i\eta x p}} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -i\eta p e^{-i\eta x p} f(x) dx \right] \\ &\quad \begin{array}{l} \text{assume} \\ \lim_{x \rightarrow \pm\infty} f(x) = 0 \end{array} \\ &= \underline{i\eta p \tilde{f}(p)} \end{aligned}$$

Using this repeatedly:

$$\hat{\mathcal{F}}_{x \rightarrow p} [f^{(n)}(x)] = \underline{(i\eta p)^n \tilde{f}^{(n)}(p)}$$

• Powers of x

$$\begin{aligned} \hat{\mathcal{F}}_{x \rightarrow p} [x^n f(x)] &= \sqrt{\frac{|\eta|}{2\pi}} \int_{-\infty}^{\infty} f(x) \underbrace{x^n e^{-i\eta x p}}_{\left(\frac{i}{-\eta}\right)^n \frac{d^n}{dp^n} e^{-i\eta x p}} dx \\ &= \left(\frac{i}{\eta}\right)^n \frac{d^n}{dp^n} \sqrt{\frac{|\eta|}{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\eta x p} dx \\ &= \underline{\left(\frac{i}{\eta}\right)^n \tilde{f}^{(n)}(p)} \end{aligned}$$

- Parseval's theorem

$$\begin{aligned} \int_{-\infty}^{\infty} f^*(x) g(x) dx &= \int_{-\infty}^{\infty} \left[\sqrt{\frac{|y|}{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p) e^{iyxp} dp \right]^* g(x) dx \\ &= \sqrt{\frac{|y|}{2\pi}} \int_{-\infty}^{\infty} \tilde{f}^*(p) \int_{-\infty}^{\infty} g(x) e^{-iyxp} dx dp \\ &= \int_{-\infty}^{\infty} \tilde{f}^*(p) \tilde{g}(p) dp \end{aligned}$$

In particular $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(p)|^2 dp$

- Real functions

If $f(x) = f^*(x)$

$$\begin{aligned} \tilde{f}^*(p) &= \left[\sqrt{\frac{|y|}{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iyxp} dx \right]^* \\ &= \sqrt{\frac{|y|}{2\pi}} \int_{-\infty}^{\infty} f(x) e^{iyxp} dx = \underline{\hat{f}(-p)} \end{aligned}$$

So

$\text{Re} \{ \tilde{f}(p) \}$ is even
 $\text{Im} \{ \hat{f}(p) \}$ is odd

- Asymptotic behavior for discontinuous functions.

If $f(x)$ is discontinuous in its m^{th} derivative at x_0 :

$$\hat{f}(p) = \sqrt{\frac{|p|}{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\eta x p} dx = \sqrt{\frac{|p|}{2\pi}} \left[\int_{-\infty}^{x_0} f(x) e^{-i\eta x p} dx + \int_{x_0}^{\infty} f(x) e^{-i\eta x p} dx \right]$$

For large p , can use the trick from edge effects in stationary phase.

$$\begin{aligned} \int_{-\infty}^{x_0} f(x) e^{-i\eta x p} dx &= \frac{i}{\eta p} f(x) e^{-i\eta x p} \Big|_{-\infty}^{x_0} - \frac{i}{\eta p} \int_{-\infty}^{x_0} f'(x) e^{-i\eta x p} dx \\ &= \left(\frac{i}{\eta p} \right) \frac{d}{dx} e^{-i\eta x p} \\ &= \dots = \sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{\eta p} \right)^{n+1} f^{(n)}(x_0^-) e^{-i\eta x_0 p} \end{aligned}$$

similarly

$$\int_{x_0}^{\infty} f(x) e^{-i\eta x p} dx = - \sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{\eta p} \right)^{n+1} f^{(n)}(x_0^+) e^{-i\eta x_0 p}$$

so

$$\begin{aligned} \tilde{f}(p) &= \sum_{n=0}^{\infty} \sqrt{\frac{|p|}{2\pi}} (-1)^{n+1} \left(\frac{i}{\eta p} \right)^{n+1} \underbrace{\left[f^{(n)}(x_0^+) - f^{(n)}(x_0^-) \right]}_{\text{these cancel for } n < m} e^{-i\eta x_0 p} \\ &\approx \sqrt{\frac{|p|}{2\pi}} \frac{e^{-i\eta x_0 p}}{(i\eta p)^{m+1}} \left[f^{(m)}(x_0^+) - f^{(m)}(x_0^-) \right] \propto \frac{e^{-i\eta x_0 p}}{p^{m+1}}, p \rightarrow \pm\infty \end{aligned}$$

Similarly, if $\tilde{f}(p)$ is discontinuous in m^{th} derivative,
 $f(x) \approx \sqrt{\frac{|p|}{2\pi}} \left(\frac{i}{\eta x} \right)^{m+1} \left[\tilde{f}^{(m)}(p_0^+) - \tilde{f}^{(m)}(p_0^-) \right] \propto \frac{1}{x^{m+1}}, x \rightarrow \pm\infty$

Discrete Fourier transform (DFT)

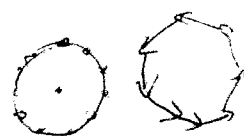
Instead of $f(x)$ we have $f_n, n=0, \dots, N-1$

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-i2\pi \frac{mn}{N}}$$

Discrete Fourier transform

Inverse: try:

$$\begin{aligned} f_{n'} &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} F_m e^{i2\pi \frac{mn'}{N}} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} f_n \underbrace{\sum_{m=0}^{N-1} e^{i2\pi \frac{(n'-n)m}{N}}}_{N \delta_{n'-n}} \end{aligned}$$



So:

$$f_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} F_m e^{i2\pi \frac{mn}{N}}$$

Inverse Discrete Fourier transform

Approximating FT with DFT

(Notice that the sums can be shifted,) if we define

$$f_{n-N} = f_n$$

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=-\lfloor \frac{N-1}{2} \rfloor}^{\lfloor \frac{N-1}{2} \rfloor} f_n e^{-i2\pi \frac{mn}{N}}$$

Let f_n be a sampling of $f(x)$:

$$f_n = f(n \Delta x)$$

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=-\lfloor \frac{N-1}{2} \rfloor}^{\lfloor \frac{N-1}{2} \rfloor} f(n \Delta x) e^{-i2\pi \frac{mn}{N}}$$

For very large N , and small Δx ,
 can approximate the sum as an integral

$$F_m \approx \frac{1}{\sqrt{N}} \int_{-x_1}^{x_2} f(x) e^{-i2\pi m x} \frac{dx}{\Delta x}$$

where $n\Delta x \rightarrow x$

$$x_1 = \lfloor \frac{N-1}{2} \rfloor \Delta x, \quad x_2 = \lfloor \frac{N}{2} \rfloor \Delta x$$

Assume $N \Delta x = \text{big} \gg \text{width of } f(x)$.
big small note $x_1 \approx x_2 \approx \frac{N\Delta x}{2} = \text{big}$.

Then

$$F_m \approx \frac{1}{\sqrt{N} \Delta x} \int_{-\infty}^{\infty} f(x) e^{-i2\pi x \left(\frac{m}{N\Delta x}\right)} dx$$

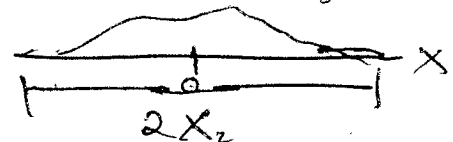
$$= \frac{\tilde{f}\left(\frac{m}{N\Delta x}\right)}{\sqrt{N} \Delta x}$$

So the sampling distance in ν is $\frac{1}{N\Delta x} \approx \frac{1}{2x_2}$

where $2x_2$ is the width over which
 we're sampling $f(x)$.

Therefore:

- To increase resolution in $\tilde{f}(\nu)$ \longrightarrow must increase range in $f(x)$

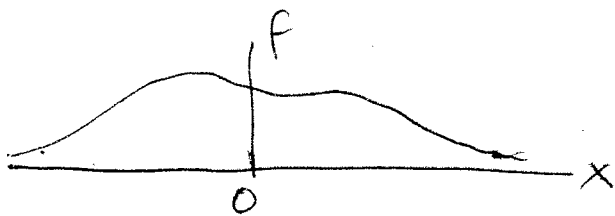


- To increase range in $\tilde{f}(\nu)$ and avoid aliasing \longrightarrow must decrease sampling spacing in $f(x)$



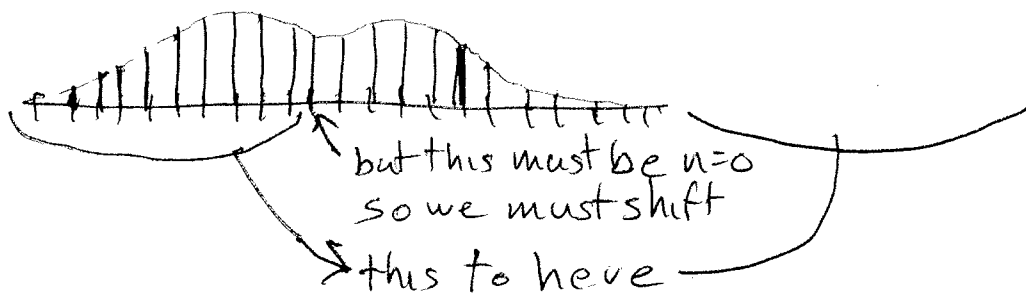
Shifting the functions.

Notice that, if we sample:

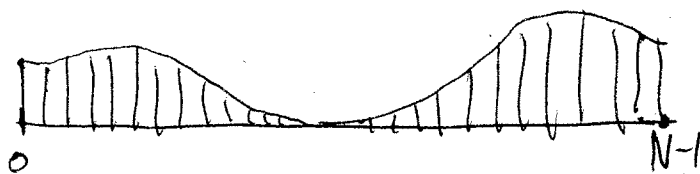


we get

f_n

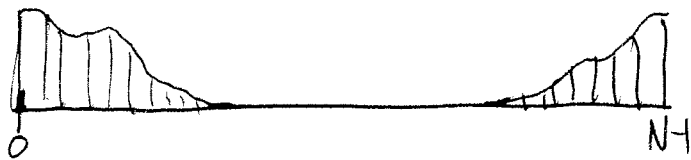


so we get



→ this is f_n

Similarly, once we get F_n , it will look like



To reconstruct $\tilde{f}(v)$ we must cut the second half and place it before the first. we also need to multiply by $\sqrt{N} \Delta x$.

Fast Fourier transform (FFT)

Notice that the, for each m , the DFT involves the sum of N terms. Since m runs from 0 to $N-1$, then N^2 must be performed. The time of computation can therefore be expected to be proportional to N^2 .

The FFT is an algorithm for performing the DFT whose time of computation is proportional to $N \log N$. While it can work for any N , its simplest form can be understood if $N = 2^M$ (so that $M = \log_2 N$):

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-i2\pi n m / N} = \frac{1}{\sqrt{N}} \left[\underbrace{\sum_{n'=0}^{N/2-1} f_{2n'} e^{-i2\pi (2n') m / N}}_{\text{terms with even } n} + \underbrace{\sum_{n'=0}^{N/2-1} f_{(2n'+1)} e^{-i2\pi (2n'+1) m / N}}_{\text{terms with odd } n} \right]$$

write as $\frac{N}{2}$

$$= \frac{1}{\sqrt{2}} \left[\underbrace{\frac{1}{\sqrt{N/2}} \sum_{n'=0}^{N/2-1} f_{2n'} e^{-i2\pi n' m / (N/2)}}_{\text{DFT of size } N/2} + e^{-i2\pi m / N} \underbrace{\frac{1}{\sqrt{N/2}} \sum_{n'=0}^{N/2-1} f_{(2n'+1)} e^{-i2\pi n' m / (N/2)}}_{\text{DFT of size } N/2} \right]$$

Each of these two sums is itself a DFT of size $\frac{N}{2}$.

They can be joined:

$$F_m = \frac{1}{\sqrt{N}} \sum_{n'=0}^{N/2-1} \left(f_{2n'} + e^{-i2\pi m / N} f_{(2n'+1)} \right) e^{-i2\pi n' m / (N/2)}$$

The same separation can be done M times.

2D DFT

$$F_{m_1, m_2} = \frac{1}{N} \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} f_{n_1, n_2} e^{-i 2\pi \frac{(m_1 n_1 + m_2 n_2)}{N}}$$

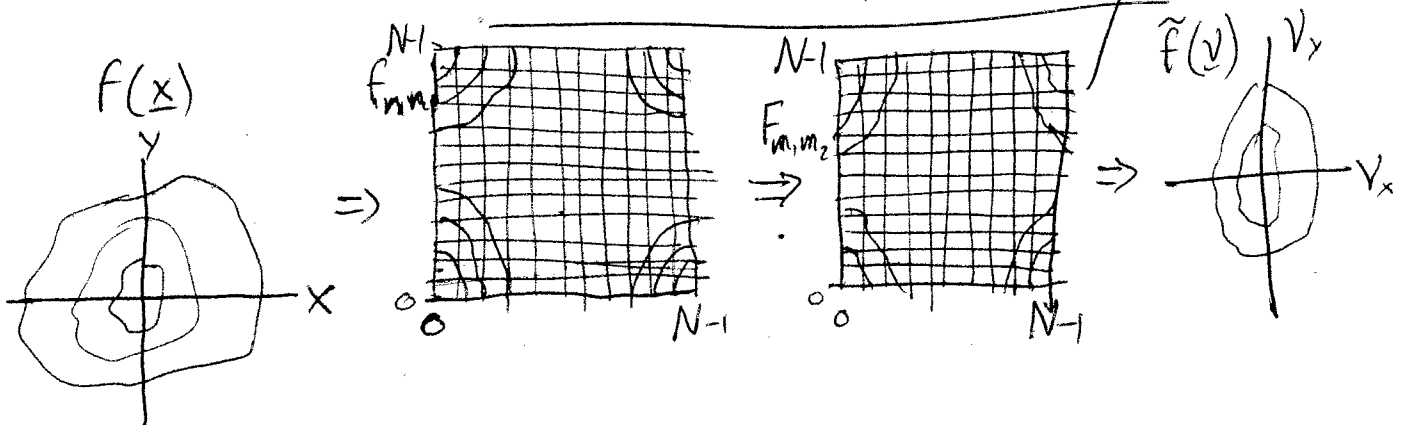
$$f_{n_1, n_2} = \frac{1}{N} \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} F_{m_1, m_2} e^{i 2\pi \frac{(m_1 n_1 + m_2 n_2)}{N}}$$

Using 2D DFT to approximate 2D FT:

if $f_{n_1, n_2} = f(n_1 \Delta x, n_2 \Delta x)$,

and $N \Delta x$ is bigger than width of f , then:

$$F_{m_1, m_2} \approx \frac{1}{N \Delta x^2} \tilde{f}\left(\frac{m_1}{N \Delta x}, \frac{m_2}{N \Delta x}\right)$$



Fast Fourier transform: time $\propto N^2 \log N$