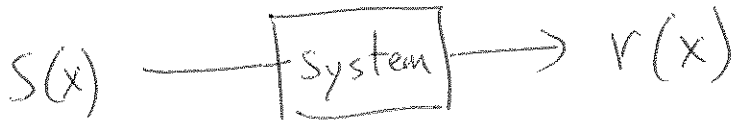


Linear Systems



$$S(x) \longrightarrow \boxplus \longrightarrow r(x)$$

Linear

$$S_1(x) + S_2(x) \longrightarrow \boxplus \longrightarrow r_1(x) + r_2(x)$$

Shift invariant

$$S(x-x_0) \longrightarrow \boxplus \longrightarrow r(x-x_0)$$

consider LSI system

if $S(x) \boxplus r(x)$, then $S'(x) \boxplus ?$

$$S'(x) = \lim_{\Delta x \rightarrow 0} \frac{S(x+\Delta x) - S(x)}{\Delta x} \boxplus \lim_{\Delta x \rightarrow 0} \frac{r(x+\Delta x) - r(x)}{\Delta x} = r'(x)$$

$$\underline{S'(x) \boxplus r'(x)}$$

$$\underbrace{S'(x) - i\nu 2\pi S(x)}_{\text{if this } = 0} \boxplus \underbrace{r'(x) - i 2\pi \nu r(x)}_{\text{then this } = 0}$$

$$S(x) = S_0 e^{i 2\pi \nu x} \boxplus r(x) = r_0 e^{i 2\pi \nu x}$$

$$e^{i 2\pi \nu x} \boxplus H(\nu) e^{i 2\pi \nu x}$$

In general

$$s(x) = \int_{-\infty}^{\infty} \tilde{s}(v) e^{i2\pi v x} dv \rightarrow \int_{-\infty}^{\infty} \underbrace{\tilde{s}(v) H(v)}_{\tilde{r}(v)} e^{i2\pi v x} dv = r(x)$$

$$\tilde{r}(v) = \tilde{s}(v) H(v) \quad H(v) = \text{transfer function.}$$

$$\begin{array}{ccc}
 s(x) & \xrightarrow{\text{box}} & r(x) \\
 \downarrow \hat{\mathcal{F}} & & \uparrow \hat{\mathcal{F}}^{-1} \\
 \hat{s}(v) & \rightarrow & \hat{r}(v) = \hat{s}(v) H(v)
 \end{array}$$

On the other hand:

$$s(x) = \int s(x') \delta(x-x') dx' \xrightarrow{\text{box}} r(x) = \int s(x') G(x-x') dx = s * G(x)$$

where $\delta(x) \xrightarrow{\text{box}} G(x)$ impulse response

so the effect of a LSI system is always a convolution, which in Fourier space is a product. Therefore $\hat{\mathcal{F}}_{x \rightarrow v} G(x) = H(v)$

The system acts as a blur, or equivalently, as a Fourier filter.

Examples: { Low pass
High pass
DC removal
derivatives (including "fractional")
combination of images



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Noise $n(x)$  random.

Fourier transform $\int_{x \rightarrow \nu} n(x) = \tilde{n}(\nu)$

often, we do not know \tilde{n} , but we do know roughly how it behaves

$$\langle |\tilde{n}(\nu)|^2 \rangle = S_n(\nu) \text{ power spectrum of the noise.}$$

average over many "realizations"

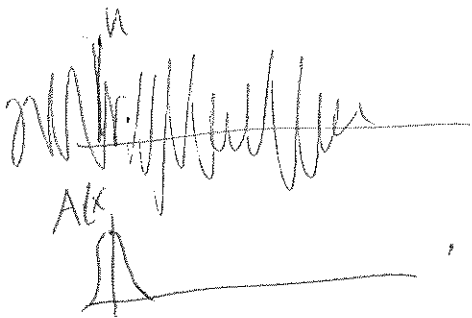
note

$$\int_{\nu \rightarrow x} S_n(\nu) e^{i2\pi\nu x} d\nu = \left\langle \int \tilde{n}^*(\nu) \tilde{n}(\nu) e^{i2\pi\nu x} d\nu \right\rangle$$

$$\left[\int n(x') e^{-i2\pi\nu x'} dx' \right]^*$$

$$= \left\langle \int \int n^*(x') \tilde{n}(\nu) e^{i2\pi\nu(x+x')} d\nu dx' \right\rangle$$

$$= \left\langle \int n^*(x') \underbrace{\int \tilde{n}(\nu) e^{i2\pi\nu(x+x')} d\nu}_{n(x+x')} dx' \right\rangle = \left\langle \underbrace{\int n^*(x') n(x+x') dx'}_{\text{autocorrelation of noise } A_n(x)} \right\rangle$$



$$\int_{x \rightarrow \nu} A_n(x) = S_n(\nu)$$

Wiener-Khinchin Theorem.

Examples if $A_n(x) \approx \delta(x)$, $S_n(\nu) \approx \text{constant}$
white noise

if $S_n(\nu) \propto \frac{1}{\nu^\alpha}$, $0 < \alpha < 2$ usually $\alpha \approx 1$.

Noise filtering & deconvolution

Consider a noisy system, where

$$r(x) = G * s(x) + n(x)$$

\uparrow response \uparrow impulse response \uparrow signal \uparrow noise

We measure $r(x)$, we know $G(x)$ (system).
 We do not know $n(x)$ but know its statistics $A_n(x)$.
 We want to estimate $s(x)$

In Fourier space:

$$\tilde{r}(\nu) = H(\nu) \tilde{s}(\nu) + \tilde{n}(\nu)$$

Propose estimate $\hat{\tilde{s}}(\nu) = W(\nu) \tilde{r}(\nu)$
 \uparrow filter.

The total error is

$$\begin{aligned} \mathcal{E} &= \int |\hat{\tilde{s}}(\nu) - \tilde{s}(\nu)|^2 d\nu = \int |WH\tilde{s} + W\tilde{n} - \tilde{s}|^2 d\nu \\ &= \int |(WH-1)\tilde{s} + W\tilde{n}|^2 d\nu \end{aligned}$$



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$$\mathcal{E} = \int |WH-1|^2 |\tilde{s}|^2 d\nu + \int |W|^2 |\tilde{n}|^2 d\nu + \int [W^* \tilde{n}^* (WH-1)\tilde{s} + c.c.] d\nu$$

Take average over possible realizations of \tilde{n} , and use $\langle \tilde{n} \rangle = 0$, $\langle |\tilde{n}|^2 \rangle = S_n(\nu)$ (noise power spectrum).

$$\langle \mathcal{E} \rangle = \int (W^* H^* - 1)(WH-1) |\tilde{s}|^2 d\nu + \int W^* W S_n(\nu) d\nu + 0$$

Now find $W(\nu)$ that minimizes error.
use variational calculus:

$$\frac{\delta}{\delta W^*(\nu)} \int W^* (WH-1) d\nu = W^* H^* - 1$$

$$\text{Force } \frac{\delta \langle \mathcal{E} \rangle}{\delta W^*(\nu)} = 0$$

$$H^* (WH-1) |\tilde{s}|^2 + W S_n = 0$$

$$\text{so } W [|H|^2 |\tilde{s}|^2 + S_n] = H^* |\tilde{s}|^2$$

$$W = \frac{H^*}{|H|^2 + S_n / |\tilde{s}|^2}$$

Wiener filter