CNR IFN Istituto di Fotonica e Nanotecnologie

POLITECNICO MILANO 1863

Oscillators: frequency stability and noise analysis

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Winter College on Optics

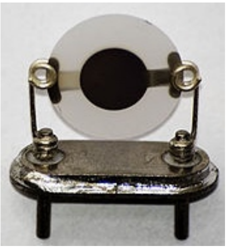
Optical Frequency Combs: from multispecies gas sensing to high precision interrogation of atomic and molecular targets

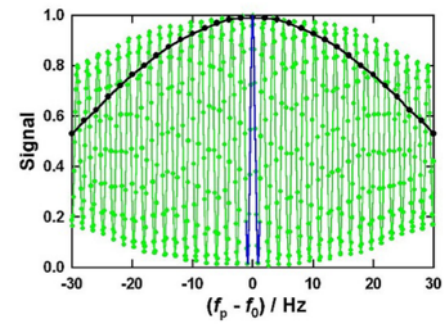
15 – 26 February 2016

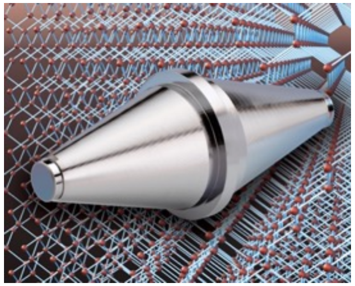
The Abdus Salam International Centre for Theoretical Physics

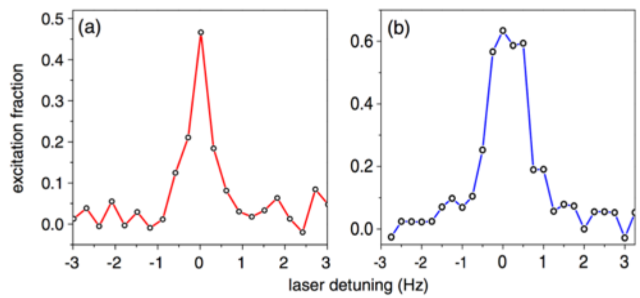
Outline

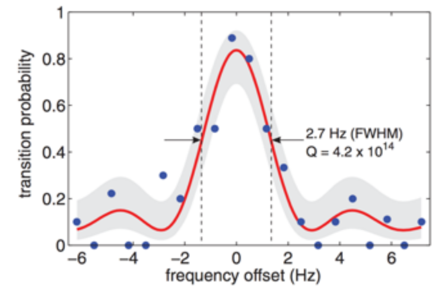
1. Frequency/Phase stability
2. Frequency domain characterization
3. Time domain characterization
4. Frequency stabilization principles











1. Clock signal – Noisy sine-wave
2. Fourier frequency domain stability
3. Time domain stability
4. Spectrum of a noisy sine-wave

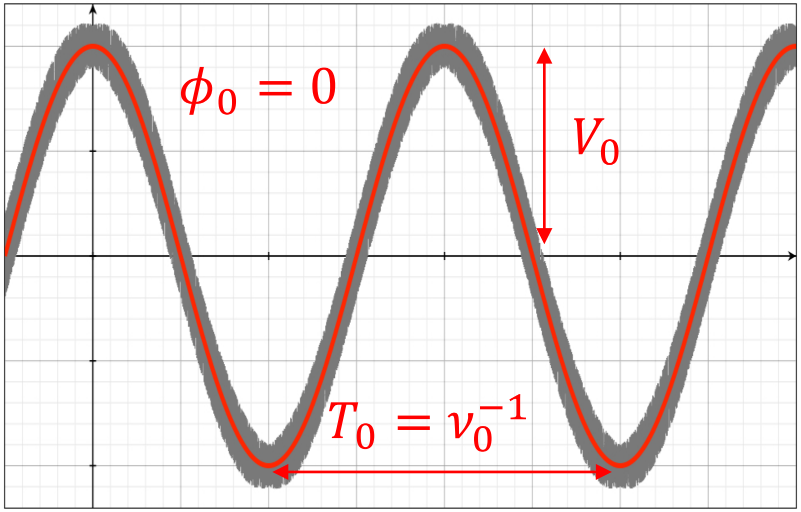
Clock signal - Noisy sinewave

$$v_{ideal}(t) = V_0 \cos(2\pi\nu_0 t + \phi_0)$$

V_0 is the amplitude [V]

ν_0 is the frequency [Hz=s^{-1}]

ϕ_0 is a constant phase [rad]



Oscillator output signal

$$v(t) = [V_0 + \varepsilon(t)] \cos[2\pi\nu_0 t + \varphi(t)]$$

$\varepsilon(t)$ is a the amplitude noise [V] $\quad \overline{\varepsilon(t)} = 0;\ |\alpha(t)| = \dfrac{|\varepsilon(t)|}{V_0} \ll 1$

$\varphi(t)$ is the phase noise [rad] $\quad \overline{\varphi(t)} = 0;\ |x(t)| = \dfrac{|\varphi(t)|}{2\pi\nu_0} \ll 1$

Not always fulfilled

$v(t) = V_0[1 + \alpha(t)] \cos[2\pi\nu_0 t + \varphi(t)]$ polar representation

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Cartesian representation

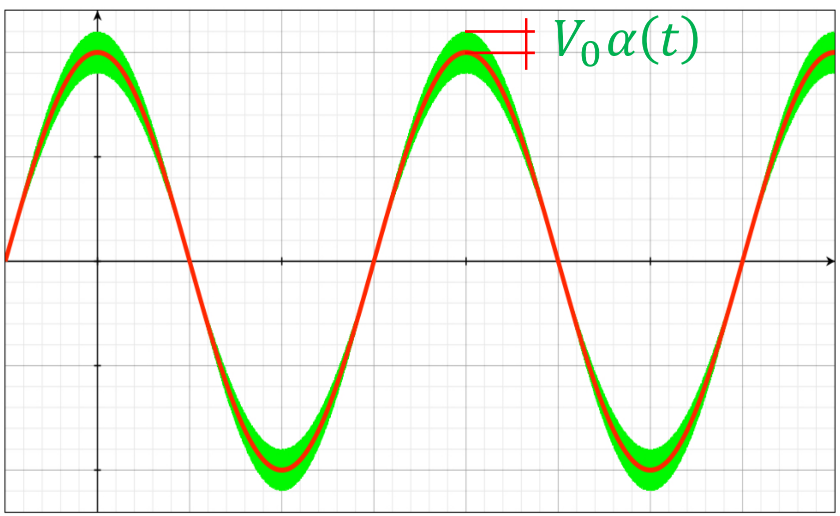
$$v(t) = V_0[1 + \alpha(t)]\cos[2\pi\nu_0 t + \varphi(t)] \quad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

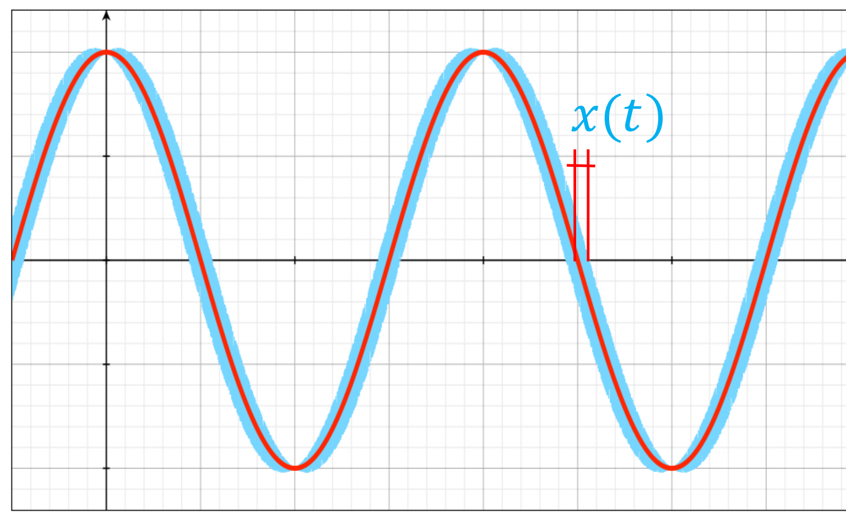
$$v(t) = V_0 \cos(2\pi\nu_0 t) + v_P(t)\cos(2\pi\nu_0 t) - v_Q(t)\sin(2\pi\nu_0 t)$$

$$v_P(t) = V_0[\cos\varphi\,(1+\alpha) - 1] \qquad v_Q(t) = V_0(1+\alpha)\sin\varphi$$

If $|\alpha(t)| \ll 1$ and $|\varphi(t)| \ll 1$ **LOW NOISE SIGNAL**

$$v(t) \cong V_0 \cos(2\pi\nu_0 t) + V_0\alpha(t)\cos(2\pi\nu_0 t) - V_0\varphi(t)\sin(2\pi\nu_0 t)$$





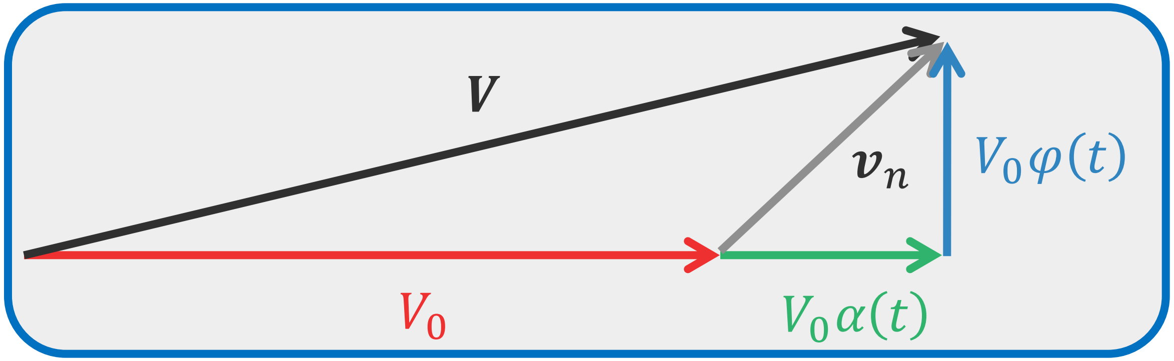
Phasor representation

$$v(t) = V_0 \cos(2\pi\nu_0 t) + V_0\alpha(t)\cos(2\pi\nu_0 t) - V_0\varphi(t)\sin(2\pi\nu_0 t)$$

$$v(t) = \mathbb{R}\left\{V_0 e^{i(2\pi\nu_0 t)} + V_0\alpha(t)e^{i(2\pi\nu_0 t)} + iV_0\varphi(t)e^{i(2\pi\nu_0 t)}\right\}$$

$$v(t) = \mathbb{R}\left\{[V_0 + V_0\alpha(t) + iV_0\varphi(t)]e^{i(2\pi\nu_0 t)}\right\}$$

$$\boldsymbol{V}(t) = V_0 + V_0\alpha(t) + iV_0\varphi(t) = V_0 + \boldsymbol{v}_n$$



Phase and frequency noise

$$v(t) = V_0[1+\alpha(t)]\cos[2\pi\nu_0 t + \varphi(t)]$$

$$\nu(t) = \frac{1}{2\pi}\frac{d}{dt}[2\pi\nu_0 t + \varphi(t)] = \nu_0 + \frac{1}{2\pi}\frac{d\varphi(t)}{dt} = \nu_0 + \frac{\dot{\varphi}(t)}{2\pi}$$

Frequency fluctuation/noise

$$\Delta\nu(t) = \nu(t) - \nu_0 = \frac{\dot{\varphi}(t)}{2\pi}\ [\mathrm{Hz}] \quad \overline{\Delta\nu(t)} = 0;\ \overline{\dot{\varphi}(t)} = 0;\ |\dot{\varphi}(t)| \ll 1$$

Normalized quantities

$$x(t) = \frac{\varphi(t)}{2\pi\nu_0}\ [\mathrm{s}]$$

Phase time Noise

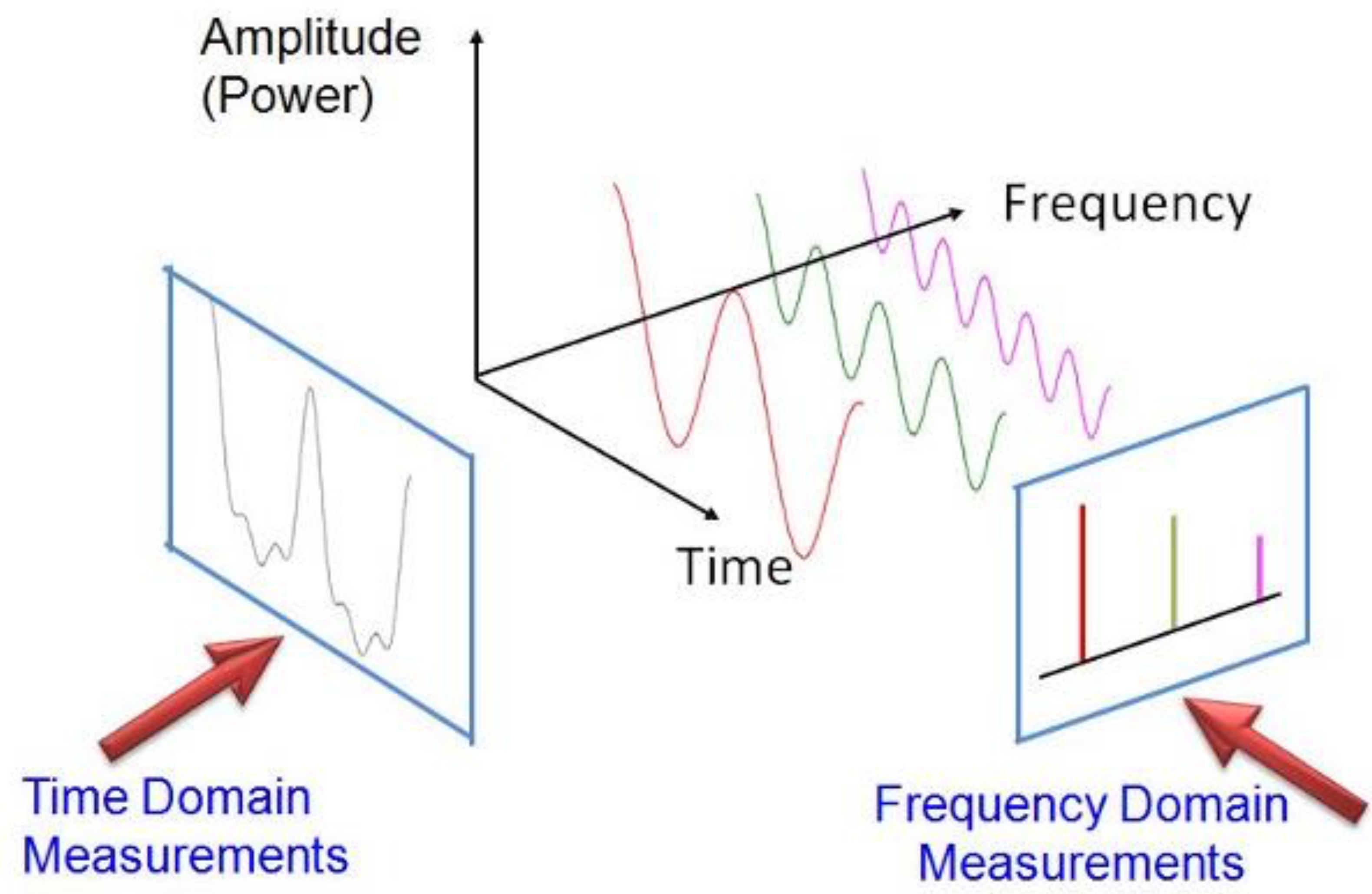
$$\frac{d}{dt} \Rightarrow$$

$$y(t) = \frac{\Delta\nu(t)}{\nu_0} = \frac{\dot{\varphi}(t)}{2\pi\nu_0} = \dot{x}(t)[1]$$

Fractional frequency noise

Invariant for phase/frequency multiplication/division

Fourier frequency domain



Power Spectral Density

Power spectral density (PSD) of a **stationary** random signal *z*(*t*)

$$S_z^{TS}(f) = \int_{-\infty}^{+\infty} R_z(\tau) e^{-i2\pi f\tau} d\tau = \mathbb{F}\{R_z(\tau)\}$$

$\mathbb{F}\{\cdot\}$ Fourier transform

$$R_z(\tau) = \mathbb{E}\{z(t)z(t+\tau)\} = \iint_{-\infty}^{+\infty} z(t)z(t+\tau)w(z,t;z,t+\tau)\,(dz)^2$$

$\mathbb{E}\{\cdot\}$ Expectation value $\qquad w\{\cdot;\cdot\}$ conjunted probability density

Assuming *z*(*t*) **ergotic** (ensemble average equal to time average)

$$R_z(\tau) = \int_{-\infty}^{+\infty} S_z^{SS}(f) e^{+i2\pi f\tau} df = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} z(t)z(t+\tau)\,dt$$

$$R_z(\tau = 0) = \int_{-\infty}^{+\infty} S_z^{SS}(f) df = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} z^2(t)\,dt = \overline{z^2}$$

Power Spectral Density

The direct measurement of $R_z(\tau)$ and $S^{TS}{}_z(f)$ needs infinite time…

A more convenient way is:

$$S_z^{TS}(f) = \lim_{T\to\infty} \frac{1}{T} |Z_T(f)|^2; \qquad Z_T(f) = \int_{-T/2}^{+T/2} z(t) e^{-i2\pi f t} dt$$

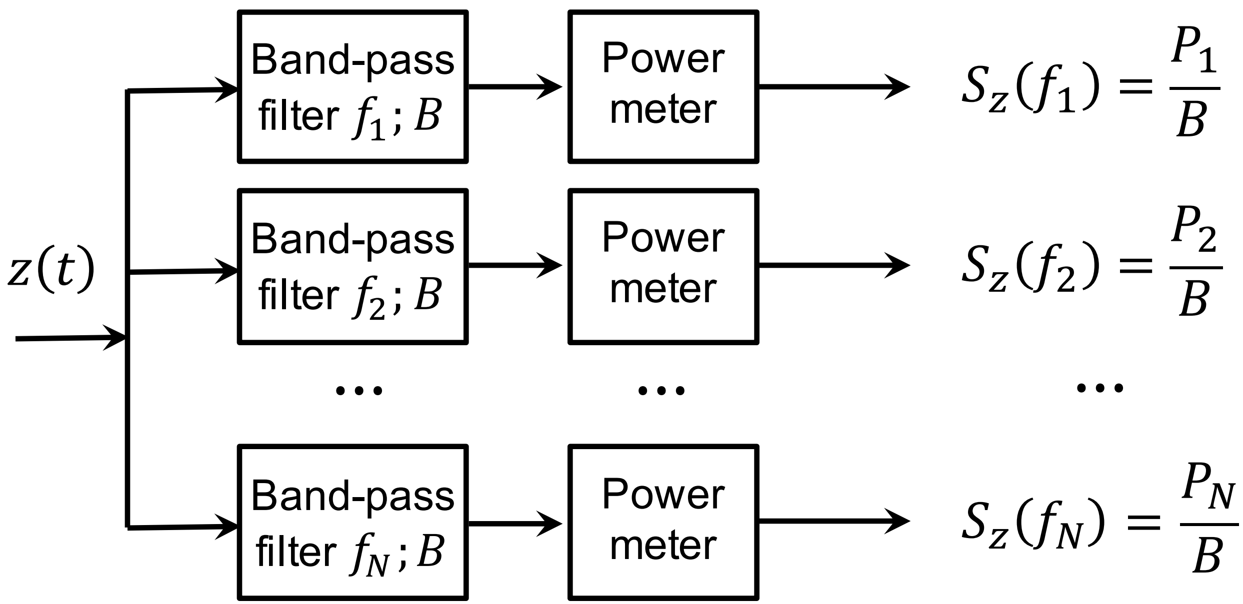
Actual waveforms $z(t)$ are real signals and $S_z^{TS}(f) = S_z^{TS}(-f)$

$$S_z^{SS}(f) = S_z(f) = \begin{cases} 2S_z^{TS}(f) & \text{for } f \geq 0 \\ 0 & \text{for } f < 0 \end{cases}$$

$$R_z(\tau = 0) = \int_0^{+\infty} S_z(f) df = \overline{z^2}$$

Physical meaning of PSD

$$\int_{f_0}^{f_0+B} S_z(f)df = P(B) = S_z(f_0)B \qquad S_z(f_0) = \frac{P(B)}{B} \quad [\frac{\text{power}}{\text{Hz}}]$$



The PSD extends the concept of root-mean-square value to the frequency domain

Power spectral density of Phase Noise

$$S_\varphi(f) = \lim_{T\to\infty} \frac{2}{T}|\varphi_T(f)|^2 \quad [\frac{\mathrm{rad}^2}{\mathrm{Hz}}]$$

$$S_x(f) = \frac{1}{(2\pi\nu_0)^2} S_\varphi(f) \quad [\frac{\mathrm{s}^2}{\mathrm{Hz}}]$$

$$x(t) = \frac{\varphi(t)}{2\pi\nu_0} \ [\mathrm{s}]$$

$$\mathcal{L}(f) = \frac{1}{2} S_\varphi(f) \quad [\frac{\mathrm{rad}^2}{\mathrm{Hz}}]$$

$$\mathcal{L}(f) = 10\log_{10}\left[\frac{1}{2} S_\varphi(f)\right] \quad [\frac{\mathrm{dB_c}}{\mathrm{Hz}}]$$

$$\varphi_{rms} = \sqrt{\overline{\varphi^2}} = \sqrt{\int_{f_L}^{f_H} S_\varphi(f)df} \quad [\mathrm{rad}]$$

$$x_{rms} = \sqrt{\overline{x^2}} = \frac{1}{2\pi\nu_0}\sqrt{\int_{f_L}^{f_H} S_\varphi(f)df} \quad [\mathrm{s}]$$

Power spectral density of Frequency Noise

$$S_{\Delta\nu}(f) = S_{\frac{\dot{\varphi}}{2\pi}}(f) = \frac{|i2\pi f|^2}{(2\pi)^2} S_{\varphi}(f) = f^2 S_{\varphi}(f) \quad [\frac{\mathrm{Hz}^2}{\mathrm{Hz}}]$$

$$y(t) = \frac{\Delta\nu(t)}{\nu_0} = \dot{x(t)} \quad [1]$$

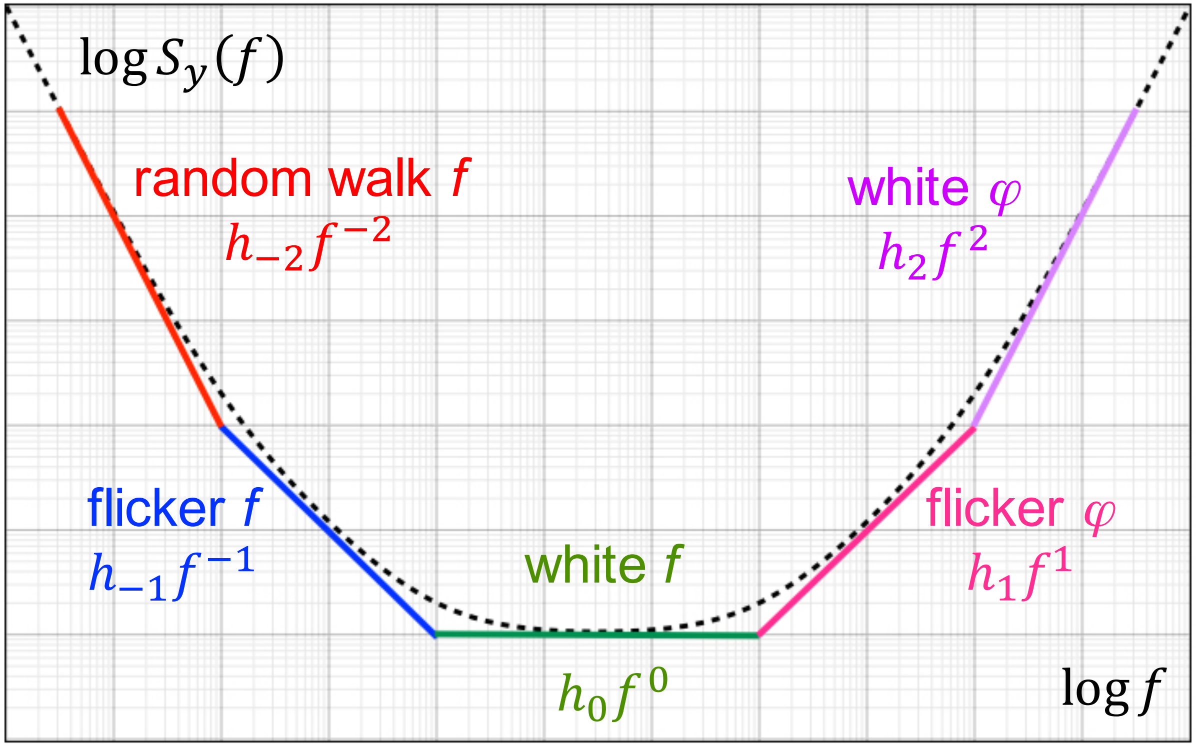
$$S_y(f) = \frac{1}{{\nu_0}^2} S_{\Delta\nu}(f) = \frac{f^2}{{\nu_0}^2} S_{\varphi}(f) = |i2\pi f|^2 S_x(f) \quad [\frac{1}{\mathrm{Hz}}]$$

$$y_{rms} = \sqrt{\overline{y^2}} = \sqrt{\int_{f_L}^{f_H} S_y(f) df} \quad [1]$$

Power law: frequency noise

Experimental evidence shows

$$S_y(f) = \begin{cases} \sum_{\alpha=-2}^{+2} h_\alpha f^\alpha & 0 \le f < f_H \\ 0 & f > f_H \end{cases}$$



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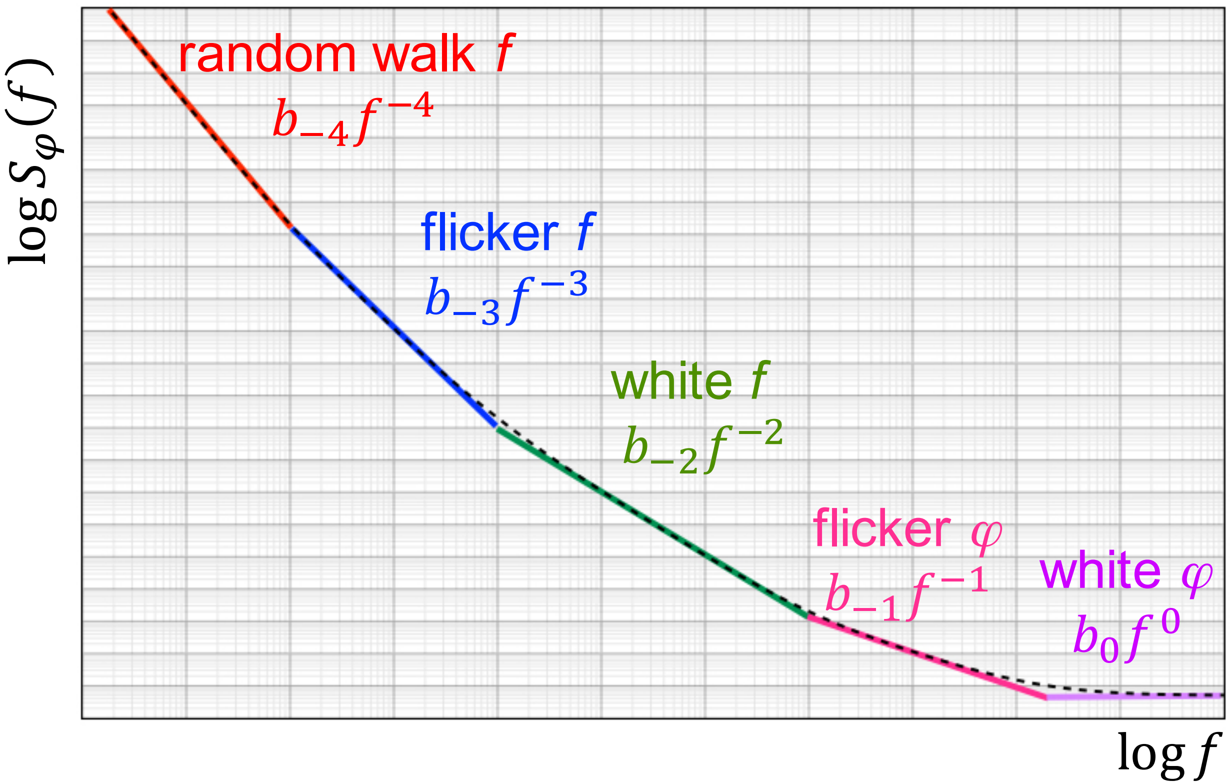
Power law: phase noise

$$S_\varphi(f) = \frac{\nu_0^2}{f^2} \sum_{\alpha=-2}^{+2} h_\alpha f^\alpha = \sum_{\alpha=-2}^{+2} \nu_0^2 h_\alpha f^{\alpha-2} = \sum_{i=-4}^{0} b_i f^i$$

$$b_i = \nu_0^2 h_{i+2}$$

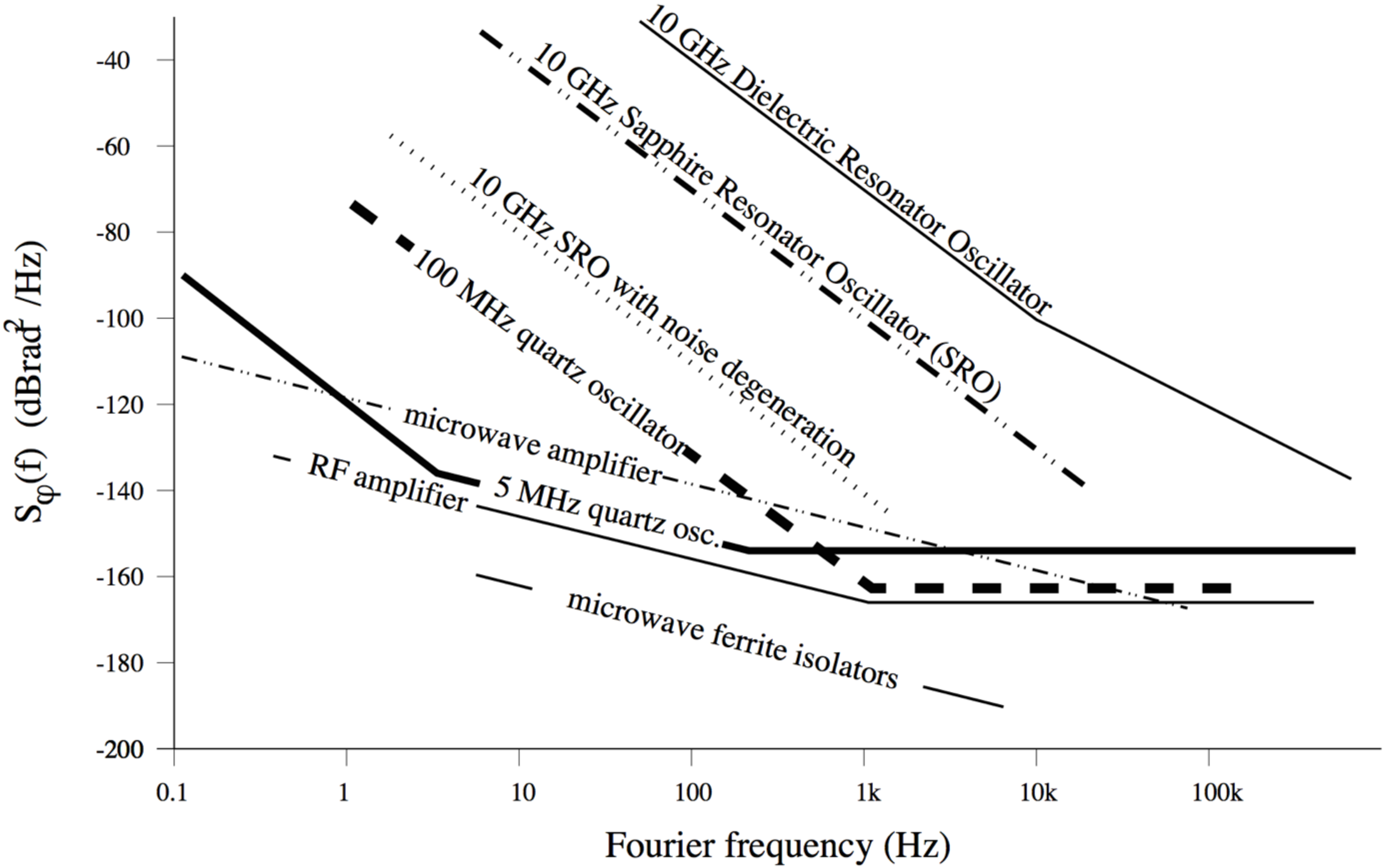


Type of noise contributions

$S_y(f)$ [Hz²/Hz]	Type of noise	Main contribution	Relation
$h_{-2}f^{-2}$	Random walk (RW) of frequency	Different unknown origin	-
$h_{-1}f^{-1}$	Flicker of frequency	Leeson effect supplied by flicker phase/resonator thermal noise	-
h_0	White frequency/ RW of phase	Resonator in band /quantum/thermal noise (Leeson effect)	$h_0 = \begin{cases} FkT/PQ^2 \\ Fh\nu_0/PQ^2 \end{cases}$
$h_1 f$	Flicker of phase	Up-conversion of the amplifier flicker noise by amplifier non-linearity	-
$h_2 f^2$	White phase	Thermal/quantum/electronics noise in amplifiers	$h_2 = \begin{cases} FkT/P\nu_0^2 \\ Fh\nu_0/P\nu_0^2 \end{cases}$

$$S_\varphi(f) = \sum_{i=-4}^{0} b_i f^{\,i}; \quad b_i = \nu_0^2 h_{i+2}$$

PSD Examples



Time domain stability: classical variance

By means of an electronic counter, the average frequency (number of counts, i.e. number of cycles) can be measured over a time interval τ

$$\overline{\nu_k}(\tau) = \frac{N_k}{\tau} = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} \nu(t)dt = \nu_0 + \frac{1}{\tau}\int_{t_k}^{t_k+\tau} \Delta\nu(t)dt$$

$$y(t) = \frac{\Delta\nu(t)}{\nu_0} = \dot{x(t)} \qquad x(t) = \frac{\varphi(t)}{2\pi\nu_0}$$

$$\overline{y_k}(\tau) = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)dt = \frac{\varphi(t_k+\tau) - \varphi(t_k)}{2\pi\nu_0\tau}$$

Frequency instability is defined as:

$$I^2(\tau) = \sigma^2[\overline{y_k}(\tau)] = \mathbb{E}\left[\overline{y_k}^{\,2}\right]$$

From classical to Allan Variance

A possible determination/definition of frequency instability is represented by the N-sample variance:

$$\sigma_y^2(N,T,\tau) = \mathbb{E}\left[\frac{1}{N-1}\sum_{i=1}^{N}\left(\bar{y}_i - \frac{1}{N}\sum_{j=1}^{N}\bar{y}_j\right)^2\right]$$

Drawbacks: dependence from *N* and it diverges for flicker and random walk noises for $\tau \to \infty$

Allan Variance (zero dead-time two-sample variance)

http://www.allanstime.com/AllanVariance/

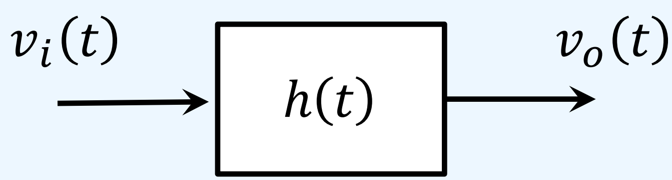
$$\sigma_y^2(\tau) = \mathbb{E}\left[\sum_{i=1}^{2}\left(\bar{y}_i - \frac{1}{2}\sum_{j=1}^{2}\bar{y}_j\right)^2\right] = \frac{1}{2}\mathbb{E}[(\overline{y_2} - \overline{y_1})^2]$$

$$\langle\sigma_y^2(\tau)\rangle_m = \frac{1}{2(m-1)}\sum_{i=1}^{m}(\overline{y_{i+1}} - \bar{y}_i)^2$$

It converges for flicker and random walk noises for $\tau \to \infty$

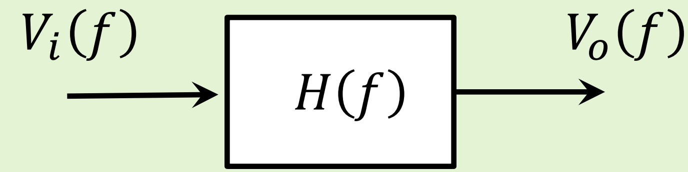
Connection between time and Fourier frequency domains

Linear and time-invariant systems (LTI)

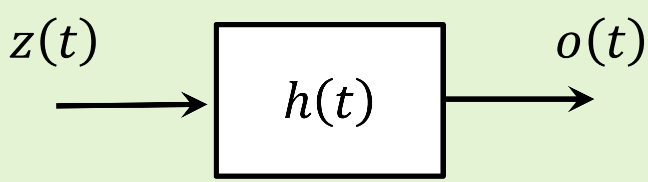


$$v_o(t) = \int_{-\infty}^{+\infty} h(t-\tau)v_i(\tau)d\tau = h(t) * v_i(t)$$

⇨



$$V_o(f) = H(f) \cdot V_i(f)$$



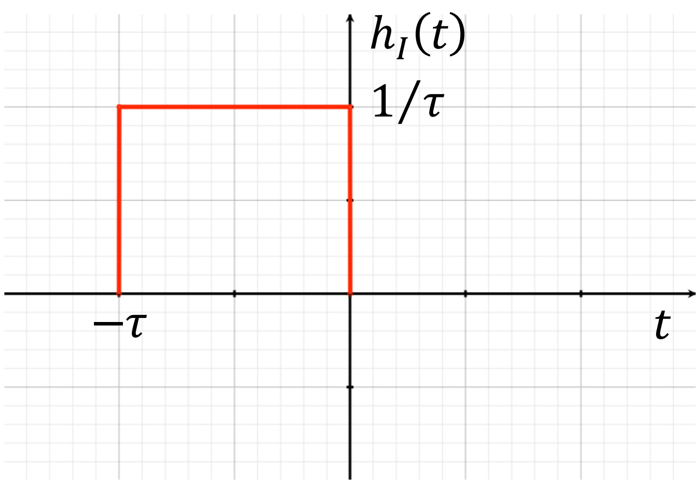
$$S_o(f) = |H(f)|^2 \cdot S_z(f)$$

Classical variance

convolution

$$I^2(\tau) = \sigma^2(\overline{y_k}) = \mathbb{E}[\overline{y_k}^{\,2}] = \mathbb{E}\left\{\left[\frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)dt\right]^2\right\} = \mathbb{E}\left\{\left[\int_{-\infty}^{+\infty} h_I(t_k - t)y(t)dt\right]^2\right\}$$

$$h_I(t) = \begin{cases} 0 & \text{elsewhere} \\ 1/\tau & t_k < t < t_k + \tau \end{cases} \quad \Rightarrow \quad H_I(f) = \frac{\sin(\pi f \tau)}{\pi f \tau}$$



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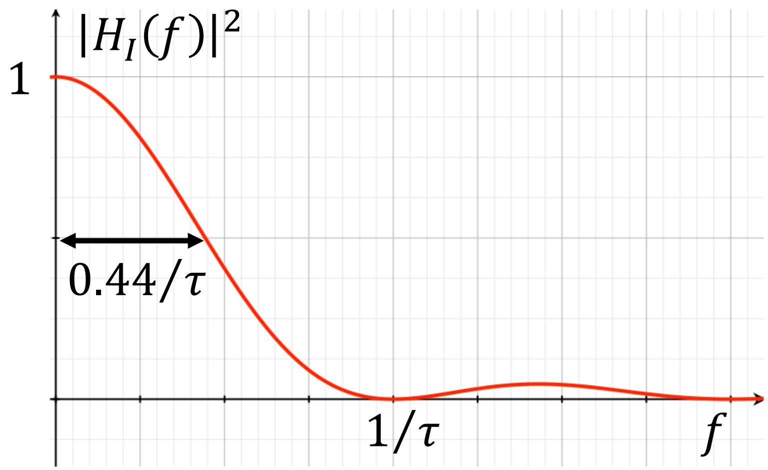


$$I^2(\tau) = \sigma^2(\overline{y_k}) = \int_0^{+\infty} |H_I(f)|^2 S_y(f)df = \int_0^{+\infty} \left|\frac{\sin(\pi f \tau)}{\pi f \tau}\right|^2 S_y(f)df$$

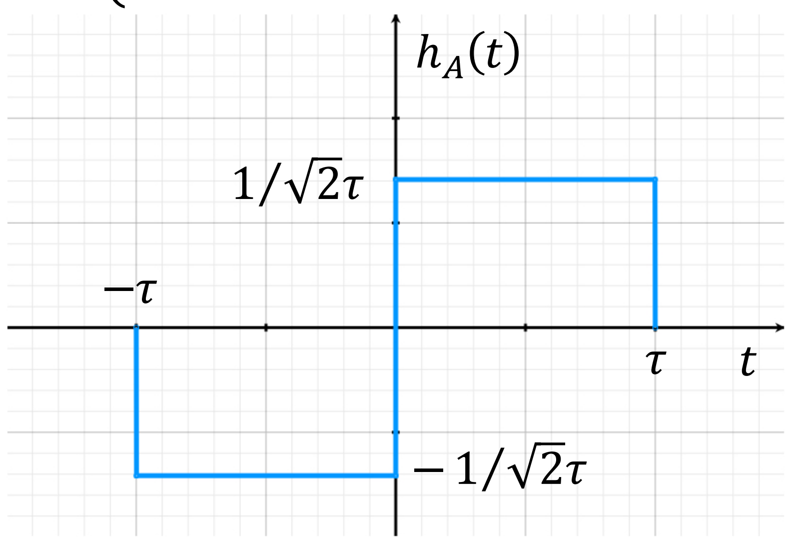
It diverges for flicker and random walk noises for *f*→ 0 (τ → ∞)

Allan variance

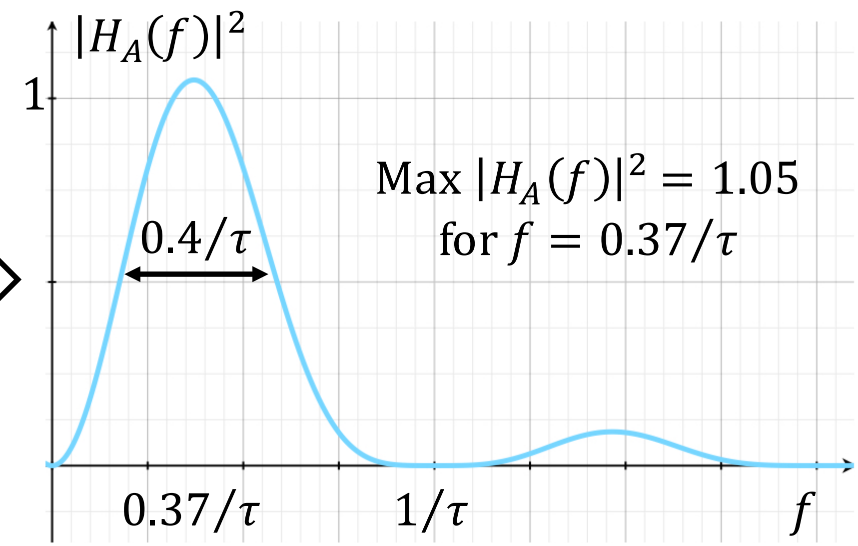
$$\sigma_y^2(\tau) = \frac{1}{2}\mathbb{E}[(\overline{y_2} - \overline{y_1})^2] = \mathbb{E}\left\{\left[\frac{1}{\sqrt{2}\tau}\int_{t_k+\tau}^{t_k+2\tau} y(t)dt - \frac{1}{\sqrt{2}\tau}\int_{t_k}^{t_k+\tau} y(t)dt\right]^2\right\}$$

$$\sigma_y^2(\tau) = \mathbb{E}\left\{\left[\int_{-\infty}^{+\infty} h_A(t_k - t)y(t)dt\right]^2\right\}$$

$$h_A(t) = \begin{cases} -1/\sqrt{2}\tau & t_k < t < t_k + \tau \\ 1/\sqrt{2}\tau & t_k + \tau < t < t_k + 2\tau \\ 0 & \text{elsewhere} \end{cases} \Rightarrow H_A(f) = \sqrt{2}\,\frac{[\sin(\pi f \tau)]^2}{\pi f \tau}$$



$\Rightarrow$



PSD to Allan variance connection

$$\sigma_y^2(\tau) = \int_0^{+\infty} |H_A(f)|^2 S_y(f) df = \int_0^{+\infty} 2\frac{[\sin(\pi f \tau)]^4}{(\pi f \tau)^2} S_y(f) df$$

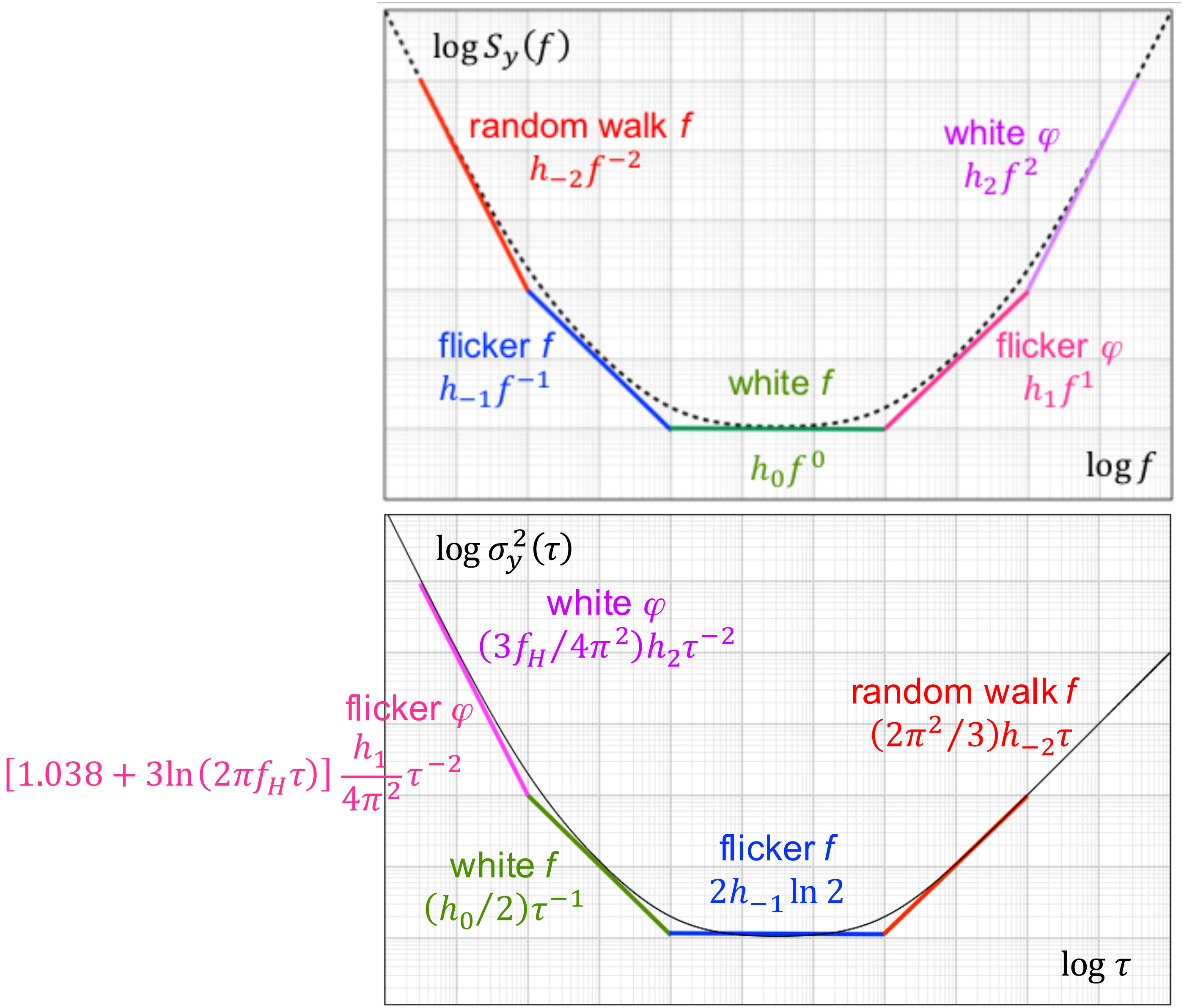
$S_y(f)$ [Hz²/Hz]	Type of noise	$\sigma_y^2(\tau)$
$h_{-2}f^{-2}$	Random walk FM	$(2\pi^2/3)h_{-2}\tau$
$h_{-1}f^{-1}$	Flicker FM	$2h_{-1}\ln 2$
h_0	White FM	$(h_0/2)\tau^{-1}$
$h_1 f$	Flicker PM	$[1.038 + 3\ln(2\pi f_H \tau)]\frac{h_1}{4\pi^2}\tau^{-2}$ *
$h_2 f^2$	White PM	$(3f_H/4\pi^2)h_2\tau^{-2}$ *

$$\sigma_y^2(\tau) \propto \tau^\mu$$

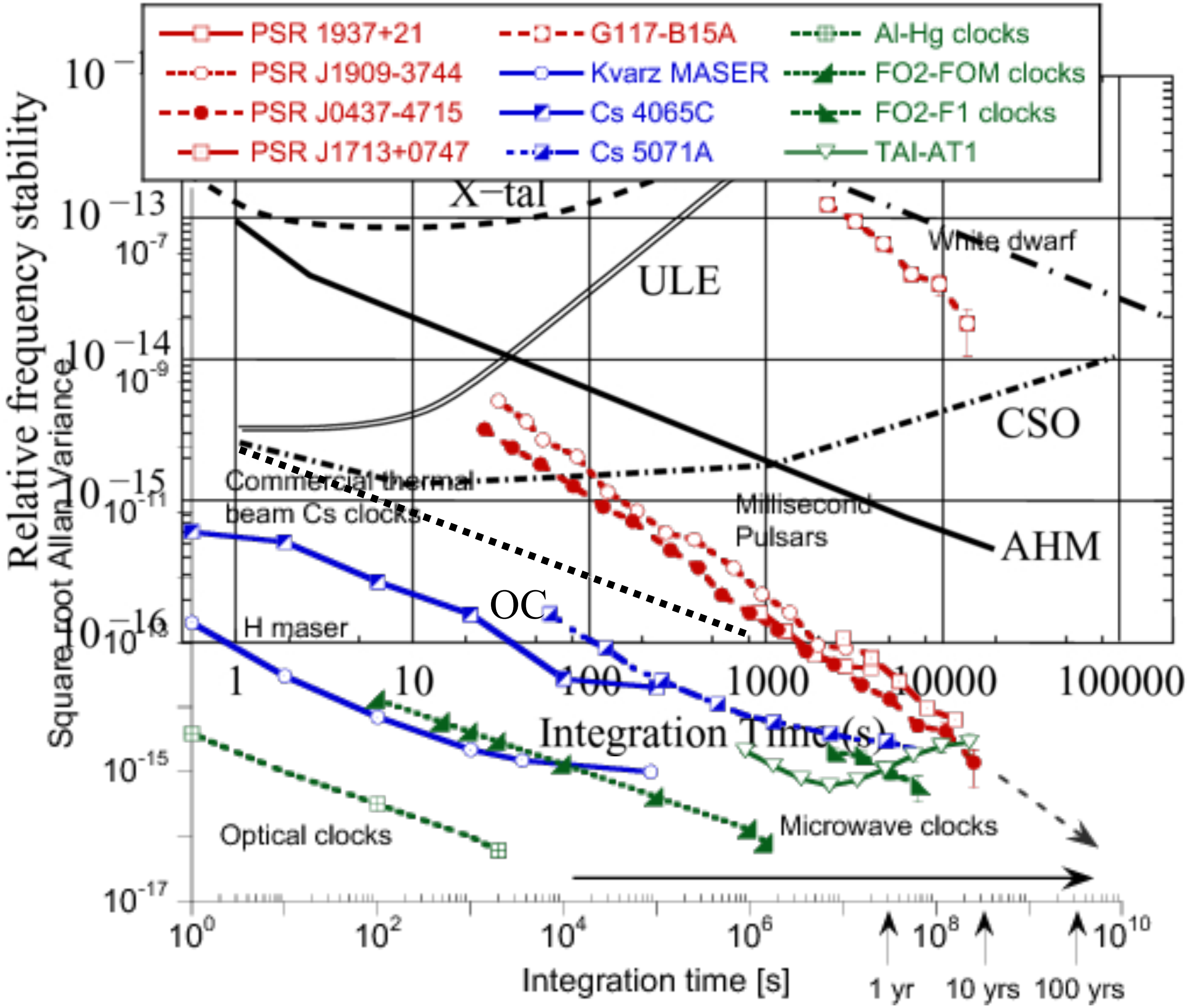
$$\mu = \begin{cases} -\alpha - 1; -2 \leq \alpha < 1 \\ -2; \alpha \geq 1 \end{cases}$$

* f_H is a high cutoff frequency, needed for the noise power to be finite ($2\pi f_H \tau >> 1$)

PSD to Allan variance connection



Allan deviation examples



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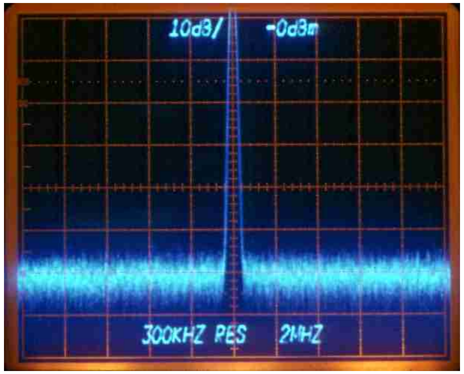
Spectrum of the clock signal

$$v(t) = V_0[1+\alpha(t)]\cos[2\pi\nu_0 t + \varphi(t)]$$

$$\overline{v^2(t)} = V_0^2[1+2\alpha(t)]$$

Noise broads the spectrum is a complex way but the random phase does not contribute to the signal power!!!

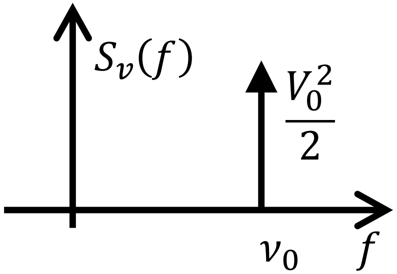
$$S_v(f,t) = \mathbb{F}\{R_v(\tau,t)\}$$



Noise-free ideal signal

$$R_v(\tau) = V_0^2\,\mathbb{E}\{\cos(2\pi\nu_0 t)\cos[2\pi\nu_0(t+\tau)]\} = \frac{V_0^2}{2}\cos(2\pi\nu_0\tau)$$

$$S_v(f) = \frac{V_0^2}{2}\delta(f-\nu_0)$$

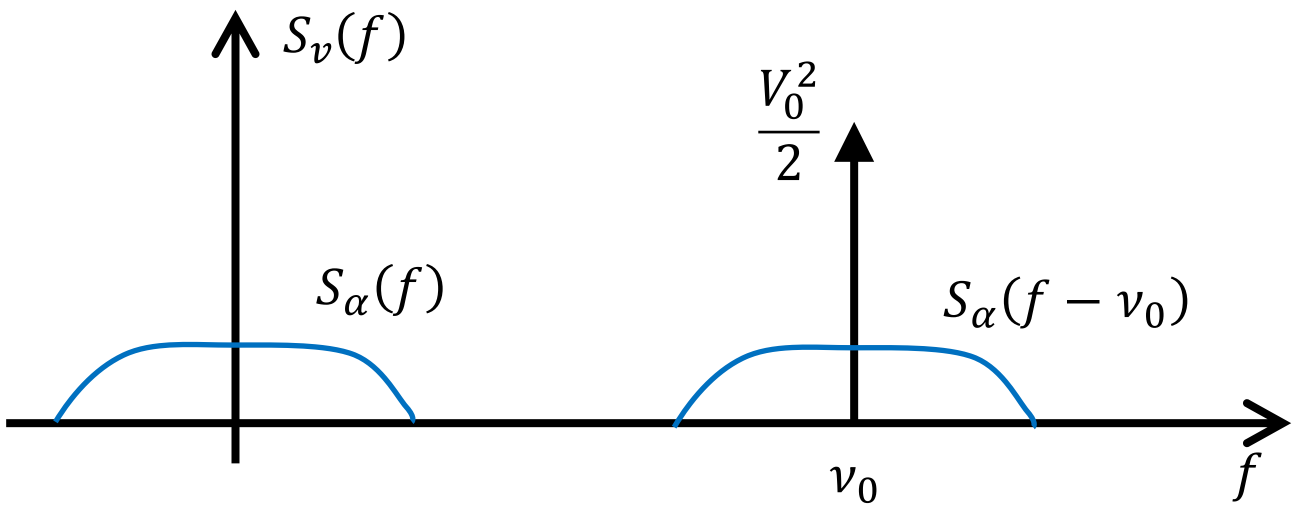


Spectrum of the clock signal

Pure AM noise signal

$$R_v(\tau) = \frac{V_0^2}{2}\cos(2\pi\nu_0\tau)[1 + R_\alpha(\tau)]$$

$$S_v(f) = \frac{V_0^2}{2}\delta(f-\nu_0) * [\delta(f) + S_\alpha(f)] = \frac{V_0^2}{2}[\delta(f-\nu_0) + S_\alpha(f-\nu_0)]$$



Spectrum of the clock signal

Pure PM noise signal

$$R_v(\tau) = \frac{V_0^2}{2}\cos(2\pi\nu_0\tau)\exp\left\{\frac{1}{2}\left[R_\varphi(0) - R_\varphi(\tau)\right]\right\}$$

$$R_v(\tau) = \frac{V_0^2}{2}\cos(2\pi\nu_0\tau)\exp\left[-2\int_0^\infty \left|\frac{\sin(2\pi f\tau)}{f}\right|^2 S_{\Delta\nu}(f)df\right]$$

$$S_v(f) = \mathbb{F}\{R_v(\tau)\}$$

Complex integral!!

White frequency noise ($S_{\Delta\nu} = \nu_0^2 h_0$)

$$S_v(f) = \frac{V_0^2}{2}\frac{(\pi\Delta f_L)^{-1}}{1 + \left[2\frac{(f-\nu_0)}{\Delta f_L}\right]^2}$$

$$\Delta f_L = \text{FWHW} = \pi\nu_0^2 h_0$$

Spectrum of the clock signal

PM and AM noise

$$R_\nu(\tau) = R_{\nu_{\Delta\nu}}(\tau)[1 + R_\alpha(\tau)]$$

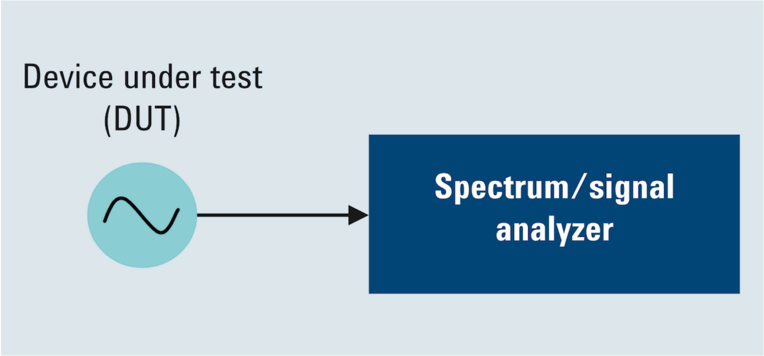
$$S_\nu(f) = \mathbb{F}\{R_\nu(\tau)\} = S_{\nu_{\Delta\nu}}(f) + S_{\nu_{\Delta\nu}}(f) * S_\alpha(f)$$

Very Complex integral!! Numerical integration is performed

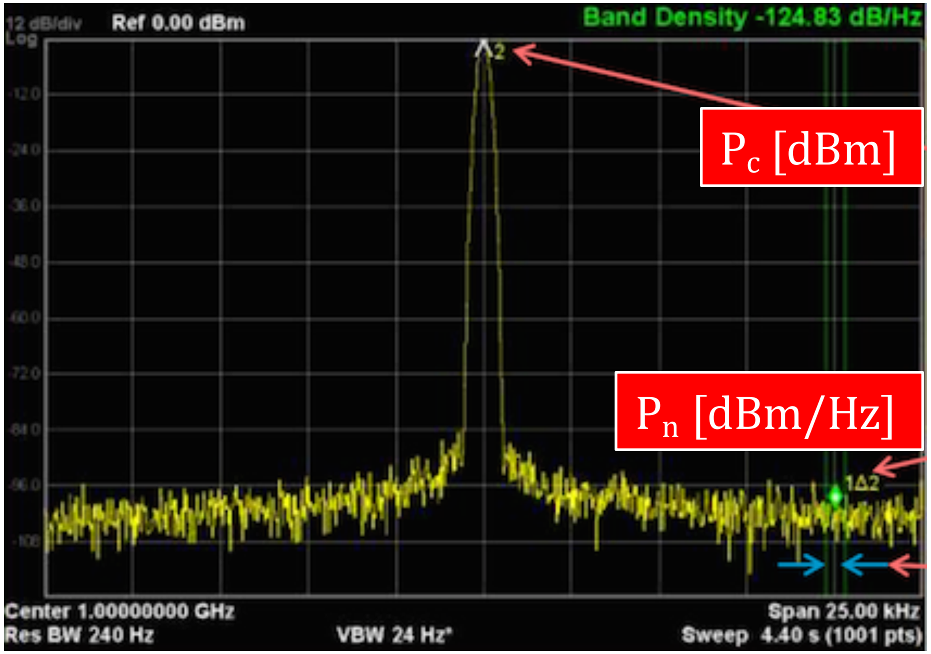
1. Direct measurement
2. Frequency discriminator
3. Phase discriminator

Direct measurement

Is the simplest and oldest method for the measurement of the phase noise power spectral density



$$\mathcal{L}(f) = \frac{\text{power in the SSB in 1 Hz}}{\text{power in the carrier}}$$

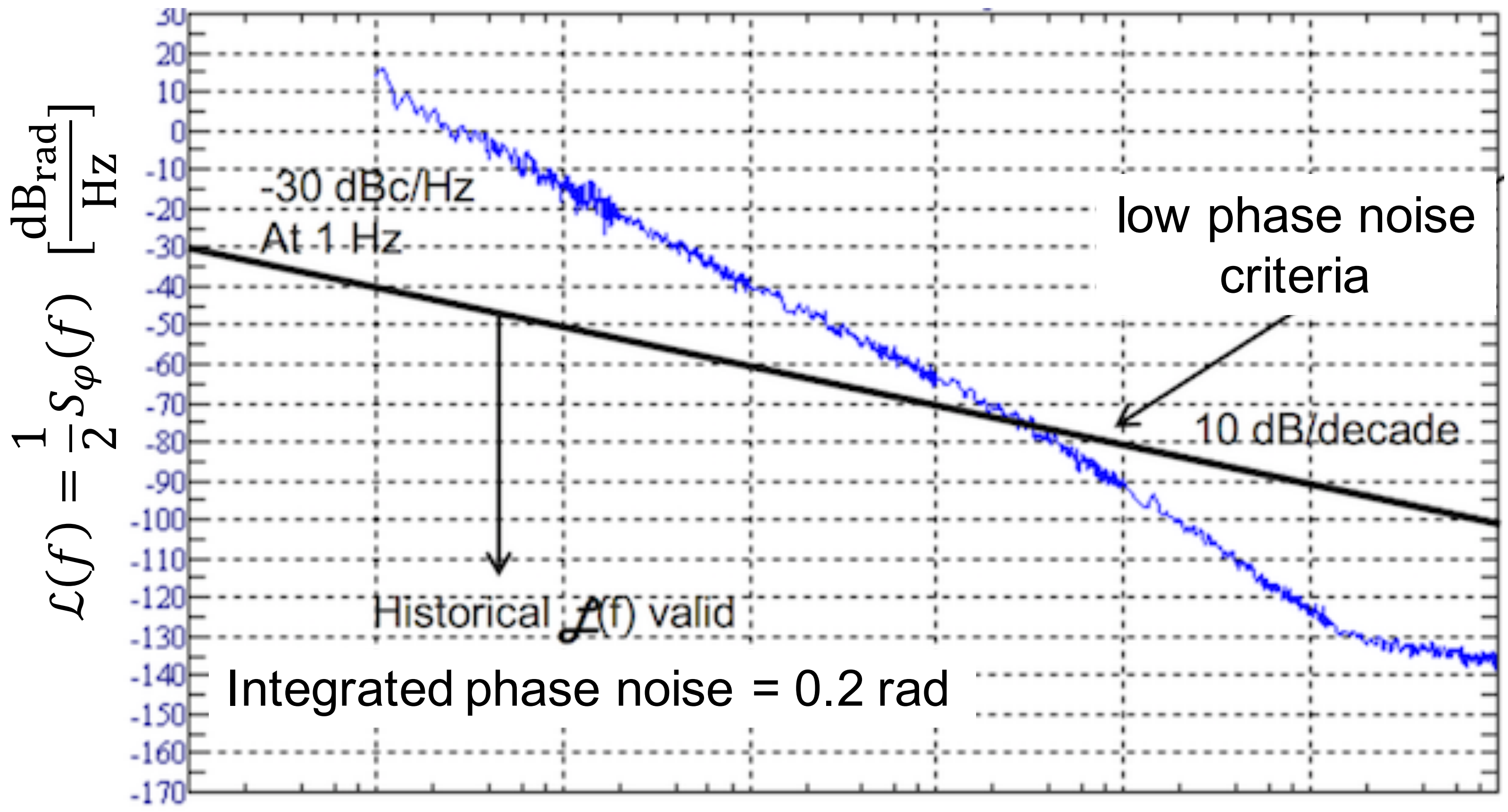


$$\mathcal{L}(f)\left[\frac{\mathrm{dB_c}}{\mathrm{Hz}}\right] = P_n\left[\frac{\mathrm{dB_m}}{\mathrm{Hz}}\right] - P_c[\mathrm{dB_m}]$$

Direct measurement limitations

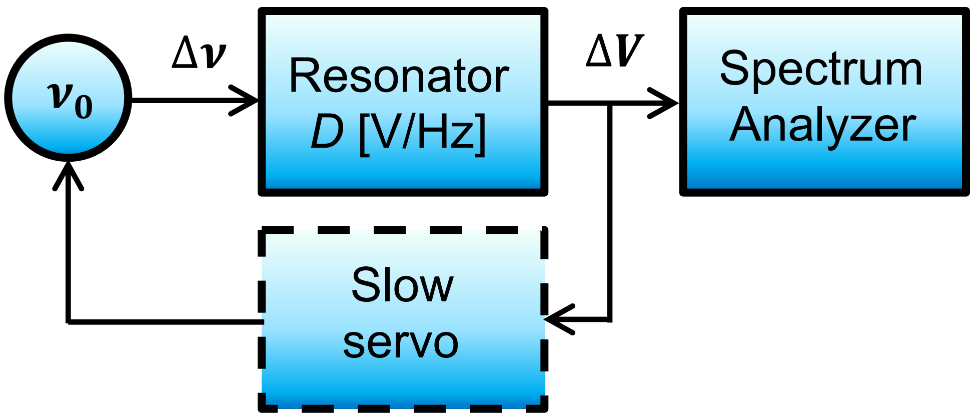
The problems with this measurement are (ambiguous results):

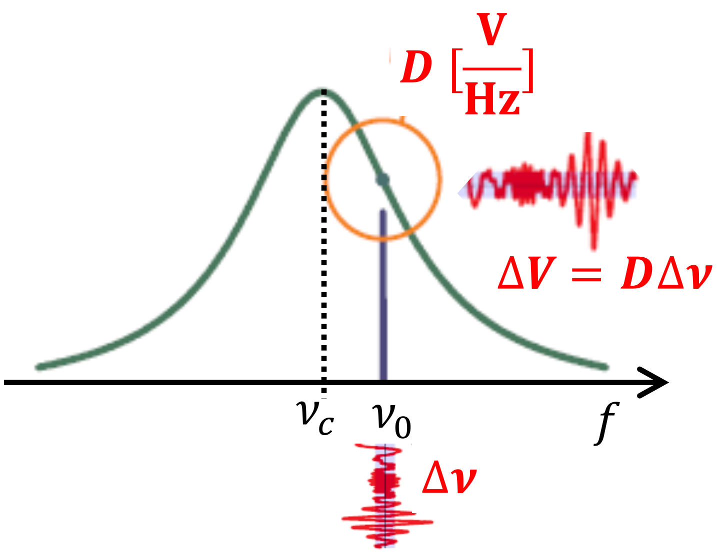
- it does not divide AM noise from PM noise
- improper value for phase noise larger than 1 rad!!



Frequency discriminator

A resonator (band pass filter) converts the frequency fluctuations into intensity/voltage





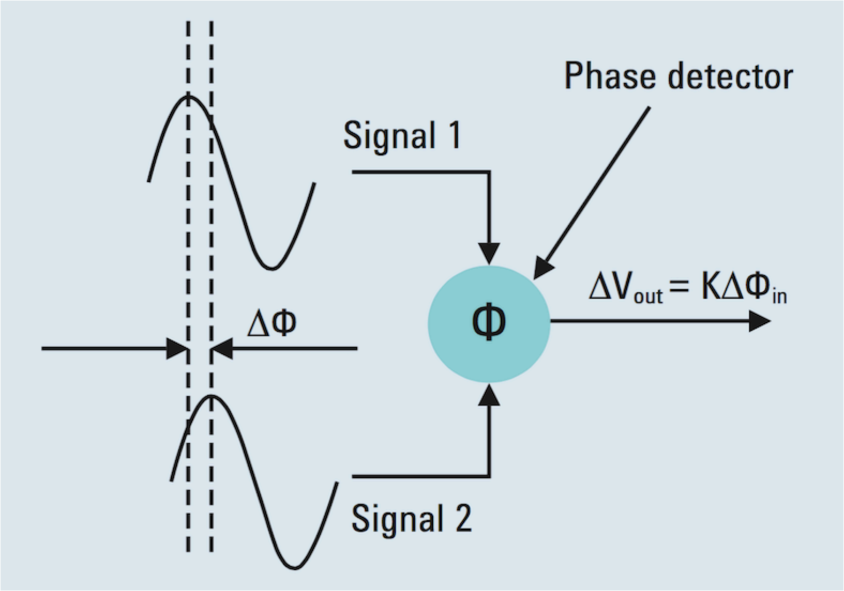
$$S_{\Delta V}(f) = D^2 S_{\Delta\nu}(f)$$

$$S_{\Delta\nu}(f) = \frac{S_{\Delta V}(f)}{D^2}$$

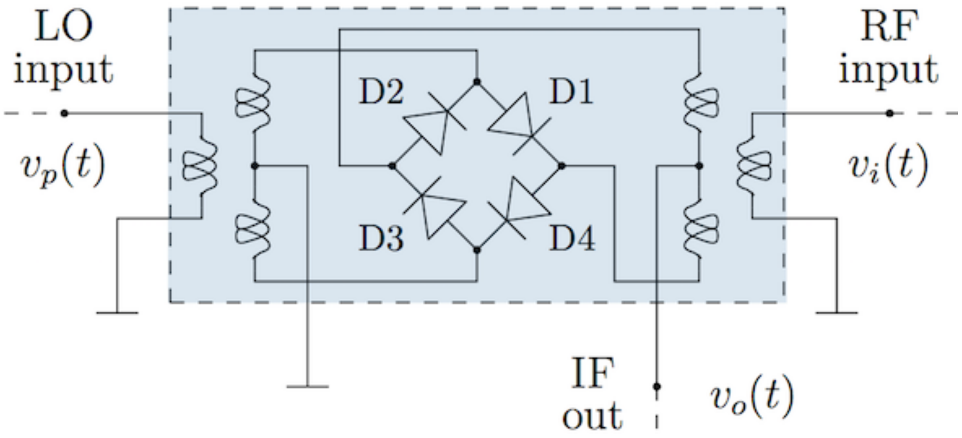
$$S_{\varphi}(f) = f^2 \frac{S_{\Delta V}(f)}{D^2}$$

Phase detector

The phase detector converts the phase difference of the two input signals into a voltage at the output of the detector



Doubled-balance mixer

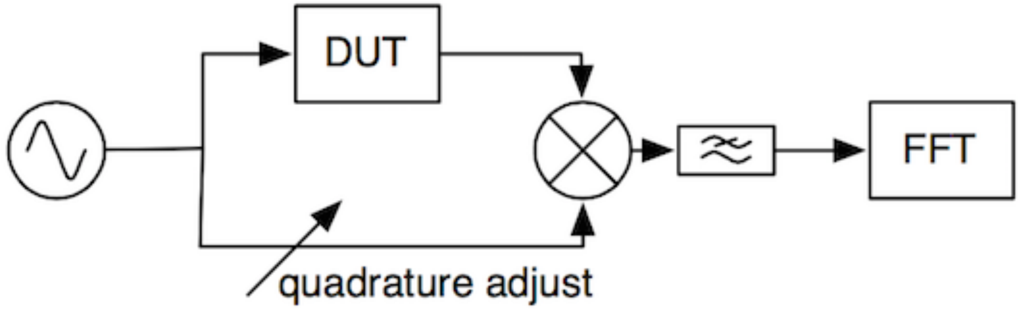


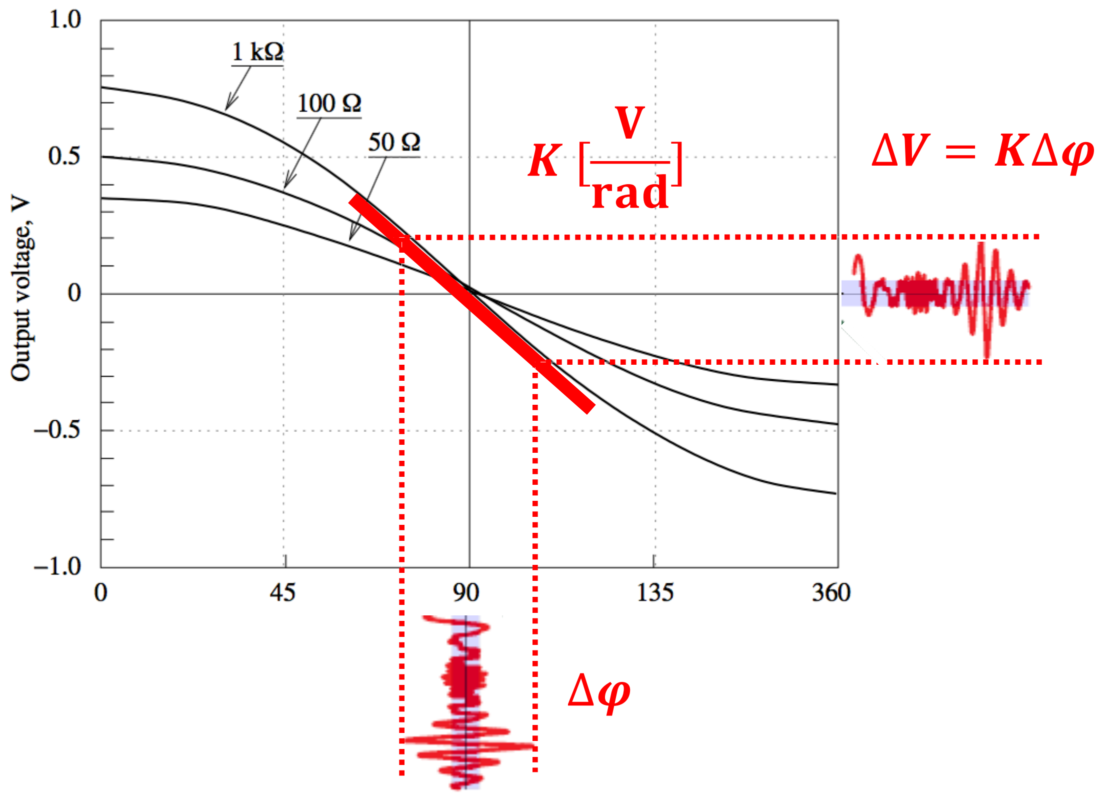
$$v_o(t) = k v_1(t) v_2(t) = k V_1 V_2 \cos(2\pi\nu_0 t + \varphi_1) \cos(2\pi\nu_0 t + \varphi_2 + \Phi)$$

$$v_o(t)\Big|_{\mathrm{DC}} = \frac{k}{2} V_1 V_2 [\cos(\varphi_1 - \varphi_2 + \Phi)]$$

$$v_o(t)\Big|_{\Phi=\frac{\pi}{2}} = \frac{k}{2} V_1 V_2 (\varphi_1 - \varphi_2) = K \Delta\varphi$$

Phase discriminator: measurement of a two port device



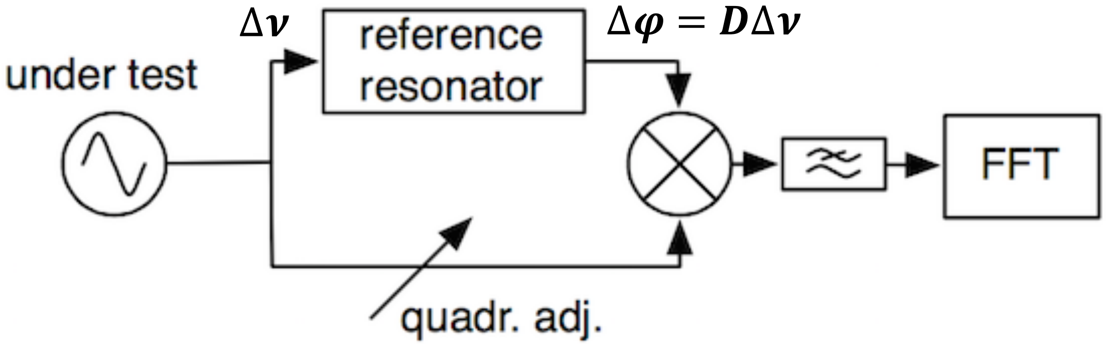


$$S_{\Delta V}(f) = K^2 S_{\Delta\varphi}(f)$$

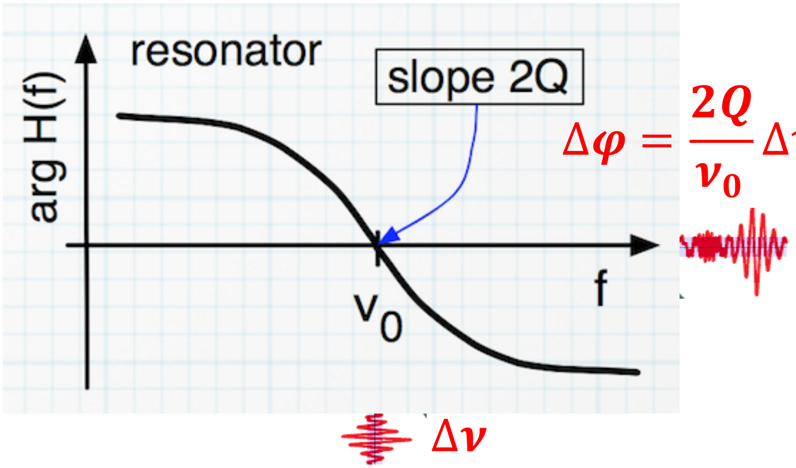
$$\boldsymbol{S_{\Delta\varphi}(f) = \frac{S_{\Delta V}(f)}{K^2}}$$

Phase discriminator: combining a frequency discriminator

A frequency discriminator, a resonator/filter or a delay line, can be used to measure the oscillator phase noise



A resonator, the imaginary part of the complex transfer function, turns a slow frequency fluctuation into a phase fluctuation



$$\Delta\varphi = \frac{2Q}{\nu_0}\Delta\nu$$

$\Delta\nu$

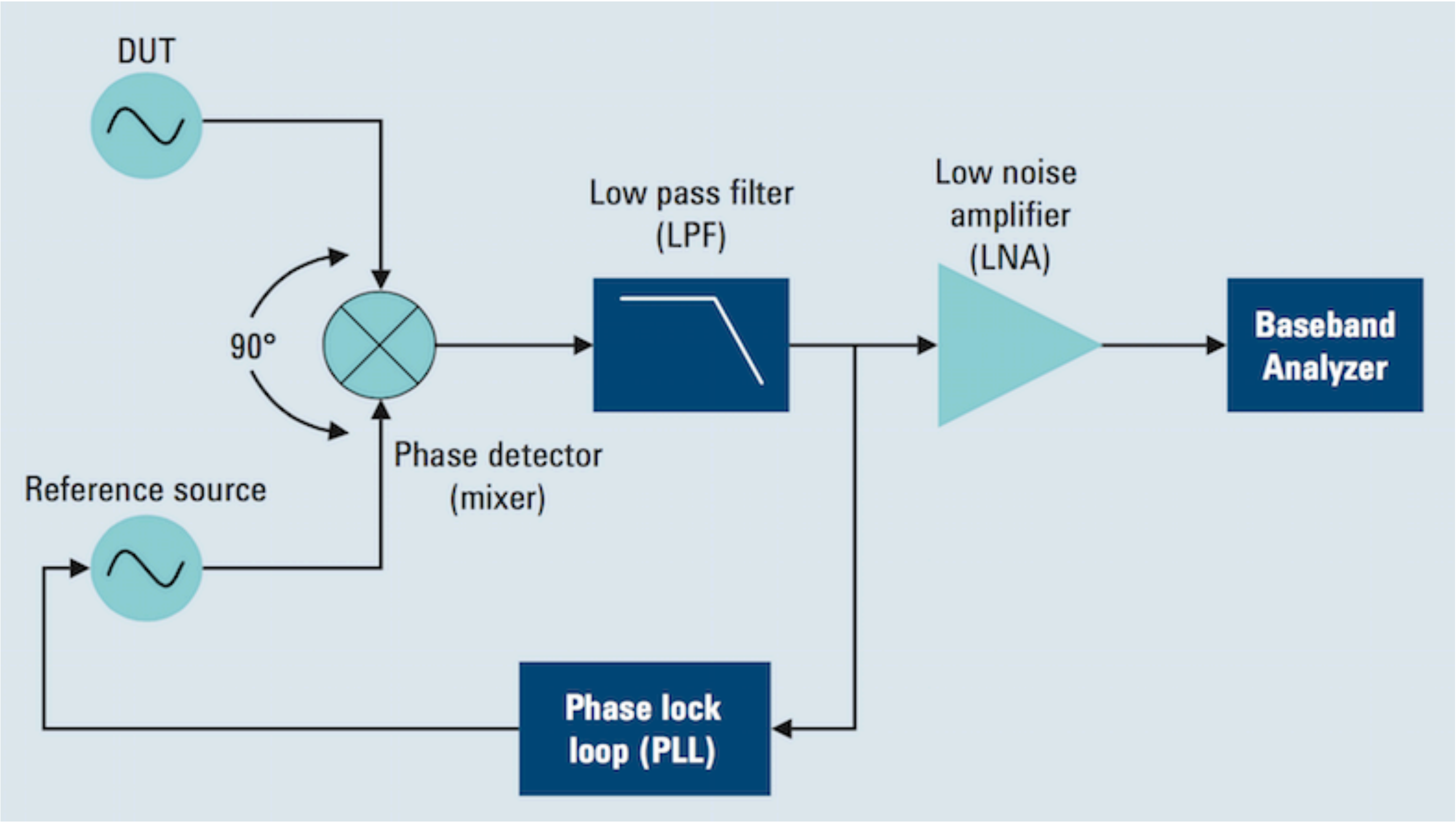
$$S_{\Delta V}(f) = K^2 S_{\Delta\varphi}(f) = 4Q^2K^2S_y(f)$$

$$S_y(f) = \frac{S_{\Delta V}(f)}{4Q^2K^2}$$

$$S_\varphi(f) = \frac{S_{\Delta V}(f)}{4Q^2K^2}\frac{f^2}{\nu_0^2}$$

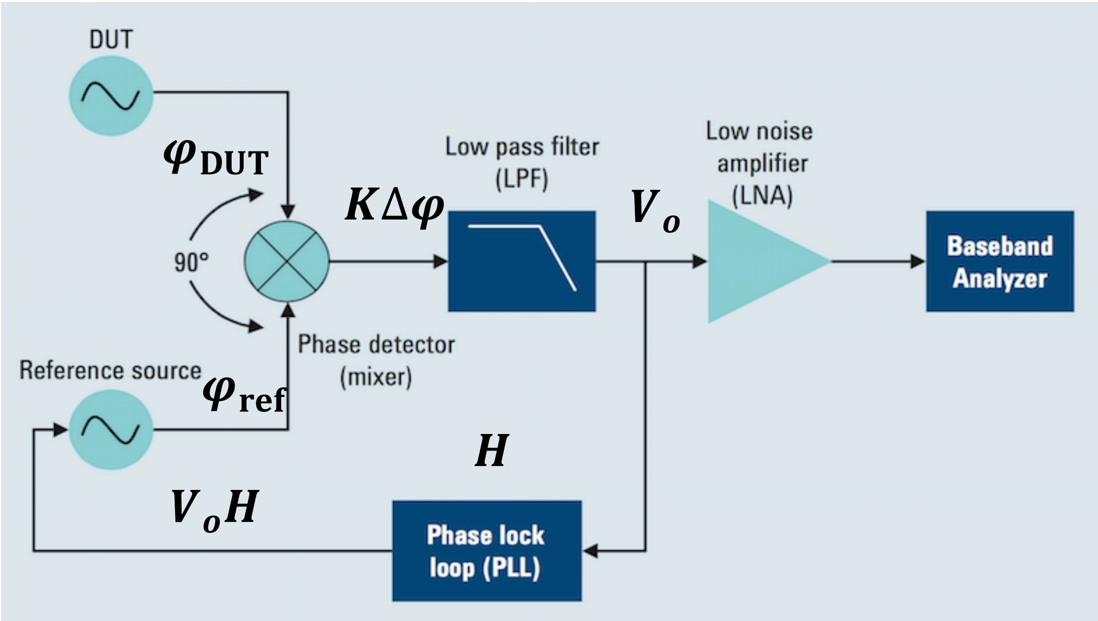
Phase discriminator: measurement of oscillator phase noise using PLL

The reference source/PLL (phase-locked-loop) is one of the most widely used methods



PLL Method

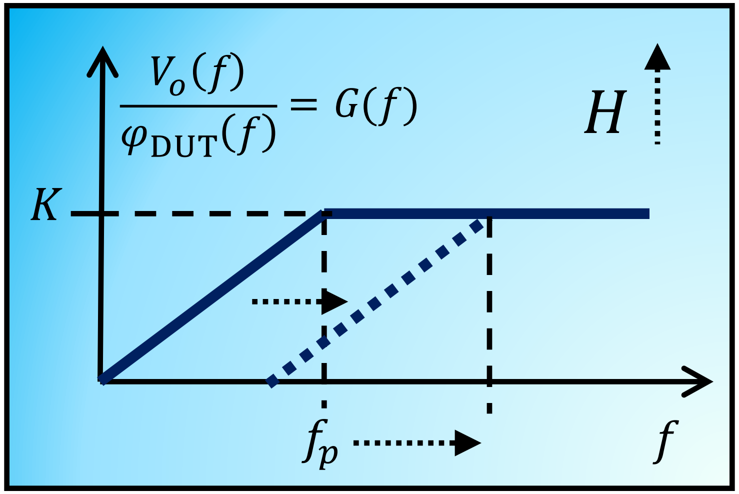
$$k_{VCO}\left[\frac{\text{rad/s}}{\text{V}}\right]$$



$$\varphi_{\text{ref}}(f) = \frac{k_{\text{VCO}} V_o H}{j2\pi f}$$

$$\frac{V_o(f)}{\varphi_{\text{DUT}}(f)} = \frac{j2\pi f K}{j2\pi f + k_{\text{VCO}} K H} = \frac{j\dfrac{f}{f_z}}{1 + j\dfrac{f}{f_p}} = G(f)$$

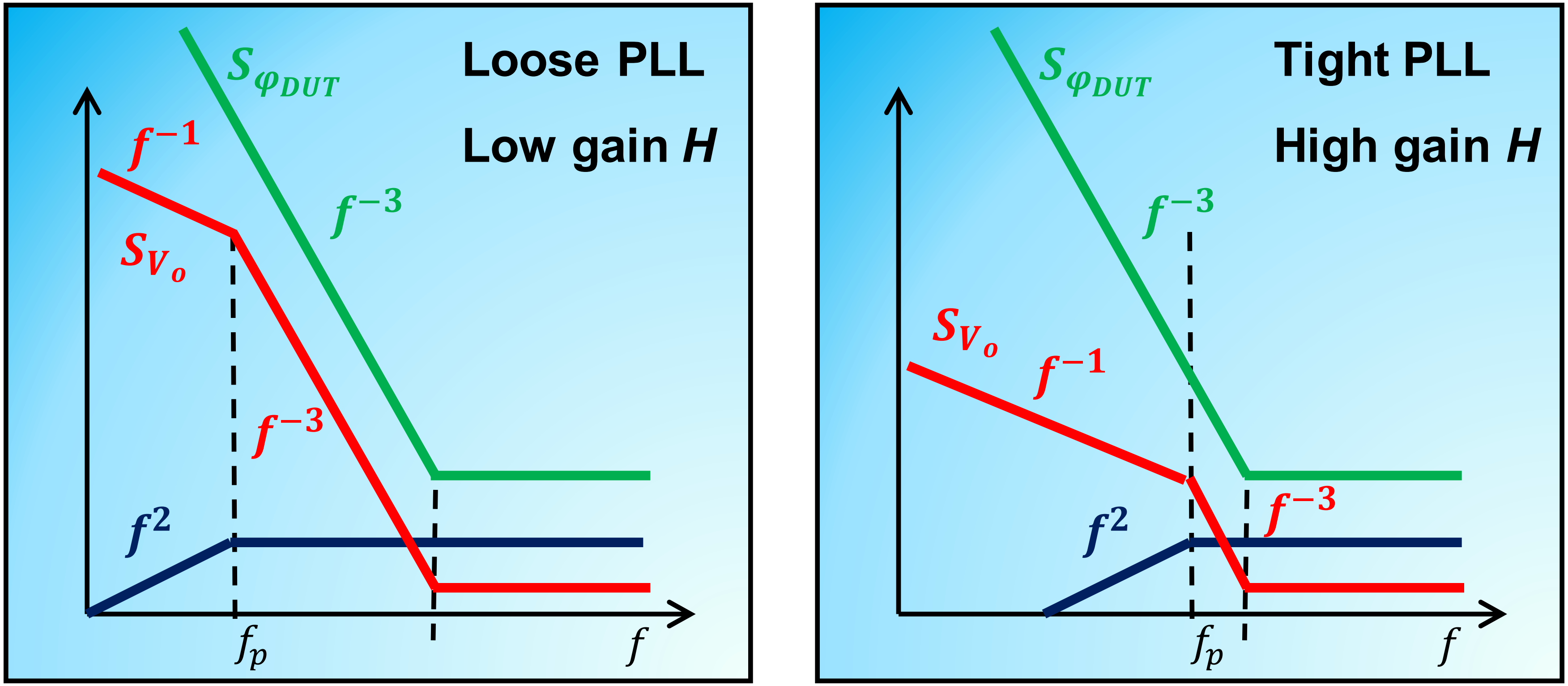
$$f_z = \frac{k_{\text{VCO}} H}{2\pi}; \quad f_p = \frac{k_{\text{VCO}} K H}{2\pi}$$



PLL Method

$$S_{V_o}(f) = |G(f)|^2 S_{\varphi_{\mathrm{DUT}}}(f) = \frac{(2\pi f K)^2}{(2\pi f)^2 + (k_{\mathrm{VCO}} K H)^2} S_{\varphi_{\mathrm{DUT}}}(f)$$

$$\boldsymbol{S_{\varphi_{DUT}}(f) = \frac{S_{V_o}(f)}{|G(f)|^2} = \frac{(2\pi f)^2 + (k_{VCO} K H)^2}{(2\pi f K)^2} S_{V_o}(f)}$$



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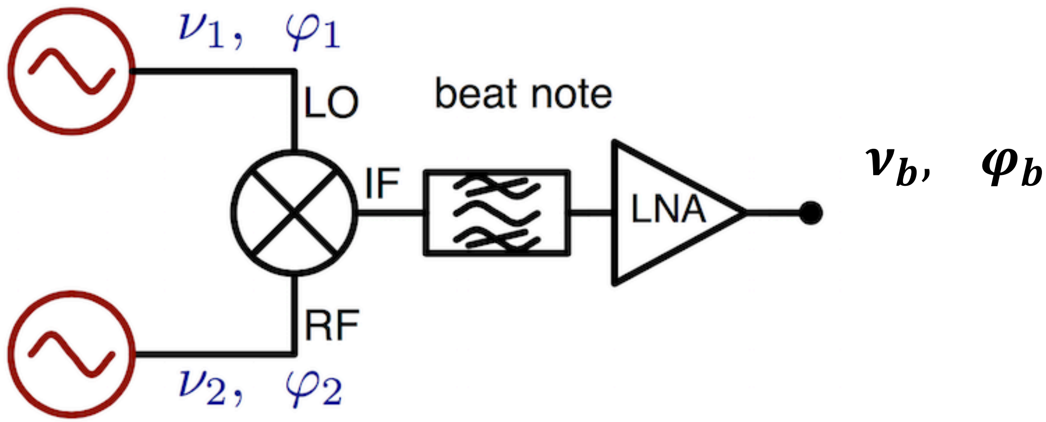
Heterodyne beatnote



$$v_b(t) = k v_1(t) v_2(t) = k V_1 V_2 \cos(2\pi\nu_1 t + \varphi_1)\cos(2\pi\nu_2 t + \varphi_2)$$

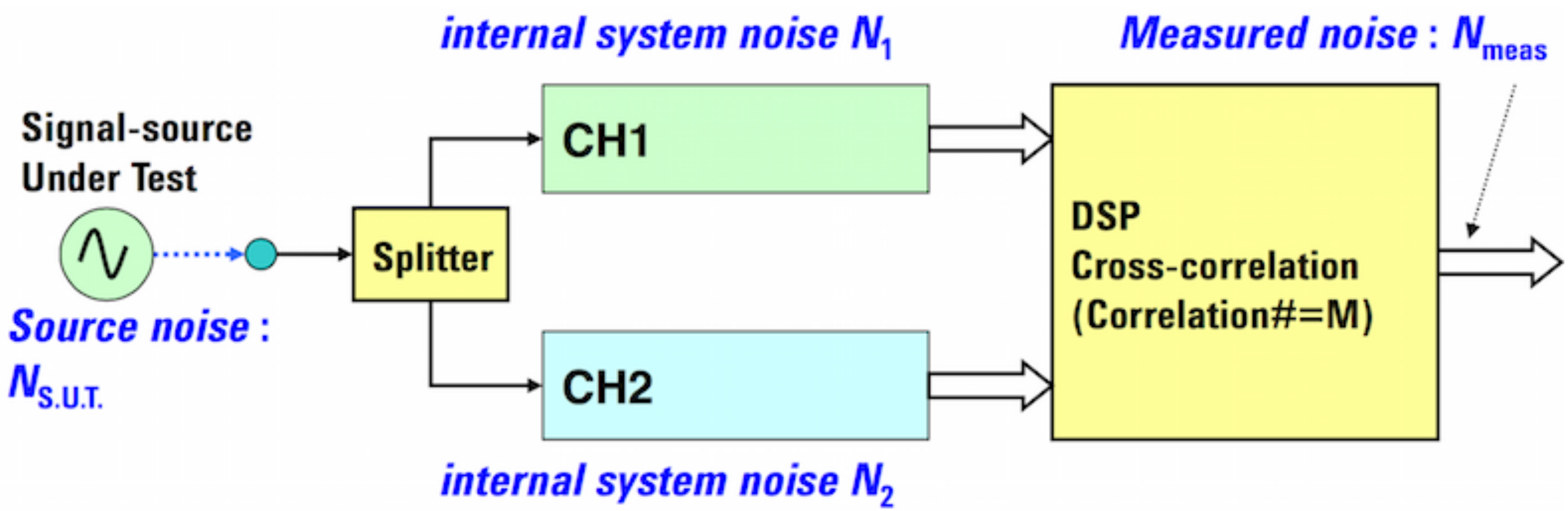
$$v_b(t)\Big|_{\mathrm{LF}} = \frac{k}{2} V_1 V_2 \cos[2\pi(\nu_1 - \nu_2)t + (\varphi_1 - \varphi_2)] = V_b \cos(2\pi\nu_b t + \varphi_b)$$

$$S_{\varphi_b}(f) = S_{\varphi_1}(f) + S_{\varphi_2}(f) = \frac{\nu_1^2}{f^2} S_{y_1}(f) + \frac{\nu_2^2}{f^2} S_{y_2}(f)$$

$$S_{y_b}(f) = \frac{f^2}{\nu_b^2} S_{\varphi_b}(f) = \frac{\nu_1^2}{\nu_b^2} S_{y_1}(f) + \frac{\nu_2^2}{\nu_b^2} S_{y_2}(f)$$

Correlation method – Noise floor reduction

The two-channel cross-correlation technique combines two duplicate single-channel reference sources/PLL systems and performs cross-correlation operations between the outputs of each channel



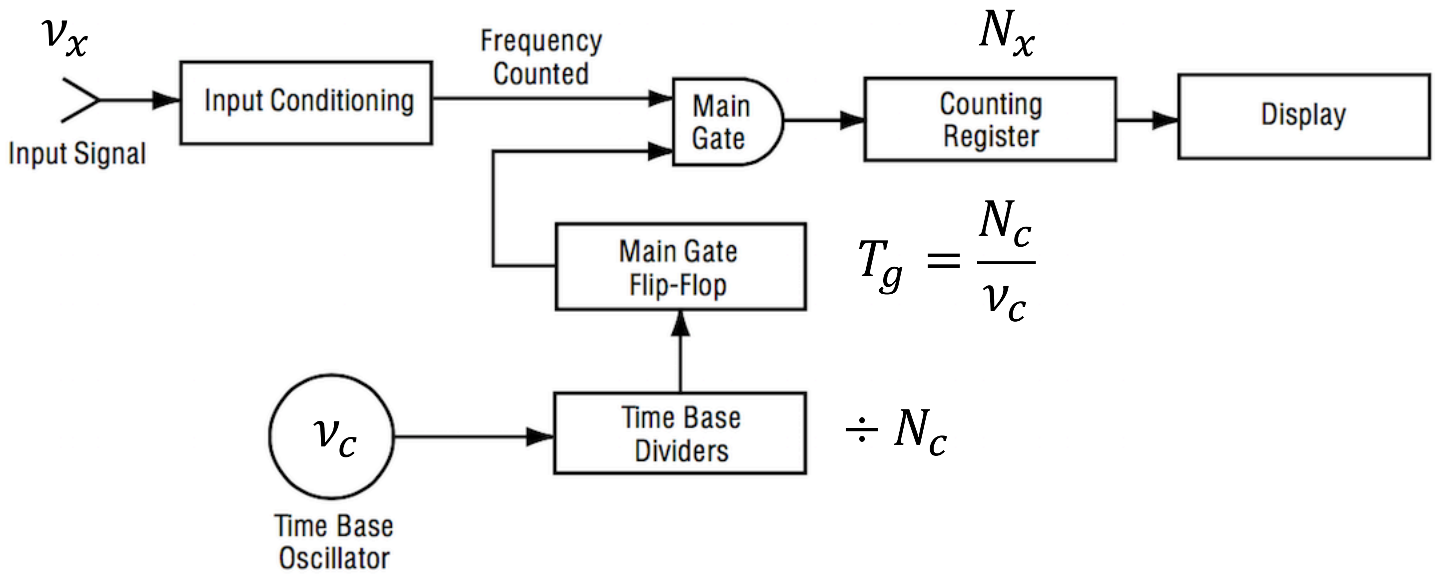
$$N_{meas} = N_{SUT} + \frac{(N_1 + N_2)}{\sqrt{M}}$$

Nunber of correlation *M*	Noise floor reduction
10^2	10 dB
10^4	20 dB

1. Classical counter (old)
2. Reciprocal counter
3. Interpolation methods

Enrico Rubiola,
FEMTO-ST Institute, CNRS and Université Franche Comté
http://www.rubiola.org/ **(also** http://rubiola.net)

Classical (old) frequency counter

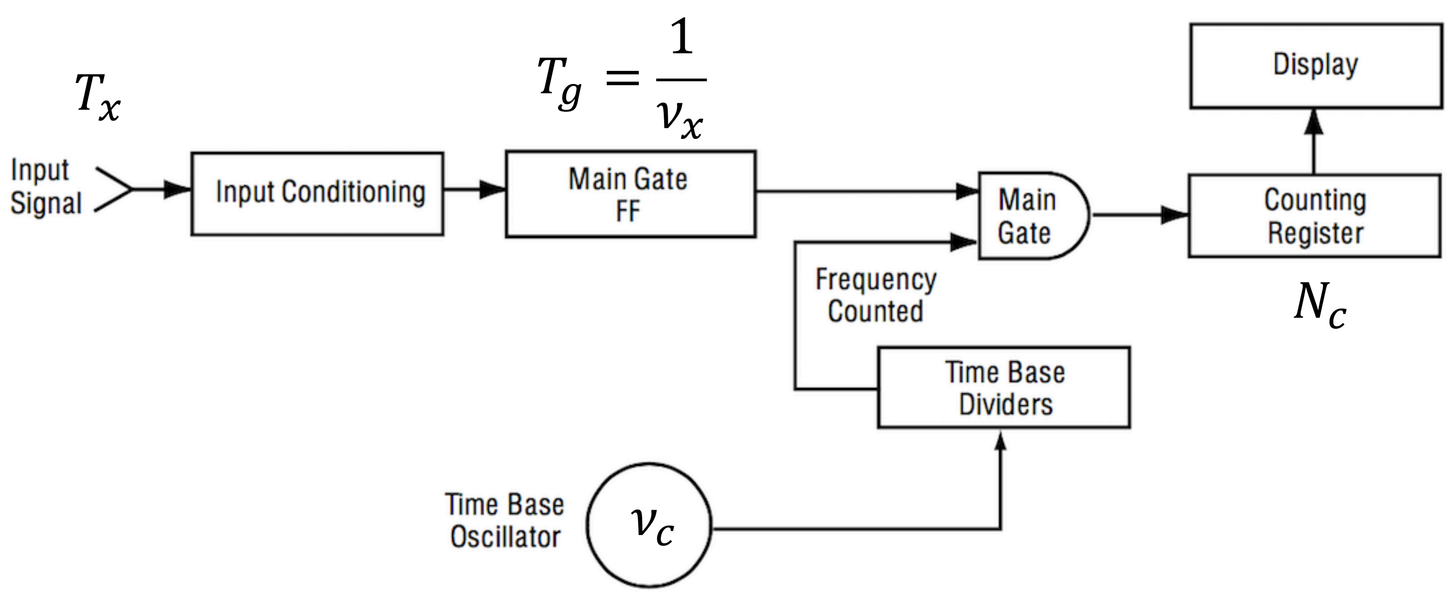


$$\nu_x = \frac{N_x}{T_g} = N_x \frac{\nu_c}{N_c} \qquad \frac{\Delta\nu_x}{\nu_x} = \frac{1}{T_g \nu_x}$$

The resolution is set by input period which can be significantly lower than reference clock

Example: ν_x =100 Hz; ν_c =10 MHz; T_g =1 s RES = 10^{-2}

Period measurement

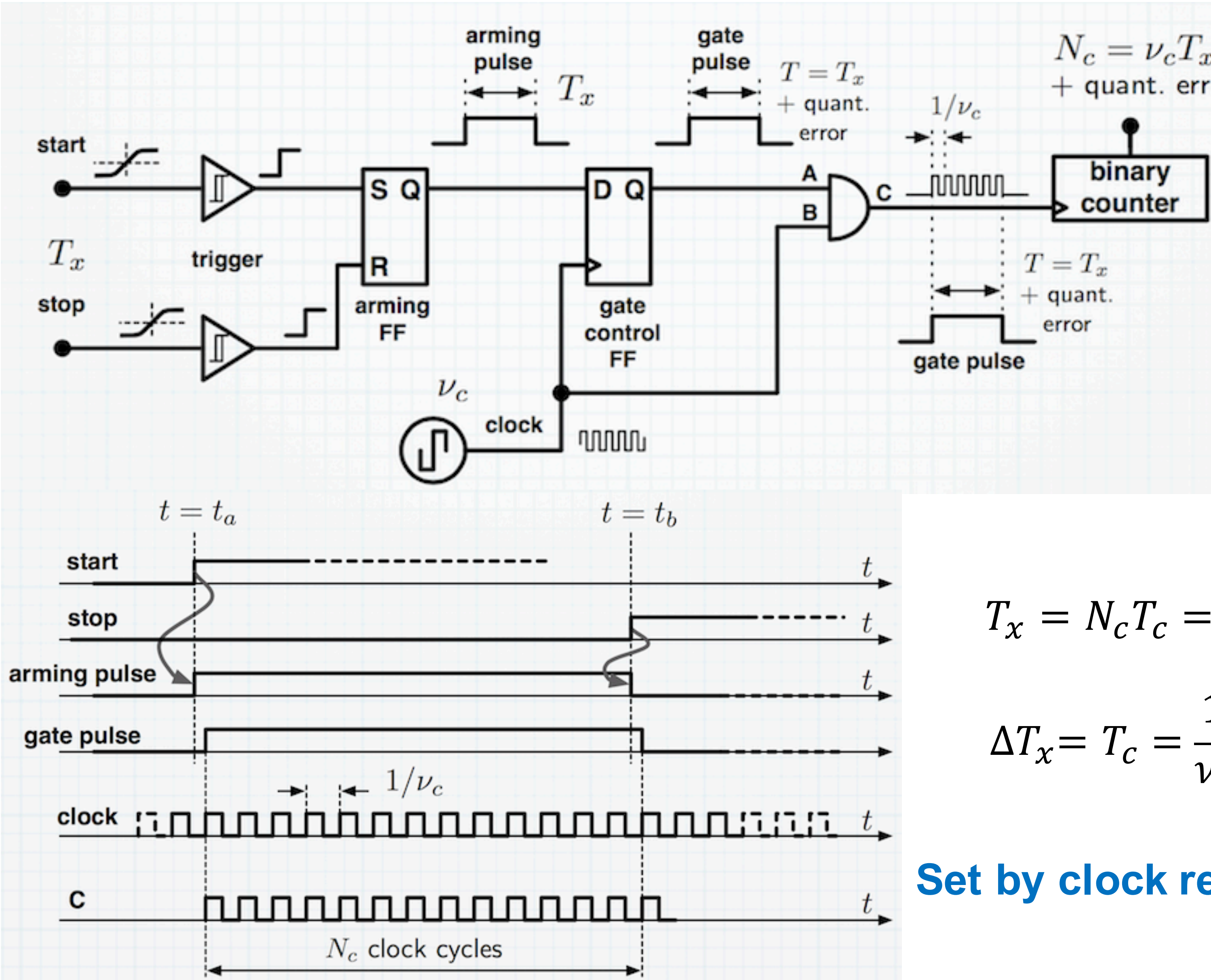


$$T_x = \frac{1}{\nu_x} = N_c T_c \qquad \frac{\Delta T_x}{T_x} = \frac{1}{T_g \nu_c}$$

The resolution is set by reference period

Example: $\nu_x = 100$ Hz; $\nu_c = 10$ MHz; $T_g = 1$ s RES = 10^{-5}

Time interval counter

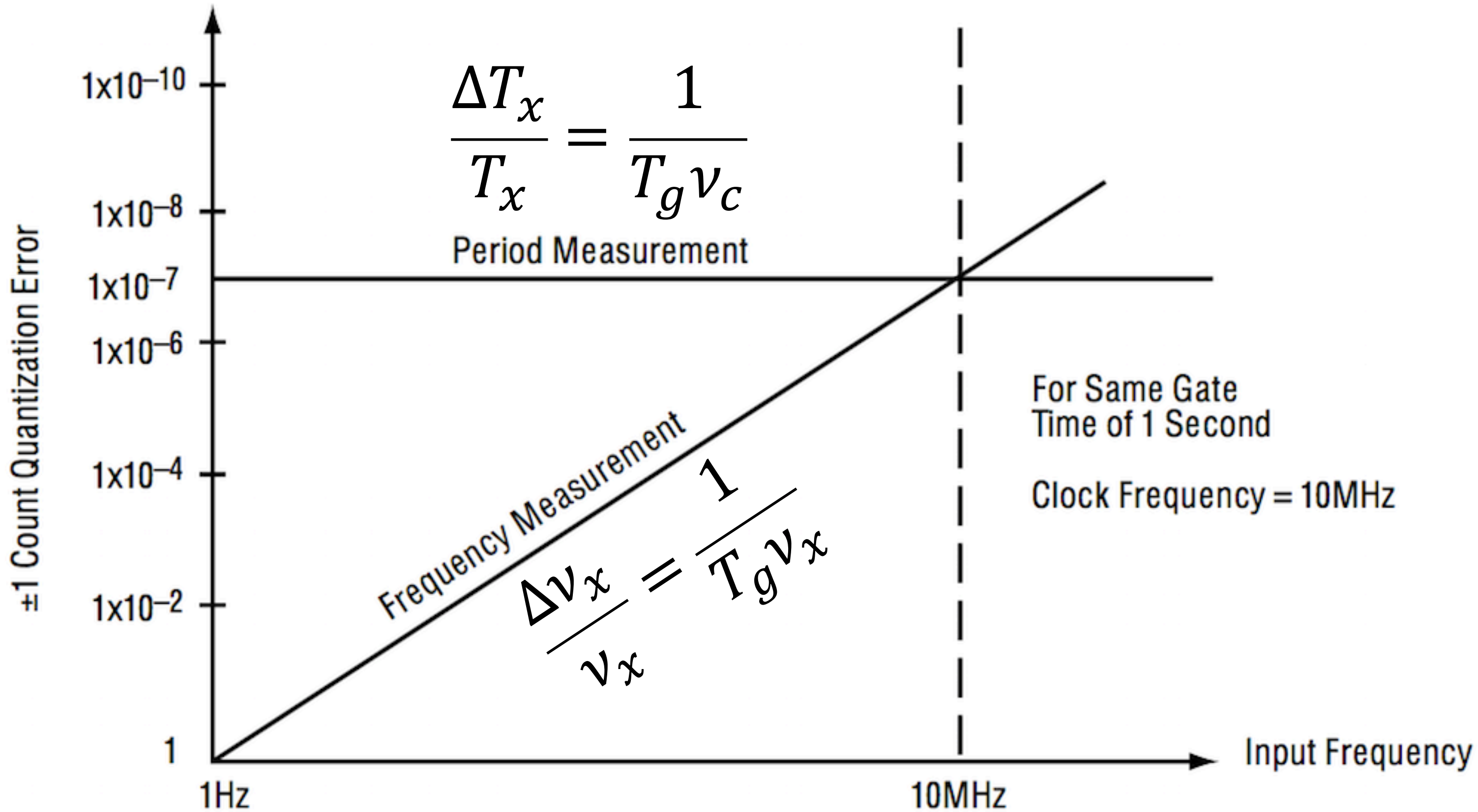


$$T_x = N_c T_c = \frac{N_c}{\nu_c}$$

$$\Delta T_x = T_c = \frac{1}{\nu_c}$$

Set by clock reference

Comparison between frequency and period resolution



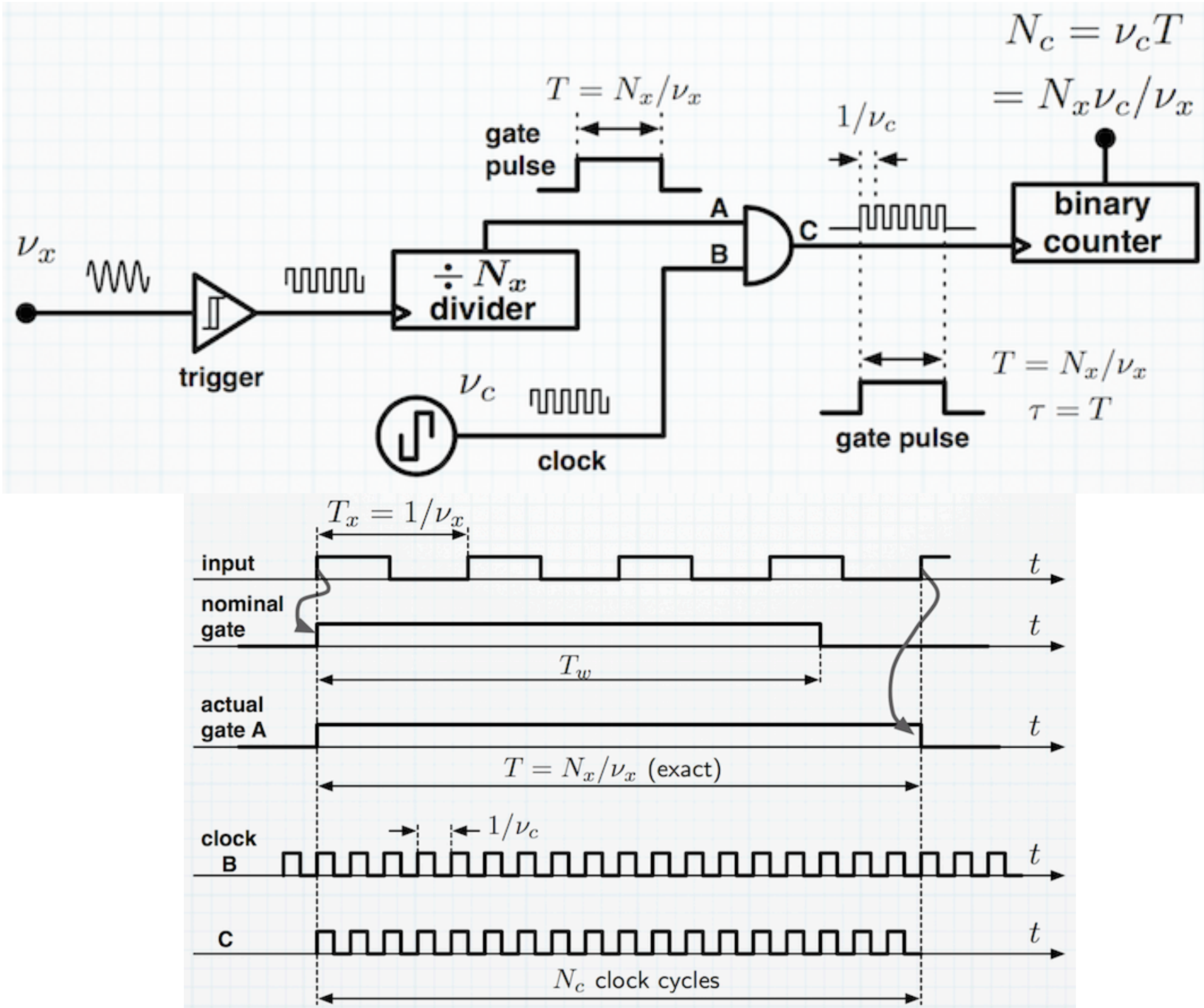
Measurement strategy

$\nu_x < \nu_c$ **period measurement**

$\nu_x > \nu_c$ **frequency measurement**

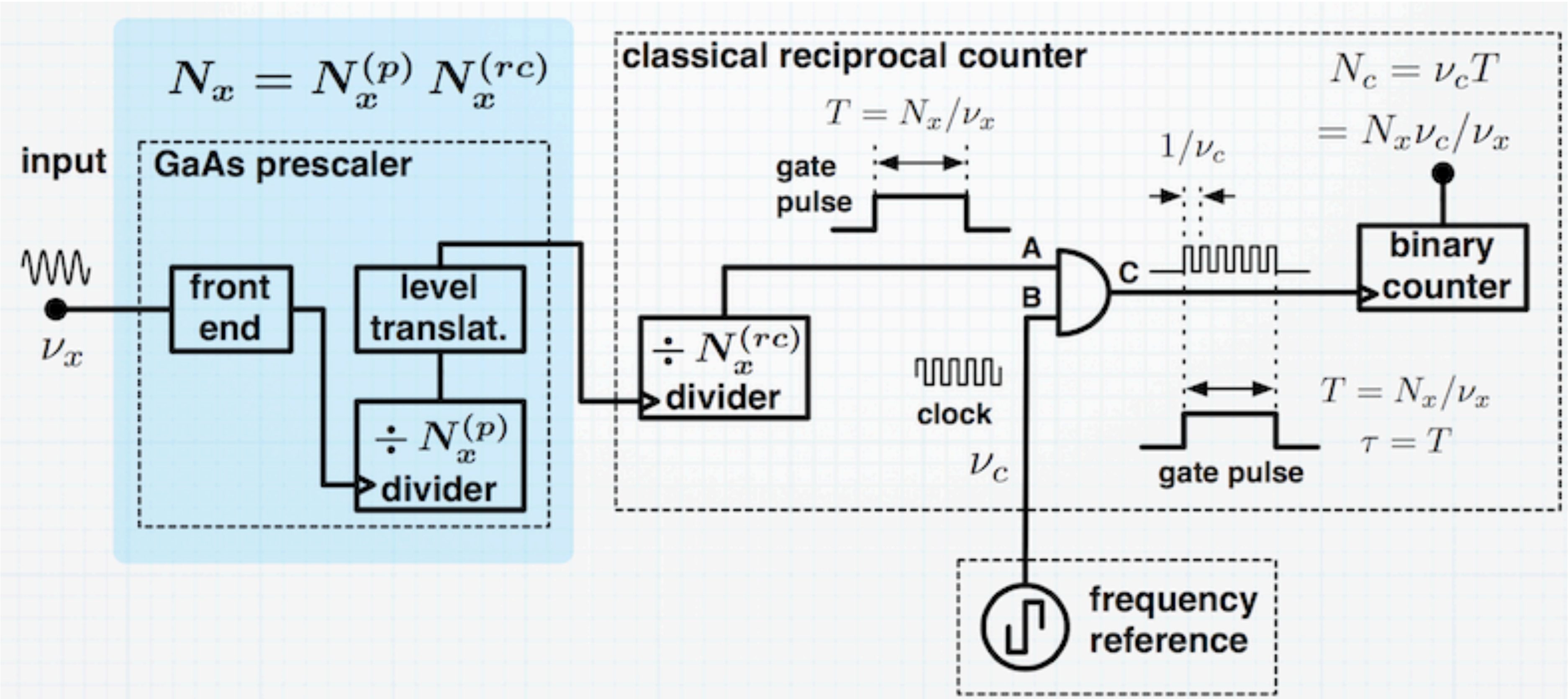
⇨ **RECIPROCAL COUNTER**

Reciprocal counter



- **Use the highest clock frequency permitted by the hardware**
- **The measurement time is a multiple of the input period**

High-frequency measurements: Prescaler

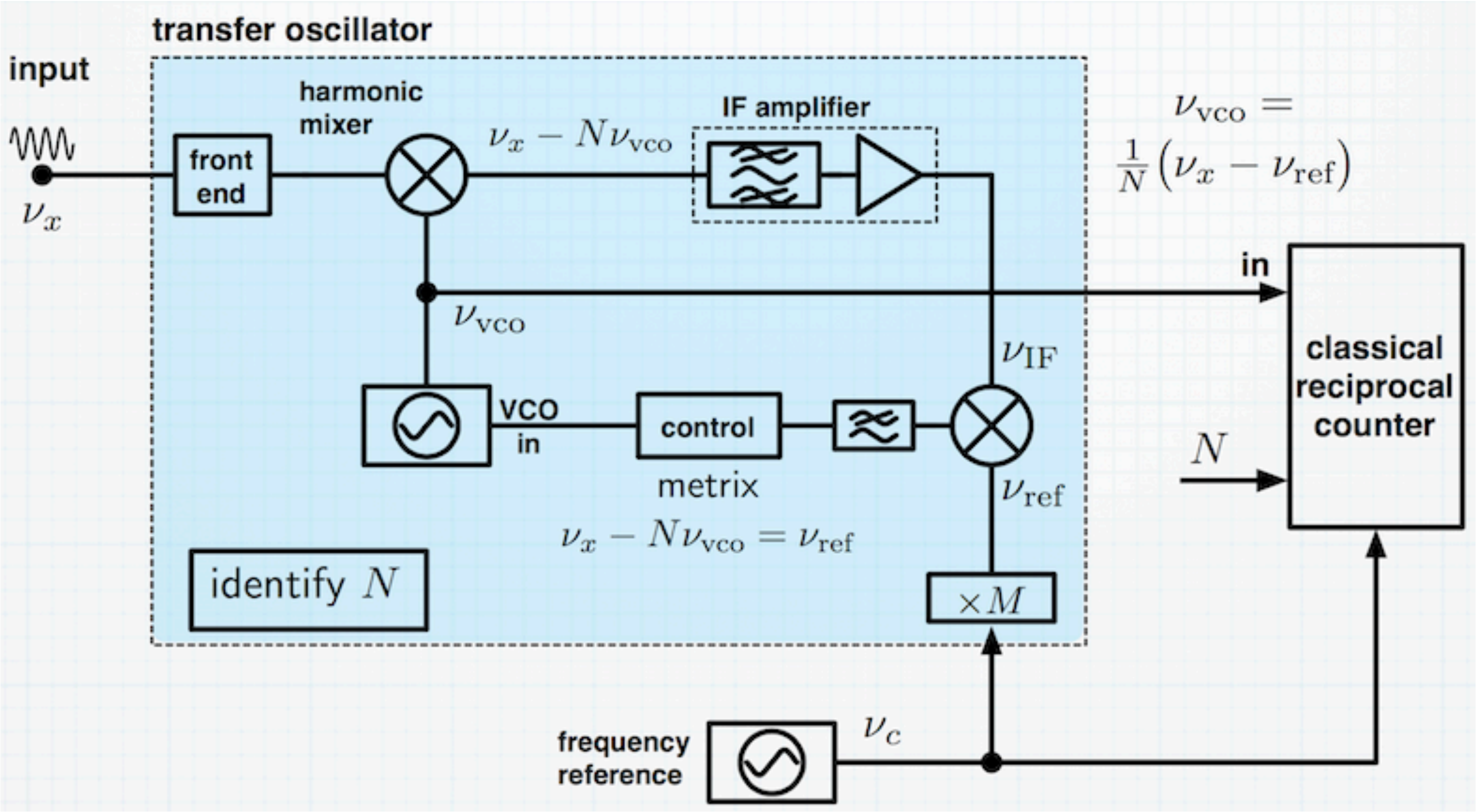


The prescaler is a n-bit binary divider $\div N_x^{(p)} = 2^n$

GaAs fast prescaler works up to 20-50 GHz

Most microwave counters use fast prescalers

High-frequency measurement: Transfer oscillator

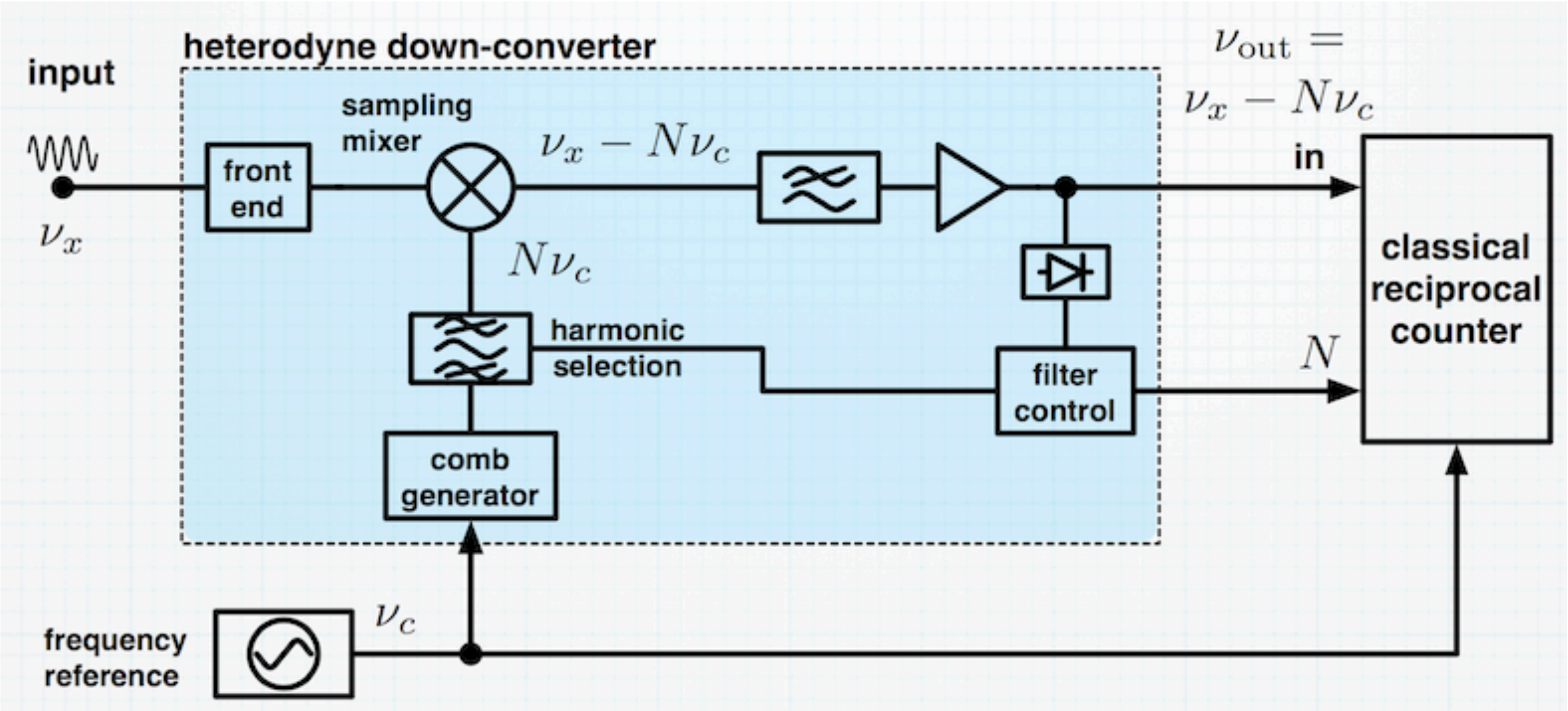


The transfer oscillator is a PLL

The counter measures the VCO frequency

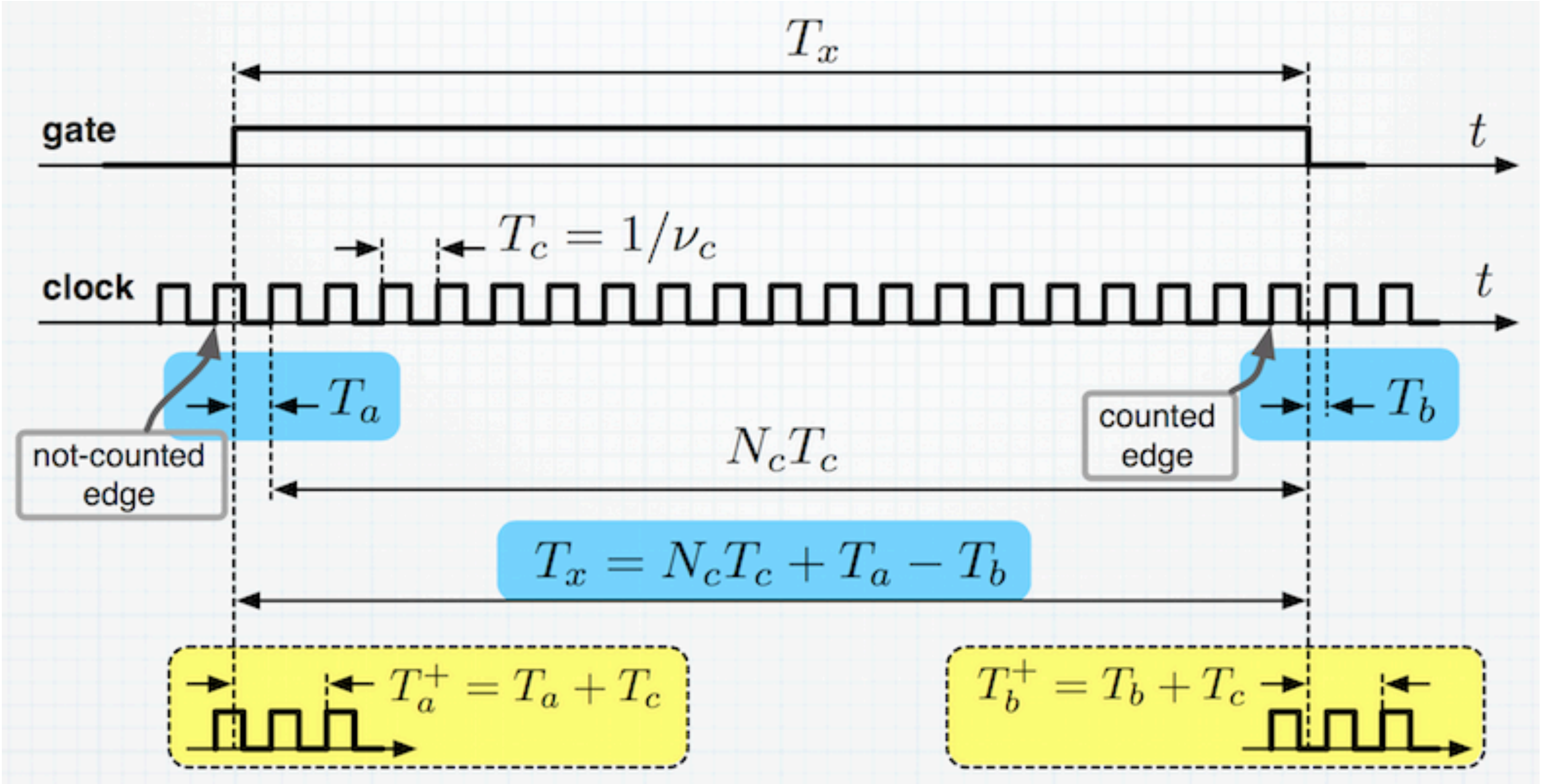
The harmonic *N* has to be measured (modulation frequency)

High-frequency measurements: Heterodyne counter



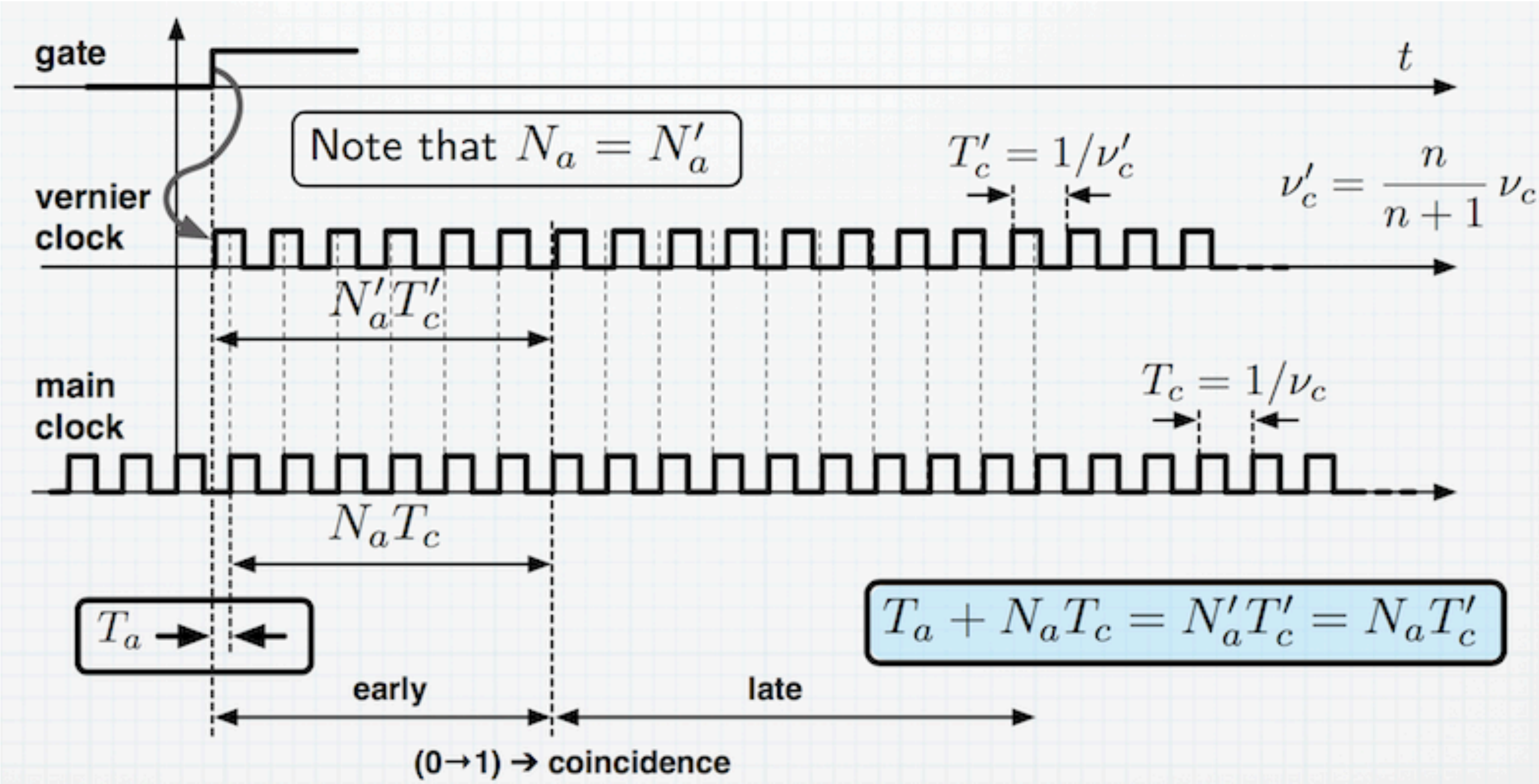
The beat frequency is in the range of the classic counters 100-200 MHz

Interpolation methods

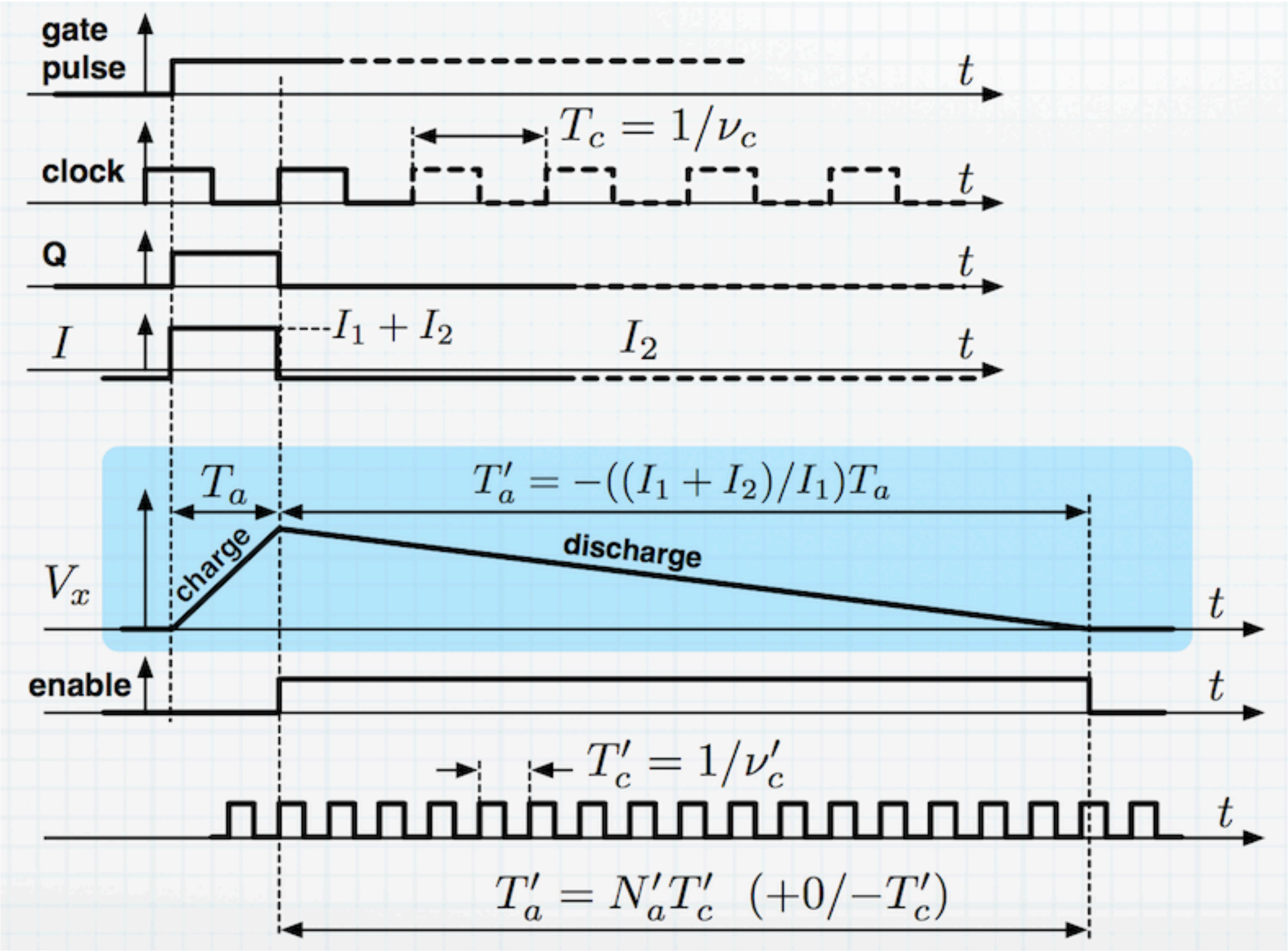


The interpolation is based on the high-stability and accuracy reference clock

Dual Vernier



Dual ramp



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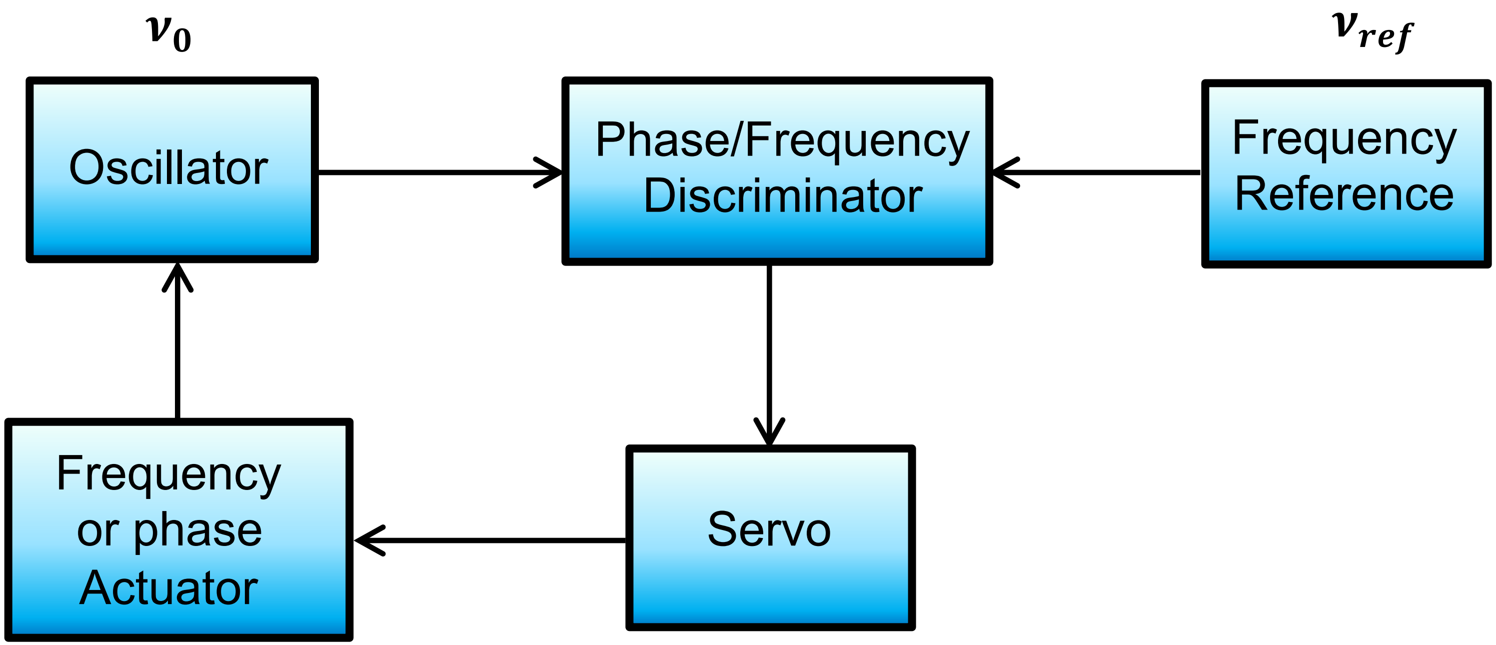
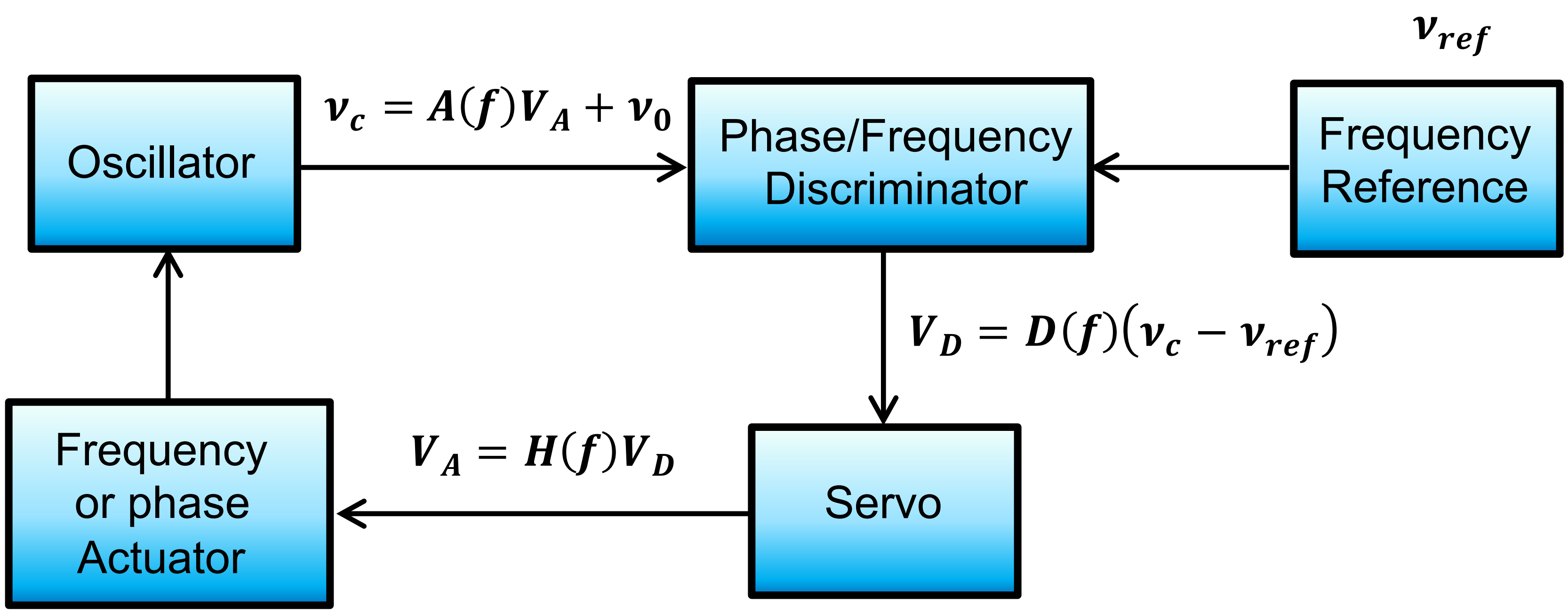
1. General scheme
2. Stability performance
3. Some examples

General stabilization scheme



Stabilization performance



$$\nu_c(f) = \frac{\nu_0(f) - \nu_{ref}(f)D(f)H(f)A(f)}{1 + D(f)H(f)A(f)} = \frac{\nu_0(f) - G_{loop}(f)\nu_{ref}(f)}{1 + G_{loop}(f)}$$

$$S_{\nu_c}(f) = \frac{S_{\nu_0}(f)}{\left|1 + G_{loop}(f)\right|^2} + \frac{S_{\nu_{ref}}(f)\left|G_{loop}(f)\right|^2}{\left|1 + G_{loop}(f)\right|^2}$$

Stabilization performance

$$S_{\nu_c}(f) = \frac{S_{\nu_0}(f)}{\left|1 + G_{loop}(f)\right|^2} + \frac{S_{\nu_{ref}}(f)\left|G_{loop}(f)\right|^2}{\left|1 + G_{loop}(f)\right|^2}$$

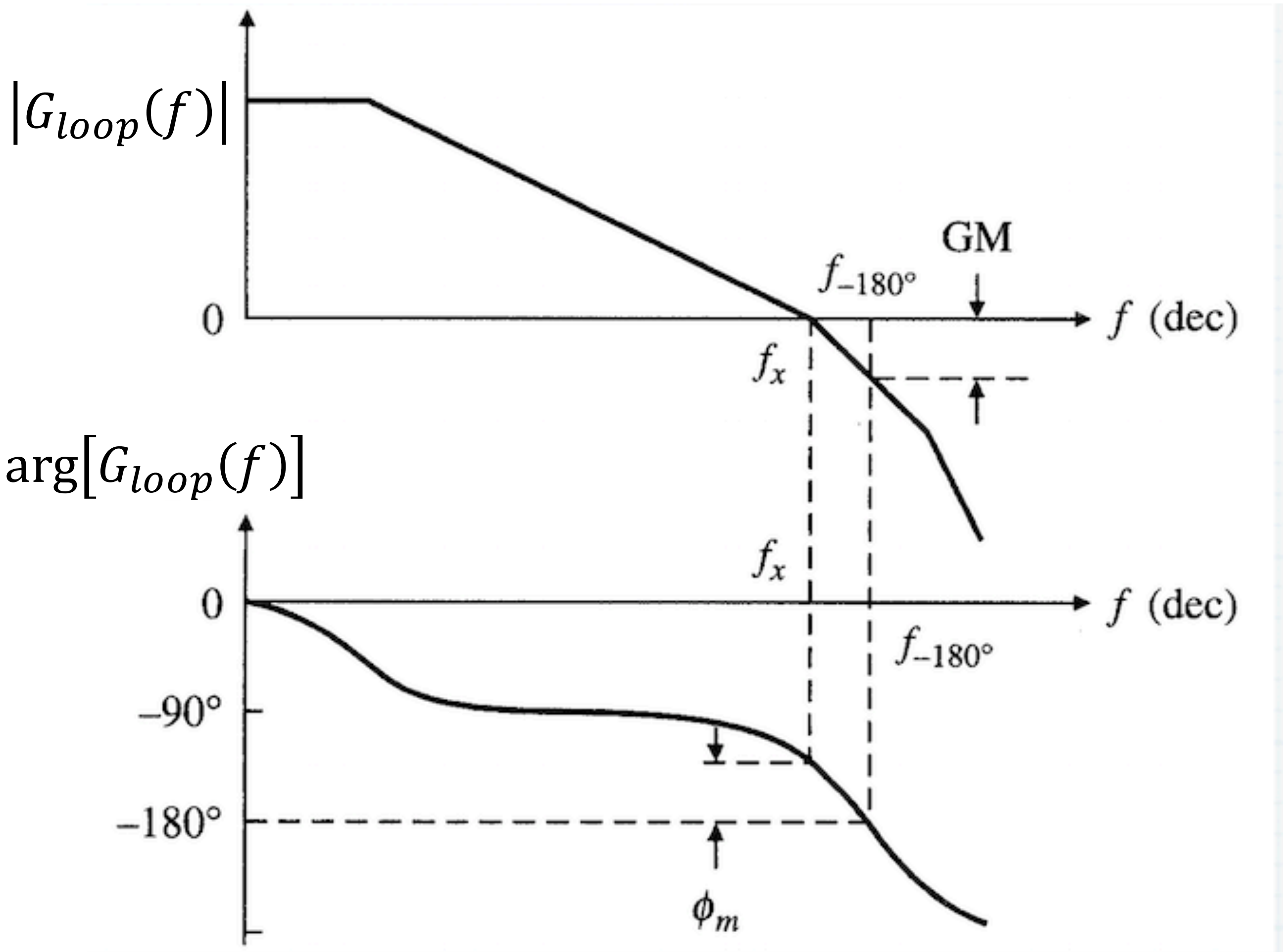
For $\left|G_{loop}(f)\right| \gg 1$

$$\boldsymbol{S_{\nu_c}(f) = \frac{S_{\nu_0}(f)}{\left|G_{loop}(f)\right|^2} + S_{\nu_{ref}}(f) \cong S_{\nu_{ref}}(f)}$$

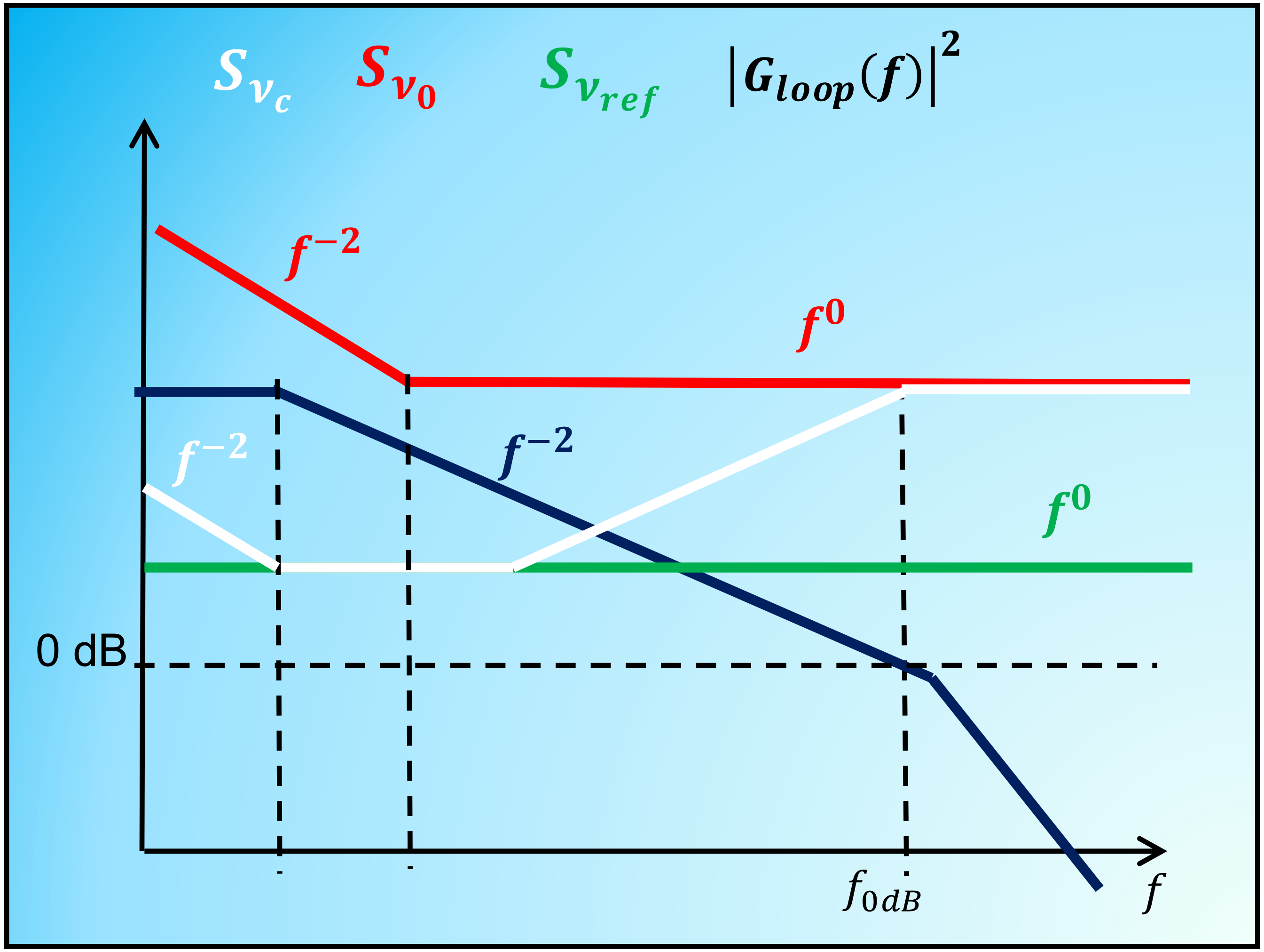
For $\left|G_{loop}(f)\right| \ll 1$

$$S_{\nu_c}(f) = S_{\nu_0}(f) + S_{\nu_{ref}}(f)\left|G_{loop}(f)\right|^2 \cong S_{\nu_0}(f)$$

Stability of the control loop



Performance

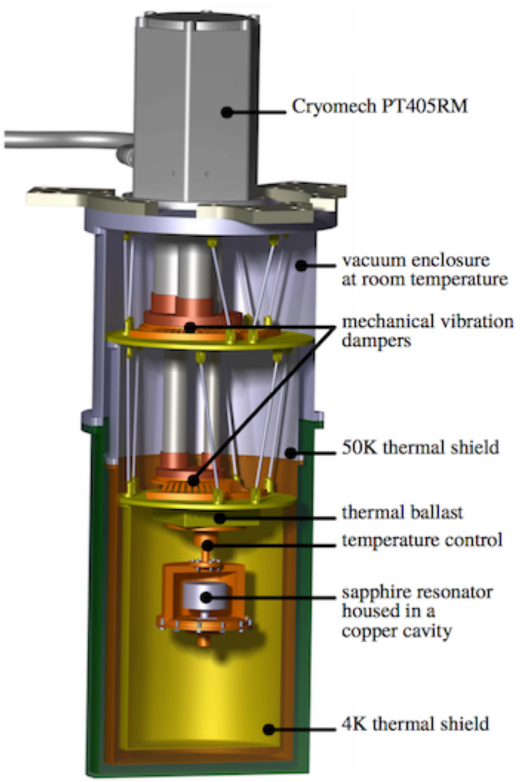


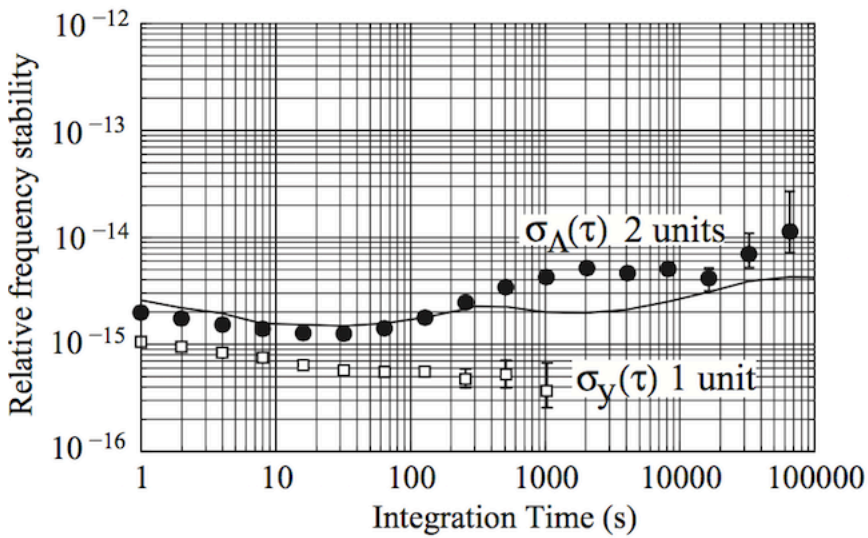
Implementation in the microwave region

AIP | Review of Scientific Instruments

New-generation of cryogenic sapphire microwave oscillators for space, metrology, and scientific applications

Vincent Giordano, Serge Grop, Benoît Dubois, Pierre-Yves Bourgeois, Yann Kersalé et al.





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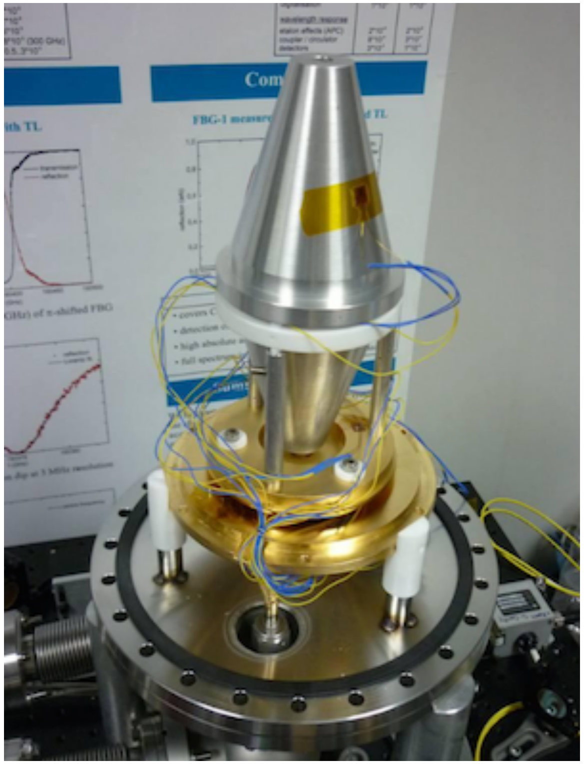
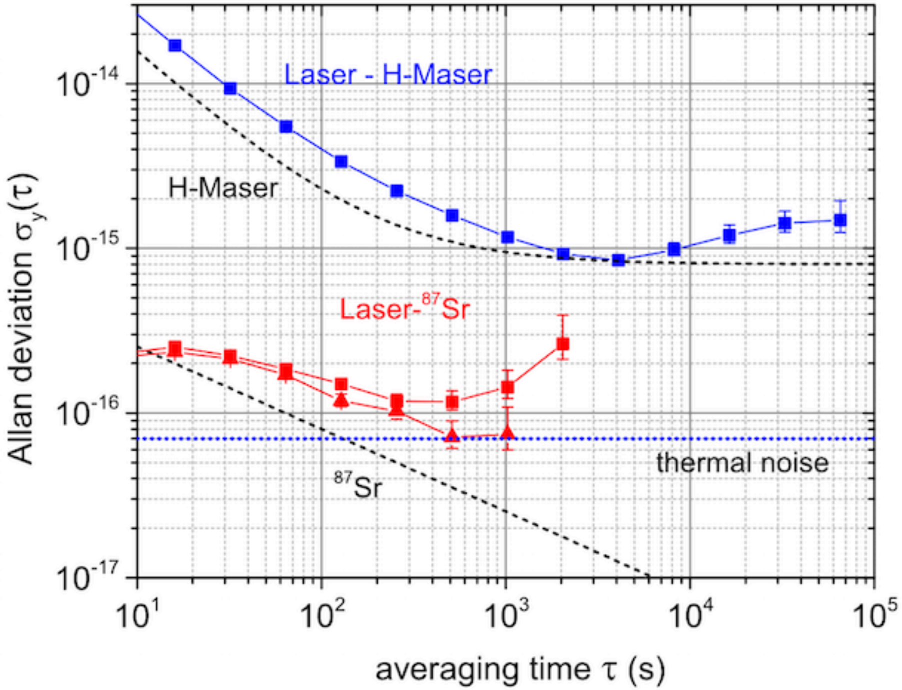
OPTICS LETTERS / Vol. 39, No. 17 / September 1, 2014

Ultrastable laser with average fractional frequency drift rate below 5×10^{-19}/s

Christian Hagemann,[1] Christian Grebing,[1] Christian Lisdat,[1] Stephan Falke,[1] Thomas Legero,[1] Uwe Sterr,[1,*] Fritz Riehle,[1] Michael J. Martin,[2] and Jun Ye[2]





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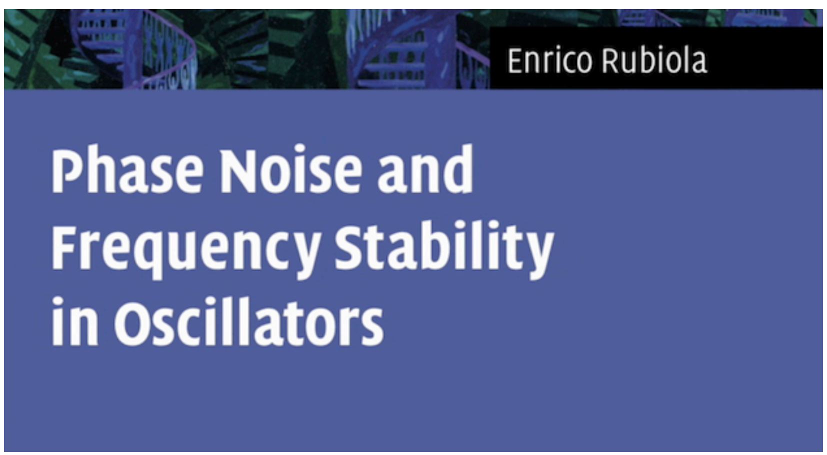

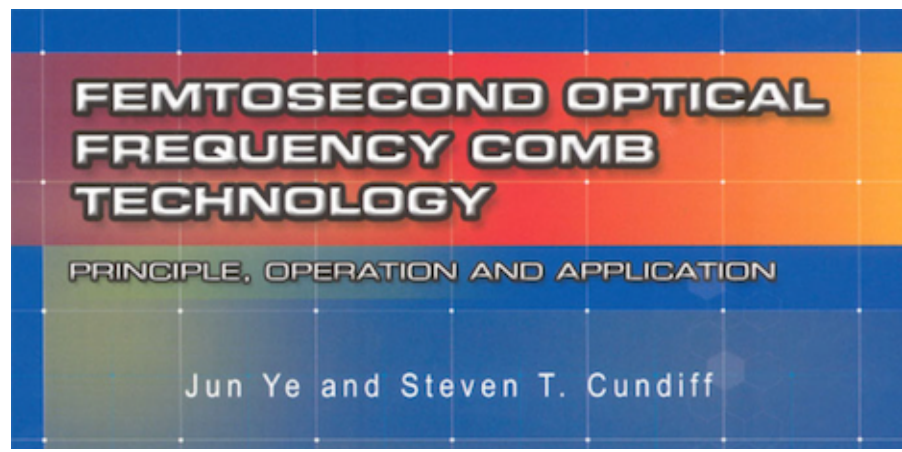

Thank you for attention

…a special thank to Elio

Washington DC, July 1998