Amplitude fluctuations in the Berezinskii-Kosterlitz-Thouless phase

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Mermin-Wagner theorem (1966)

No spontaneous breaking of continuous symmetries in $d=2$ at $T>0$.

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No long range order (LRO)

Special situation: *U(1)* symmetry group

No LRO, but (at T sufficiently small):

- Algebraic decay of order-parameter correlations
- Order-parameter stiffness
- Spin-wave (phase) excitations

At T sufficiently large:

- The usual behavior (exponential decay of correlations)

The Berezinskii-Kosterlitz-Thouless (BKT) transition

- Driven by topological excitations (vortices)
- The free energy is a smooth (but not analytical) function
- Universal jump of order-parameter stiffness
- Essential singularity of the correlation length

Amplitude fluctuations in the BKT phase

Standard approach: "Amplitude fluctuations are suppressed"

Effective Hamiltonian for (only) phase fluctuations

$$H_{eff} = \frac{K}{2} \int d^2r \, (\nabla\theta)^2$$

Governs the low-T phase

Higher T - vortices

BKT from FRG (ϕ^4 model)

Graeter, Wetterich, PRL 1995

Gersdorff, Wetterich, PRB 2001

PJ, Dupuis, Delamotte, PRE 2014

PJ, Eberlein, PRE 2016

PJ, Metzner, arXiv:1606.04547

Common feature:

Vortices not captured

Amplitude fluctuations kept…

Examine the role of the amplitude mode.

Motivation:

Coupling between the amplitude and phase modes

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Strong renormalization of the amplitude mass

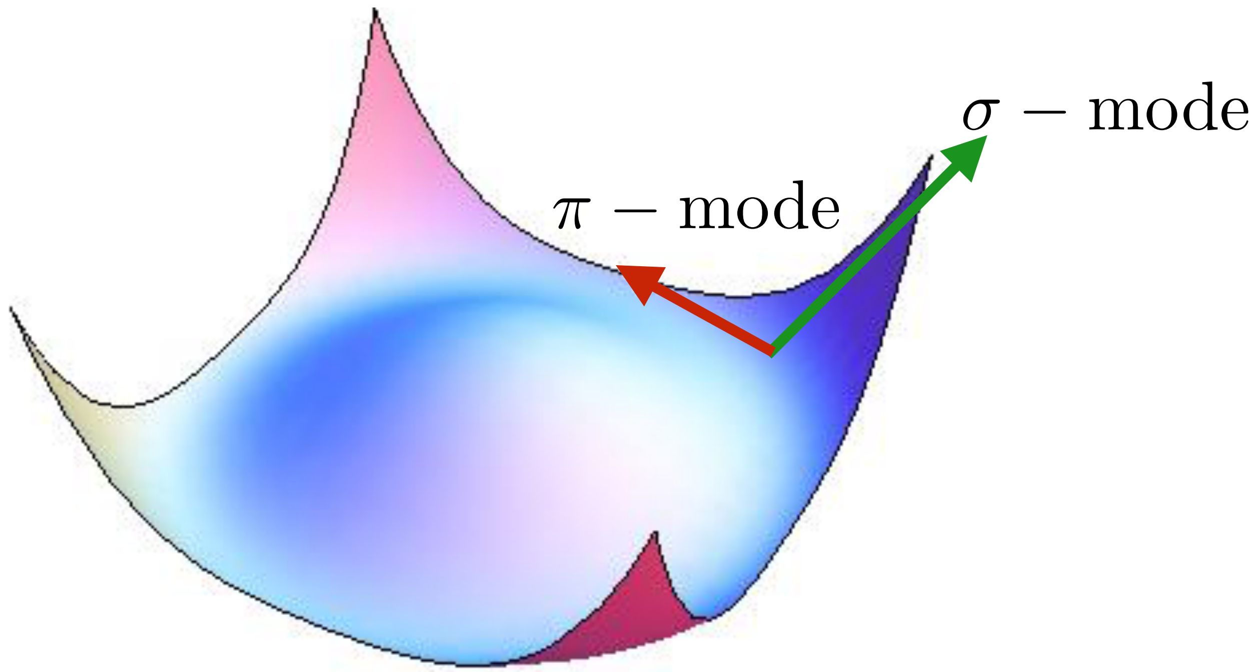
E.g. for the ground state of the interacting Bose gas the amplitude mass is reduced to **zero**…

e.g. *Castellani et al (1997)*

Effect of the *effectively* massless amplitude mode on quasi long-range order unknown ?

Model:

$$S[\phi] = \beta \left[\frac{u_0}{8} \int d^2r \left(|\phi(\mathbf{r})|^2 - \alpha_0^2 \right)^2 + \frac{Z_0}{2} \int d^2r |\nabla \phi(\mathbf{r})|^2 \right]$$



$$\mathbf{r} \in \mathbb{R}^2$$

$$\phi(\mathbf{r}) \in \mathbb{C}$$

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S[\phi]}$$

Field decomposition: $\phi(\mathbf{r}) = \alpha + \sigma(\mathbf{r}) + i\pi(\mathbf{r})$

The radial mode is supposed to be irrelevant in the standard BKT theory.

Is this really true?

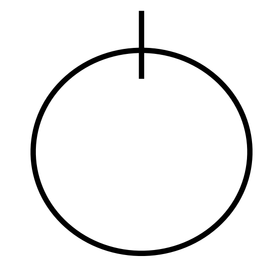
Functional renormalization group (1PI scheme)

$$\lim_{\Lambda\to 0}\Gamma_\Lambda[\phi] = F[\phi]$$

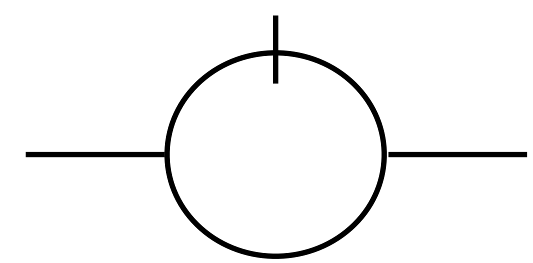
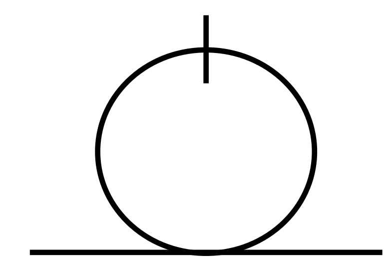
$$\lim_{\Lambda\to \Lambda_{UV}}\Gamma_\Lambda[\phi] = S[\phi]$$

Wetterich equation

$$\partial_\Lambda \Gamma_\Lambda[\phi] = \frac{1}{2}\mathrm{Tr}\left\{\partial_\Lambda R\left[\Gamma^{(2)}[\phi] + R\right]^{-1}\right\}$$

$\partial_\Lambda \Gamma_\Lambda \sim$ 

$$\partial_\Lambda \Gamma^{(2)}_\Lambda[\phi] = \dots$$

$\partial_\Lambda \Gamma^{(2)}_\Lambda \sim$  + 

$$\rho(\mathbf{r}) = |\phi(\mathbf{r})|^2$$

Derivative expansion:

$$\Gamma_\Lambda[\phi] = \beta\int d^2\mathbf{r}\left\{U_\Lambda\left(\rho(\mathbf{r})\right) + \frac{1}{2}Z_\Lambda\left(\rho(\mathbf{r})\right)|\nabla\phi(\mathbf{r})|^2 + \frac{1}{8}Y_\Lambda(\rho(\mathbf{r}))\left(\nabla|\phi(\mathbf{r})|^2\right)^2\right\}$$

Derivative expansion + vertex expansion:

$$\Gamma_\Lambda[\phi] = \beta\int d^2\mathbf{r}\left\{\frac{u_\Lambda}{8}\left[|\phi\left(\mathbf{r}\right)|^2 - \alpha_\Lambda^2\right]^2 + \frac{Z_\Lambda}{2}|\nabla\phi(\mathbf{r})|^2 + \frac{Y_\Lambda}{8}\left[\nabla|\phi(\mathbf{r})|^2\right]^2\right\}$$

Flow equations:

$$\partial_\Lambda \Gamma_\Lambda[\phi] = \frac{1}{2}\mathrm{Tr}\left\{\partial_\Lambda R\left[\Gamma^{(2)}[\phi] + R\right]^{-1}\right\}$$

Full derivative expansion:

$$\partial_\Lambda U_\Lambda(\rho) = \dots$$
$$\partial_\Lambda Z_\Lambda(\rho) = \dots$$
$$\partial_\Lambda Y_\Lambda(\rho) = \dots$$

Closed system of nonlinear PDEs

Derivative expansion + vertex expansion:

$$\partial_\Lambda \alpha_\Lambda = \dots$$
$$\partial_\Lambda u_\Lambda = \dots$$
$$\partial_\Lambda Z_\Lambda = \dots$$
$$\partial_\Lambda Y_\Lambda = \dots$$

Closed system of nonlinear ODEs

Radial mass: $m_\Lambda^2 = u_\Lambda \alpha_\Lambda^2$

Reproducing the standard BKT phase:

PJ, Metzner, arXiv:1606.04547

$$\partial_\Lambda \alpha_\Lambda = \dots$$
$$\partial_\Lambda u_\Lambda = \dots$$
$$\partial_\Lambda Z_\Lambda = \dots$$
$$\partial_\Lambda Y_\Lambda = \dots$$

$$m_\Lambda^2 = u_\Lambda \alpha_\Lambda^2$$

Discard amplitude fluctuations →

$$\partial_\Lambda \alpha_\Lambda = \dots$$
$$\partial_\Lambda Z_\Lambda = \dots$$

Solution:

$$\alpha_\Lambda^2 \sim \Lambda^\eta$$
$$Z_\Lambda \sim \Lambda^{-\eta}$$

Stiffness: $J_\Lambda = Z_\Lambda \alpha_\Lambda^2$

$$\partial_\Lambda J_\Lambda = 0 \quad \text{for} \quad \Lambda \to 0$$

$$\eta \approx \frac{T}{2\pi J} \quad \text{for} \quad T \ll 4\pi J$$

This reproduces the line of BKT fixed points in the low-T phase

But what about the amplitude fluctuations?

Power counting:

At the BKT fixed point:

$$\alpha_\Lambda^2 \sim \Lambda^\eta$$

$$Z_\Lambda \sim \Lambda^{-\eta}$$

$$\partial_\Lambda m_\sigma^2 = \ldots \xrightarrow{\text{(Discard amplitude fluctuations)}} m_\Lambda^2 \sim \Lambda^{2-\eta}$$

Other quantities:

$$u_\Lambda \sim \Lambda^{2-2\eta}$$

$$Y_\Lambda \sim \Lambda^{-2\eta}$$

Conclusion: Phase and amplitude modes equally relevant for $\Lambda \to 0$

Justification for dropping the amplitude mode is illusive

Coupled amplitude and phase fluctuations

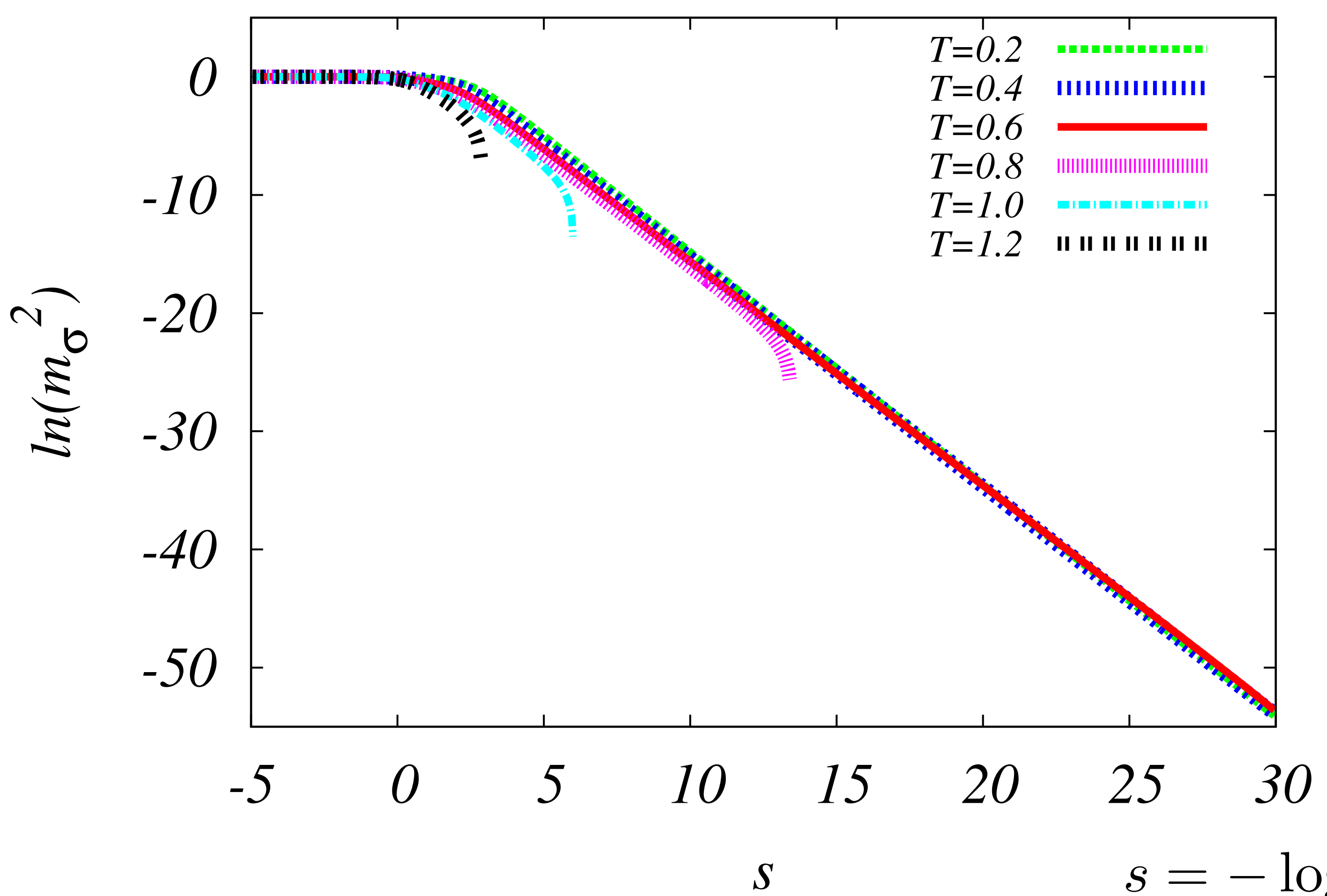
$$\partial_\Lambda \alpha_\Lambda = \dots$$
$$\partial_\Lambda u_\Lambda = \dots$$
$$\partial_\Lambda Z_\Lambda = \dots$$
$$\partial_\Lambda Y_\Lambda = \dots$$

$$m_\Lambda^2 = u_\Lambda \alpha_\Lambda^2$$

BKT fixed point is (marginally) destabilised by amplitude fluctuations

Logarithmic flow of amplitude stiffness

RG flow ultimately ends up in the phase with vanishing stiffness (even if this happens at very low scales)



$s = -\log(\Lambda/\Lambda_{UV})$

Coupled amplitude and phase fluctuations

$$\partial_\Lambda \alpha_\Lambda = \ldots$$
$$\partial_\Lambda u_\Lambda = \ldots$$
$$\partial_\Lambda Z_\Lambda = \ldots$$
$$\partial_\Lambda Y_\Lambda = \ldots$$

$$m_\Lambda^2 = u_\Lambda \alpha_\Lambda^2$$

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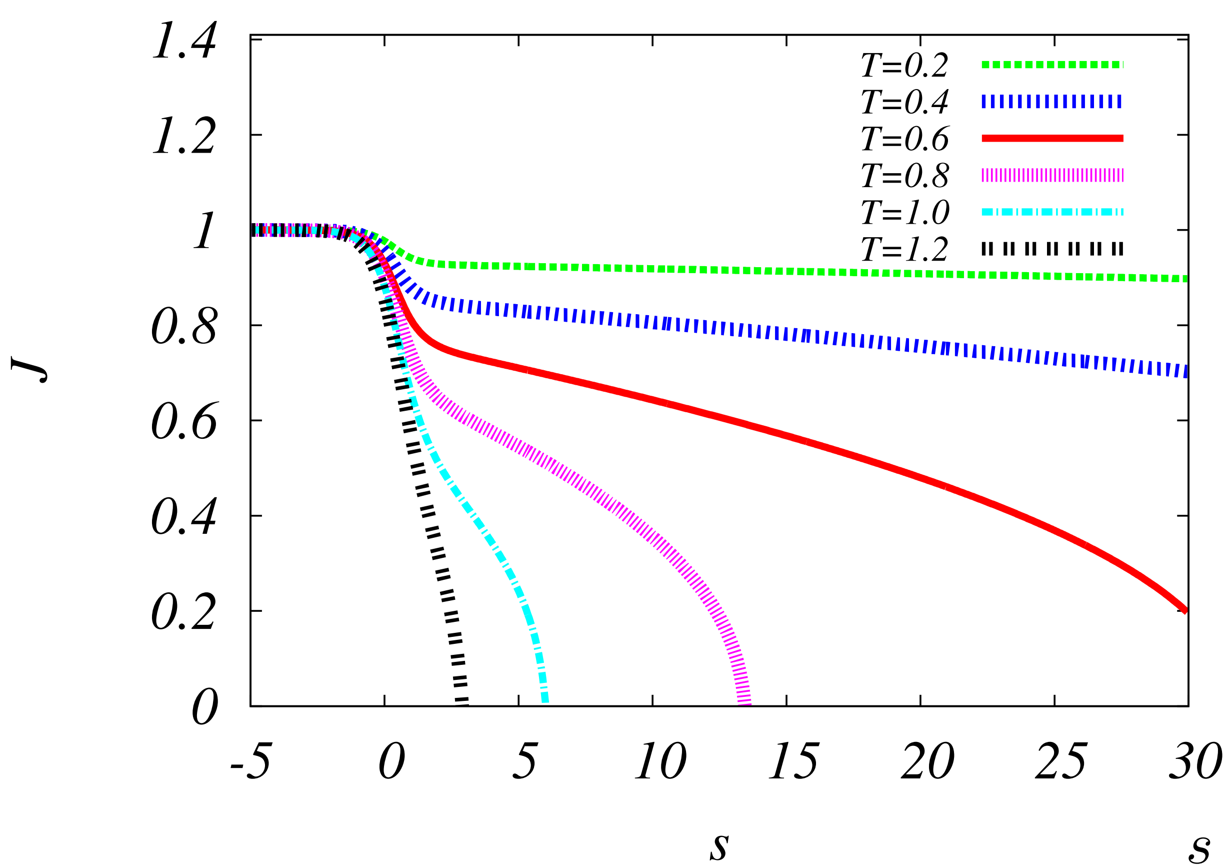
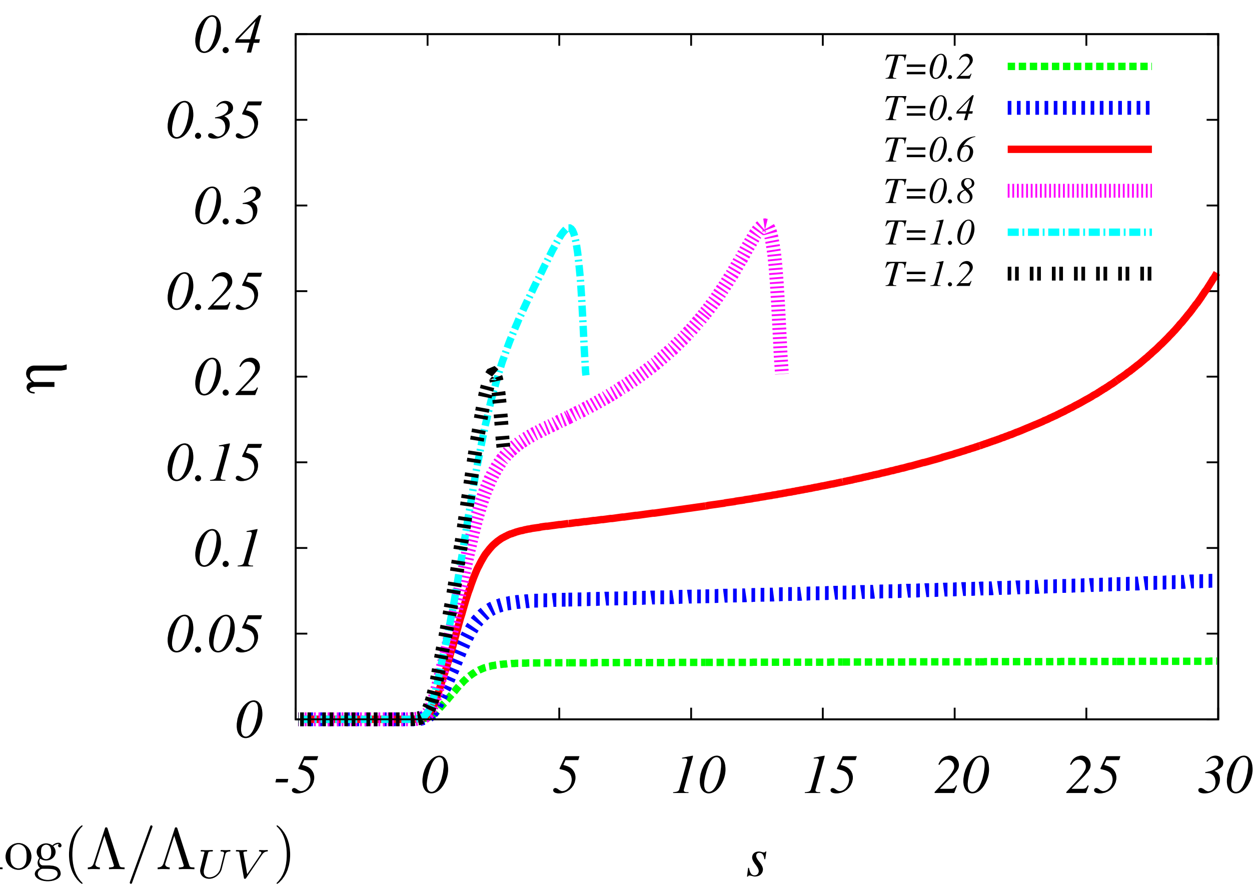
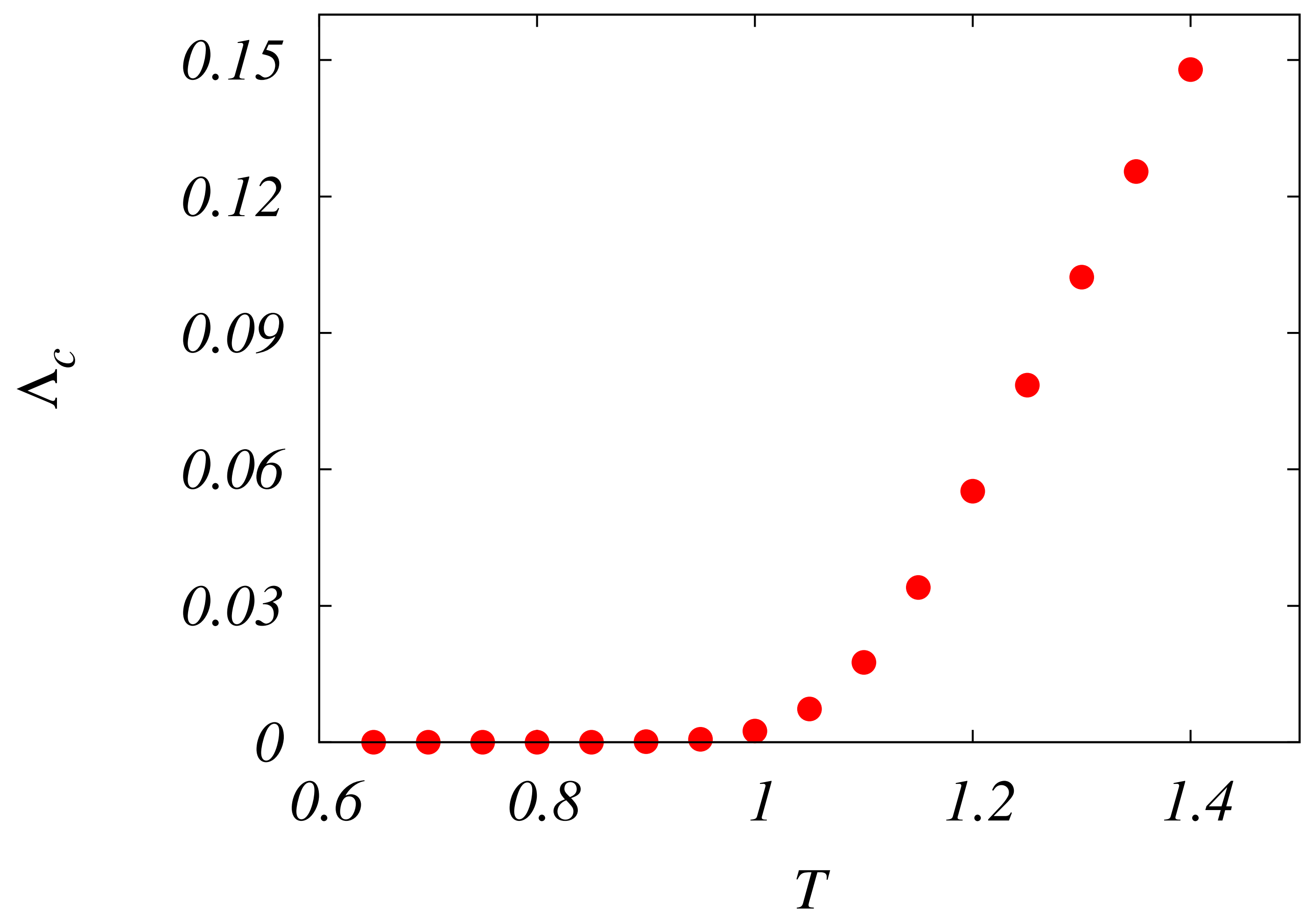




$$s = -\log(\Lambda/\Lambda_{UV})$$

Critical scale:

$$\Lambda_c \sim \xi^{-1}$$

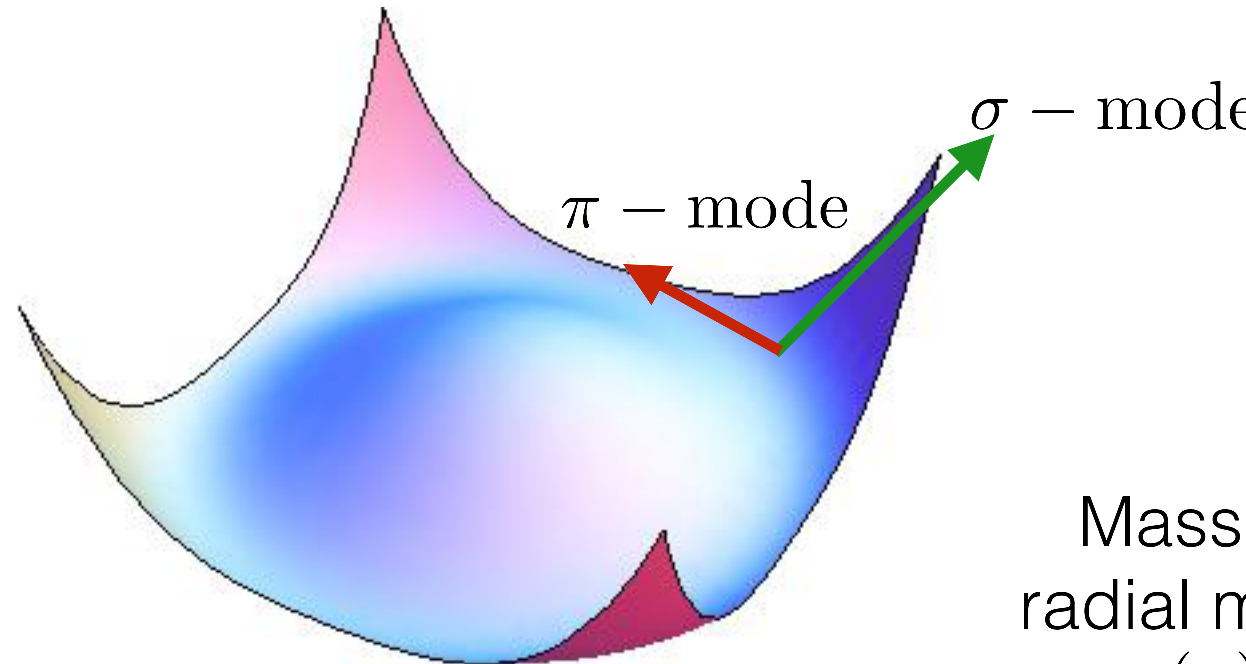


Exponential decrease upon lowering *T*

$$\Lambda_c \approx \Lambda_{UV} e^{-const/T^2}$$

ξ practically infinite for *T* small

Summarizing picture:



"Bare" model:	Massless Goldstone mode (π)	Massive radial mode (σ)

Interactions between the modes → Softening of the radial mode

Feedback of the softened radial mode on stiffness → Collapse of quasi long-range order at any *T>0*.

Absence of BKT transition in *2d* systems hosting the amplitude mode → Instead a very sharp crossover

BKT phase may still exist in systems without the amplitude mode

No contradiction to rigorously established results

Derivative expansion at order ∂^2

$$\Gamma_\Lambda[\phi] = \beta \int d^2\mathbf{r} \left\{ U_\Lambda(\rho(\mathbf{r})) + \frac{1}{2} Z_\Lambda(\rho(\mathbf{r})) |\nabla\phi(\mathbf{r})|^2 + \frac{1}{8} Y_\Lambda(\rho(\mathbf{r})) \left(\nabla|\phi(\mathbf{r})|^2\right)^2 \right\}$$

- No fixed points at low T
- Integrating the flow at T low not possible
- Limited analytical insight
- Presence of quasi-plateaus

Gersdorff, Wetterich, PRB 2001

PJ, Dupuis, Delamotte, PRE 2014

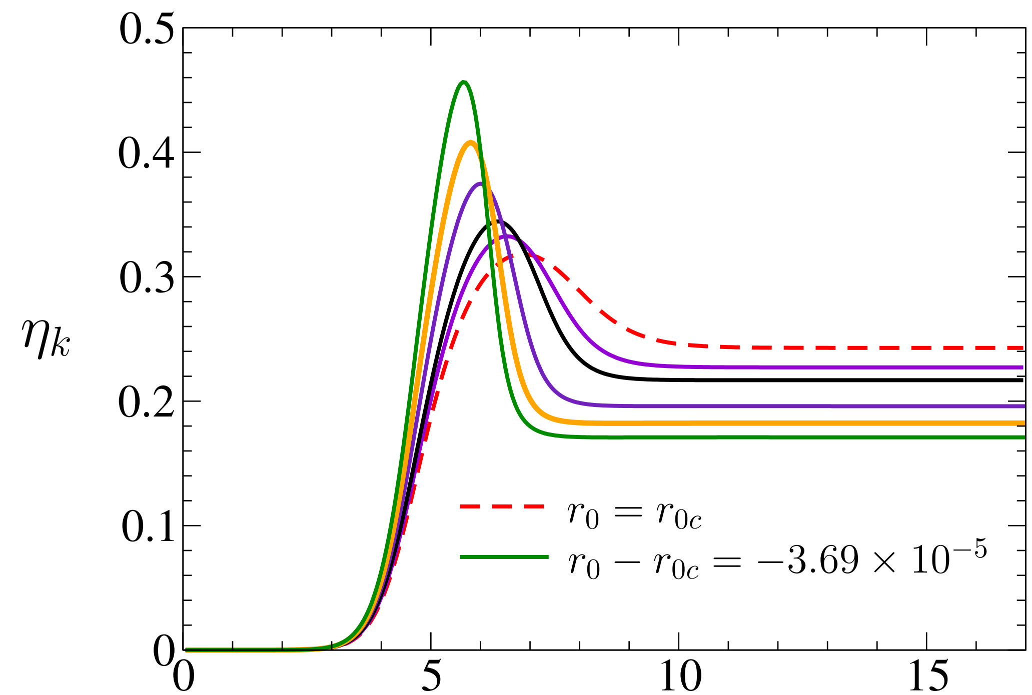
PJ, Eberlein, PRE 2016

BKT phase not restored by any higher-order terms in ϕ

But….

possibility of tuning the regulator to obtain a true BKT-type behavior
(for some choices of renormalization point, some *T*-dependent cutoffs, only for sufficiently high *T*…)

agreement on the values of η and ρ_s despite the absence of vortices…



On the other hand… other choices of the regulator yield absence of any fixed-point…

Summary:

G-W type calculation:

Reproduces the BKT f-p line
(upon neglecting the amplitude fluctuations).

Leads to a marginal instability of quasi long-range order due to the amplitude mode.

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No contradiction with rigorous results, but implies a reinterpretation of the whole BKT theory.

Full DE:

Has the potential of enforcing quasi long range order in a range of temperatures close to T_{KT} by manipulating the cutoff.
(In a somewhat unnatural way.)

At T_{KT} agrees with the standard BKT theory at a quantitative level.
(Despite the crucial difference in the mechanisms driving the transition…)