Integrable versus ergodic approaches to describe quasi-stationary states

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A.C. Ribeiro-Teixeira, F.P.C.B., R. Pakter & Y. Levin, *Phys. Rev. E* **89** (2014)

The HMF model Konishi & Kaneko (1992), Antoni & Ruffo (1995)

- N spins
- Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} [1 - \cos(\theta_i - \theta_j)]$$

- Equation of motion

$$\dot{\theta}_i = p_i$$

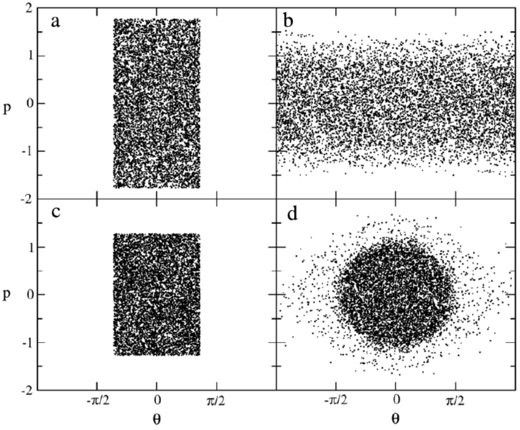
$$\dot{p}_i = -M \sin\theta_i$$

$$M = \frac{1}{N} \sum_{i=1}^{N} \cos\theta_i$$

- Thermodynamic limit $N \to \infty$: Vlasov (collisionless) dynamics

Quasi-stationary states

Two phases:



- paramagnetic ($M = 0$)
- ferromagnetic ($M > 0$)

Virial and non-virial initial conditions

Virial theorem:

$$-\left\langle \sum_{i=1}^{N} F(\mathbf{q}_i) \cdot \mathbf{q}_i \right\rangle = 2 \langle \mathcal{K} \rangle$$

$\mathcal{K}$ = kinetic energy

- Ex: 3D gravity $\rightarrow 2\mathcal{K} = -\mathcal{U}$.

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Initial conditions do not satisfy virial theorem

- Strong mean-field oscillations
- Resonances, core-halo distribution (Friday's talk by Y. Levin)

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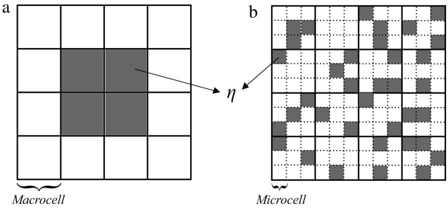
Initial conditions satisfy virial theorem

- Minimal mean-field oscillations
- Quasi-stationary potential

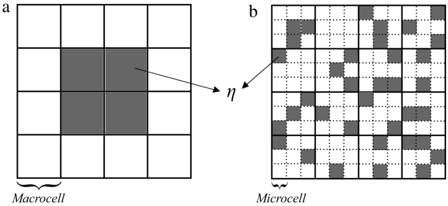
Lynden-Bell statistics



Initial distribution: $f_0(\mathbf{r}, \mathbf{v}) = \eta\Theta(r_m - |\mathbf{r}|)\Theta(v_m - |\mathbf{v}|)$
Number of microstates W:

- Phase space $\rightarrow$ macrocells and microcells
- Incompressible dynamics: number of occupied microcells is preserved
- Boltzmann counting

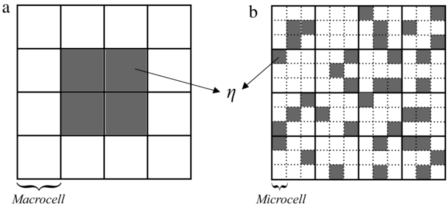
Lynden-Bell (LB) Statistics



Coarse-grained entropy: $s_{\mathrm{LB}} = k_B \ln W$
Coarse-grained distribution:

$$f_{\mathrm{LB}}(\mathbf{r}, \mathbf{v}) = \frac{\eta}{1 + \exp\left[\beta(\epsilon(\mathbf{r}, \mathbf{v}) - \mu)\right]}$$

Lynden-Bell (LB) Statistics



Assumption:

- Equal probability of phase elements occupying any microcell $\rightarrow$ ergodicity and mixing

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$$-\left\langle \sum_{i=1}^{N} F(\mathbf{q}_i)\cdot\mathbf{q}_i \right\rangle = 2\left\langle \mathcal{K}\right\rangle , \qquad \mathcal{K} = \text{ kinetic energy}$$

- Ex: 3D gravity $\rightarrow 2\mathcal{K} = -\mathcal{U}$.

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Initial conditions satisfy virial theorem

$$\left.\begin{array}{l} \blacktriangleright \text{ Minimal mean-field oscillations} \\ \blacktriangleright \text{ Quasi-stationary potential} \\ \blacktriangleright \text{ Lynden-Bell statistics} \end{array}\right\} ?$$

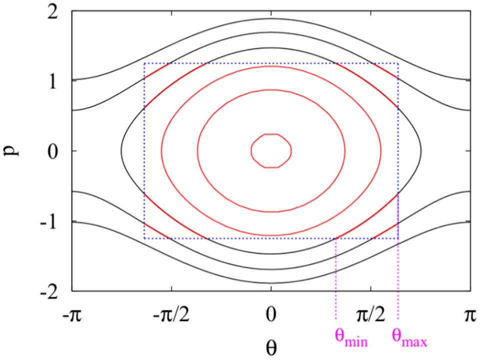
Uncoupled pendula approach

QSS:

- Quasi-static field $M = \langle \cos\theta \rangle$
- Equation of motion $\ddot{\theta} = -M \sin\theta$

The model – de Buyl et al, *PRE* **84** (2011):

- External field H
- Uncoupled particles, equation of motion $\ddot{\theta} = -H \sin\theta$
- $f(\varepsilon) = n(\varepsilon)/g(\varepsilon)$ is conserved
- Integrable dynamics $\rightarrow$ "integrable model" (IM)



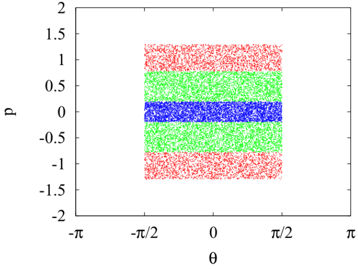
A.C. Ribeiro-Teixeira et al, PRE **89** (2014)

$$f(\varepsilon; H) = \frac{\int f_0(\theta, p)\delta[\varepsilon - \epsilon(\theta, p, H)]d\theta dp}{\int \delta[\varepsilon - \epsilon(\theta, p, H)]d\theta dp}$$

$$P(\theta; H) = \int f[\varepsilon(\theta, p); H]dp$$

Association with HMF: $H = \int \cos\theta P(\theta; H)d\theta$

Application to the HMF



- Initial conditions:

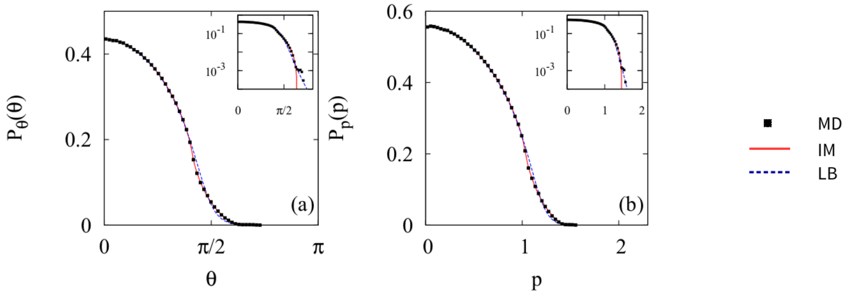
$$f_0(\theta, p) = \Theta(\theta_m - |\theta|) \times \sum_{i=1}^{L} \eta_i \Theta(|p| - p_{i-1}) \Theta(p_i - |p|)$$

- Analytical equation for $f(\varepsilon)$

Comparison with molecular dynamics (MD) and Lynden-Bell (LB)

Angle and momentum distributions

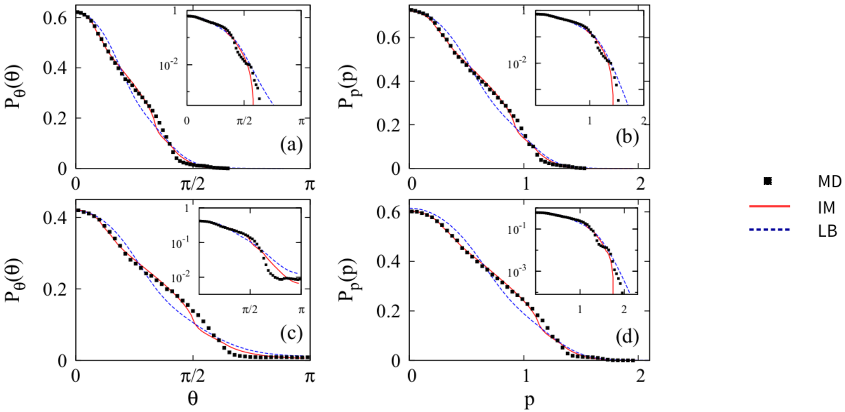
$L = 1$



Comparison MD, IM, LB

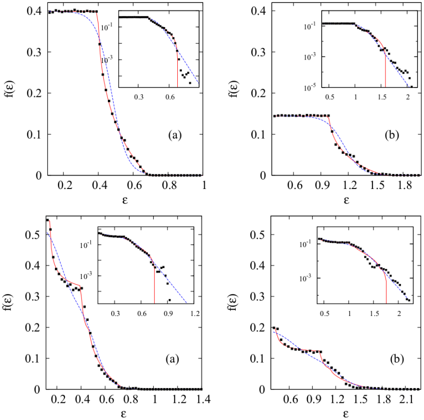
Angle and momentum distributions

$L = 3$



Comparison MD, IM, LB

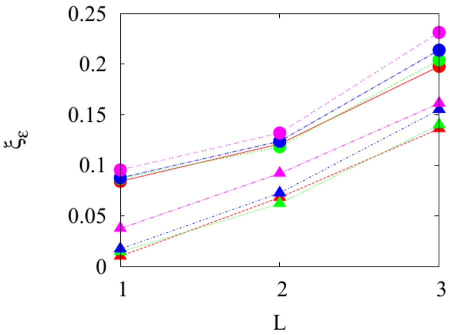
Energy distribution



- $L = 1$
- $L = 2$

Comparison MD, IM, LB

RMS deviation of energy distributions
(triangles: IM-MD deviation, circles: LB-MD deviation)



Summary

- IM: Uncoupled particles subject to external field $H = M$
- LB: Ergodic, mixing approach; new statistical method
- $H = M$, $M \rightarrow$ virial magnetization
- IM (integrable) better results than LB (ergodic) for MD with initial multilevel waterbag distributions
- Agreement decreases when number of levels increases
- IM can be used for other long-range systems, i.e. 3d self-gravitating systems (FPCB, A.C Ribeiro-Teixeira, R. Pakter & Y. Levin, *PRL* **113** 2014)

Comparison with molecular dynamics (MD)