Self-gravitating systems and cosmological structure formation

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2 Lecture Plan

PART I (1.5 lectures) : **A brief introduction to the problem of SF in cosmology**

PART II (0.5 lectures): **1D toy models for astrophysics and cosmology**

Caveat: References

All of Part I is a review for a broad physics audience of “the basic essentials” of this field, or at least what I myself consider these to be. It is based on many different sources, and only the perspective and emphasis are my own. **I have not attempted to provide a full bibliography or cite the original references exhaustively, and indeed references are given only for specific results which are less part of the standard “canon” of the field.** Although written now over 30 years ago, Peebles “The Large Scale Structure of the Universe” (Princeton 1980) remains a basic reference the analytical theory described; fairly up to date reviews of N body numerical simulations are easy to find; for halo models the review of Cooray and Sheth (Phys. Rep. 2002) is a good starting point.

In Part II on the other hand I provide more extensive (but not exhaustive) references, as the topic is a more circumscribed one where this is easily done.

PART I

Intro to cosmological structure formation

Things I would hope an uninitiated listener may "take away":

Things I would hope an uninitiated listener may "take away":

- What the “**problem of structure formation**” (SF) is
- Why the **Newtonian limit** of purely self-gravitating matter is a good approx.
- What the problem of SF then reduces to **formally (equations!)**
- How **initial conditions** are described and derived
- Some **basic analytical results** about SF

- -

- What **numerical methods** for simulation of SF are
- What **main qualitative results** are for non-linear SF
- A few major **open issues**, some connections to other LR systems

The problem of cosmological structure formation

Modern cosmology

Homogeneous and isotropic universe in GR → FRW solutions of gravity

"Real universe":

perturbed FRW metric coupled to perturbed matter/energy content

→ "The standard cosmological model"

\+ **4 parameters** specifying FRW model

(radiation, baryons, dark matter, dark energy)

\+ **2 parameters** specifying initial initial fluctuations

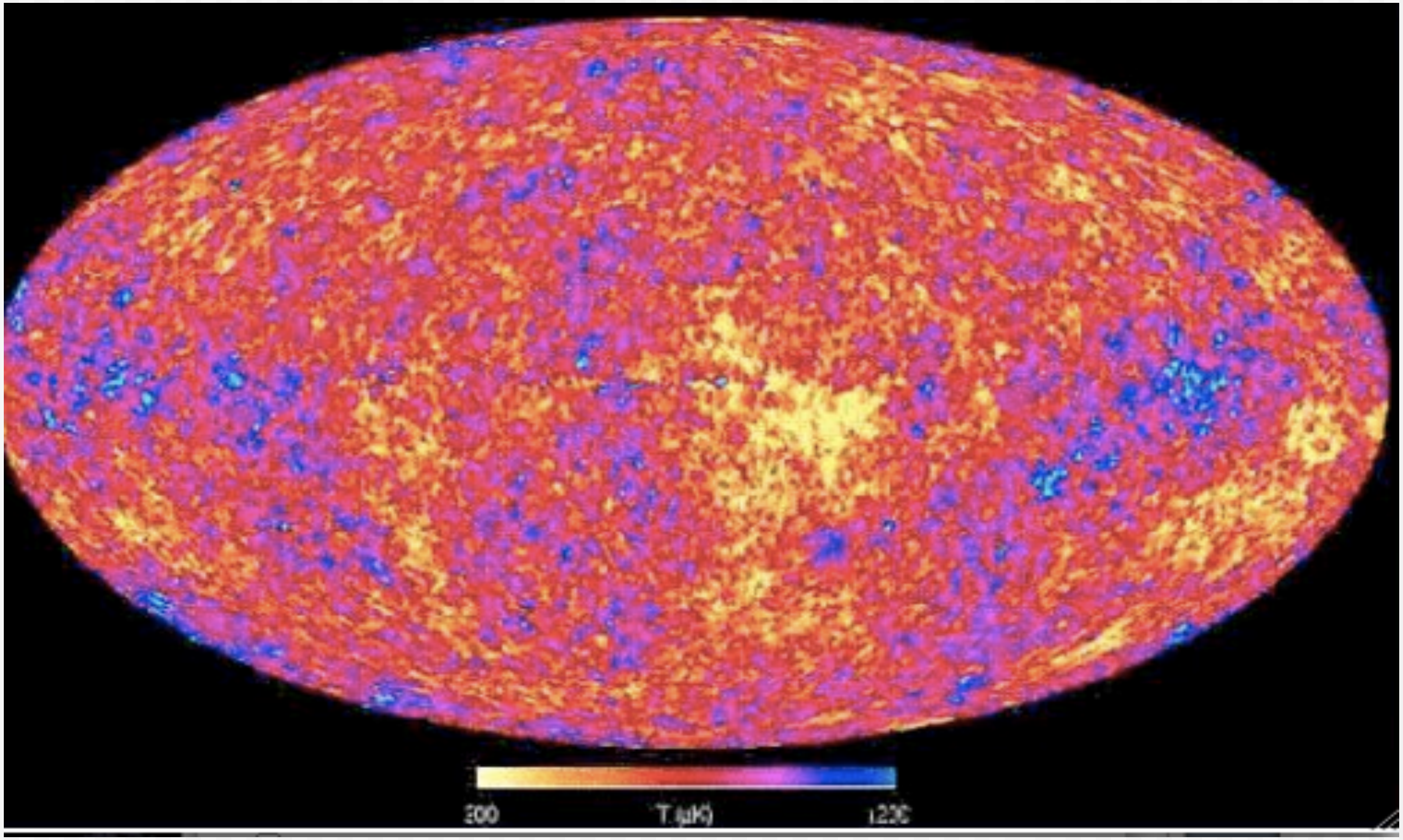
(+"standard physics")

→ Predictions for evolution of universe!!

Linearized version for small perturbations is impressively successful…

Cosmology

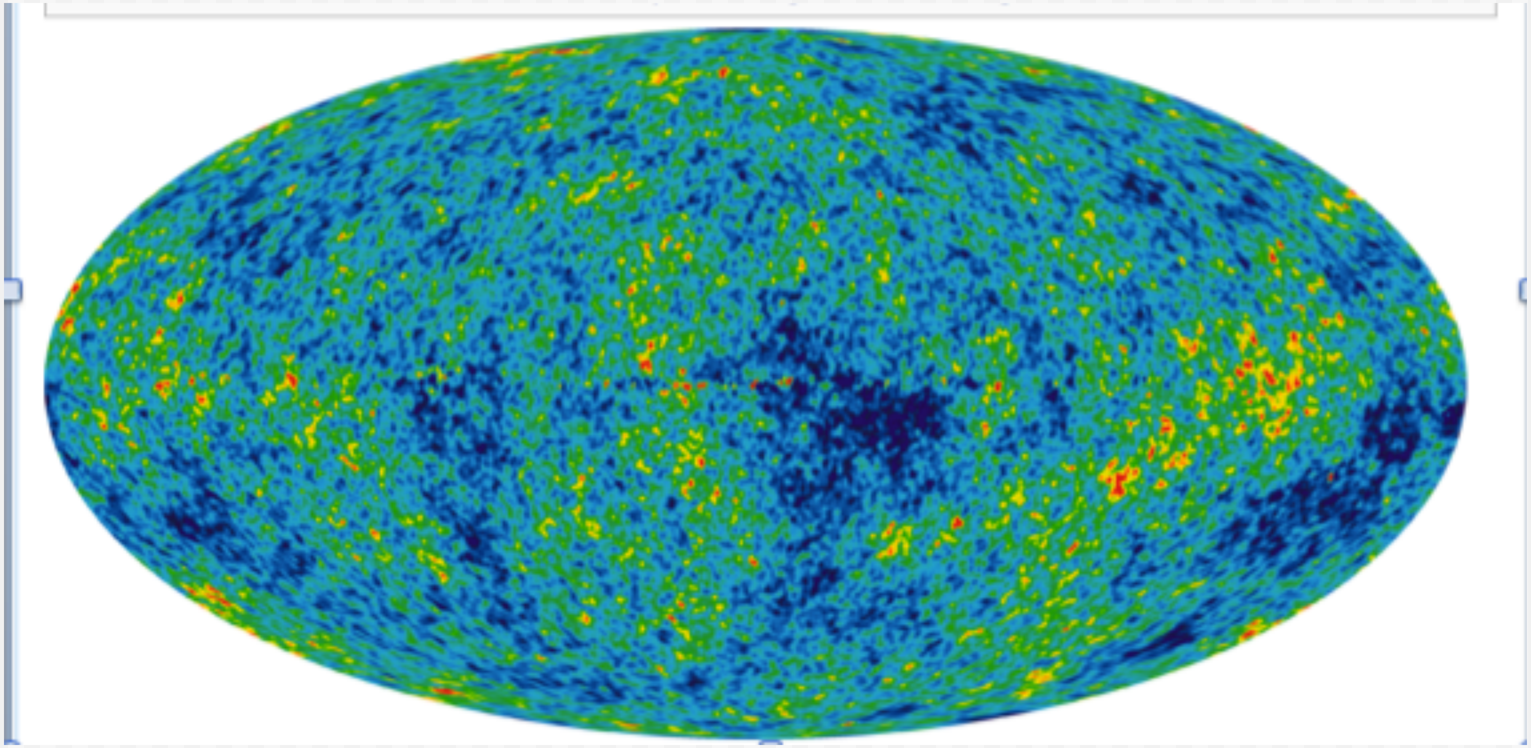
"WMAP" : the universe at ~10^5 years…



Density fluctuations ~**10^{-4} to 10^{-5}**

Cosmology

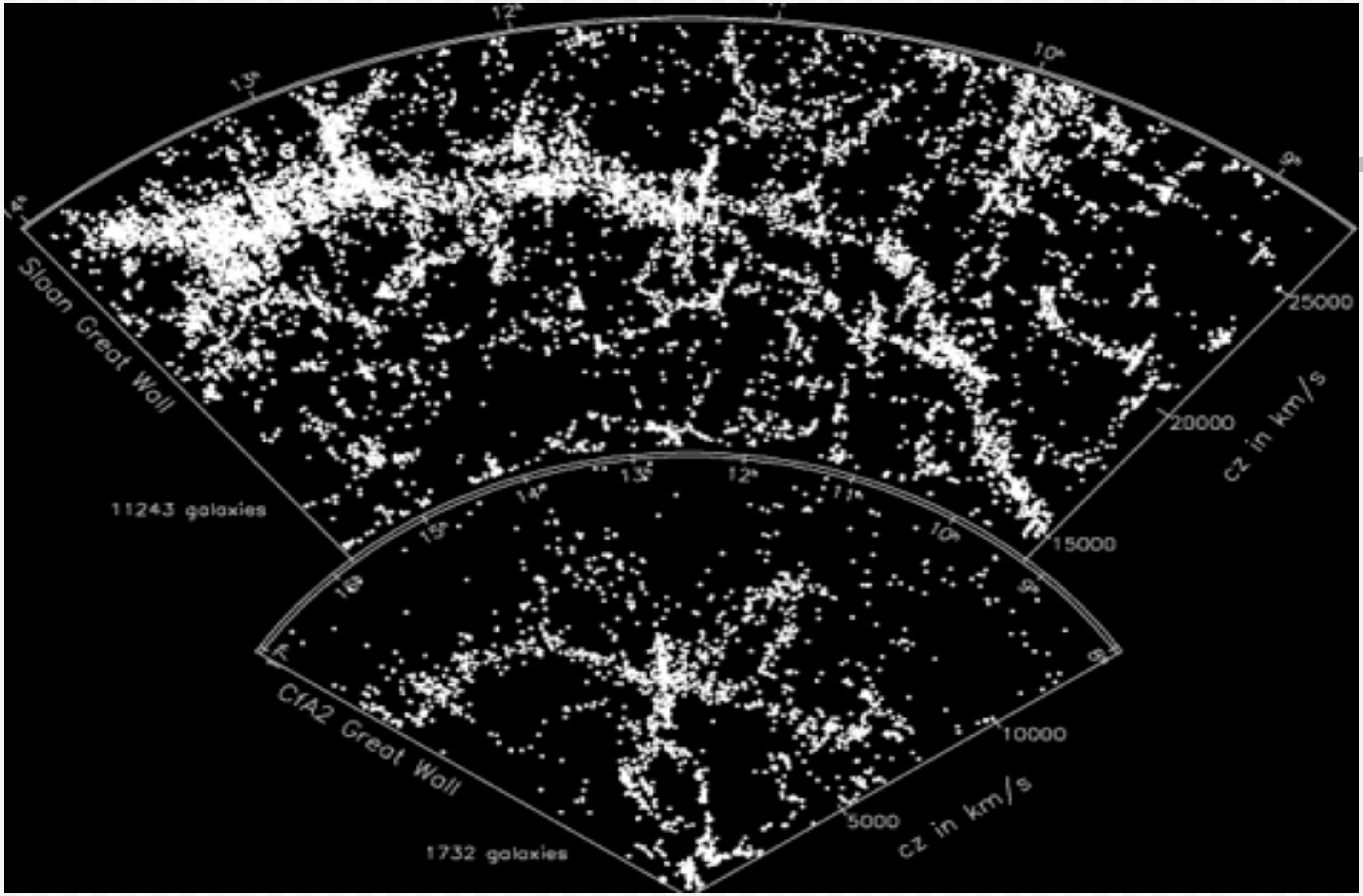
"PLANCK" : the universe at ~10^5 years…



Density fluctuations ~**10^{-4} to 10^{-5}**

Cosmology

“SDSS” : the universe “today” (10^{10} years)



Fluctuations >> **1** at corresponding scales

The problem of structure formation

How do we get **from tiny fluctuations in “primordial universe” to large fluctuations today ?**

What is full **quantitative theoretical prediction** for observations?

The problem of structure formation: simplifying (valid) approximations

Simplification of this very complex non-linear problem

- Non-linearity becomes important essentially only in **"matter dominated era",** dominant matter component is **non-relativistic**

- After "decoupling" non-gravitational forces can be neglected except at "small" **(< galaxy)** scales

- Perturbed FRW metric remains a good approximation (**weak fields**)

→ **Treatment in Newtonian limit of purely self-gravitating system**

Newtonian approximation: time and length scales

(Very roughly) valid for

Length scales: from galaxy scales (~ 0.01 Mpc) to “horizon” scales (~ 10^4 Mpc)

Time scales: from ~ 10^5 years (“matter domination”) to today (~ 10^{10} years)

Cosmological structure formation in the Newtonian limit

The Newtonian limit of FRW cosmology

What “Newtonian limit” ?

Newtonian gravity is badly defined in the infinite system limit..!

Newtonian gravity: Finite and infinite

$$\ddot{\mathbf{r}}_i = -Gm \sum_{j \neq i} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

Finite system: N particles *in a finite region* of (infinite) space

Infinite system: an infinite number of particles distributed *throughout space*

For latter case **the sum is badly defined** !

The Newtonian limit of FRW cosmology

It is GR which prescribes how to regularize Newtonian gravity

GR has well defined (FRW cosmologies) for an infinite matter distribution

→ Regularize Newtonian sum to obtain these !

FRW cosmology in the Newtonian limit

Self-gravitating particles distributed uniformly in infinite space

Calculate force ***summing in spheres about a chosen centre***

$$\ddot{\mathbf{r}}_i = -Gm \lim_{R_s \to \infty} \sum_{j \neq i, j \in S(R_s, \mathbf{r}_0)} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

Uniform mass density: → force proportional to distance from centre

→ homologous expanding/contracting solutions:

$$\mathbf{r}_i(t) = a(t)\,\mathbf{r}_i(0) \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho(t)$$

a(t) obeys the Friedmann equation for scale factor as in GR

Note: **relative motion of particles is independent of choice of centre!**

Perturbed FRW cosmology in the Newtonian limit

Same equations as above, but now **not** exact FRW initial conditions

i.e. infinite distribution of mass which is

close to uniform and close to Hubble flow above some finite scale

Convenient to change to "comoving coordinates"

$$\mathbf{x}_i(t) = \frac{\mathbf{r}_i}{a(t)} \qquad \dot{\mathbf{x}}_i(t) = \frac{1}{a(t)}[\dot{\mathbf{r}}_i - H\,\mathbf{r}_i] \qquad H = \frac{\dot{a}}{a}$$

$\dot{\mathbf{r}}_i - H\,\mathbf{r}_i$ is "peculiar velocity", i.e. velocity relative Hubble flow

Perturbed FRW cosmology in the Newtonian limit

In these **comoving coordinates** equations of motions become

$$**** \quad \frac{d^2\mathbf{x}_i}{dt^2} + 2H\frac{d\mathbf{x}_i}{dt} = \frac{1}{a^3}\mathbf{F}_i^{\mathrm{REG}} \quad **** \qquad H = \frac{\dot{a}}{a}$$

where

$$\mathbf{F}_i^{\mathrm{REG}} = -\lim_{R_s \to \infty} \sum_{j \neq i, j \in S(R_s, \mathbf{r}_0)} \left[\frac{Gm(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3} - \frac{4\pi G}{3}\rho_0 \mathbf{x}_i \right]$$

This **"regulated force" is zero for the case of an infinite uniform density** ["Jeans regulated force"]

The motion in comoving coordinates is due only to inhomogeneities

Regulated Newtonian force in an infinite system

The regulated force can be written formally in different ways

1) As the limit of a **symmetrical sum about each point**

$$\mathbf{F}_i^{\mathrm{REG}} = - \lim_{R_s \to \infty} \sum_{j \neq i, j \in S(R_s, \mathbf{r}_i)} \frac{Gm(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3}$$

2) As the limit of a **screened gravitational force** (cf. Kiessling 1999):

$$\mathbf{F}_i^{\mathrm{REG}} = - \lim_{\mu \to 0} \sum_{j \neq i} \frac{Gm(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3} e^{-\mu |\mathbf{x}_i - \mathbf{x}_j|}$$

3) In terms of a “Jeans regulated” potential

$$\mathbf{F}_i^{\mathrm{REG}} = -a \nabla_{\mathbf{x}} \Phi^{REG}(\mathbf{x} = \mathbf{x}_i)$$

$$\nabla_{\mathbf{x}}^2 \Phi^{REG}(\mathbf{x}) = \frac{4\pi G}{a} [\sum_{j \neq i} m \delta^3(\mathbf{x} - \mathbf{x}_j) - \rho_0]$$

Remark: Expansion and fluid damping

Redefining the time variable as $\tau = \int \frac{dt}{a^{3/2}}$ equations can be rewritten as

$$\frac{d^2\mathbf{x}_i}{d\tau^2} + \Gamma \frac{d\mathbf{x}_i}{d\tau} = \mathbf{F}_i^{\mathrm{REG}}$$

where

$$\Gamma = \sqrt{2\pi G \rho_0 / 3}$$

→ Dynamics of inhomogeneities in comoving coordinates is equivalent to that of **self-gravitating particles in a static universe subject to a fluid damping**..

An important remark on "physical coordinates"

Suppose **S a finite subsystem of the infinite uniform mass distribution**

The force on a particle in S can be written as

Force relative to CM due to mass in S + Force on CM (due to mass outside S)
+ (tidal) forces due to mass outside S

If the substructure is "sufficiently dense and far from other mass" so that the tidal forces can be neglected it follows that

Equation of motion for the particles in S relative to its centre of mass are, *in physical coordinates,* those in an isolated self-gravitating system

Thus e.g. stars in a galaxy, or planets in the solar system are "decoupled" from the Hubble flow. In comoving coordinates they "shrink".

Dynamics of self-gravitating matter in an expanding universe

Dynamics of self-gravitating matter in an expanding universe: continuum limit

In cosmology use a continuum description of matter: particles are microscopic
Just want to determine e.g. phase space density
→**Vlasov-Poisson equations**

In physical coordinates: "usual" VP equations + **infinite system regularisation**
In comoving coordinates this gives

$$\partial_t f + \vec{v}\cdot\partial_{\vec{x}} f + [-\frac{1}{a^2}\partial_{\vec{x}}\Phi - 2H(t)\vec{v}]\cdot\partial_{\vec{v}} f = 0$$

$$\partial_{\vec{x}}^2\Phi = \frac{4\pi G}{a}[a^6\int f(\vec{x},\vec{v})d^3v - \rho_0] \quad \left(\vec{v} = \frac{d\vec{x}}{dt} \right)$$

Remark: different writings of this equation abound → different conventions for f
Here f defined by

$$dm = f d^3r d^3v_{phys} = (fa^6)d^3x d^3v$$

"Linear theory": Evolution of **small perturbations** about FRW

Zeroth moment of VP**: continuity equation**

First moment of VP : **Euler equation**

- Neglect "velocity dispersion" term (→cold matter)
- Linearize in

$$\delta(\vec{x},t) = \frac{\rho(\vec{x},t) - \rho_0}{\rho_0}$$

mass density fluctuation

and

$$\vec{u}(\vec{x},t) = \frac{\int \vec{v} f(\vec{x},\vec{v}) d^3v}{\int f(\vec{x},\vec{v}) d^3v}$$

bulk velocity

Pressureless linearized fluid equations

$$\frac{d\delta}{dt} + \vec{\nabla}_x \cdot \vec{u} = 0$$

$$\frac{d\vec{u}}{dt} + 2H\vec{u} = -\vec{\nabla}_x \cdot \Phi$$

$$\nabla_x^2 \Phi = -\frac{4\pi G\rho_0}{a}\delta$$

→ **Matter density fluctuations have a growing mode**

"Linear theory": Evolution of **small perturbations** about FRW

"Linear amplification", just gravitational instability!

$$\delta(\vec{x},t) = a(t)\delta(\vec{x},0)$$

$$\tilde{\delta}(\vec{k},t) = a(t)\tilde{\delta}(\vec{k},0)$$ (Fourier Transform)

NOTE*: The amplification is scale-independent (property of Newtonian gravity)*

Irrotational component of physical "peculiar velocity field" is amplified

$$i\,\vec{k}\cdot\tilde{\vec{u}}(\vec{k},t) = \dot{\tilde{\delta}}(\vec{k},t)$$

[BUT **gravitational potential Φ is not amplified,** it is constant **and weak!**]

Lagrangian formulation: “Zeldovich approximation”

Fluid equations can be cast in Lagrangian formalism

$$\vec{x}(t) = \vec{q} + \vec{u}(\vec{q}, t)$$

where q Lagrangian coordinate = initial position of the fluid element

At linear order, the displacement field in the growing mode obeys

$$\vec{u}(\vec{q}, t) = a(t)\vec{u}(\vec{q}, t_0)$$ where $$\vec{u}(\vec{q}, t_0) \propto \vec{\nabla}_q \Phi(\vec{q}, t_0)$$

i.e. **just motion parallel to gravitational field** (thus irrotational)

Initial conditions
for cosmological structure formation

Cosmological initial conditions: origin and description

Some physical process in “primordial” universe

→ Initial low amplitude metric/matter/energy fluctuations to FRW
(e.g. amplification of quantum fluctuations during “inflation”)

Fluctuation fields are **realizations of a** (statistically translation and rotation invariant) **stochastic process,** which is (usually) assumed **gaussian**

Evolve with **linearized** but fully relativistic theory (of all fields and interactions..) until the **matter dominated era**

(Note: Linear evolution propagates gaussianity trivially)

Initial conditions for structure formation in matter dominated era

"Cold dark matter fluid", in "growing mode"

$$f(\vec{x},\vec{v},t=t_i) = \rho(\vec{x},t_i)\,\delta^{(3)}(\vec{v}-\vec{u}(\vec{x},t_i))$$

Density field : $\rho(\vec{x},t_i)$ realization of a *correlated gaussian process*

Fully characterized by power spectrum P(k)

e.g. "CDM" or "LambdaCDM" spectrum or variants

Velocity field derived assuming growing mode of linear theory.

Initial conditions for structure formation: power spectrum

Standard cosmological model assumes (and e.g. inflation produces) a so-called **"scale invariant** spectrum":

variance of potential fluctuations is (almost) independent of scale

When "processed" through cosmological evolution, it gives, at matter domination a power spectrum for matter fluctuations:

$$P(k) = AT^2(k)k$$

where T(k) is "transfer function", and A a constant.

T(k)=1 corresponds to "primordial spectrum" : P(k)=Ak

Note: Measurements of CMB fluctuations fix (in particular) the amplitude

Initial conditions for structure formation: “transfer function” for standard CDM

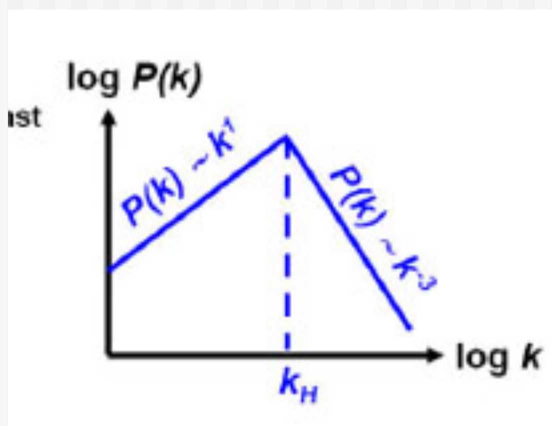
Numerical fit to the “transfer function” of “standard CDM”

$$T\left(q \equiv k/\Gamma h\,\mathrm{Mpc}^{-1}\right) = \frac{\ln[1+2.34q]}{(2.34q)}\left[1+3.89q+(16.2q)^2+(5.47q)^3+(6.71q)^4\right]^{-0.25}$$

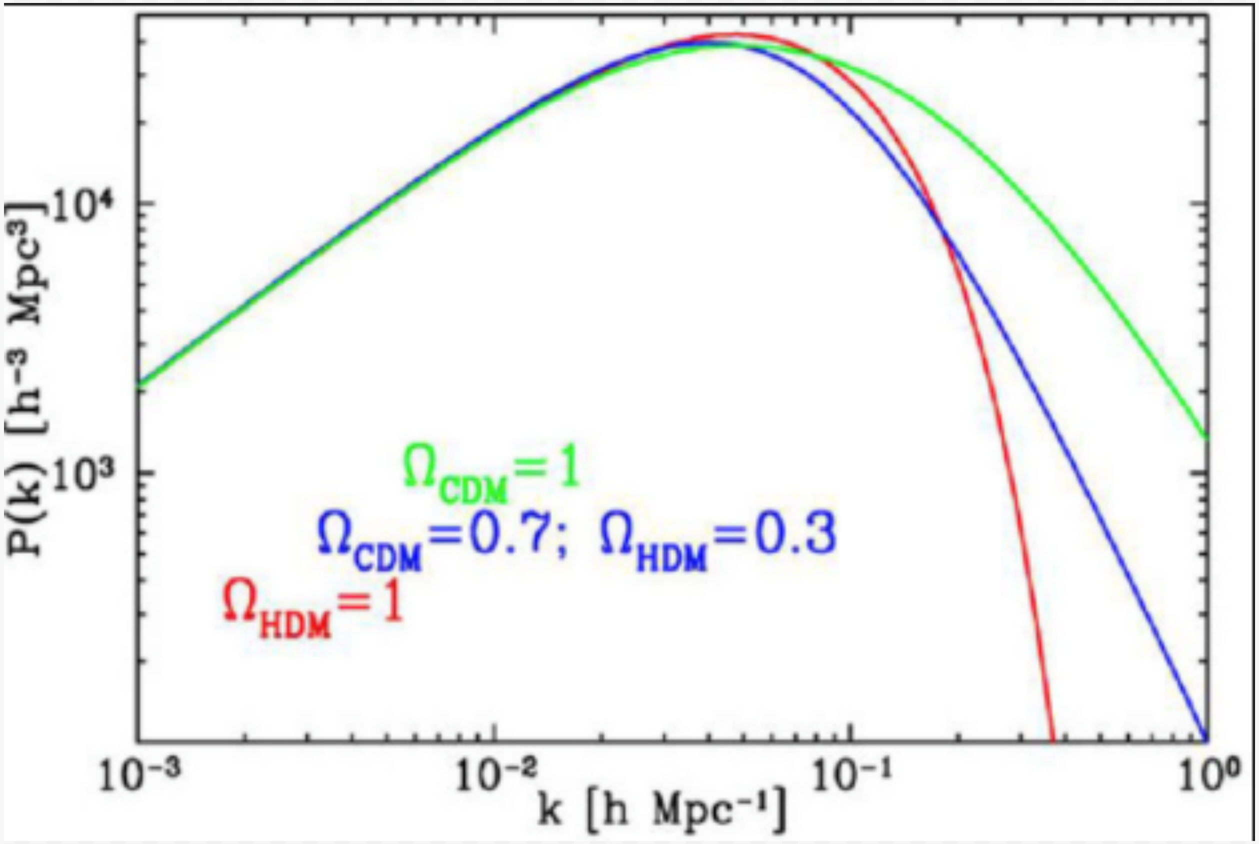
(see e.g.)

Γ is constant determined by the ratio of matter/radiation, fixes “turnover scale”

Schematically:



Initial conditions for structure formation: power spectra for different models



Fluctuations in real space

Define **volume averaged relative mass fluctuation**

$$\delta(V) = \frac{1}{V}\int_V \delta(\vec{r})\, d^3r$$

Its variance is related to power spectrum by

$$\sigma^2(V) = \langle \delta^2(V) \rangle = \int P(k) |\tilde{W}_V(\vec{k})|^2\, d^3k$$

where $\tilde{W}_V(\vec{k})$ is FT of window function for volume V

Cosmological initial conditions: averaged density fluctuation in a sphere

V = sphere of radius R, and $P(k) = Ak^n$ (and n < 1)

$$\sigma^2(R) \sim k^3 P(k)\big|_{k=\pi/R} \sim \frac{A}{R^{3+n}}$$

For **all standard type cosmological models** -3 < d(ln P)/d(ln k) < 1
→
density fluctuations are a monotonically decreasing function of scale

[Note: also true for n>1, some subtleties in relation of real and k space]

From the linear to the non-linear regime: some analytical approaches

Linear theory (LT)

The **most impressive observational successes** of the standard cosmological model are in the linear regime, i.e., where linear perturbation theory applies

Notably

→ Fluctuations in CMB (WMAP, Planck and many others..)

→ **very** large scale structure in galaxies (“**baryon acoustic oscillations**”)

Latter described to a very good approx. by **linear theory applied up to today..**

Breakdown of linear theory?

Assumption of LT: "small density fluctuations", "small velocity dispersion"

Criterion for its validity? In general depends on full spectrum of fluctuations

→
LT valid for density/velocity field smoothed on some scale R if density/velocity fluctuations on this scale, and larger scales, are small

….."**provided not too much fluctuations below scale R**"

Taking $P(k) \propto k^n$ expect on simple grounds that it is **sufficient to have n<4**

(Zeldovich/Peebles)

Linear evolution at a given scale is then negligibly affected by non-linear fluctuations at smaller scales

Evolution of non-linearity in cold matter: "hierarchical structure formation"

For cosmological spectra, smoothed density/velocity field is a monotonically decreasing function of scale

→ LT itself then prescribes **scale at which LT break downs as function of time**
e.g. for power law spectra $P(k) \propto k^n$ obtain $R_{\mathrm{NL}}(a) \propto a^{\frac{2}{3+n}}$

Cold matter with cosmological spectra → **"hierarchical structure formation"** :

- monotonically growing **non-linearity scale driven by linear amplification**,
- **time of non-linearity for each scale essentially independent of all others**

What happens to a given scale when it "goes non-linear"?

Evolution of non-linearity in cold matter: “hierarchical structure formation”

Cold matter with cosmological spectra → **"hierarchical structure formation"** :

- monotonically growing “**non-linearity scale” driven by linear amplification**,
- **time of non-linearity for each scale essentially independent of all others**

What happens to a given scale when it "goes non-linear"?

A guide for non-linear evolution: The "spherical collapse model"

"Spherical collapse model":

spherical "top-hat" over-density in an otherwise uniform expanding universe

Exact analytical solution, for **comoving radius R(a)** (in parametric form):

$$\frac{R(a)}{R_0} = \left(\frac{4}{3}\right)^{\frac{2}{3}} \frac{1-\cos\theta}{(\theta-\sin\theta)^{\frac{2}{3}}}$$

$$a\delta_0 = \delta_L(a) = \frac{3}{5}\left(\frac{3}{4}\right)^{\frac{2}{3}}(\theta-\sin\theta)^{\frac{2}{3}}$$

$\delta_L(a)$ is linearly extrapolated amplitude at a, δ_0 is initial amplitude ($a \rightarrow 0$, $R \rightarrow R_0$)

The “spherical collapse model”: linear density fluctuation at singularity

$$\frac{R(a)}{R_0} = \left(\frac{4}{3}\right)^{\frac{2}{3}} \frac{1-\cos\theta}{(\theta-\sin\theta)^{\frac{2}{3}}}$$

$$a\delta_0 = \delta_L(a) = \frac{3}{5}\left(\frac{3}{4}\right)^{\frac{2}{3}}(\theta-\sin\theta)^{\frac{2}{3}}$$

$\delta_L(a)$ is linearly extrapolated amplitude at a, δ_0 is initial amplitude (a→0, R→R_0)

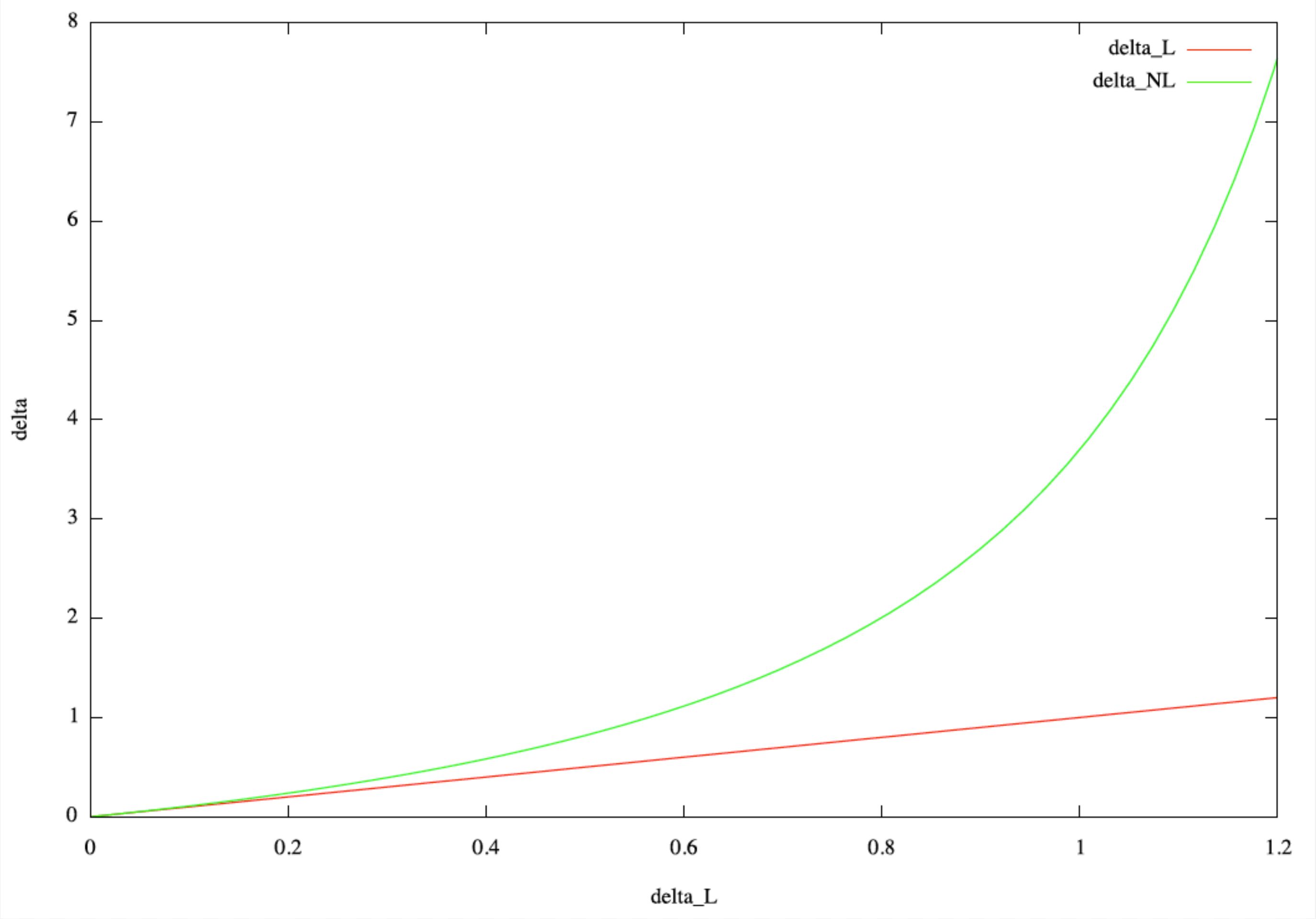
→ **Linear evolution at low amplitude**

→ **Singularity in a finite time depending only on initial fluctuation amplitude**
[R(a)=0 when θ=2π, i.e. $\delta_L = \frac{3}{5}\left(\frac{3\pi}{2}\right)^{\frac{2}{3}} \approx 1.68$]

Thus **non-linear collapse is “more efficient” than linear amplification**

Spherical collapse model: evolution of density fluctuation

Exact density fluctuation as a function of linear evolved density fluctuation:



Spherical collapse model: extrapolation beyond collapse

With additional assumptions SC model can give further predictions

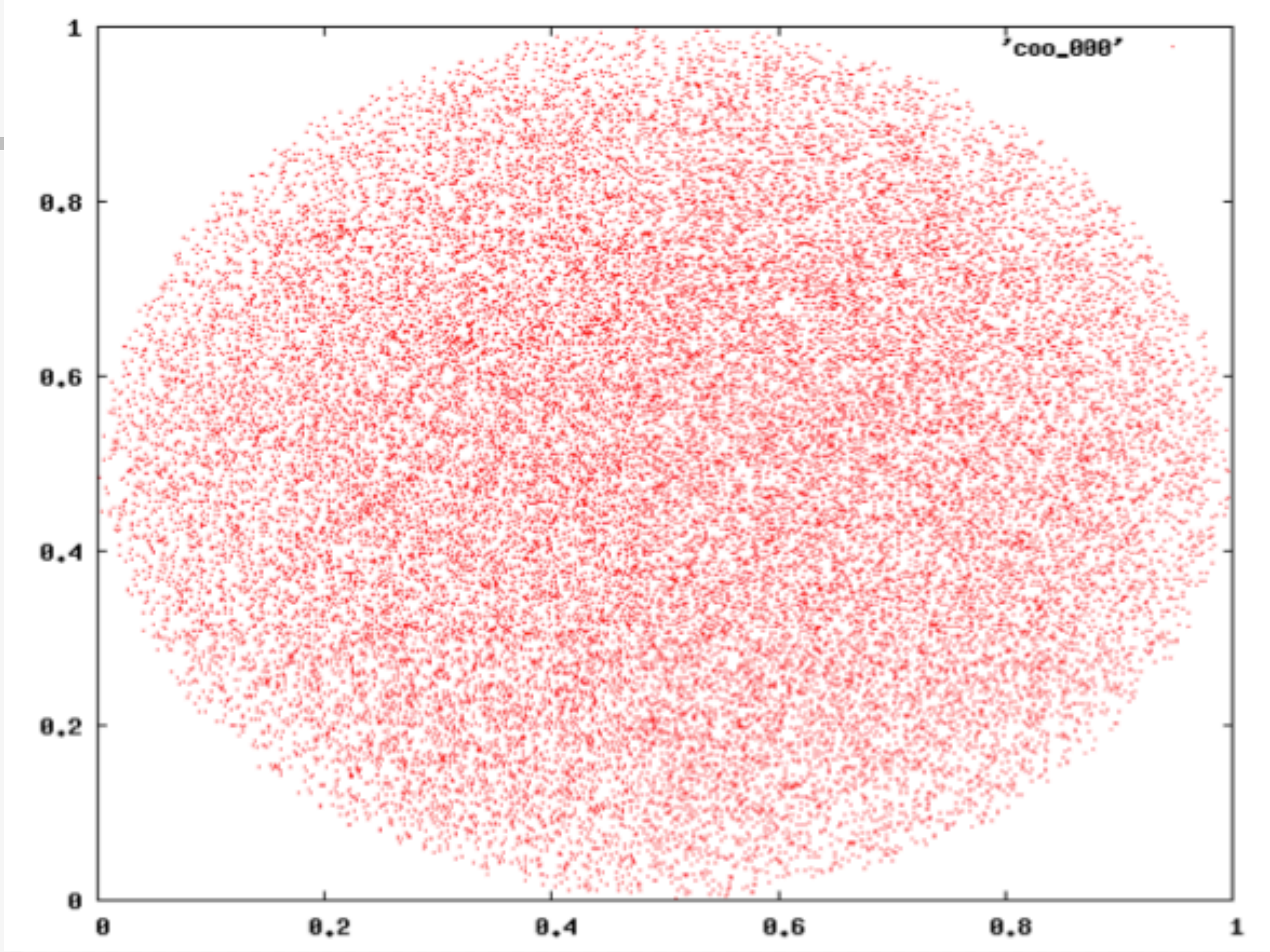
Model defines a time of "turnaround" (at $\theta=\pi$) when **physical velocity is zero**

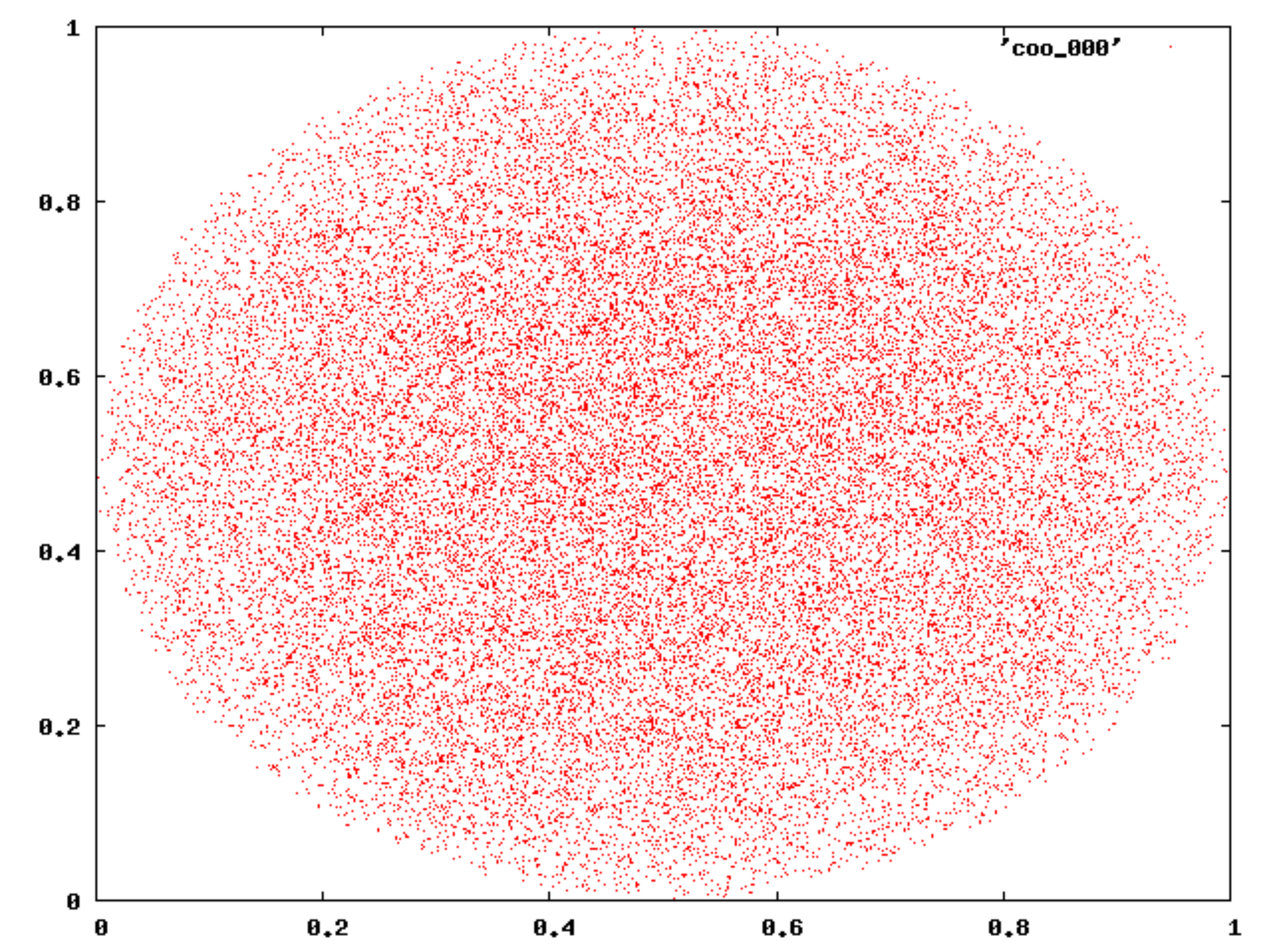
$\rightarrow$ from this time evolution of a "cold" **isolated** uniform sphere (in these coords)

Real system: collapse not singular because of fluctuations

Instead **obtain a finite stationary (and virialized) system**

Evolution of a cold quasi uniform sphere





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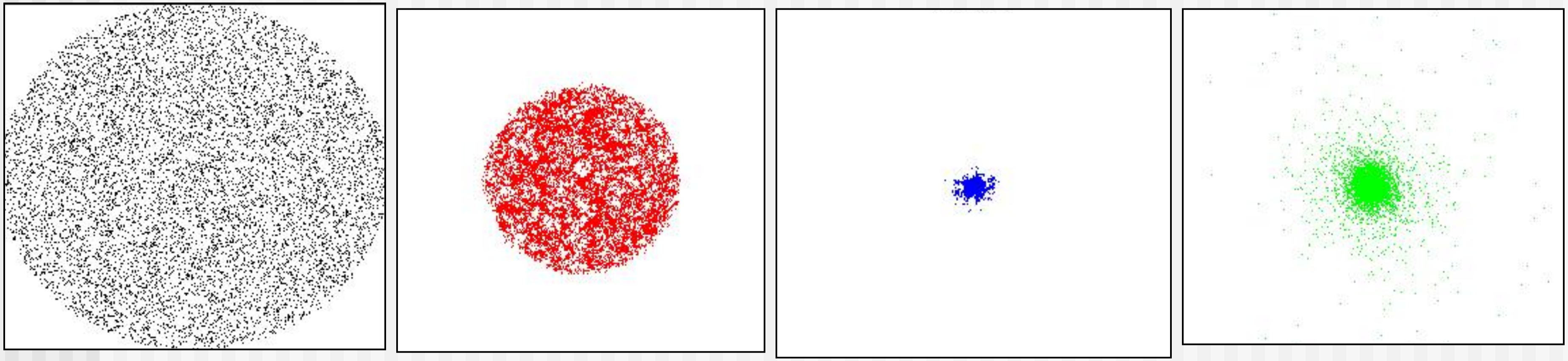
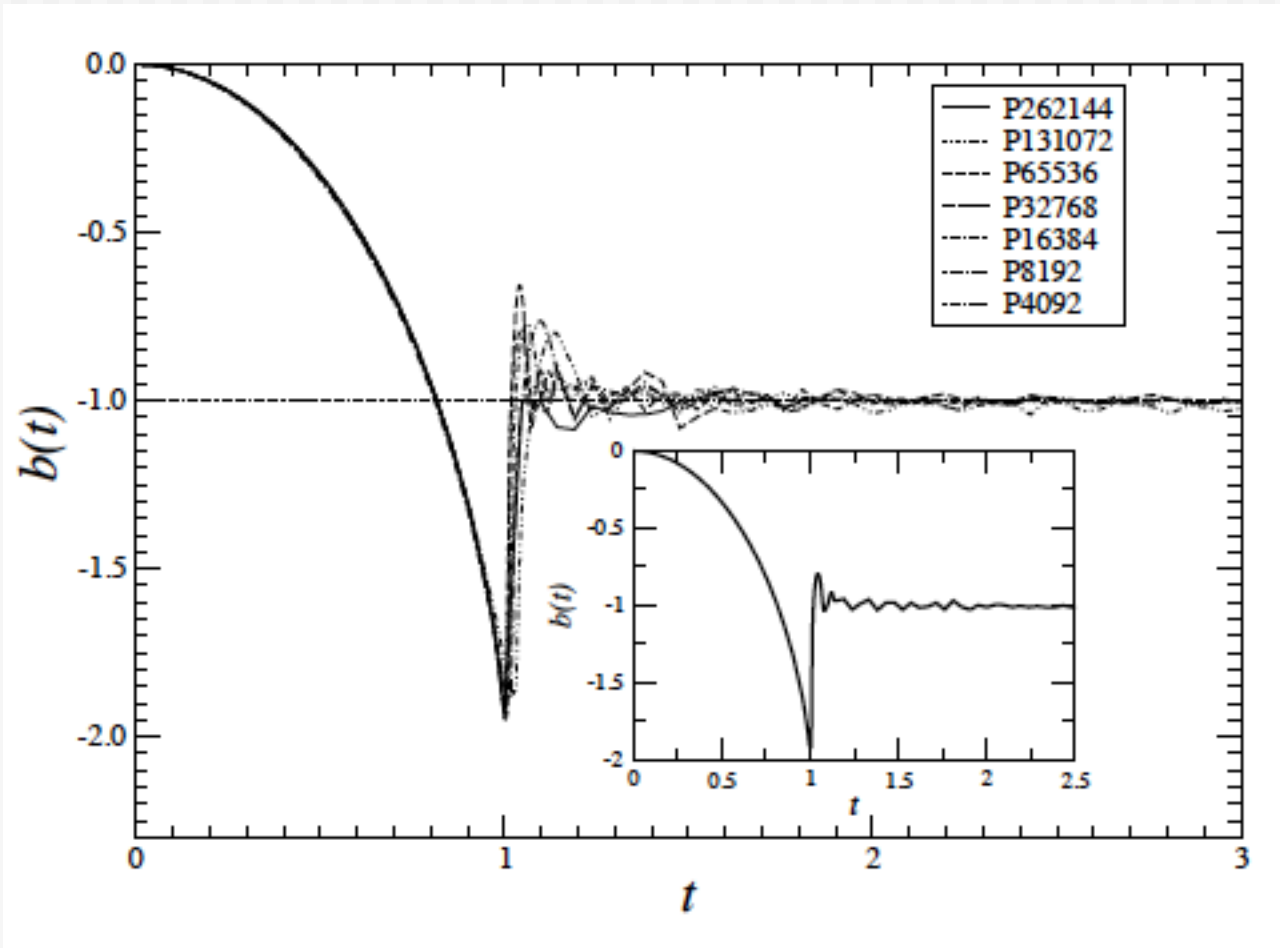
Collapse and virialization of a cold quasi-uniform sphere



Collapse and virialization of a cold quasi-uniform sphere



Extending the SC model: mean density/size at virialization

Assuming

- virialization at time of theoretical singularity
- energy conservation (?)

→ Simple estimate of **characteristic mean density of systems at virialization**

$$\delta_{VIR} \approx 200$$

Thus virialized structure **about 1/6 of initial (comoving) size of collapsed mass**

Extending the SC model: "Press-Schecter" formalism

Using

+ SC model's **"linear threshold for virialization" ($\delta \approx 1.68$)** [a region will give rise to a virialized structure when its extrapolated linear amplitude is 1.68]

+ initial power spectrum of fluctuations [statistics of regions with given initial δ]

→

prediction for **number density of virialized systems of given mass at any time**, or so-called **"mass function"**

+ many refinements/modifications..

[Dark matter clumps virializing "today" → large galaxy clusters]

Beyond collapse and virialization: the stable clustering approximation

Assume that these **virialized clumps then evolve like isolated systems**

They “decouple from Hubble flow” and are “stable”

→ just **“shrink” as 1/a** in comoving coordinates

→ Fluctuations at a given scale is then a calculable function of initial fluctuations at a larger scale in linear regime..

In practice expect non-linear structures of different sizes to interact, and even merge…only numerical simulation can tell us how much!

Scale free models and "self-similarity"

Initial power spectrum $P(k) = Ak^n$ (+UV cut-off)

+ a(t) which is power law

→ No characteristic scale other that non-linear scale

→ If structure formation is UV insensitive, clustering must be "self-similar" e.g. 2 point correlation function

$$\xi(x,a) = \xi_0\left(\frac{x}{R_s(a)}\right)$$ where $$R_s = a^{\frac{2}{3+n}}$$ (from linear theory)

Non-linear clustering in scale-free models

If non-linear clustering is also assumed **stable,** it must then be scale-free e.g.

$$\xi(x) \sim x^{-\gamma_{sc}}$$

The exponent $\boldsymbol{\gamma_{sc}}$ can be determined analytically

$$\gamma_{sc} = \frac{3(3+n)}{5+n}$$

(Davis and Peebles 1977)

→ Testable analytical predictions **for such models**

Numerical simulations of cosmological structure formation

Numerical simulation of structure formation: equations

The equations one ***would like to*** solve are the VP equations

In practice use "N-body method": solve the N body particle problem!

$$\ddot{\mathbf{x}}_i + 2\frac{\dot{a}}{a}\dot{\mathbf{x}}_i = -\frac{Gm}{a^3}\sum_{j\neq i}\frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} W_\varepsilon(|\mathbf{x}_i - \mathbf{x}_j|)$$

where regularisation of sum in **infinite periodic system** is left implicit here

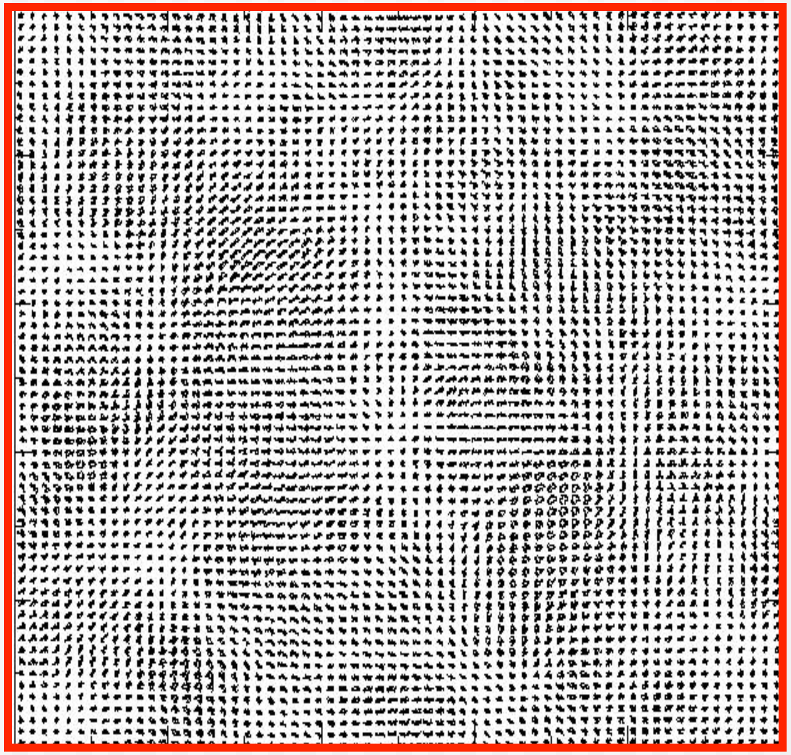
W_ε : regularisation of interaction when $|x_i - x_j| \to 0$

N body particles are "softened macro-particles"

[Direct solution of VP?

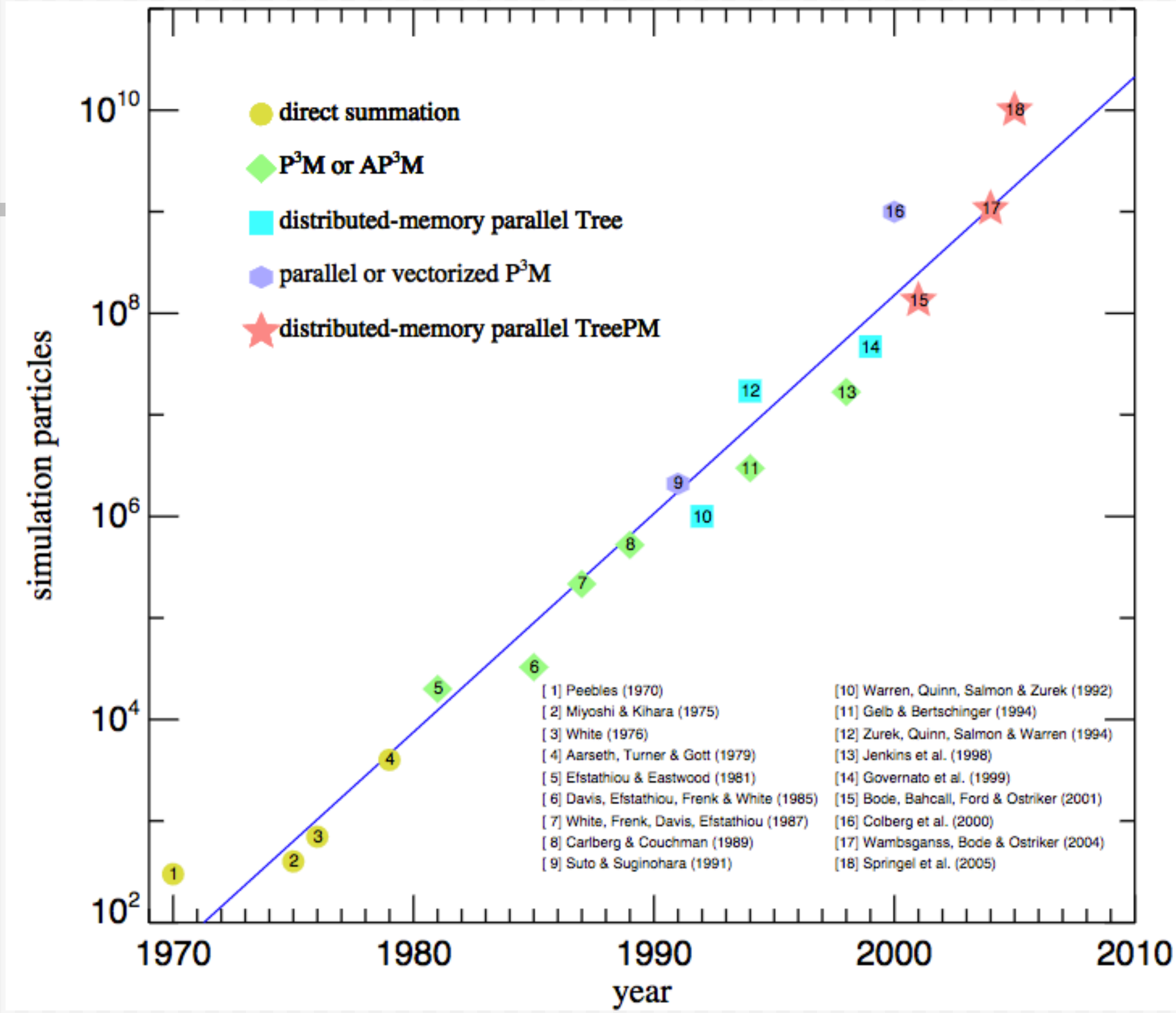
See Yoshikawa K. et al., MNRAS (2013), Colombi et al., MNRAS (2015)]

Initial conditions of NBS



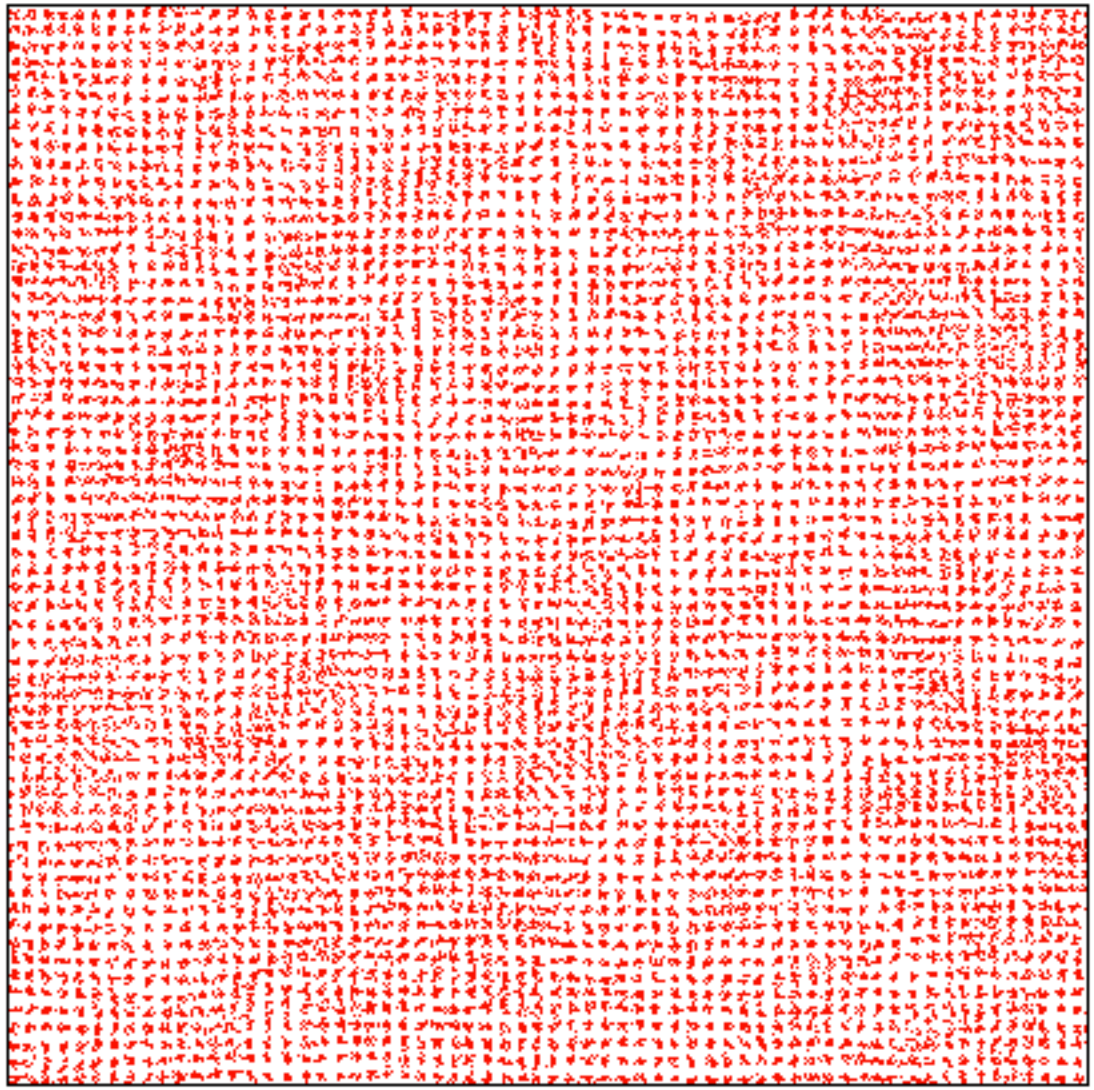
Particles **displaced off a lattice** (or "glass") to produce desired density field

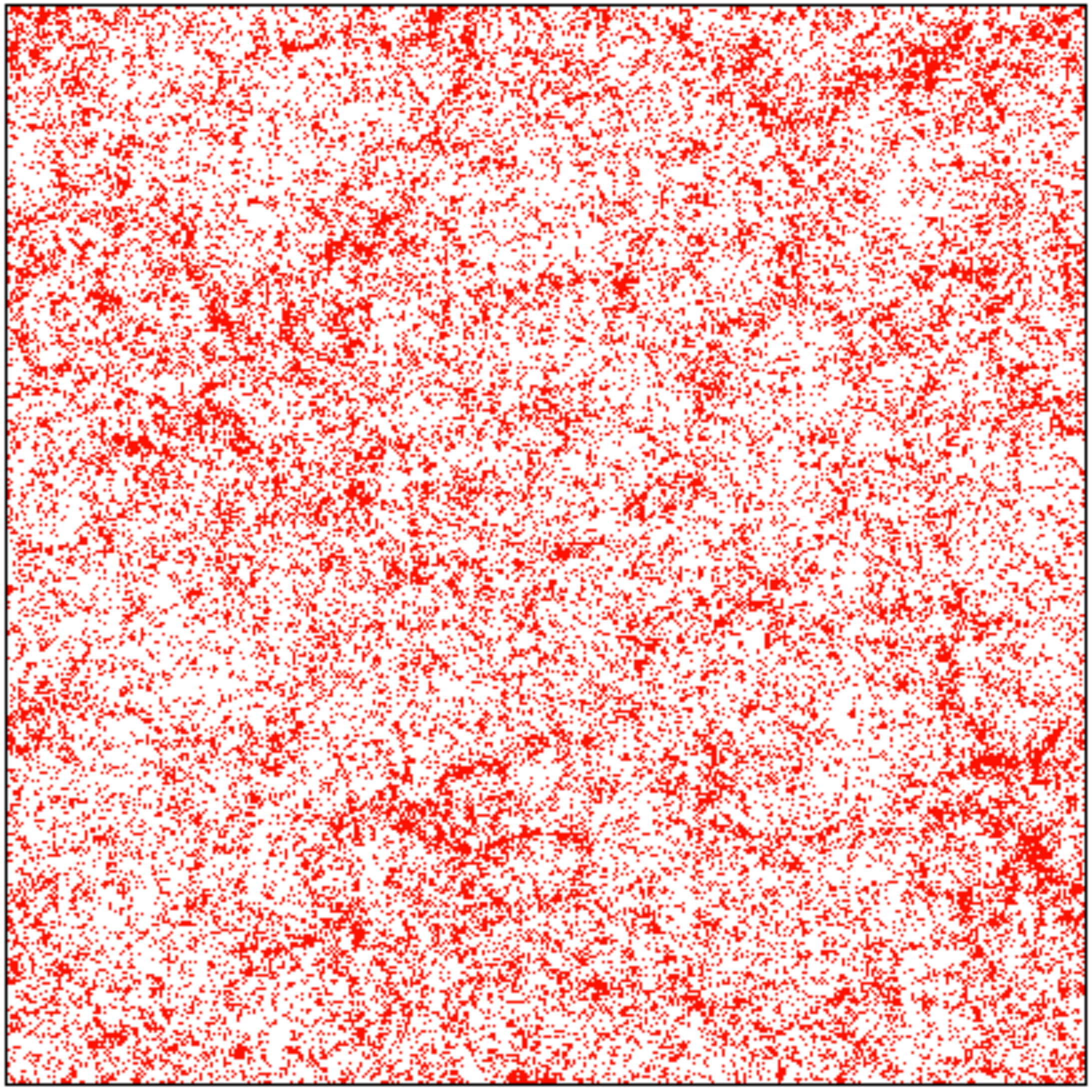
[+velocities as prescribed by LT growing mode ("Zeldovich Approx")]

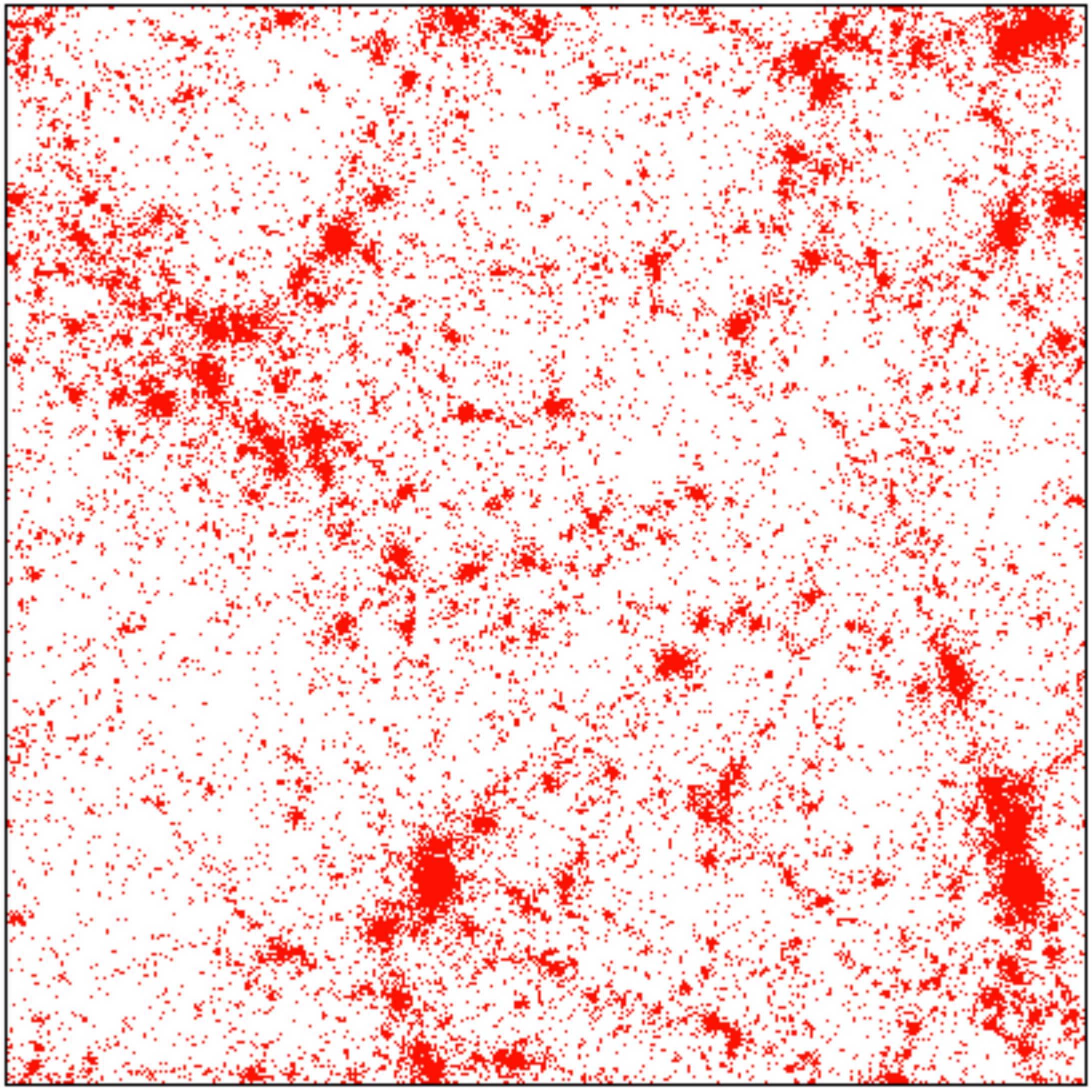


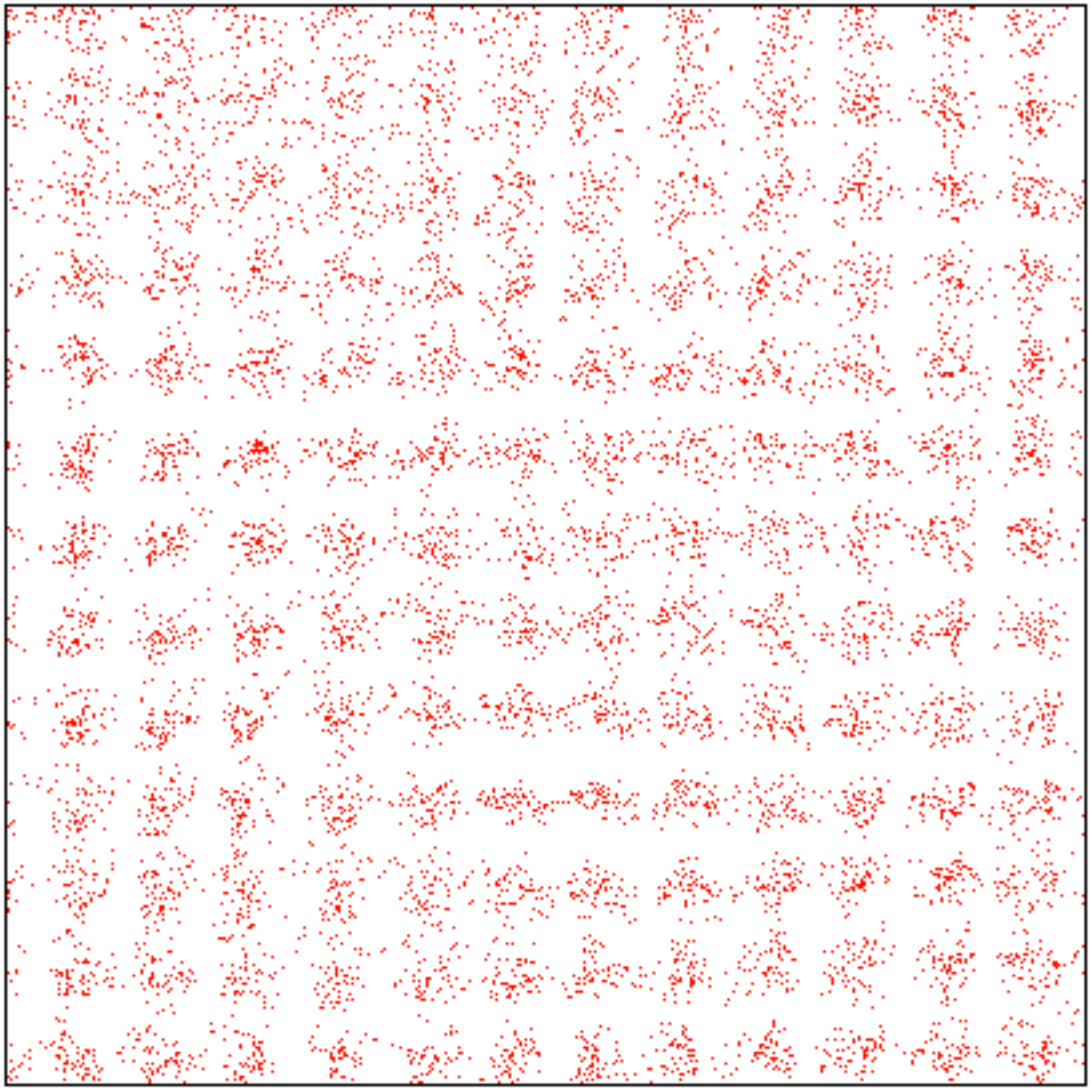
(From V. Springel et al., Nature 2005)

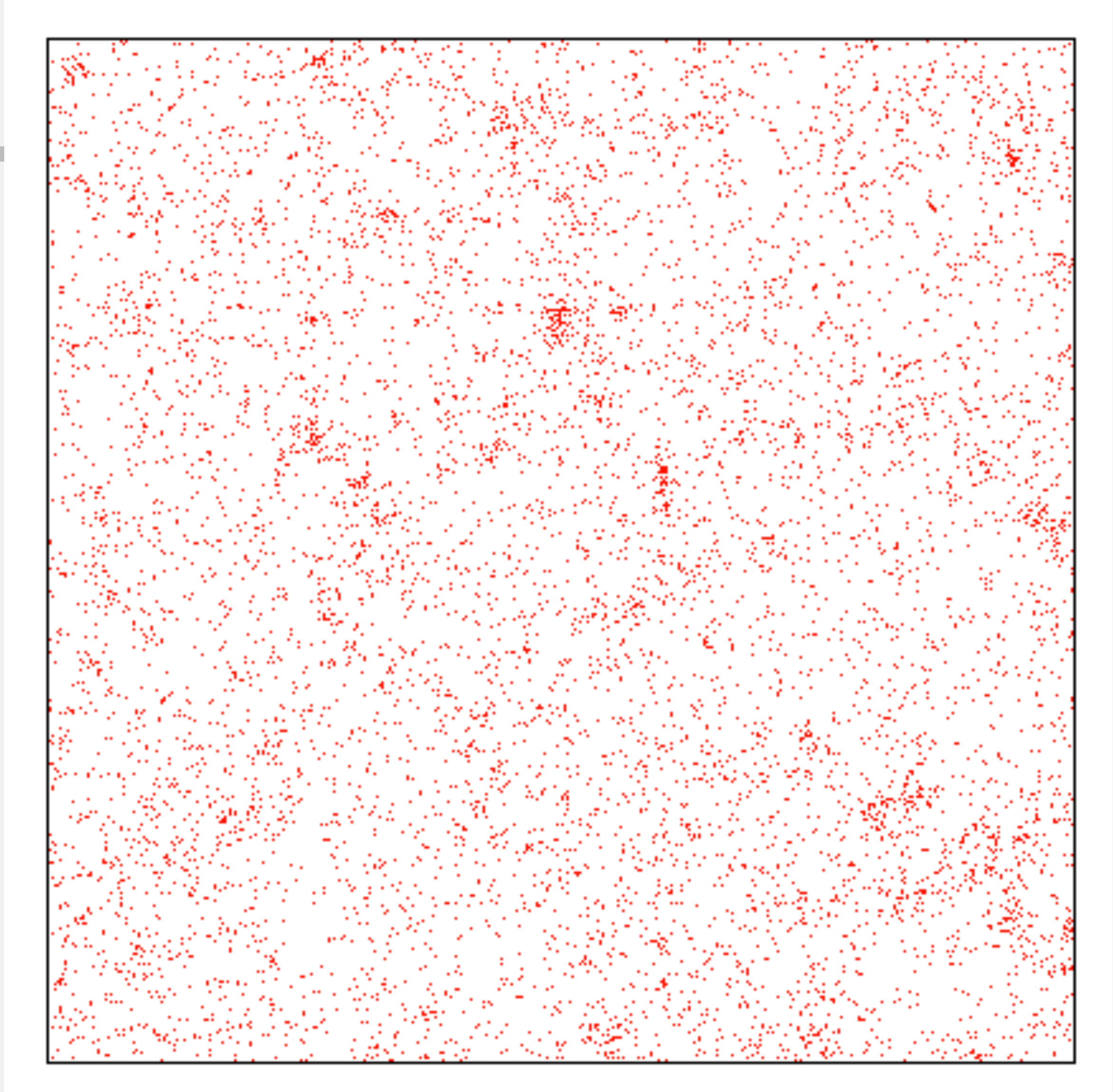
Initial power spectrum $P(k) \propto k^{-1}$ +velocities as prescribed by “Zeldovich Approx”

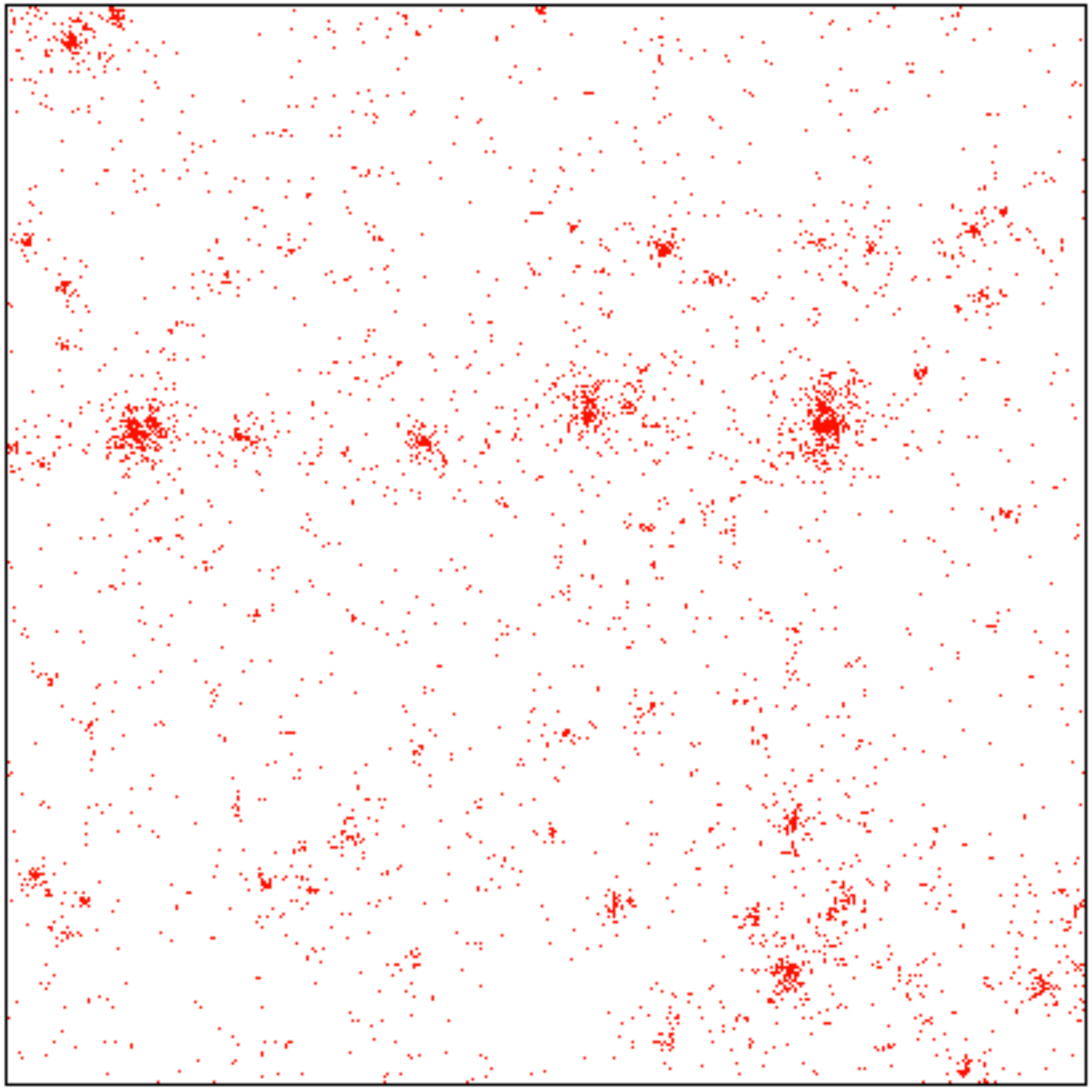


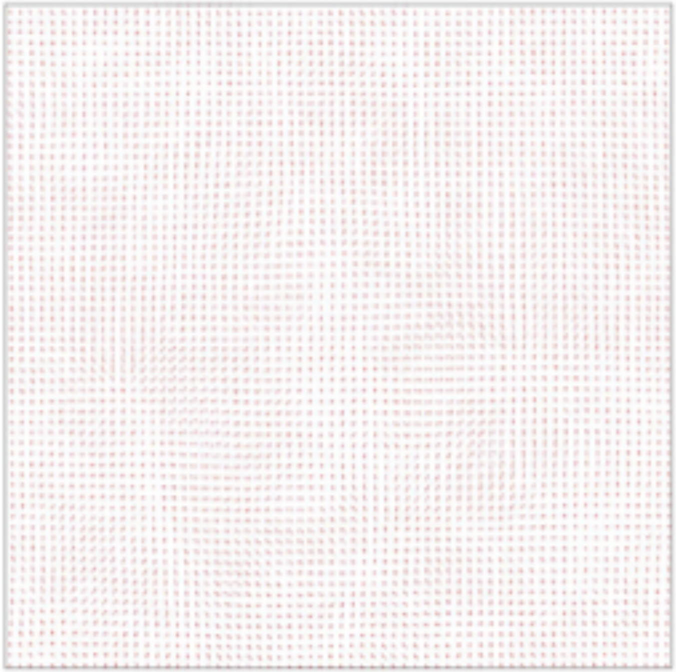


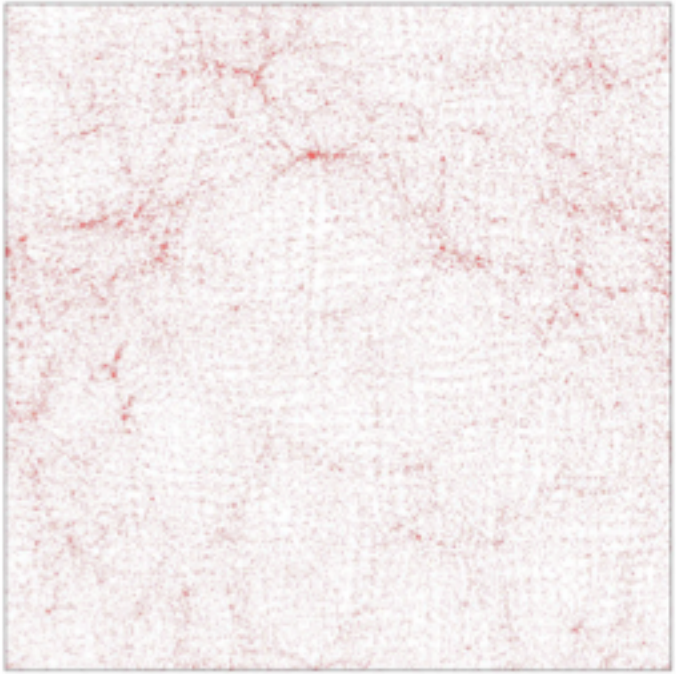




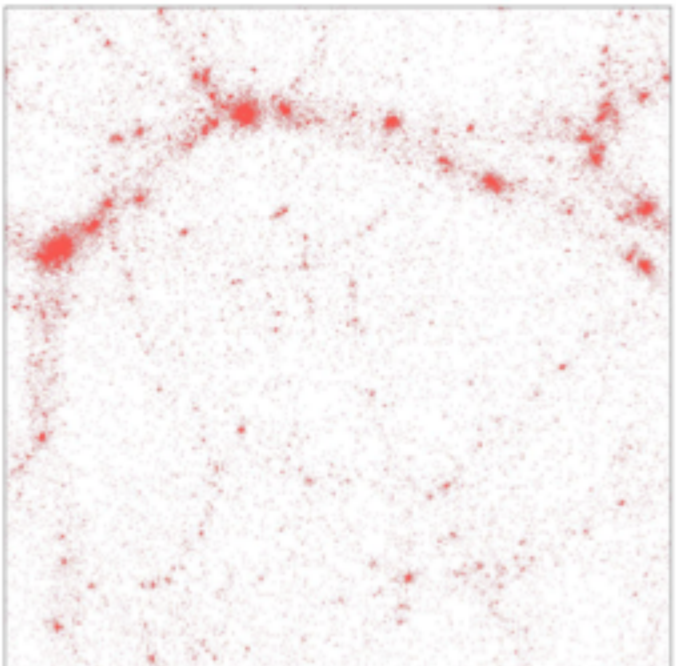








$$P(k) \propto k^{-2}$$



Simulating the joint evolution of quasars, galaxies and their large-scale distribution

Volker Springel[1], Simon D. M. White[1], Adrian Jenkins[2], Carlos S. Fr
Naoki Yoshida[3], Liang Gao[1], Julio Navarro[4], Robert Thacker[5], Darre
John Helly[2], John A. Peacock[6], Shaun Cole[2], Peter Thomas[7], Hugh
August Evrard[8], Jörg Colberg[9] & Frazer Pearce[10]

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[6]*Institute of Astronomy, University of Edinburgh, Blackford Hill, Edinburgh EH9 3HJ, U*
[7]*Dep. of Physics & Astron., University of Sussex, Falmer, Brighton BN1 9QH, UK*
[8]*Dep. of Physics & Astron., Univ. of Michigan, Ann Arbor, MI 48109-1120, USA*
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The cold dark matter model has become the leading theoretical para mation of structure in the Universe. Together with the theory of cos model makes a clear prediction for the initial conditions for structu predicts that structures grow hierarchically through gravitational in this model requires that the precise measurements delivered by gala compared to robust and equally precise theoretical calculations. Here framework for the quantitative physical interpretation of such survey the largest simulation of the growth of dark matter structure ever car techniques for following the formation and evolution of the visible com that baryon-induced features in the initial conditions of the Universe a torted form in the low-redshift galaxy distribution, an effect that can be the nature of dark energy with next generation surveys.

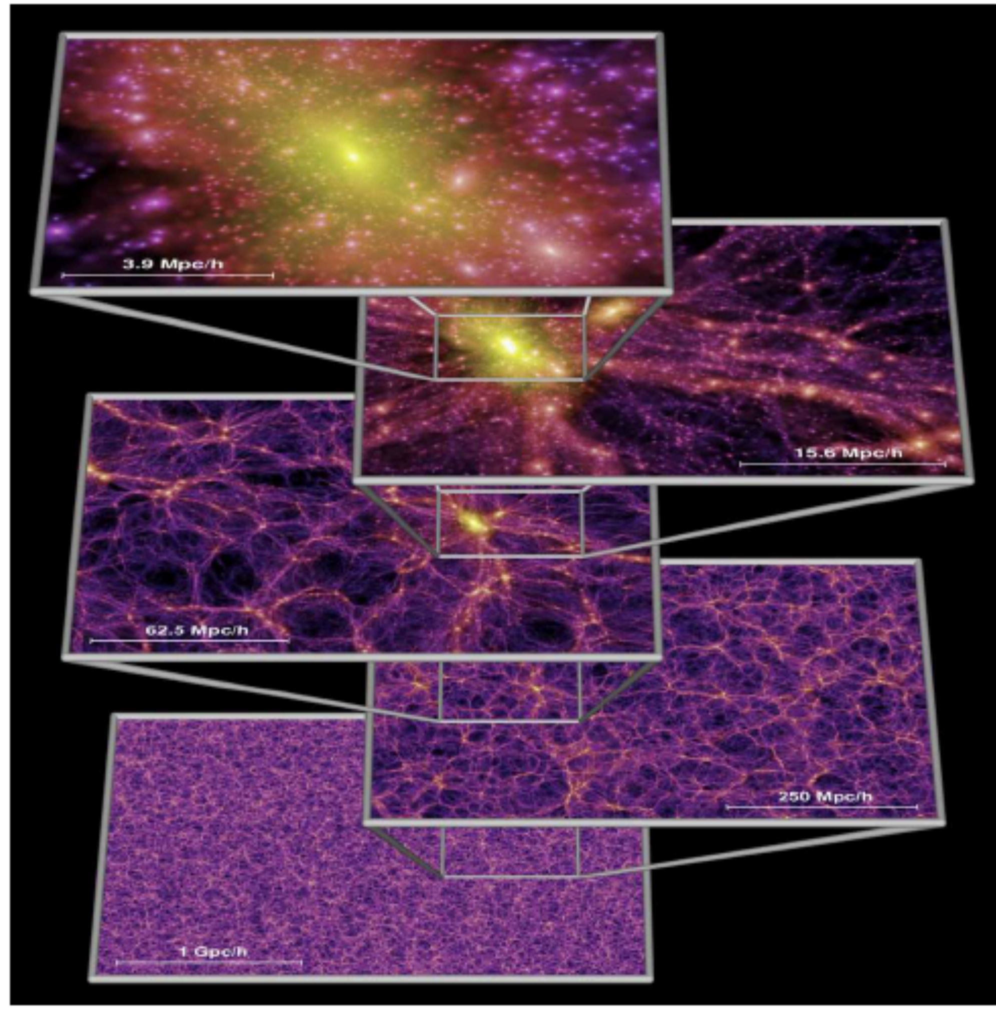
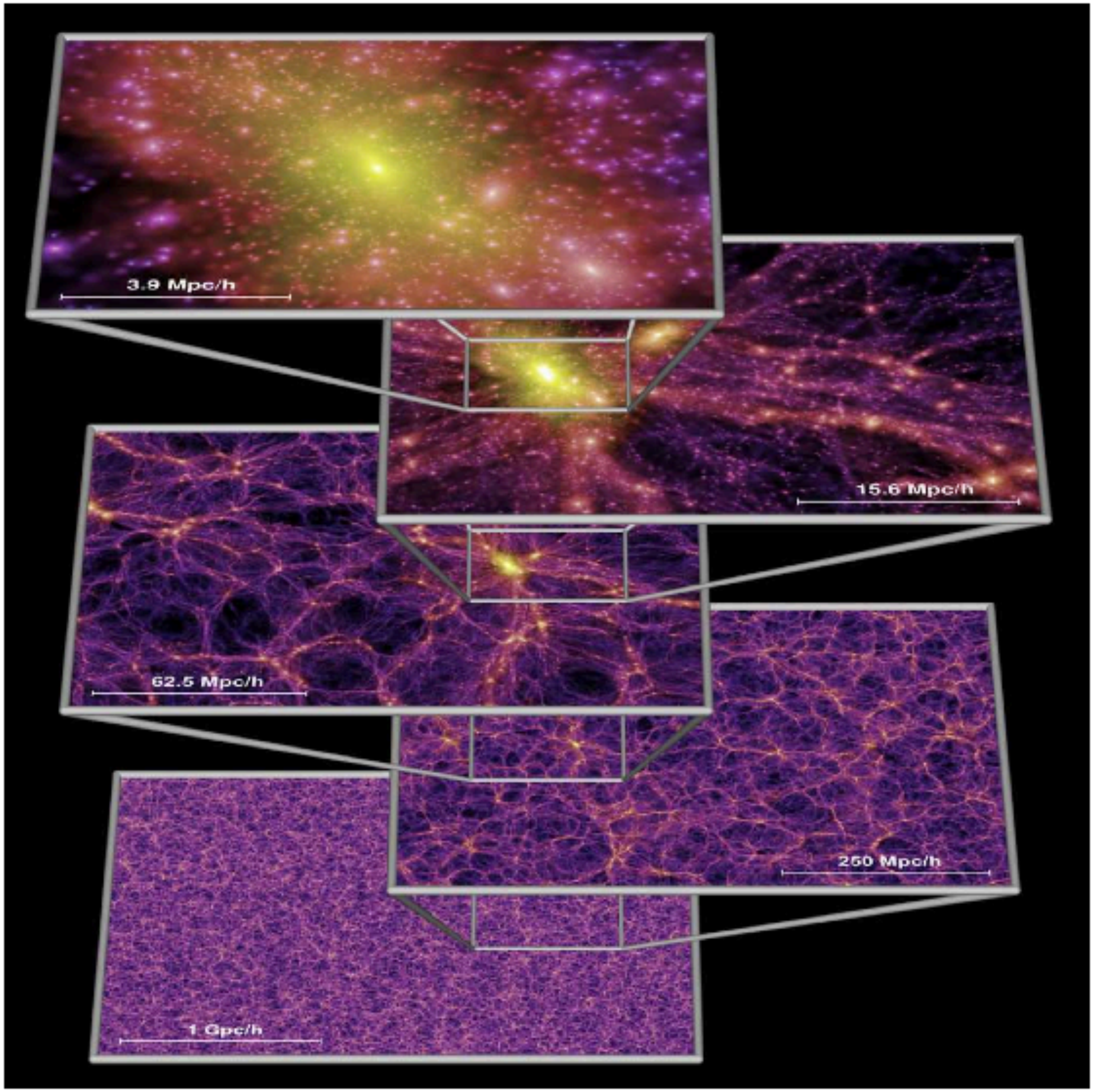


Figure 1: The dark matter density field on various scales. Each individual image shows the projected dark matter density field in a slab of thickness $15h^{-1}$Mpc (sliced from the periodic simulation volume at an angle chosen to avoid replicating structures in the lower two images), colour-coded by density and local dark matter velocity dispersion. The zoom sequence displays consecutive enlargements by factors of four, centred on one of the many galaxy cluster halos present in the simulation.



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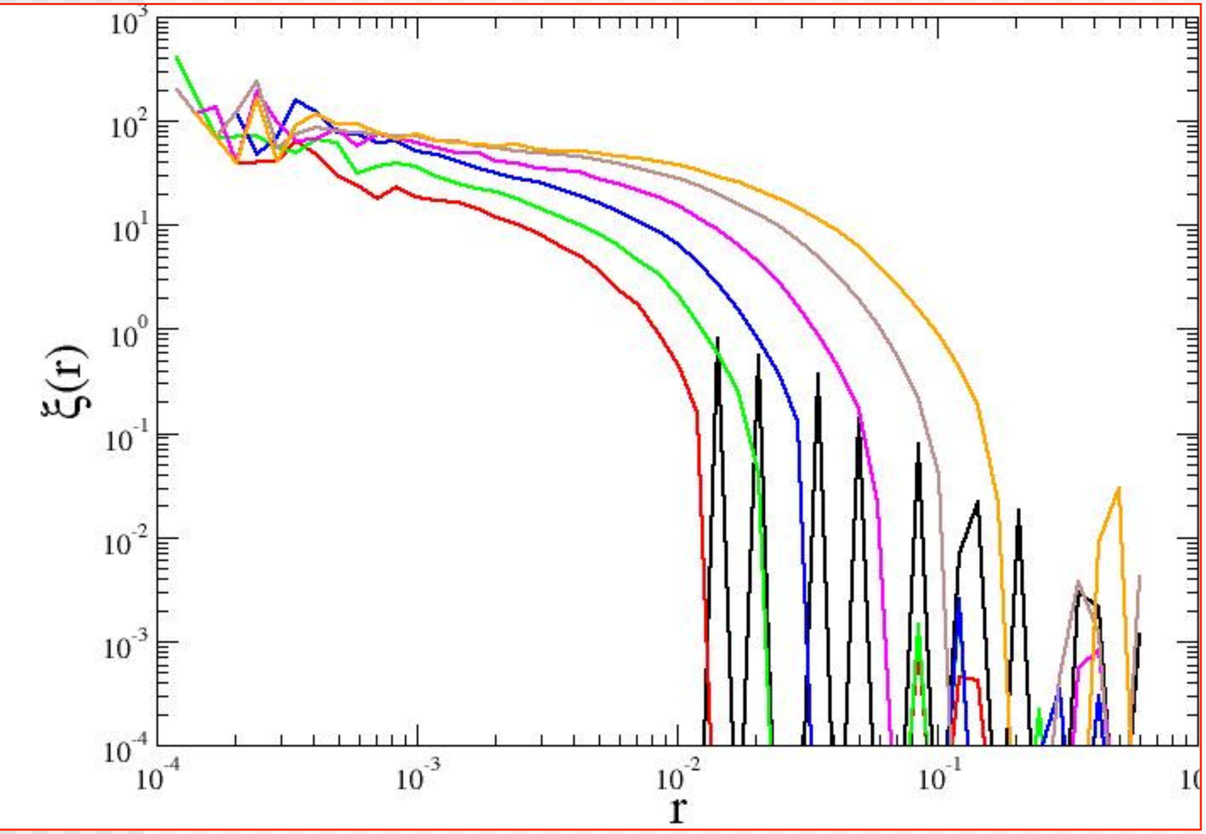
Evolution of 2 point correlations: schematic



ξ(r, t) > 1 strong correlation

ξ(r, t) < 1 weak corrélation

Defines Scale

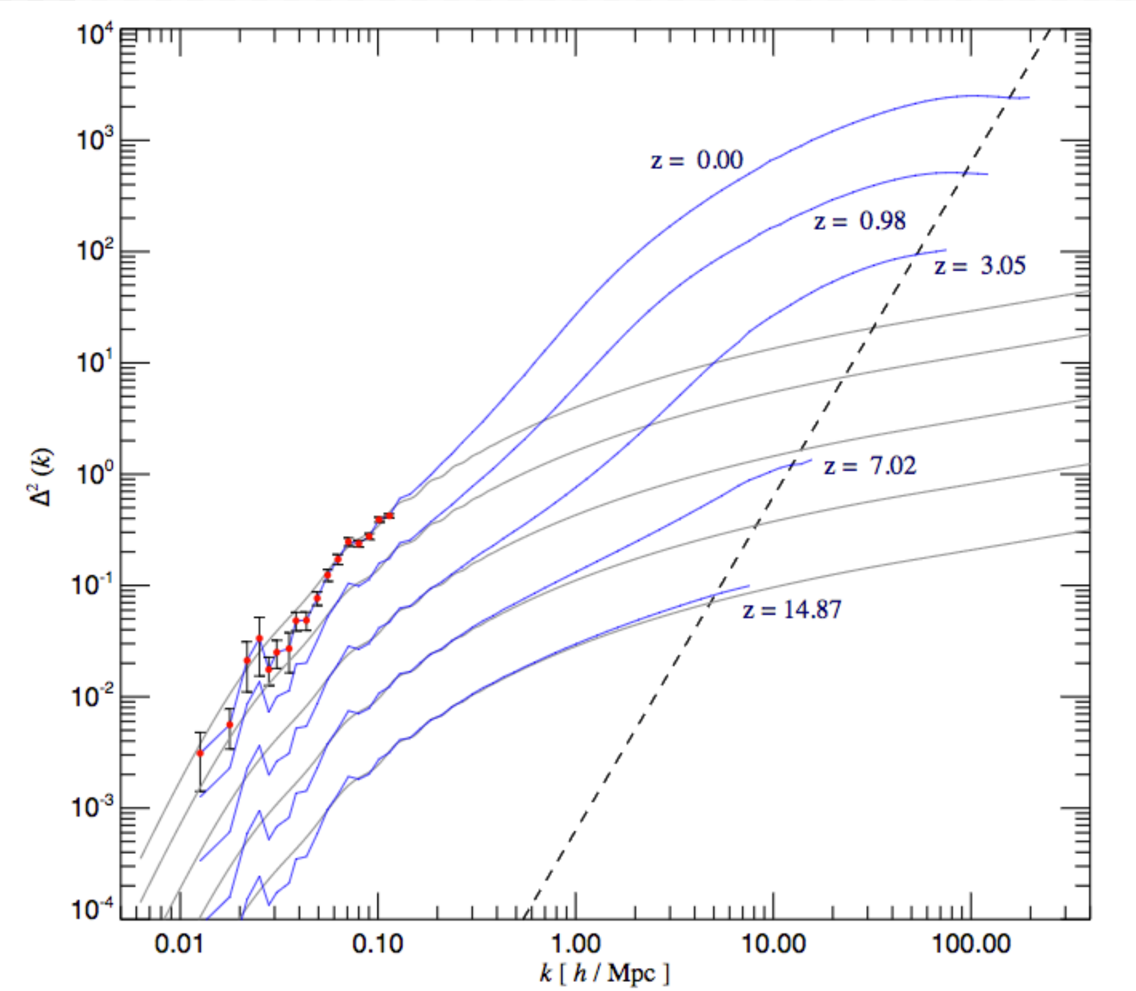
$$\boldsymbol{\xi (\lambda (t) , t) = 1}$$

λ(t): scale of non-linearity

which increases with time

Evolution of power spectrum

(e.g. "LambdaCDM", V. Springel et al., Nature 2005)



$$\Delta^2(k) \equiv \frac{1}{2\pi^2} k^3 P(k) \approx \sigma^2(R = \pi/k)$$

Clustering in cold dark matter simulations: “Hierarchical structure formation”

- Linear theory describes evolution well at sufficiently large scales (small k)
- Non-linearity scale grows monotonically at a *rate predicted by linear theory*
- In non-linear regime “flow of power” <u>**from large to small scales**</u>
  (via collapse dynamics exemplified by “spherical collapse model”)

This is **“HIERARCHICAL STRUCTURE FORMATION”**

Clustering in cold dark matter simulations: non-linear regime

Distribution of masses of **largest** "non-linear clumps" ("mass function") is **roughly** as predicted by spherical collapse model **+** "improved" Press Schecter

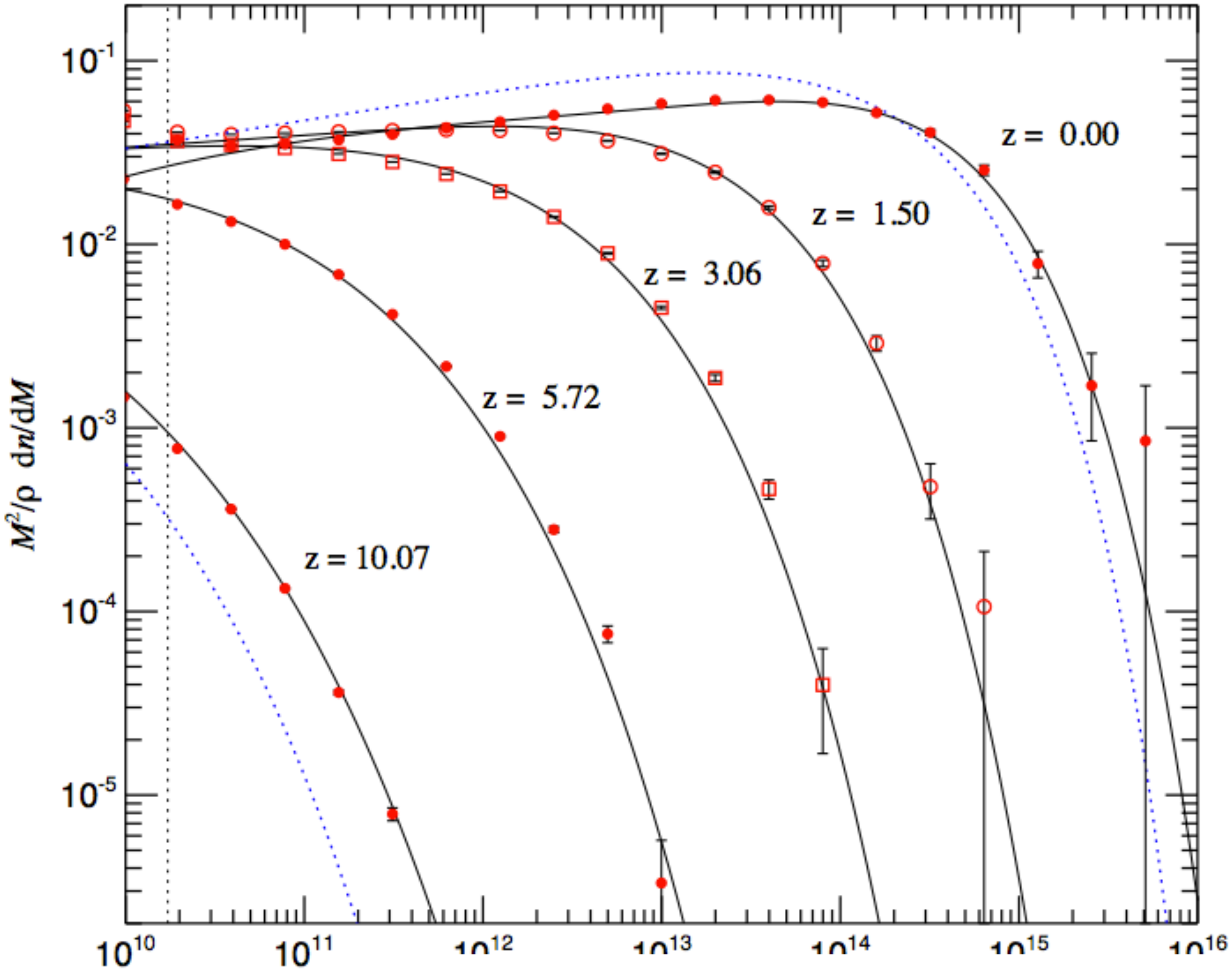


Figure 2: Differential halo number density as a function of mass and epoch. The function $n(M,z)$ gives the comoving number density of halos less massive than M. We plot it as the halo multiplicity function $M^2\rho^{-1}\,\mathrm{d}n/\mathrm{d}M$, where ρ is the mean density of the universe. Groups of particles were found using a friends-of-friends algorithm[6] with linking length equal to 0.2 of the mean particle separation. The fraction of mass bound to halos of more than 20 particles (vertical dotted line) grows from 6.42×10^{-4} at $z = 10.07$ to 0.496 at $z = 0$. Solid lines are predictions from an analytic fitting function proposed in previous work[11], while the dashed lines give the Press-Schechter model[14] at $z = 10.07$ and $z = 0$.

Clustering in non-linear regime: halos

Distribution of masses of **largest** "non-linear clumps" ("mass function") is **roughly** as predicted by spherical collapse model + "improved" Press Schecter

These **halos** have some substructure but are smooth to good approximation ["Stable clustering" breaks down (see e.g. Smith et al., MNRAS 2006)]

Halos are (putatively) approximately virialized finite systems
i.e. quasi-stationary states, stationary solution of Vlasov-Newton Eqs.

Halos have apparently "universal" properties (i.e. independent of cosmology and initial conditions), notably

- Density profiles (e.g. "NFW")
- "Phase space density" profiles

The non-linear regime as now seen (understood?) by cosmologists

Huge (N > 10^{10} !) studies focussed on “realistic” cosmological IC

Increasing N → increasing range of scale resolved in non-linear regime

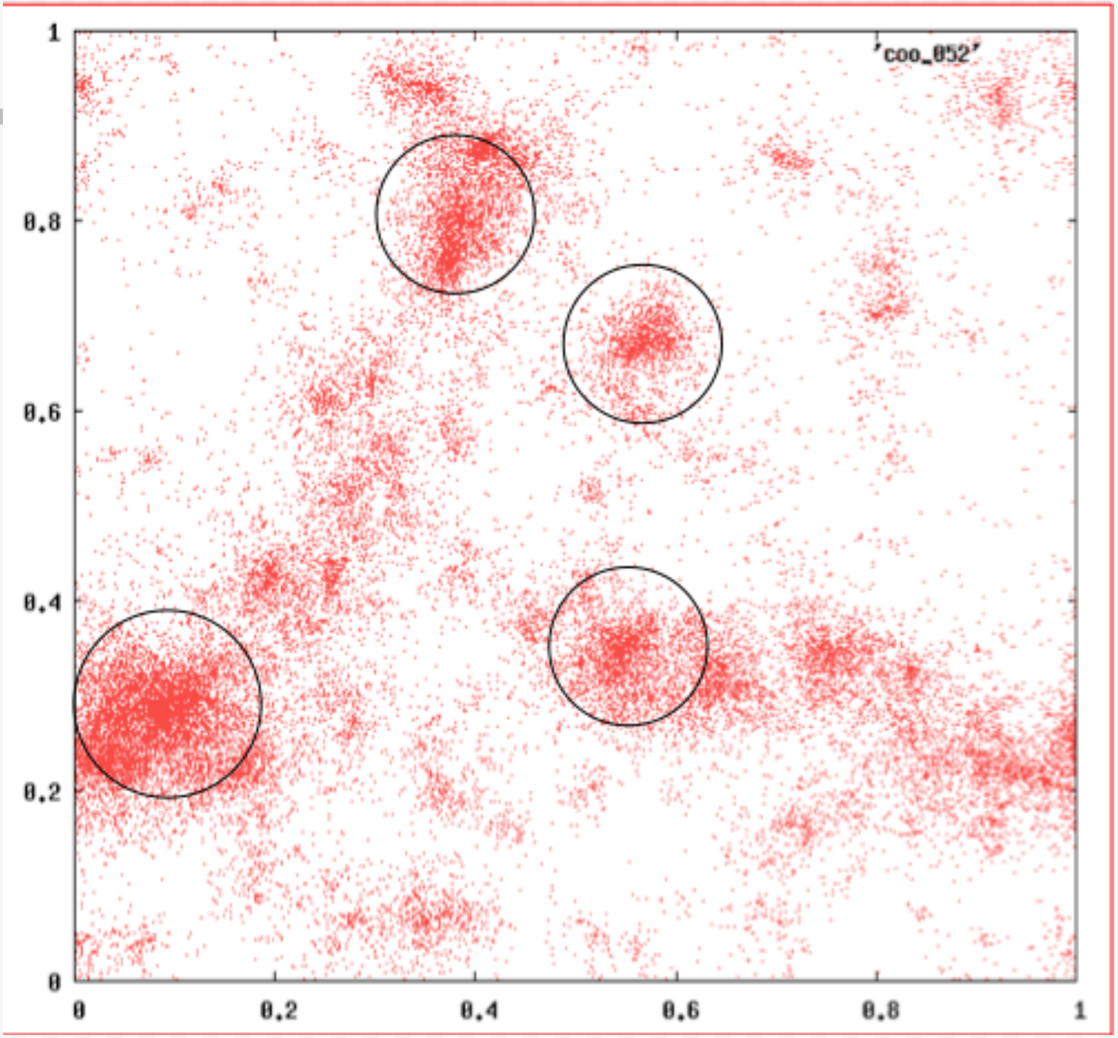
→ increasing resolution of interior of largest clumps

→reveals “nested substructure” but most of mass smoothly distributed

→ **phenomenological descriptions of non-linear regime in terms of these clumps**

These are so called **“halo models”**

“Halo models” of non-linear clustering



Matter density field ≈

collection of (non-overlapping) spherical *smooth* virialized structures

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A UNIVERSAL DENSITY PROFILE FROM HIERARCHICAL CLUSTERING

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Received 1996 November 13; accepted 1997 July 15

ABSTRACT

We use high-resolution N-body simulations to study the equilibrium density profiles of dark matter halos in hierarchically clustering universes. We find that all such profiles have the same shape, independent of the halo mass, the initial density fluctuation spectrum, and the values of the cosmological parameters. Spherically averaged equilibrium profiles are well fitted over two decades in radius by a simple formula originally proposed to describe the structure of galaxy clusters in a cold dark matter universe. In any particular cosmology, the two scale parameters of the fit, the halo mass and its characteristic density, are strongly correlated. Low-mass halos are significantly denser than more massive systems, a correlation that reflects the higher collapse redshift of small halos. The characteristic density of an equilibrium halo is proportional to the density of the universe at the time it was assembled. A suitable definition of this assembly time allows the same proportionality constant to be used for all the cosmologies that we have tested. We compare our results with previous work on halo density profiles and show that there is good agreement. We also provide a step-by-step analytic procedure, based on the Press-Schechter formalism, that allows accurate equilibrium profiles to be calculated as a function of mass in any hierarchical model.

Subject headings: cosmology: theory — dark matter — galaxies: halos — methods: numerical

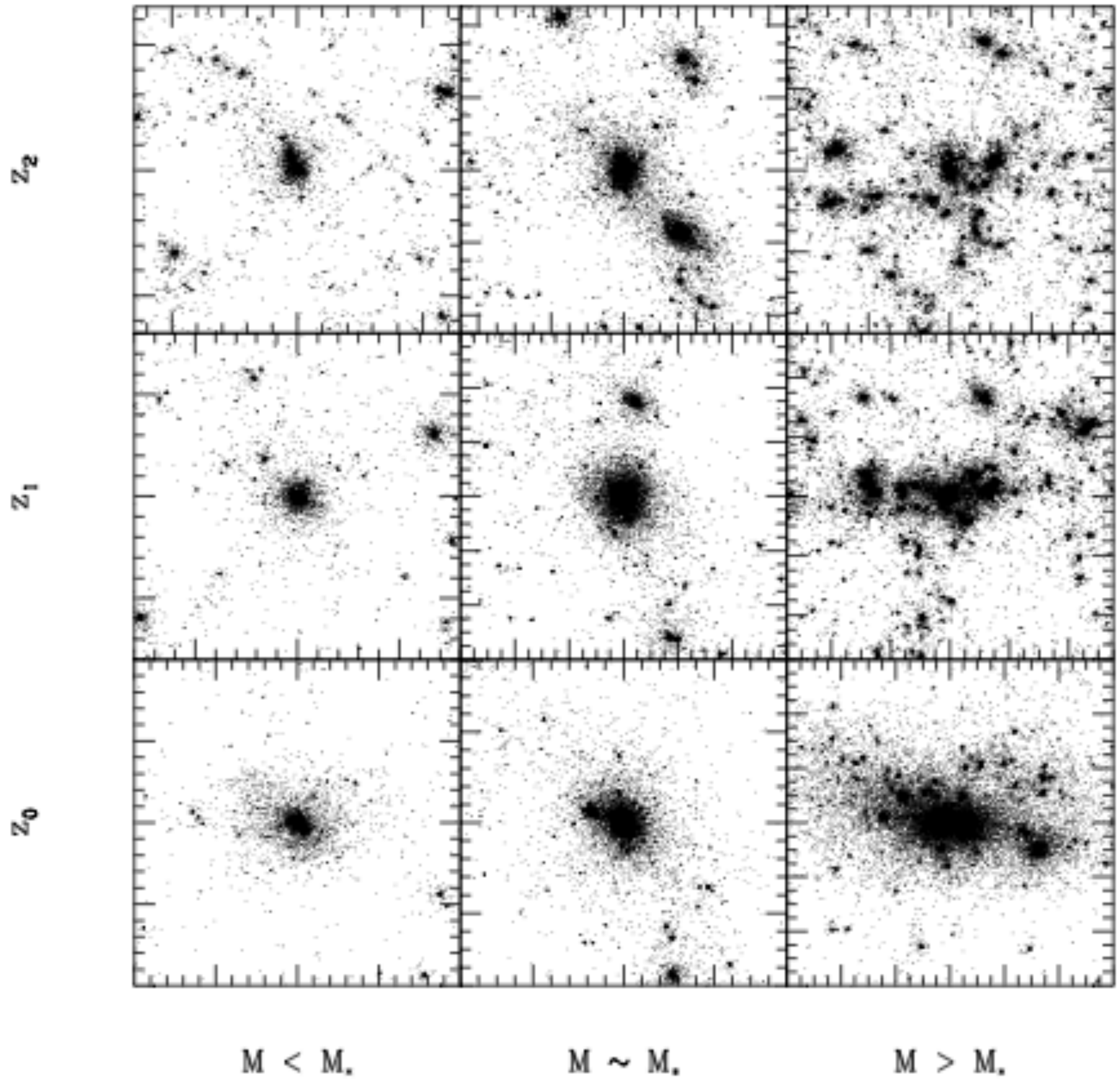

FIG. 1.—Particle plots illustrating the time evolution of halos of different mass in an $\Omega_0 = 1$, $\Lambda = 0$, and $n = -1$ cosmology. The box sizes of each column are chosen so as to include approximately the same number of particles. At $z_0 = 0$, the box size corresponds to about $6r_{200}$. Time runs from top to bottom. Each snapshot is chosen so that M_* increases by a factor of 4 between each row. Low-mass halos assemble earlier than their more massive counterparts. This is true for every cosmological scenario in our series.

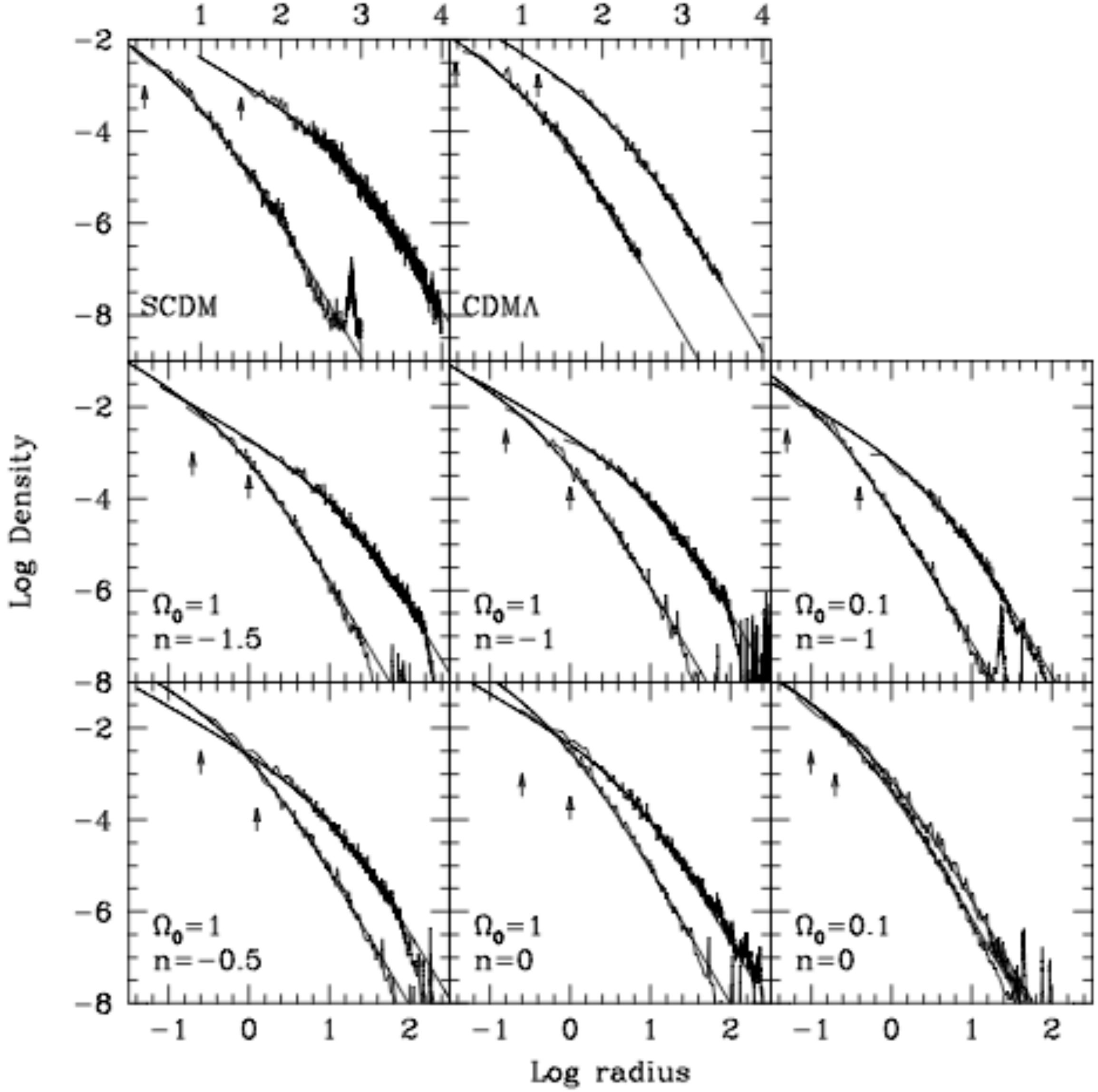

FIG. 2.—Density profiles of one of the most massive halos and one of the least massive halos in each series. In each panel, the low-mass system is represented by the leftmost curve. In the SCDM and CDMΛ models, radii are given in kiloparsecs (*scale at top*), and densities are in units of 10^{10} $M_\odot$ kpc^{-3}. In all other panels, the units are arbitrary. The density parameter, Ω_0, and the value of the spectral index, n, are given in each panel. The solid lines are fits to the density profiles using eq. (1). The arrows indicate the value of the gravitational softening. The virial radius of each system is in all cases 2 orders of magnitude larger than the gravitational softening.

The Density and Pseudo-Phase-Space Density Profiles of CDM halos

arXiv:1102.0002v1 [astro-ph.CO] 31 Jan 2011

Aaron D. Ludlow[1,⋆], Julio F. Navarro[2], Michael Boylan-Kolchin[3,4], Volker Springel[5,6], Adrian Jenkins[7], Carlos S. Frenk[7], and Simon D. M. White[3],

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[6] *Zentrum für Astronomie der Universität Heidelberg, ARI, Mönchhofstr. 12-14, 69120 Heidelberg, Germany*
[7] *Institute for Computational Cosmology, Dept. of Physics, Univ. of Durham, South Road, Durham DH1 3LE, UK*

2 February 2011

ABSTRACT

Cosmological N-body simulations indicate that the spherically-averaged density profiles of cold dark matter halos are accurately described by Einasto profiles, where the logarithmic slope is a power-law of adjustable exponent, $\gamma \equiv d\ln\rho/d\ln r \propto r^{\alpha}$. The pseudo-phase-space density (PPSD) profiles of CDM halos also show remarkable regularity, and are well approximated by simple power laws, $Q(r) \equiv \rho/\sigma^3 \propto r^{-\chi}$. We show that this is expected from dynamical equilibrium considerations, since Jeans' equations predict that the pseudo-phase-space density profiles of Einasto halos should resemble power laws over a wide range of radii. For the values of α typical of CDM halos, the inner Q profiles of equilibrium halos deviate significantly from a power law only very close to the center, and simulations of extremely high-resolution would be needed to detect such deviations unambiguously. We use an ensemble of halos drawn from the Millennium-II simulation to study which of these two alternatives describe best the mass profile of CDM halos. Our analysis indicates that at the resolution of the best available simulations, both Einasto and power-law PPSD profiles (with adjustable exponents α and χ, respectively) provide equally acceptable fits to the simulations. A full account of the structure of CDM halos requires understanding how the shape parameters that characterize departures from self-similarity, like α or χ, are determined by evolutionary history, environment or initial conditions.

Key words: cosmology: dark matter – methods: numerical

“Halo profiles” :

(see e.g. Cooray and Sheth, Phys. Rep. 2002)

- Density profiles of these “halos” fitted by “universal” form, e.g.,

“NFW profile” $$\rho(r) = \frac{\rho_0}{(r/r_s)(1 + r/r_s)^2}$$ (or e.g. “Einasto profile”)

2 parameters fitted from simulation: usually, **halo mass m and “concentration”** defined by

$$c \equiv \frac{r_v}{r_s}$$

where r_v is halo radius or “virial radius”, where density is 200 x mean density

Physical origin? Extensive literature, no definitive answer..

"Halo models" : ingredients

(see e.g. Cooray and Sheth, Phys. Rep. 2002)

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\+ “Mass function” n(m) for halos
\+ “Mass- concentration relation”
\+ Correlation properties of halo centres (~ linear theory at large distances)

Halo model example: 2 point correlations

(see e.g. Cooray and Sheth, Phys. Rep. 2002)

- Measured (deterministic) mass concentration relation
- Density profiles $\rho(r|m,c) = m u(r|m)$ where $4\pi \int_0^{r_v} u(r|m,c)\, r^2\, dr = 1$

+ statistics of halo (centre) distribution: mass function n(m), correlation fns.

We have

$$\langle \rho(\vec{x})\rho(\vec{y})\rangle = \langle \sum_i \sum_i m_i m_j u(r|m_i) u(r|m_j) \rangle$$

Two point correlation function of mass density

$$\xi_m(|\vec{x}-\vec{y}|) = \frac{1}{\bar{\rho}^2}\langle \rho(\vec{x})\rho(\vec{y})\rangle - 1 \qquad \langle \rho(\vec{x})\rangle = \bar{\rho}$$

divides into **"one-halo term"** (i=j) and **"two halo term"** (i ≠ j)

2 point correlations in halo model: two contributions

One halo term depends only on average mass function and density profiles:

$$\frac{1}{\bar{\rho}^2}\int d^3z \int dm\, m^2 n(m) u(\vec{x}-\vec{z}|m) u(\vec{y}-\vec{z}|m)$$

This describes the **strongly non-linear regime**

Two halo term depends also on spatial correlation properties of halos:

$$\frac{1}{\bar{\rho}^2}\int d^3z_1 \int d^3z_2 \int dm_1\, m_1 n(m_1) \int dm_2\, m_2 n(m_2)$$
$$\times u(\vec{x}-\vec{z}_1|m) u(\vec{y}-\vec{z}_2|m) \left[\frac{\langle n(m_1,\vec{z}_1) n(m_2,\vec{z}_2)\rangle}{n(m_1) n(m_2)} - 1\right]$$

To a reasonable approximation this can be just be approximated by linear regime

Halo models : exploitation

These models give **analytical forms for n-point correlation properties** (real and k space) in terms of a finite number of parameters measured in simulations..

These are then used in making observational predictions (e.g. lensing)

Galaxy distributions are constructed positing Prob(galaxy|m)
(with numerous free parameters then adjusted to observations..)

Halos models can be "refined" to model e.g. fraction of "substructure", more complex mass-concentration relations, **at price of additional fit parameters..**

"Halo bias": relation between correlation of halos and those of all matter

Cosmological structure formation: Some open issues

General questions about the “non-linear regime”

- **How is non-linear clustering best characterized ?**

  (mathematical tools..)

- **How does non-linear clustering depend on initial conditions and cosmology?**

  (and can we understand and precisely characterize this..)

Both questions are also of fundamental importance **observationally**

Halo models: open problems…

Problems with “halo model” approach

- “Halos” are poorly defined objects..
- The approximation of smoothness is problematic; increased resolution has revealed layer after layer of “substructure”..
- Unclear what “universality” means, what is its origin if it exists..

(**Huge** literature on these issues..)

Resolution of N body simulations

How accurately does discrete NBS reproduce clustering of underlying continuum physical model (VP limit)?

i.e. What are finite N effects?

Practically:

what is "resolution scale" R(a) ?

i.e. above which a given clustering stat is measured with desired precision?

The resolution/discreteness problem

N Body method introduces several non-physical parameters

- **Λ**: **mean interparticle distance** ("mass resolution")
- **ε**: **force softening length** ("force resolution")

[+ others: Box size L, starting red-shift, choice "pre-initial" configuration (grid/glass…)]

How does R(a) depend on Λ, ε, a ? On model simulated ?

Why is there a '?' ?

Numerical convergence studies do not in practice resolve the question..

+ Prima facie problem:

Naively might expect condition:

$$R \gg \max\{\Lambda, \varepsilon\}$$

However N-body simulations typically use $\Lambda \gg \varepsilon$

$$R(\text{final}) \sim \varepsilon$$

i.e. resolution is given by the smoothing length, even when $\varepsilon \ll \Lambda$

Example: “Millenium” simulation

Simulating the joint evolution of quasars, galaxies and their large-scale distribution

Volker Springel[1], Simon D. M. White[1], Adrian Jenkins[2], Carlos S. Frenk[2], Naoki Yoshida[3], Liang Gao[1], Julio Navarro[4], Robert Thacker[5], Darren Croton[1], John Helly[2], John A. Peacock[6], Shaun Cole[2], Peter Thomas[7], Hugh Couchman[5], August Evrard[8], Jörg Colberg[9] & Frazer Pearce[10]

$N=2058^3$, $L=500\ h^{-1}$ Mpc

THUS

$\mathbf{\Lambda \approx 0{,}25\ h^{-1}\ Mpc}$

$\varepsilon\ \mathbf{\approx 5\ h^{-1}\ kpc}$

i.e. $\boldsymbol{\varepsilon/\Lambda} \approx 0.02$

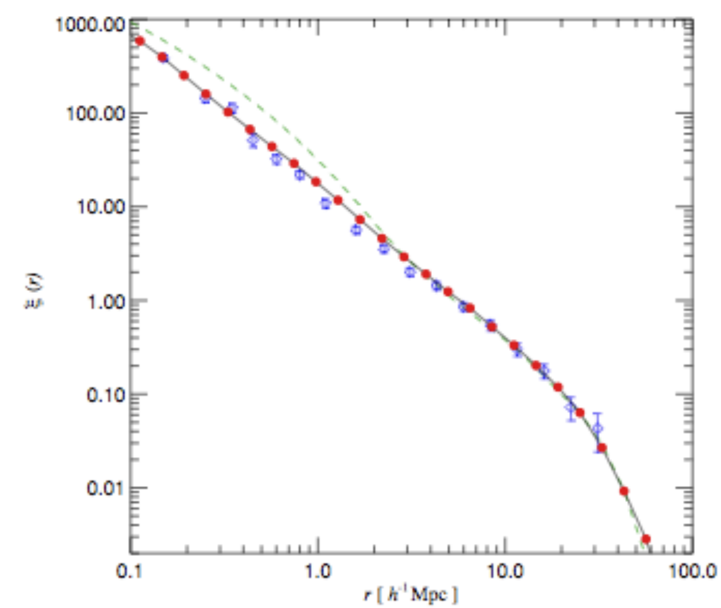
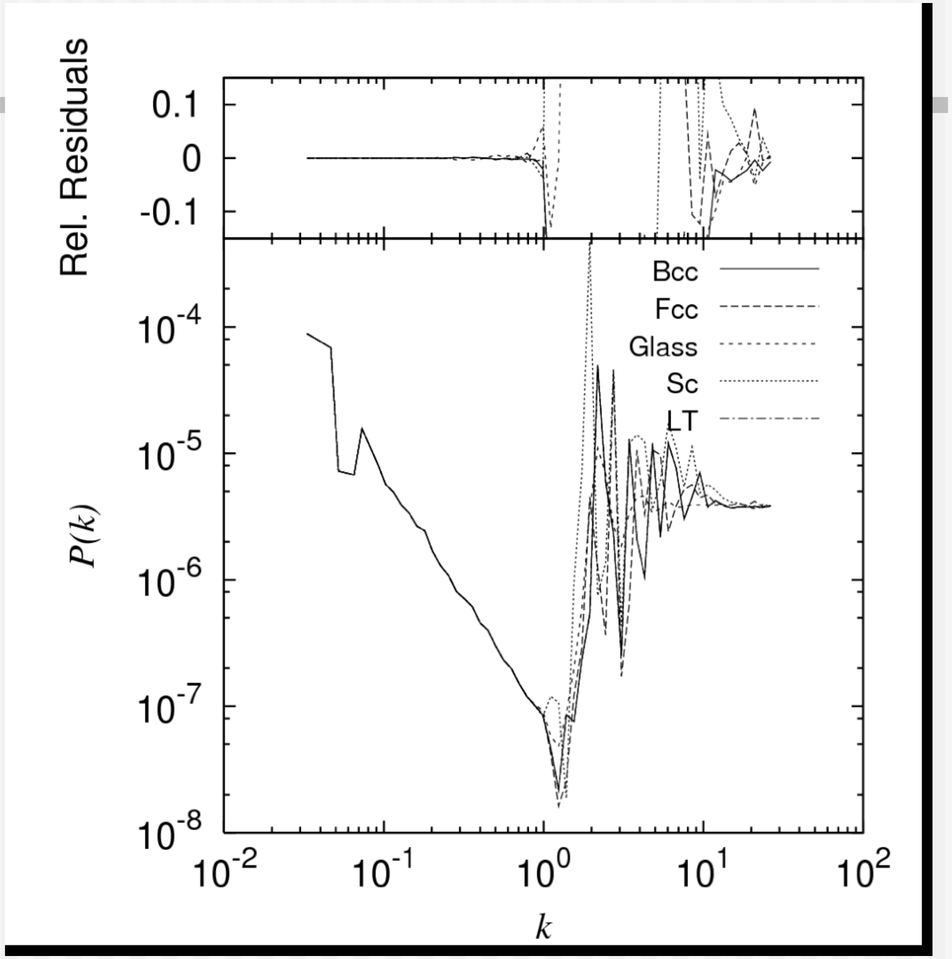


Figure 4: Galaxy 2-point correlation function at the present epoch. Red symbols (with vanishingly small Poisson error-bars) show measurements for model galaxies brighter than $M_K = -23$. Data for the large spectroscopic redshift survey 2dFGRS[28] are shown as blue diamonds. The SDSS[34] and APM[31] surveys give similar results. Both, for the observational data and for the simulated galaxies, the correlation function is very close to a power-law for $r \lesssim 20\,h^{-1}$Mpc. By contrast the correlation function for the dark matter (dashed line) deviates strongly from a power-law.

N.B: a large part of the non-linear regime is in the range of scales

$$\boldsymbol{\varepsilon < r < \Lambda}$$

Resolution at starting time



Initial (small) **fluctuations of model accurately reproduced for scales > Λ**
Large **fluctuations due to discreteness for scales < Λ**

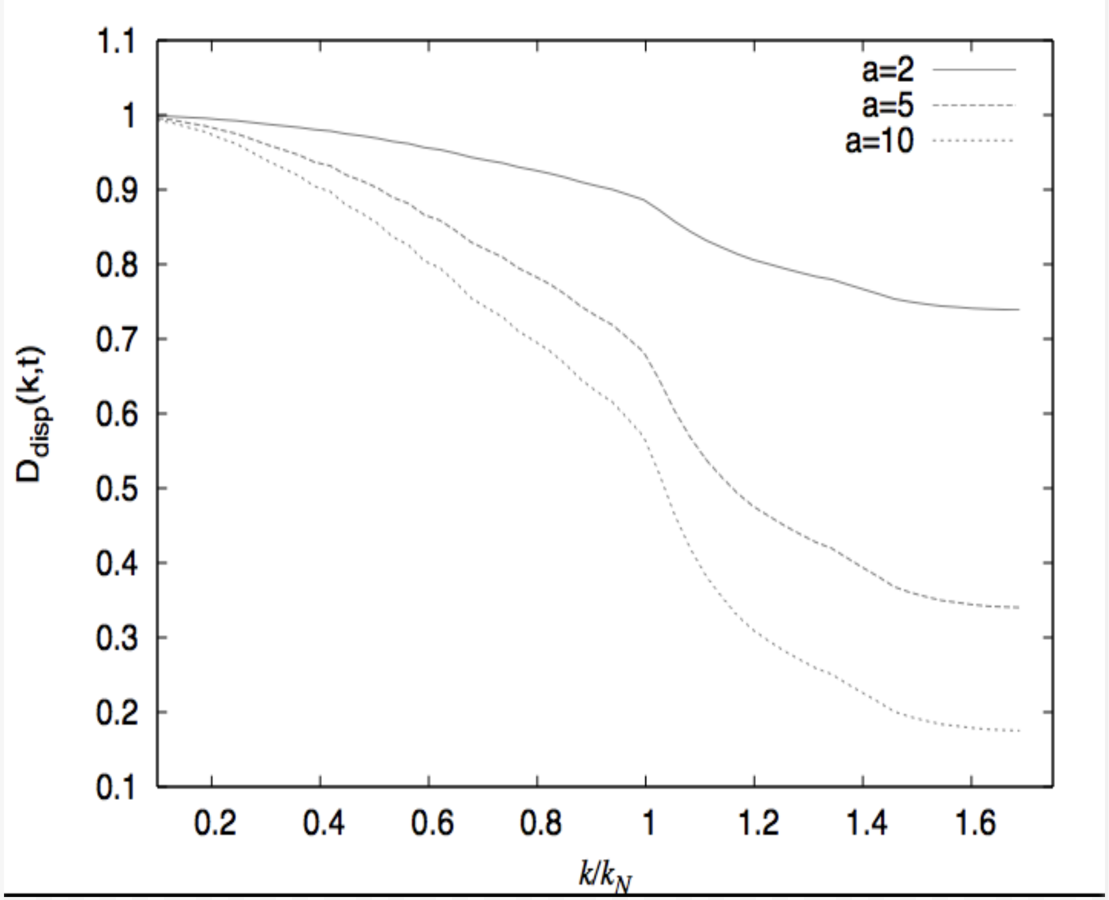
Evolution of resolution in linear regime

MJ, B. Marcos, A. Gabrielli, T. Baertschiger, F. Sylos Labini
Gravitational evolution of a perturbed lattice and its fluid limit,
Phys. Rev. Lett. 95:011334 (2005)]:

Small displacements from an infinite periodic lattice:
Evolution can be calculated exactly ! It's just an eigenmode problem
→ **"Particle Linear Theory"**

Linear evolution of power on a lattice

M. Joyce and B. Marcos,
Quantification of discreteness effects in cosmological N body simulations. II: Early time evolution
Phys. Rev. D76:103505 (2007)



$k/k_N = k\Lambda/\pi$

- Simulation begins at a=1
- ***Deviation from unity is the discreteness effect***

Resolution in the non-linear regime

Modes now couple.....

Role of “missing power”? Role of added (discrete) power ?

Claim: R(a) decreases strongly and “follows” non-linear clustering

Justification: Non-linear gravitational clustering

“**efficiently transfers power from large scale to small scale**”

cf. spherical collapse model

→ At sufficiently long times all **memory of initial conditions at “missing” scales is lost**

Despite “convergence studies” basic questions remain..

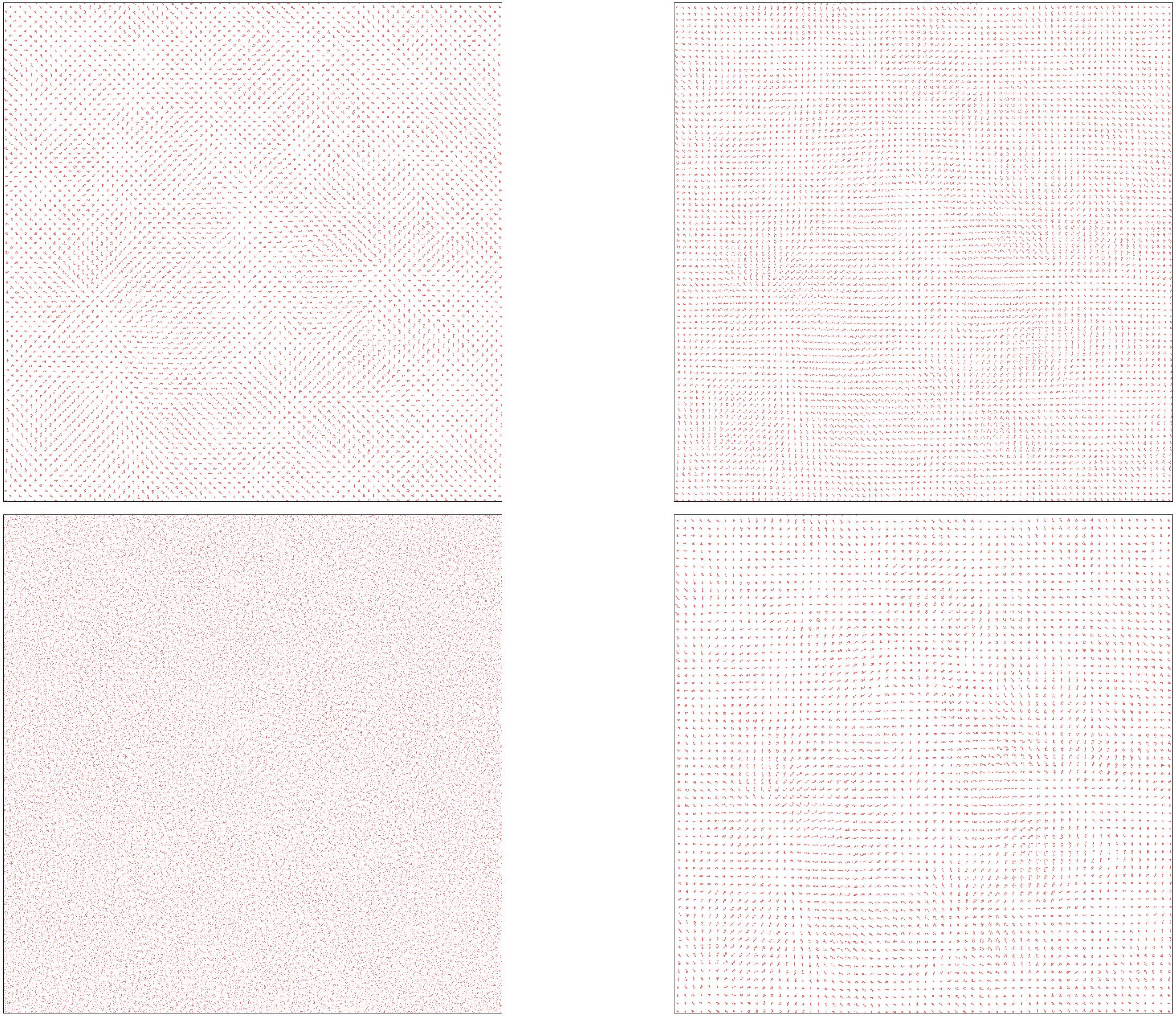
How efficient is transfer power from large scale to small scale?

How much does it really “wipe out” dependence on discreteness in IC?

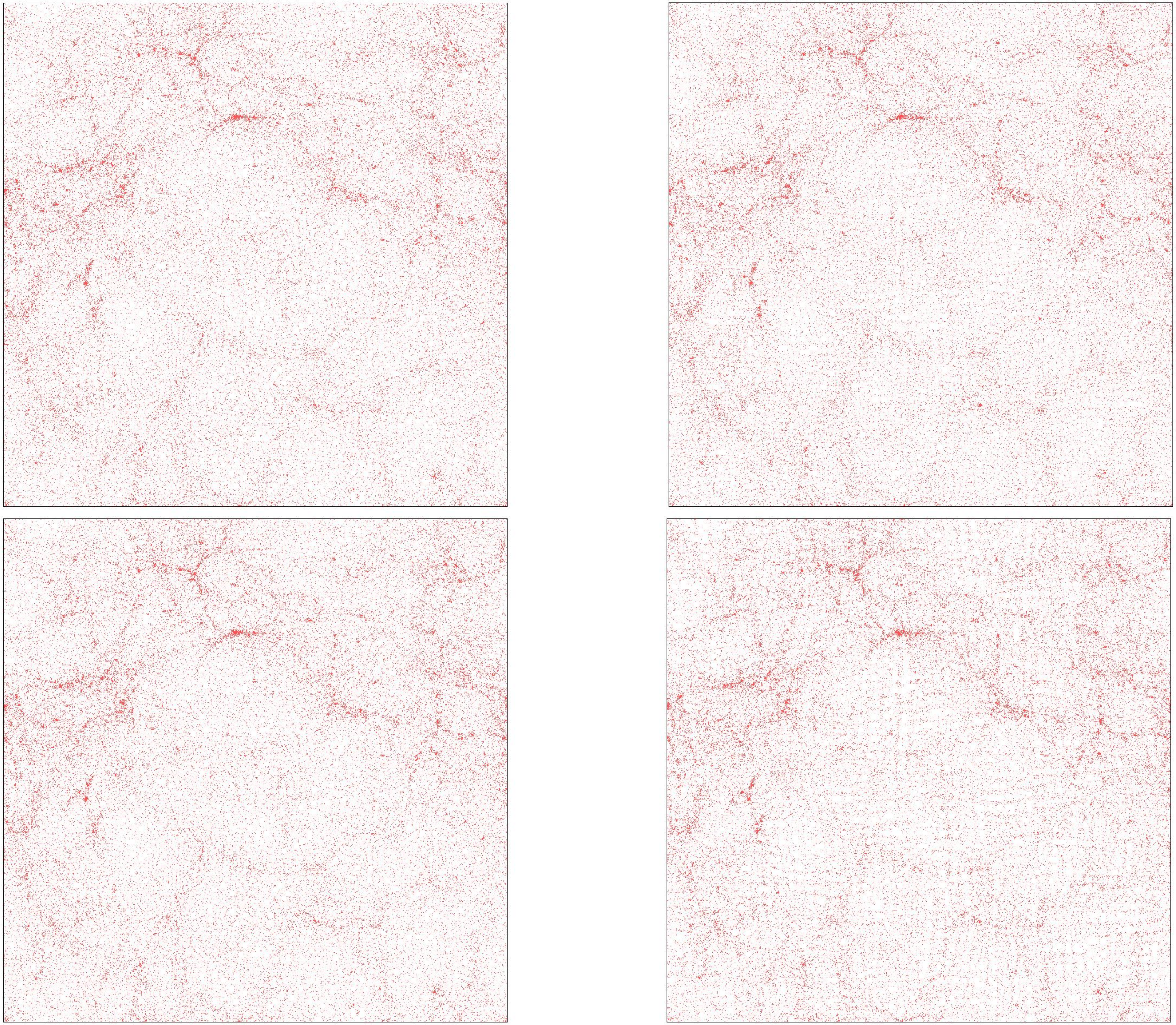
→

Can we quantify R(a) ? Is it model dependent ?

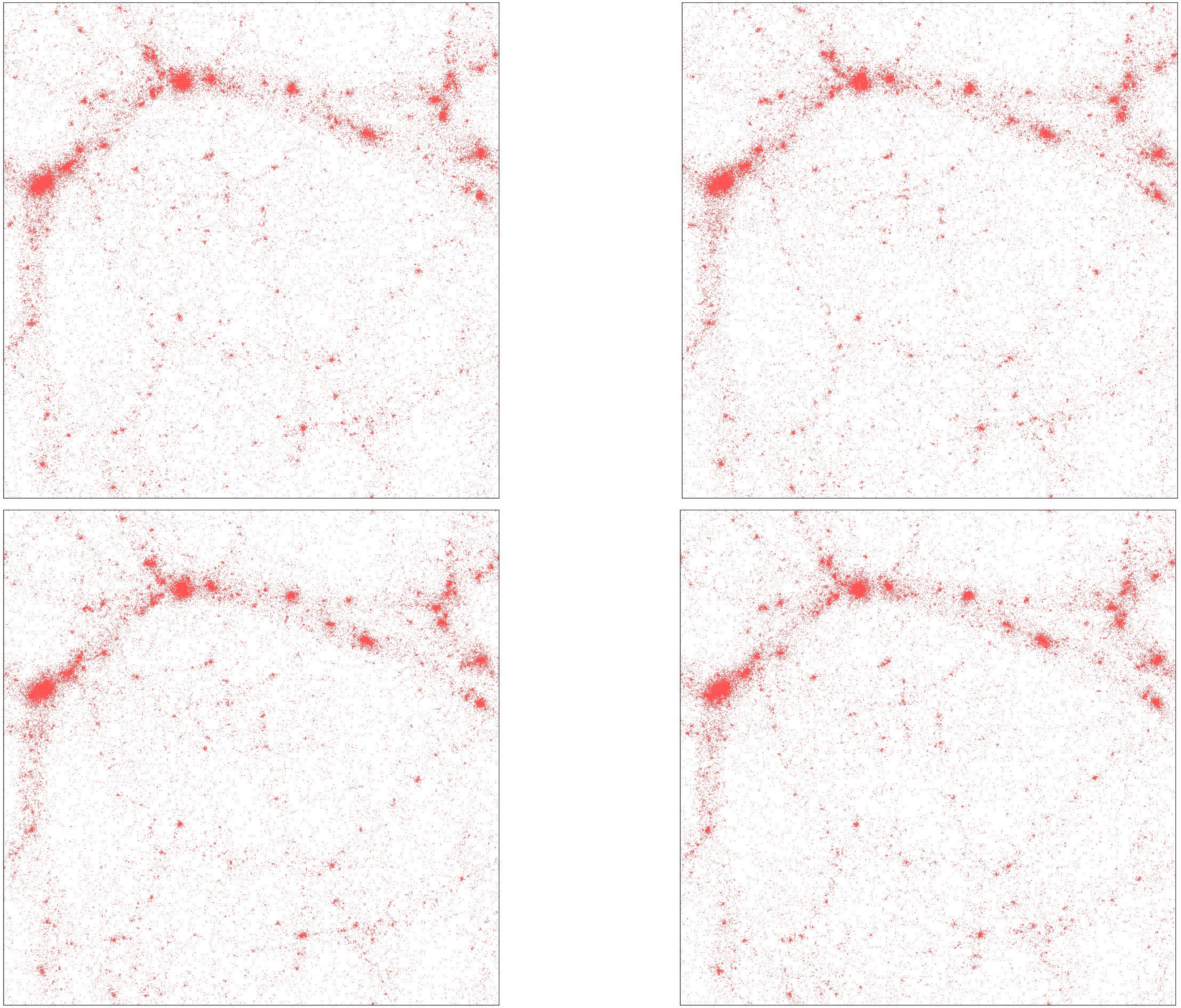
Initial conditions a=1



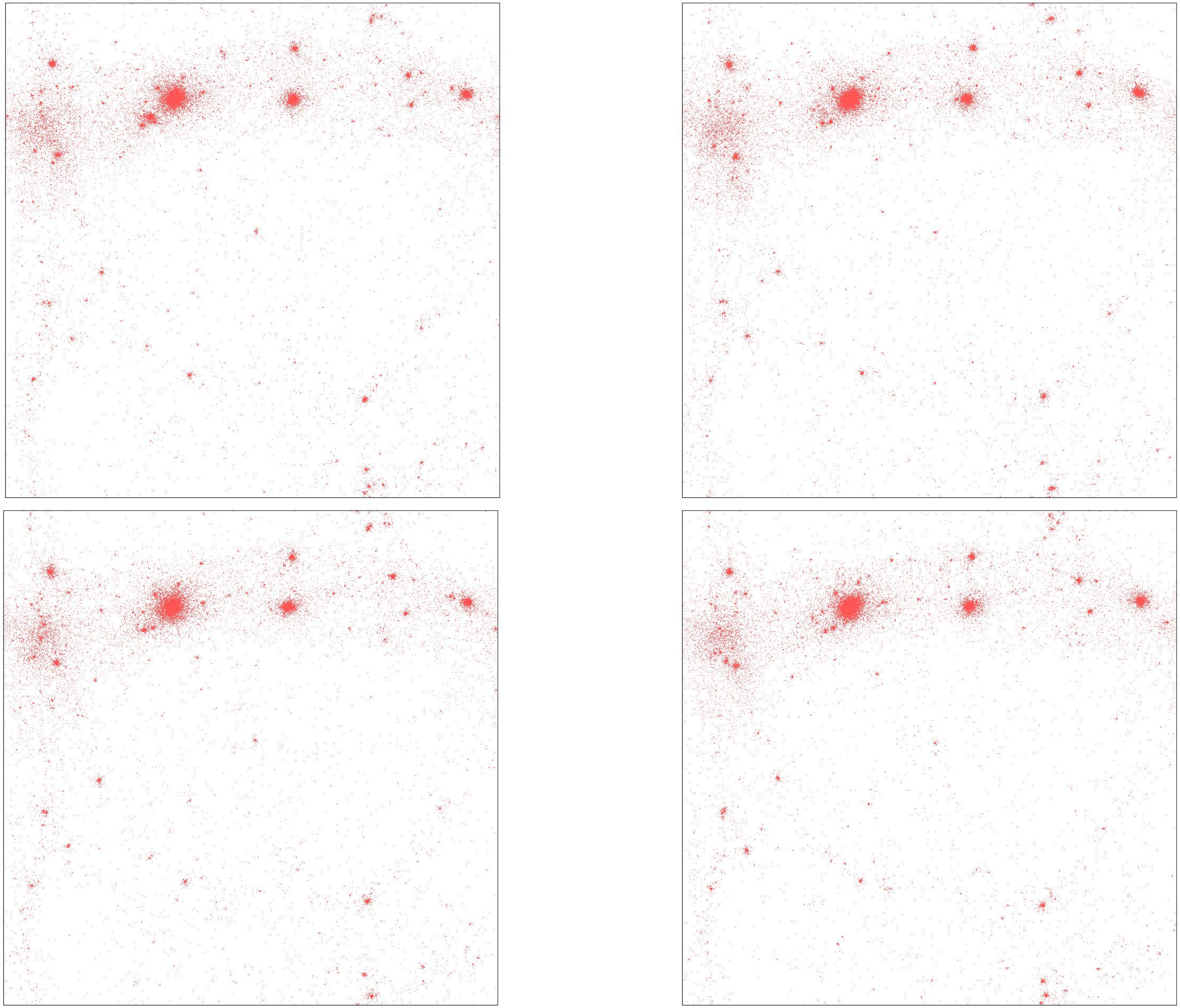
Evolved to $a=2^3$

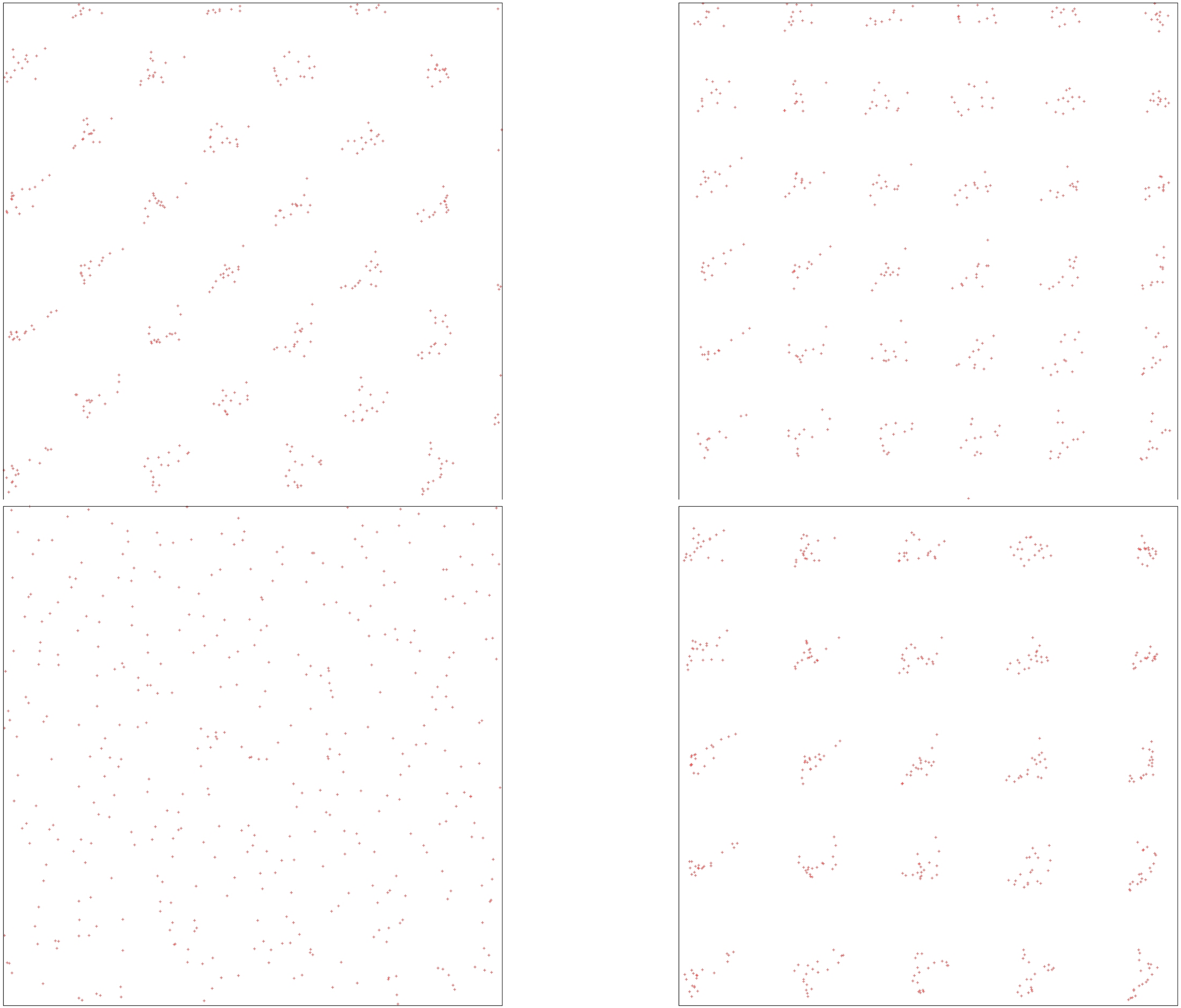


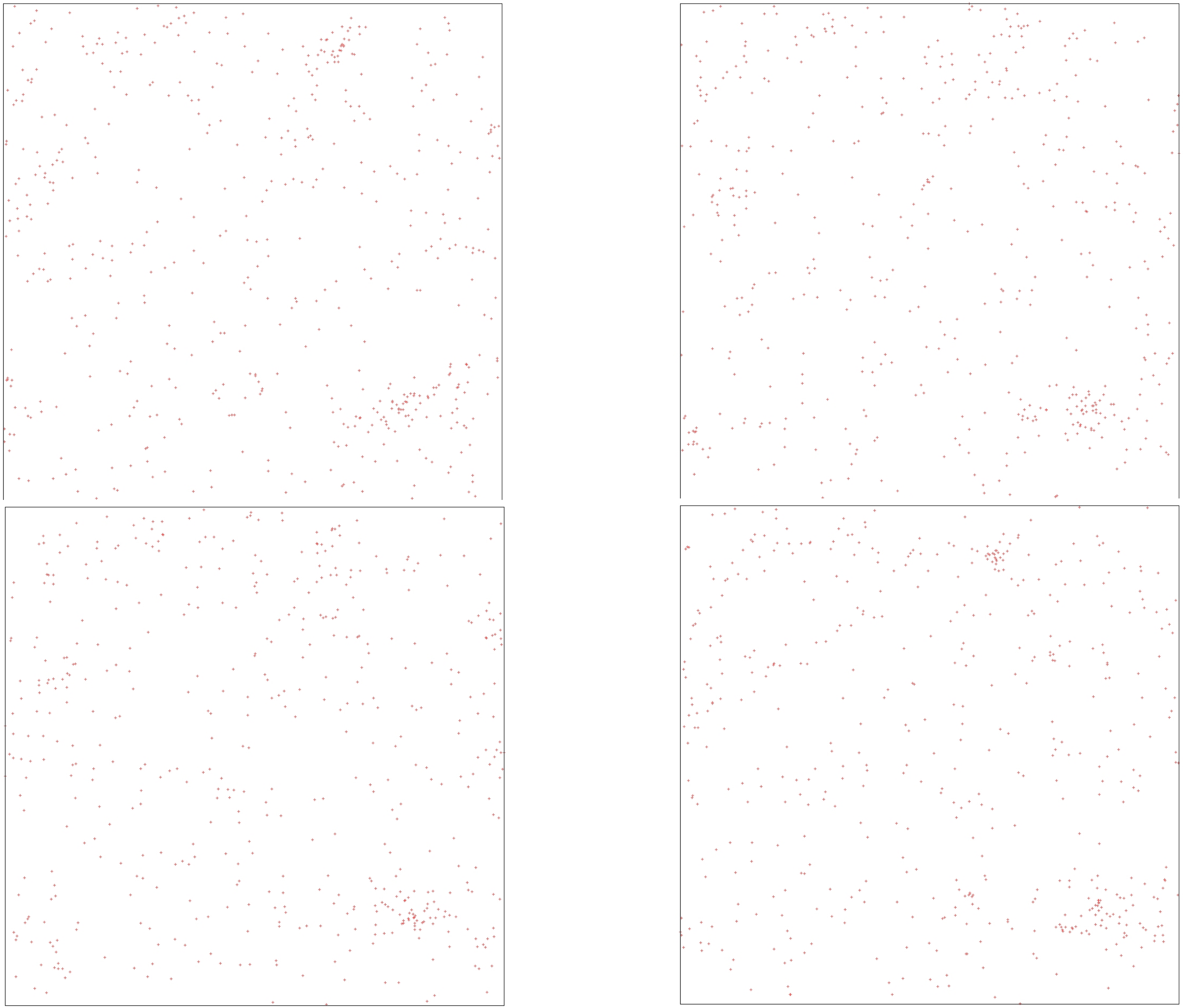
Evolved to $a=2^5$

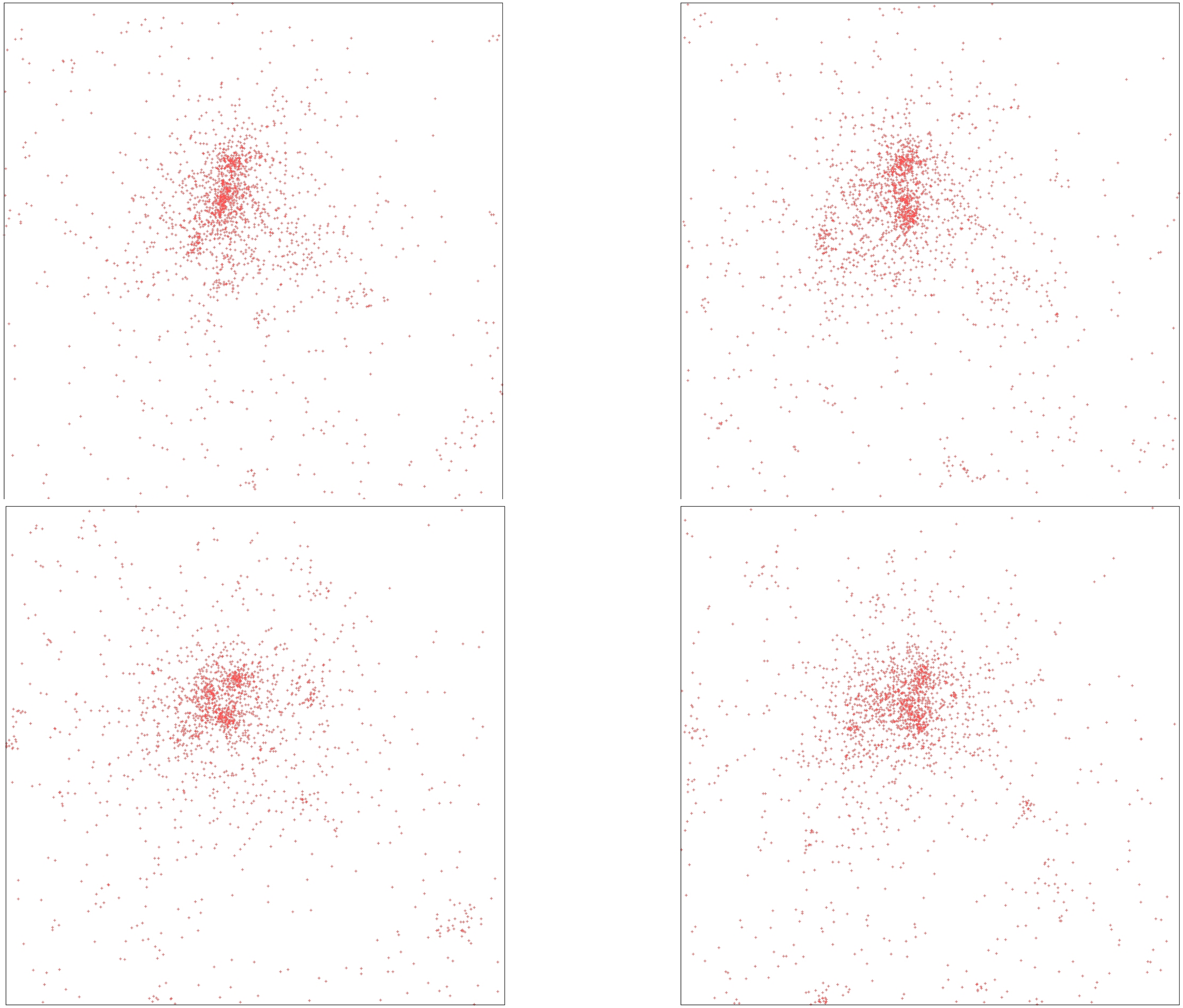


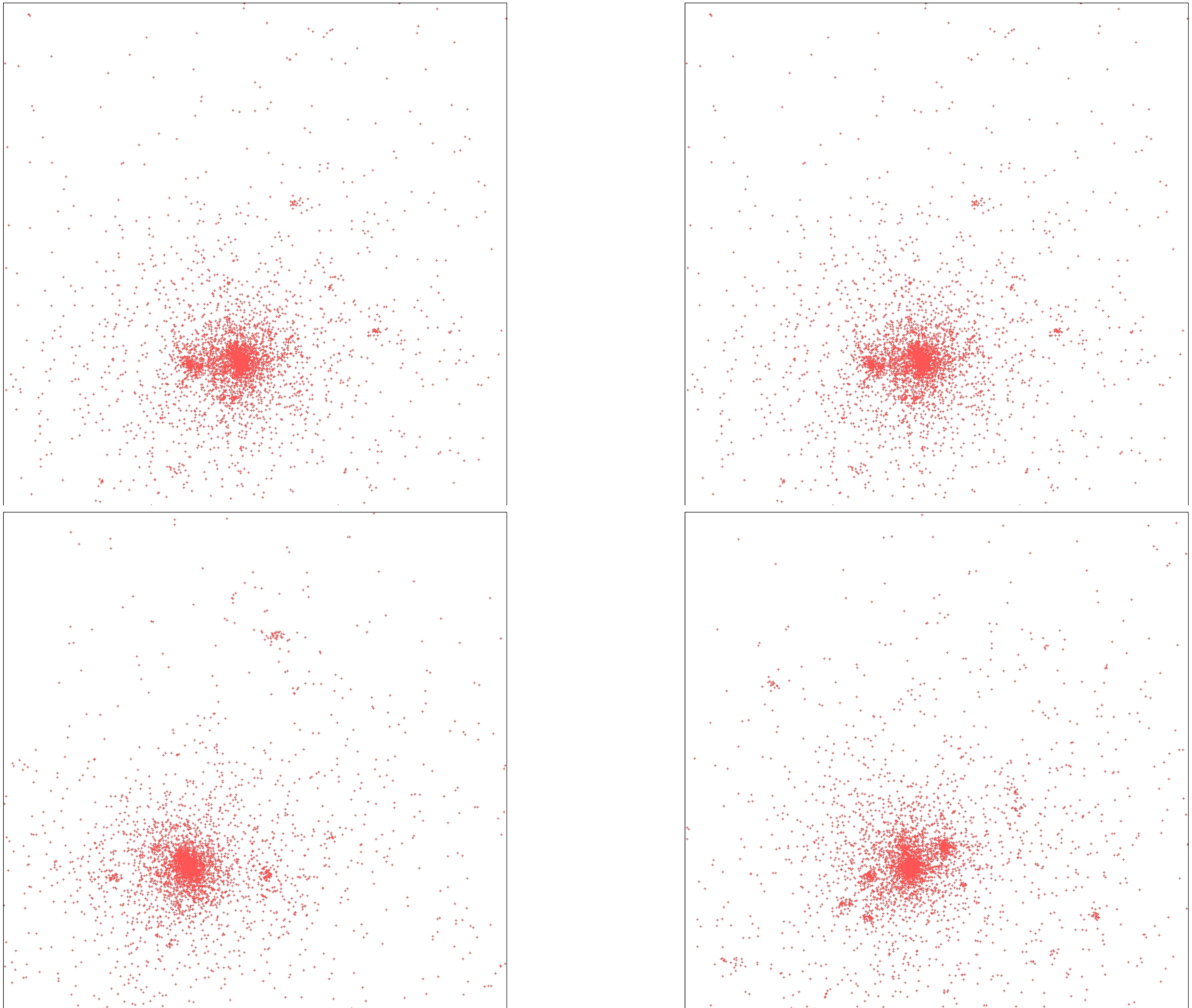
Evolved to $a=2^7$











END PART I !

PART II:

1D toy models for cosmology (and astrophysics)

Newtonian Gravity in 1D

Poisson equation in 1D
→
Attractive pair force independent of separation

Equations of motion:

$$\ddot{x}_i = -g \sum_{j \neq i} \mathrm{sgn}(x_i - x_j)$$

→Forces are constant except at crossing
→"Exact" numerical integration using an event driven algorithm

Newtonian Gravity in 1D

Many studies in literature going back to 50s at least, for references see e.g.
M. Joyce and T. Worrakitpoonpon, J. Stat. Mech., P10012 (2010)

Also now a relevant model of a real laboratory system:

Long-range one-dimensional gravitational-like interaction in a neutral atomic cold gas
Chalony, M.; Barré, J.; Marcos, B.; Olivetti, A.; Wilkowski, D.

Phys. Rev. A (2013)

1D gravity: thermal equilibrium

1D analytical solution (Spitzer 1942, Rybicki 1971)

$$f\left(p, x\right) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sigma\Lambda}\mathrm{sech}^2(\frac{x}{\Lambda})e^{-(\frac{p}{\sigma})^2}$$

where

$$\sigma^2 = \frac{4m^2E}{3M} \quad \text{and} \quad \Lambda = \frac{4E}{3gM^2}$$

A simple diagnostic of macroscopic evolution

M. Joyce and T. Worrakitpoonpon

Relaxation to thermal equilibrium in the self-gravitating sheet model, J. Stat. Mech., P10012 (2010)

To monitor macroscopic evolution useful to consider e.g.

$$\phi_{11} = \frac{\langle |x||v| \rangle}{\langle |x| \rangle \langle |v| \rangle} - 1 = \frac{\int |x||v| f\, dx dv}{\int |x| f\, dx dv \int |v| f\, dx dv} - 1$$

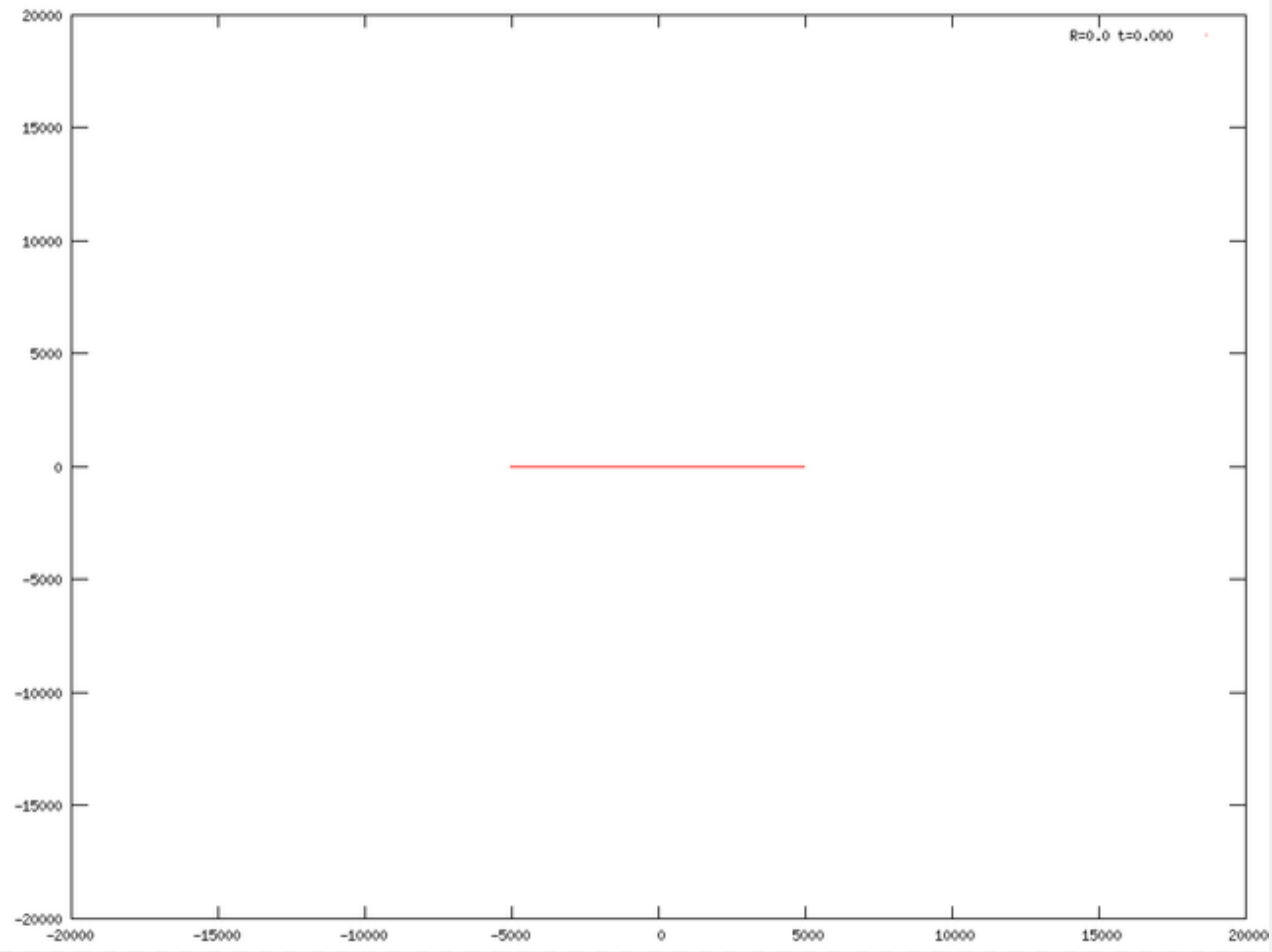
→ Measure of "phase space entanglement":

It is

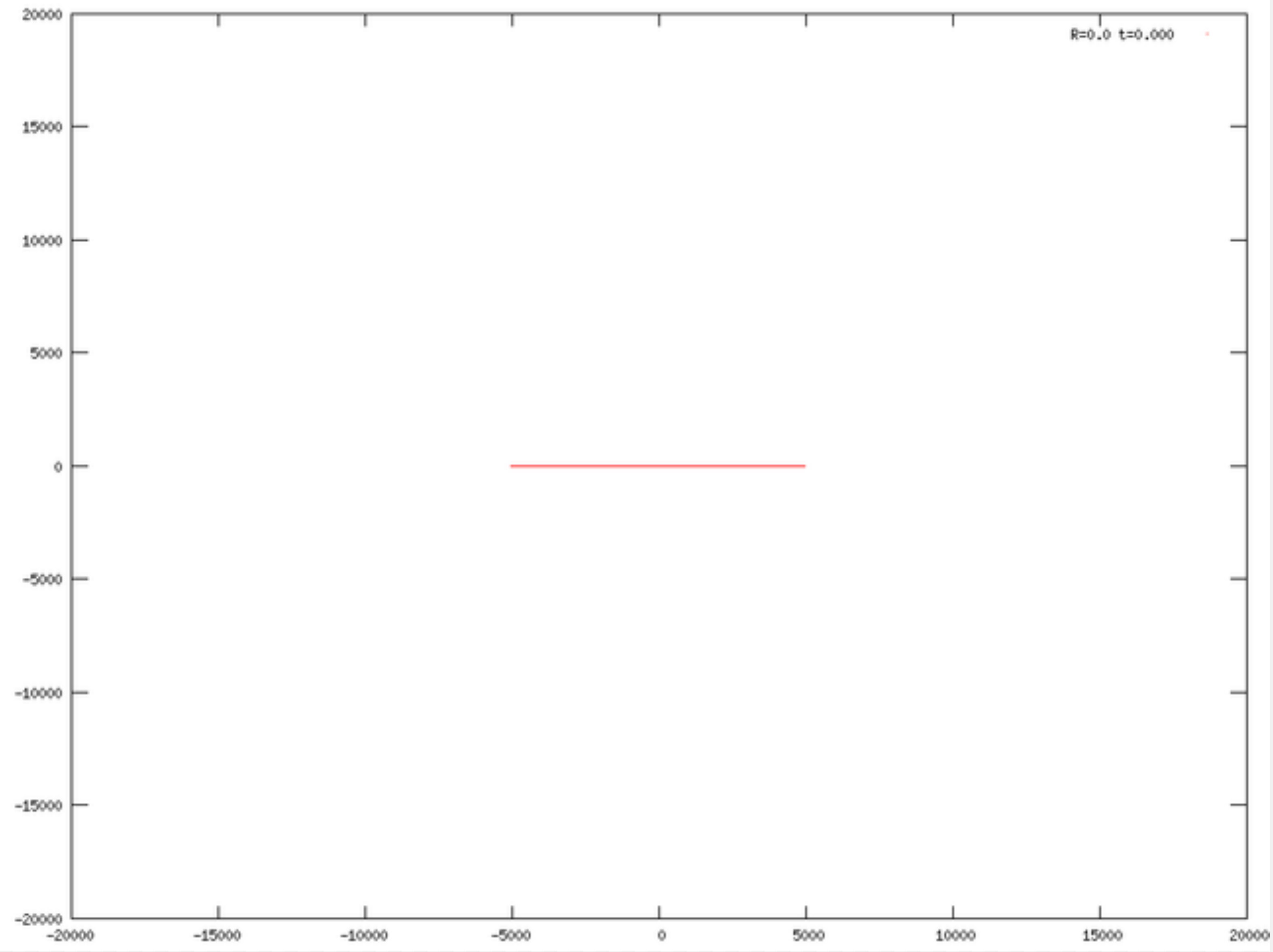
•**zero** in thermal equilibrium

•**non-zero and constant** in QSS

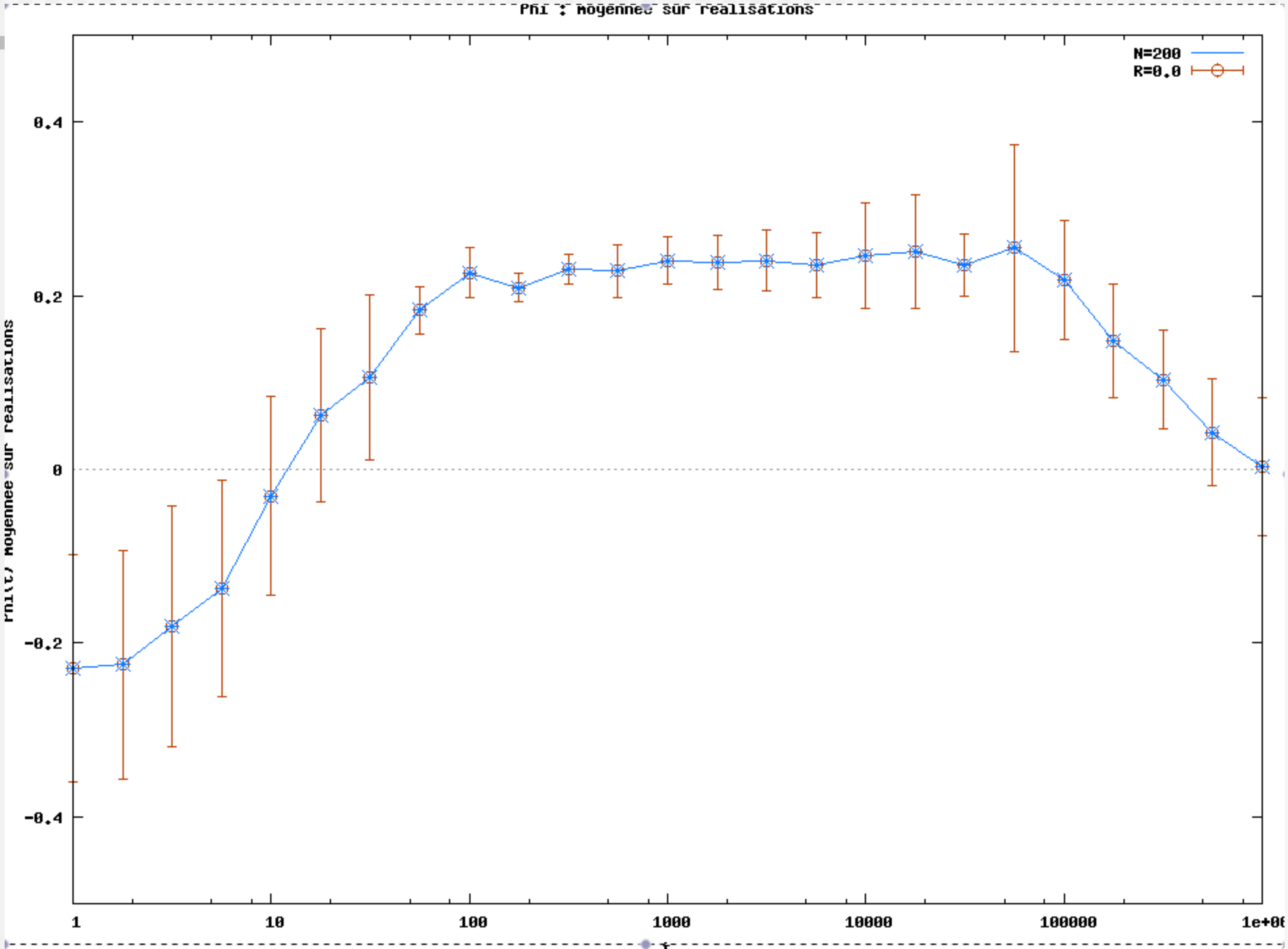
Cold collapse and virialization in 1D



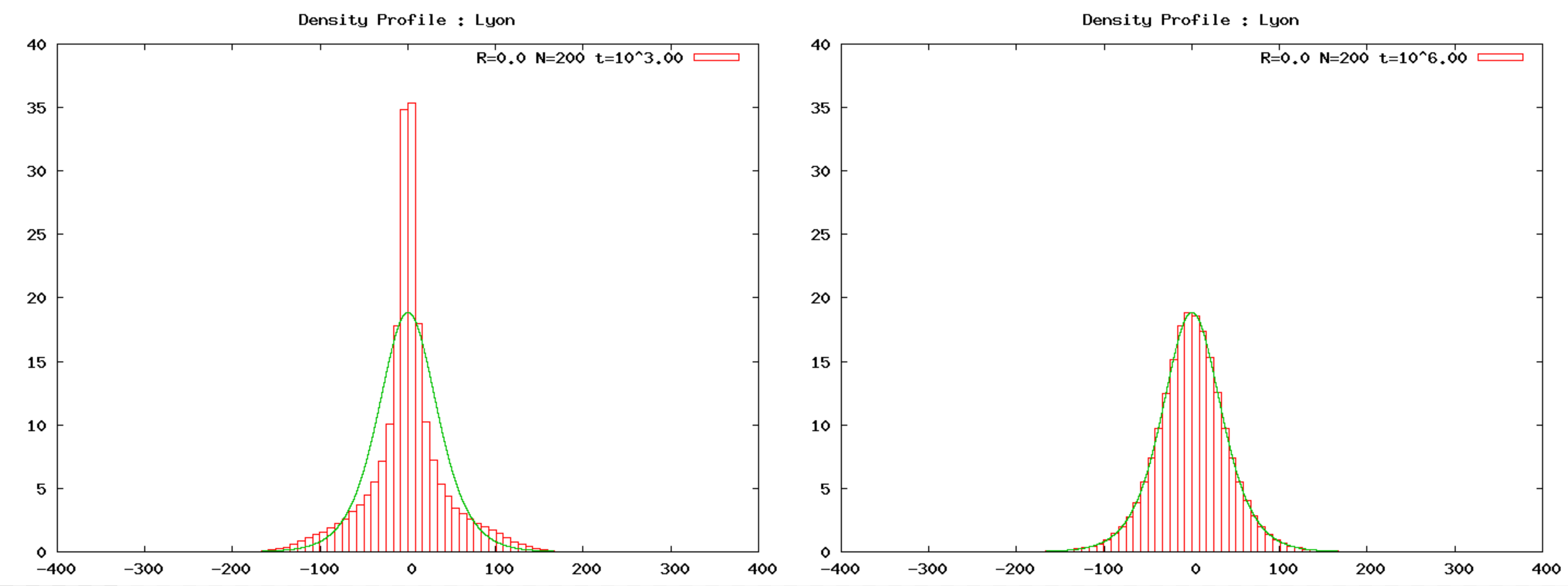
Cold collapse and virialization in 1D



Evolution of a 1D self- gravitating system

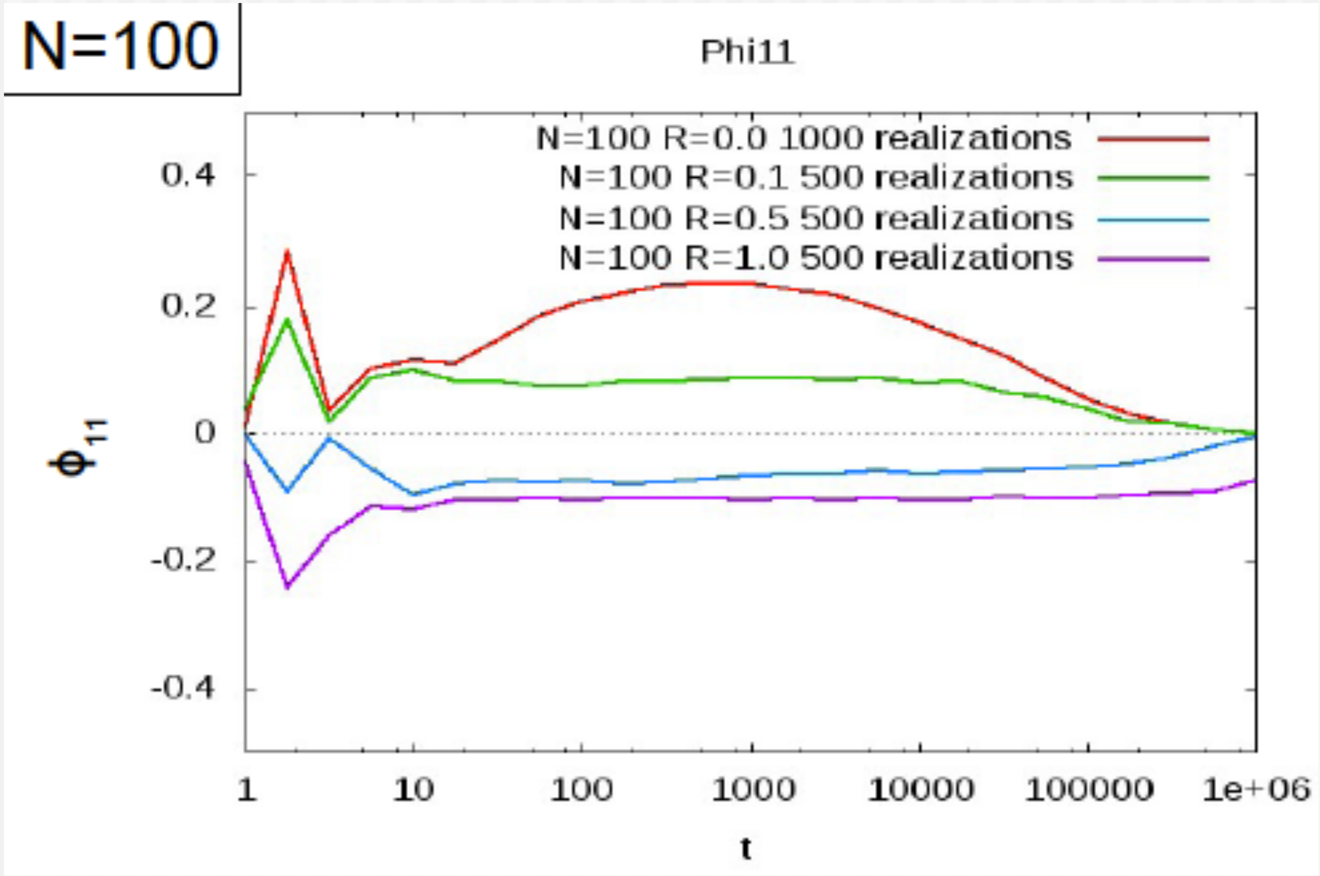


Evolution of density profiles



Green curve: Thermal equilibrium (Rybicki 1971)

Evolution of 1D gravitating systems : different IC



$T_{rel} \sim (10^2\text{-}10^3)\ N\ T_{mf}$

1D models of cosmological structure formation

1D gravity: infinite system limit ?

Just as in 3D the sum

$$\ddot{x}_i = -g \sum_{j \neq i} \mathrm{sgn}(x_i - x_j)$$

is not defined for an infinite uniform distribution..

→ Proceed as in 3D??

Gravitational dynamics in a 1D “expanding” universe

1D gravity **does not** have expanding universe solutions analagous to 3D!

-> Work directly with comoving coordinates,
Just replace **by hand** 3D forces with 1D forces

$$\frac{d^2\vec{x}_i}{dt^2} + 2H\frac{d\vec{x}_i}{dt} = -\frac{1}{a^3}\Big[\lim_{\mu\to 0}\sum_{j\neq i}\frac{Gm(\vec{x}_i-\vec{x}_j)}{|\vec{x}_i-\vec{x}_j|^3}e^{-\mu|\vec{x}_i-\vec{x}_j|}\Big]$$

by

$$\frac{d^2x_i}{dt^2} + 2H\frac{dx_i}{dt} = -\frac{1}{a^3}\Big[\lim_{\mu\to 0}\sum_{j\neq i} g\,\mathrm{sgn}(x_i-x_j)e^{-\mu|x_i-x_j|}\Big]$$

where a(t) is the **3D expansion** (but particle motion in 1D!)

Regulated 1D gravitational force

The expression

$$F_i^{REG} = -\lim_{\mu \to 0} \sum_{j \neq i} g \,\mathrm{sgn}(x_i - x_j) e^{-\mu |x_i - x_j|}$$

can be calculated exactly in an **infinite periodic system**

In 1D position of a particle i at any time can be written as its displacement u_i from a nearby lattice site (without overlapping).
The force is

$$F_i^{REG} = 2gn_0[u_i - \langle u \rangle]$$

where <u> is average value of u_i

1D gravity in an expanding universe: damped inverted oscillators

Like in 3D one can change to time variable to obtain equations as

$$\frac{d^2 u_i}{d\tau^2} + \Gamma \frac{du_i}{d\tau} = 2gn_0 u_i$$

where «damping » is

$$\Gamma = \sqrt{gn_0/3}$$

→ **D**ynamics of an infinite set of **damped** inverted harmonic oscillators displaced off a regular lattice (and which bounce elastically when they collide)

Motion is exactly integrable between crossings → similar exact event driven methods as for finite system

1D gravity in an expanding universe: a family of models

It is natural to consider

$$\frac{d^2 u_i}{d\tau^2} + \Gamma \frac{du_i}{d\tau} = 2 g n_0 u_i$$

where «damping » is a free parameter.

$$\Gamma = \kappa \sqrt{g n_0 / 3}$$

This corresponds to taking a 3D expansion law derived from

$$\frac{\ddot{a}}{a} = -\kappa^2 \frac{4\pi G}{3a^3} \rho_0$$

i.e. « speeding up » expansion by a factor of κ^2

Cosmology in a 1D universe: an "historical note"

In **cosmological literature**: studies of model $\kappa = 1$ by Melott (1982) PRL
Yano & Gouda (1998) Astrophys. J. Supp.

In **Stat. Phys. literature**

- **"RF" Model,** corresponding to $\kappa = \sqrt{3}$ introduced by Rouet et al. (1990) , studied extensively by Miller et al. (e.g. PRE 2002, 2007)
  These authors also studied **"static" model** with $\kappa = 0$

- **"Quintic" Model** corresponding to $\kappa = 1$ introduced by
  Aurell & Fanelli (2002) Astron. Astrophys.

Exhaustive study of family of models (range of κ) by
Sicard & Joyce, MNRAS (2010), Benhaeim et al, MNRAS (2012)

Recent work: VP simulations of RF and Q model by Manfredi et al. 2015((PRE)

Linear theory in 1D models

Results for linear theory in 3D carry over to 1D model

Prior to crossing, growing mode of particle motion is exactly that of fluid element in Zeldovich approximation:

$$u_i \propto e^{2\alpha\Gamma\tau} = a^{\alpha} \quad \text{where} \quad \alpha = \frac{1}{4}[-1 + \sqrt{1 + \frac{24}{\kappa^2}}]$$

→ Scale independent amplification of density fluctuations

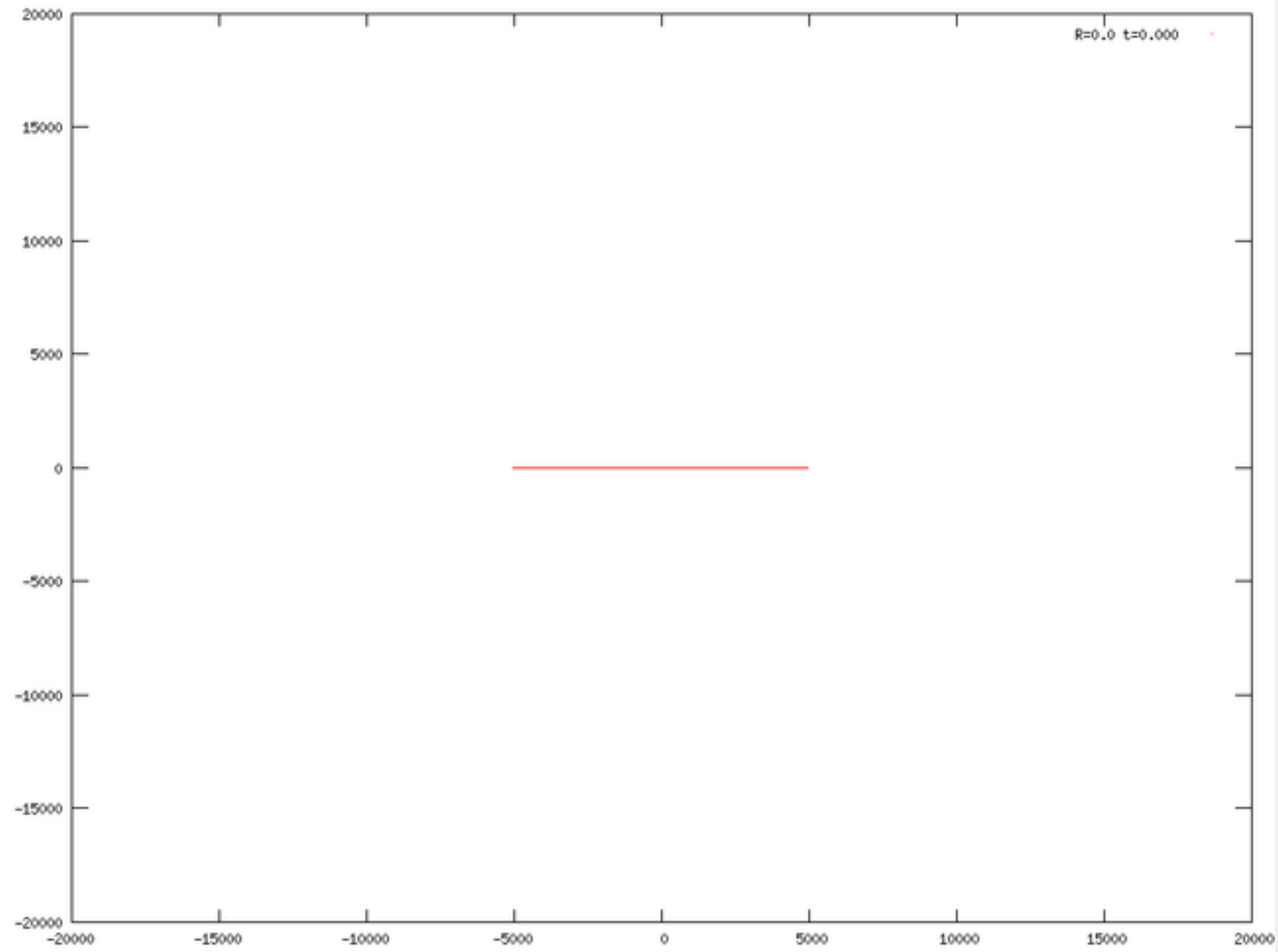
“Spherical collapse” model in 1D

Results analagous to 3D:

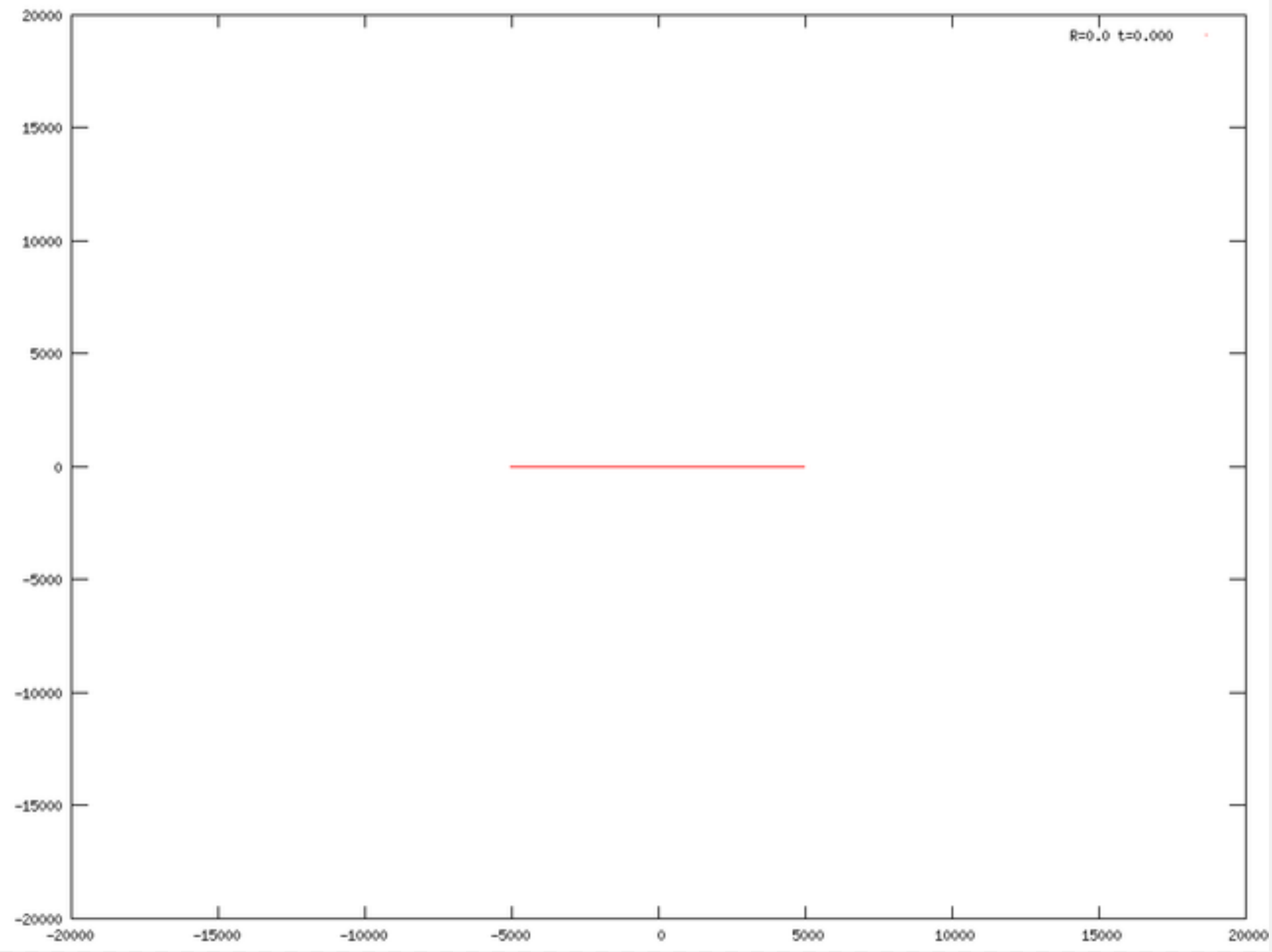
Collapse to singularity in finite time, independent of size

In presence of initial fluctuations singularity is regularized, system virializes (albeit somewhat less efficiently than in 3D)

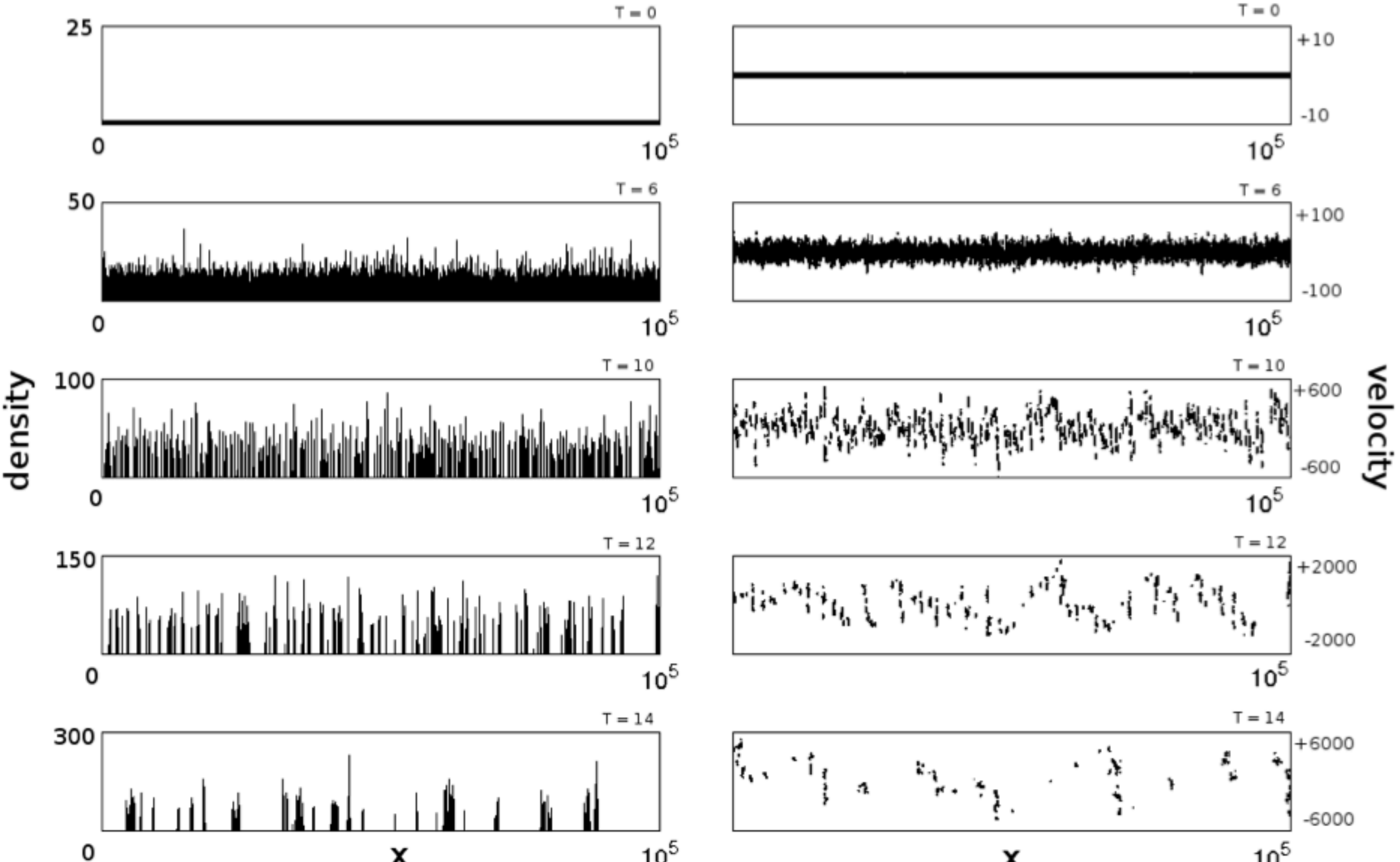
Cold collapse and virialization in 1D



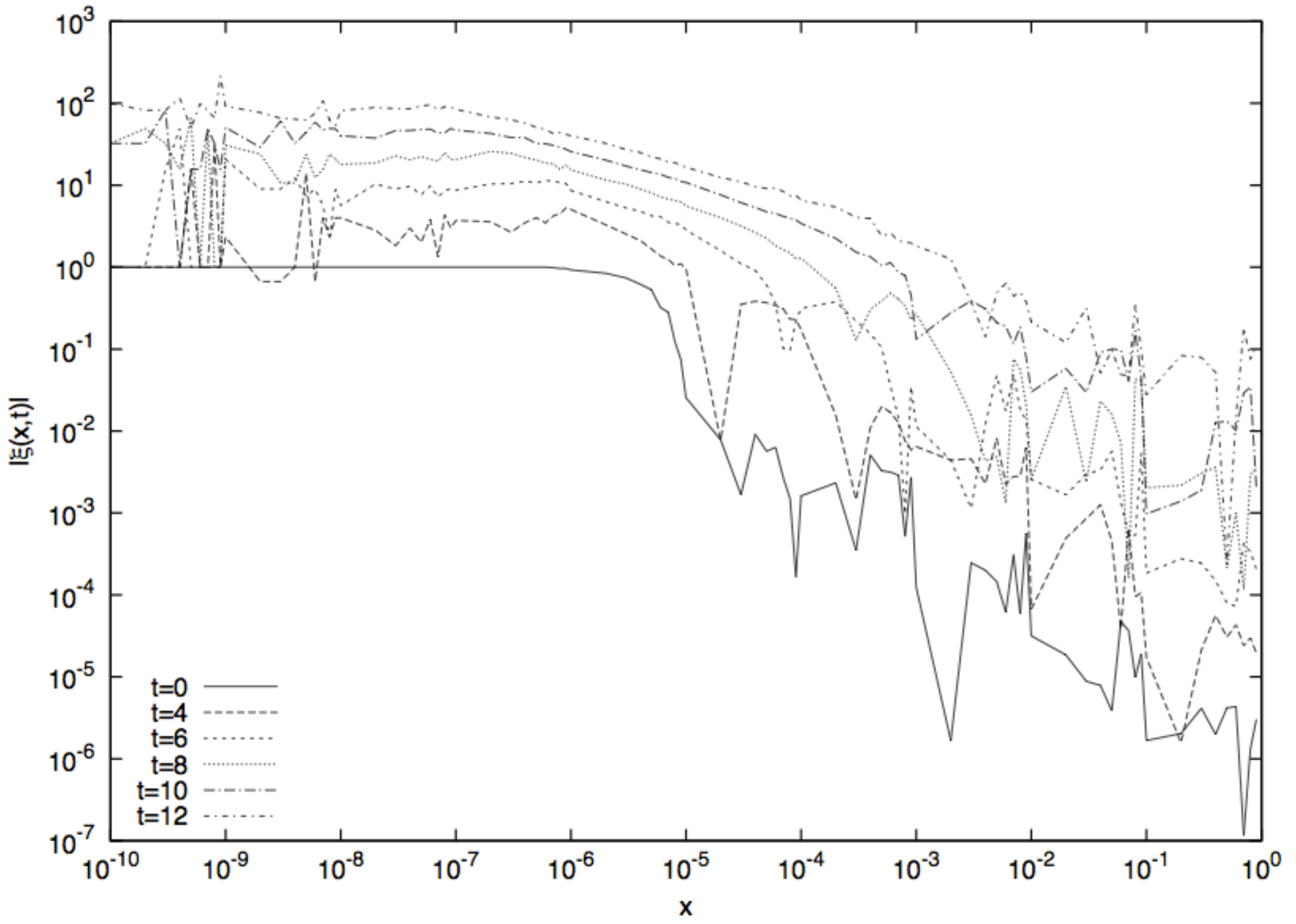
Cold collapse and virialization in 1D



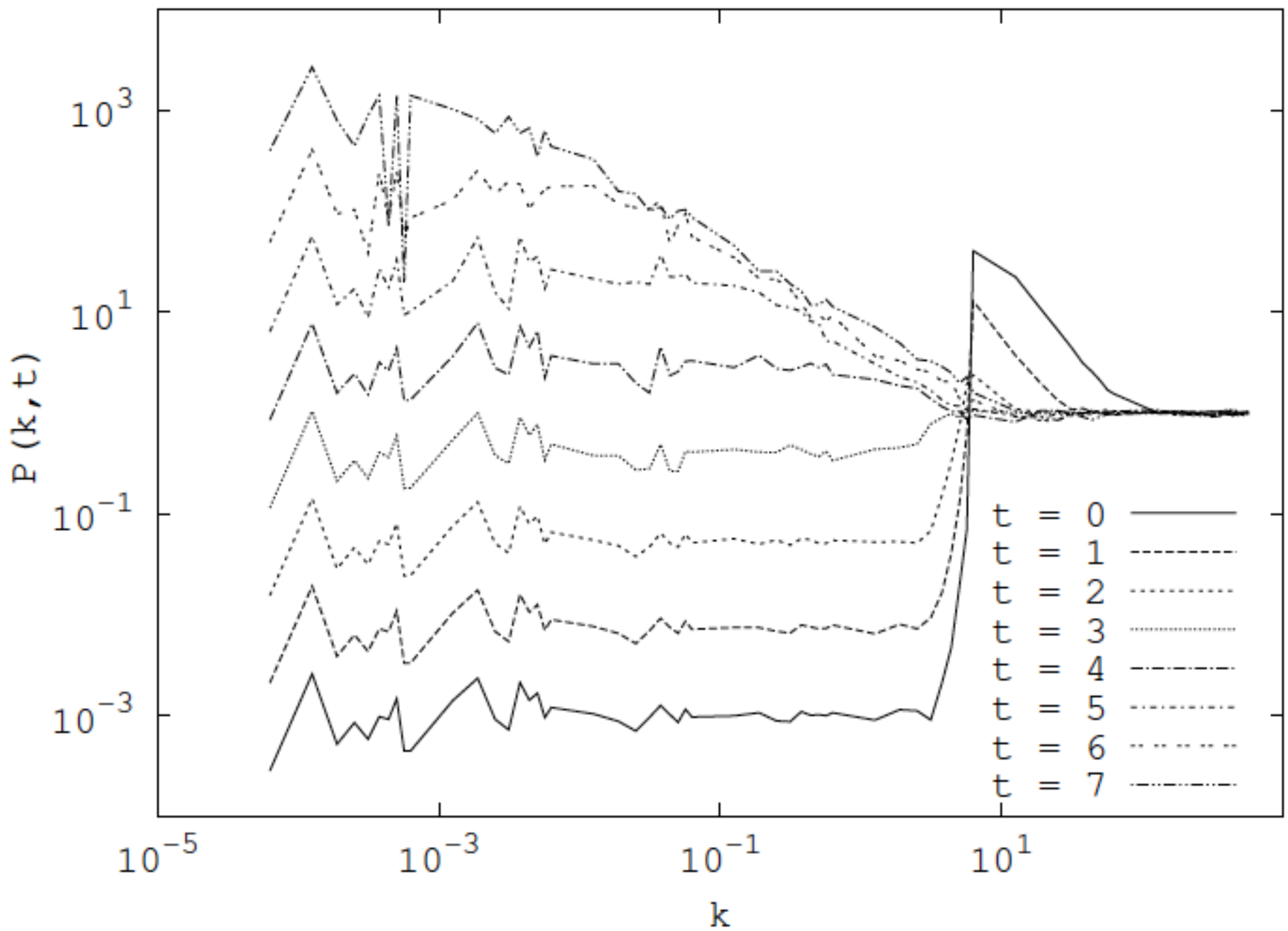
Results: clustering in a 1D universe



Evolution of correlation function (1D)
(n=0, κ=1)



Evolution of power spectrum (n=0, κ=1)



1D clustering from cold initial conditions:

Quantitative analyse reveals **behaviour completely analagous to 3D**

→ Hierarchical clustering (linear amplification + collapse)

Growth of non-linearity scale driven by linear amplification

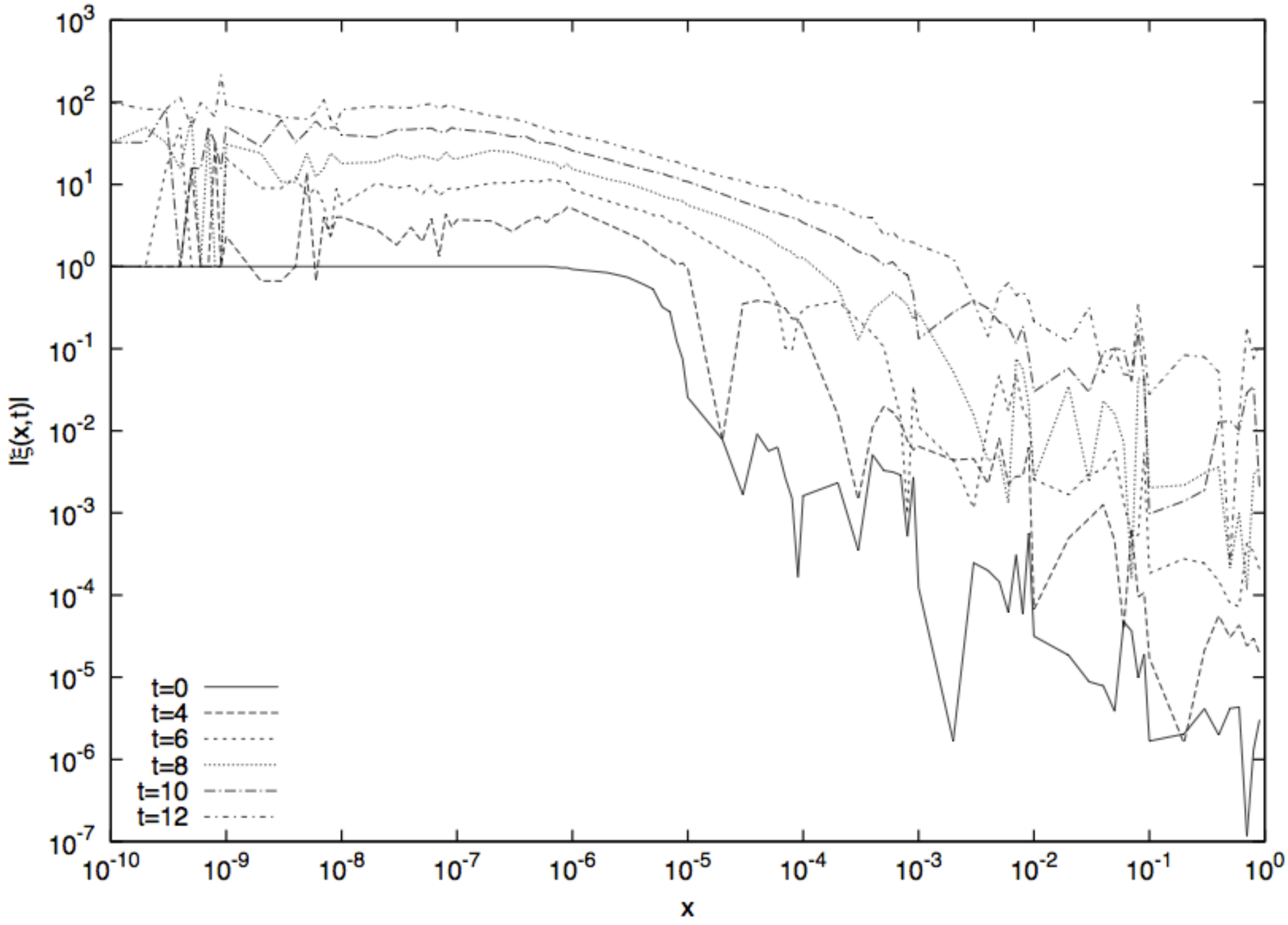
Prediction for scale-free 1D models

For PS $P(k)=Ak^n$ **self-similarity** (same assumptions as in 3D)

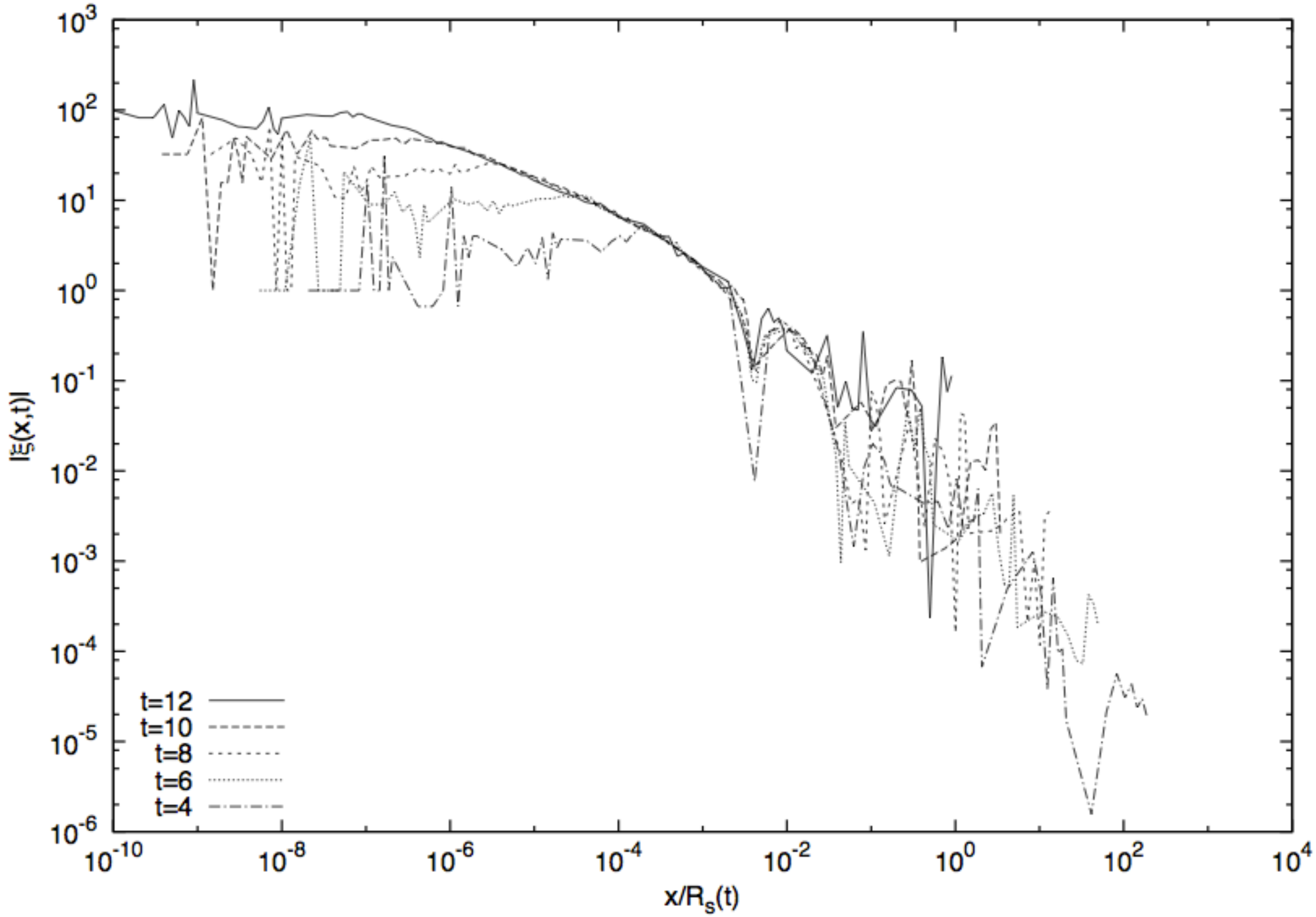
$$\xi(x,a) = \xi_0\left(\frac{x}{R_s(a)}\right)$$

$$R_s = e^{\frac{4\alpha}{1+n}\Gamma\tau} = a^{\frac{2\alpha}{1+n}}$$

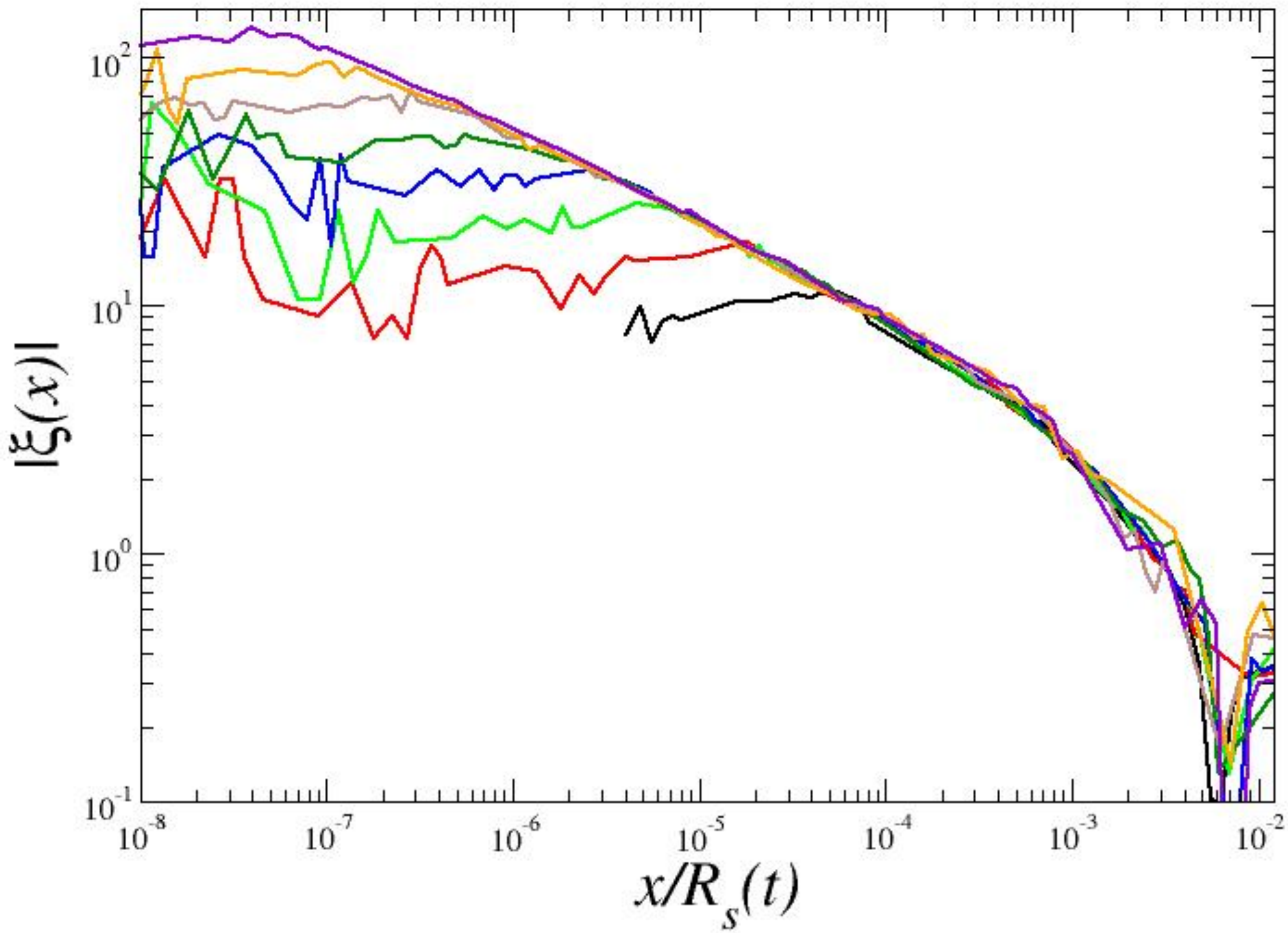
Evolution of correlation function (1D)



Self-similarity (1D): correlation function

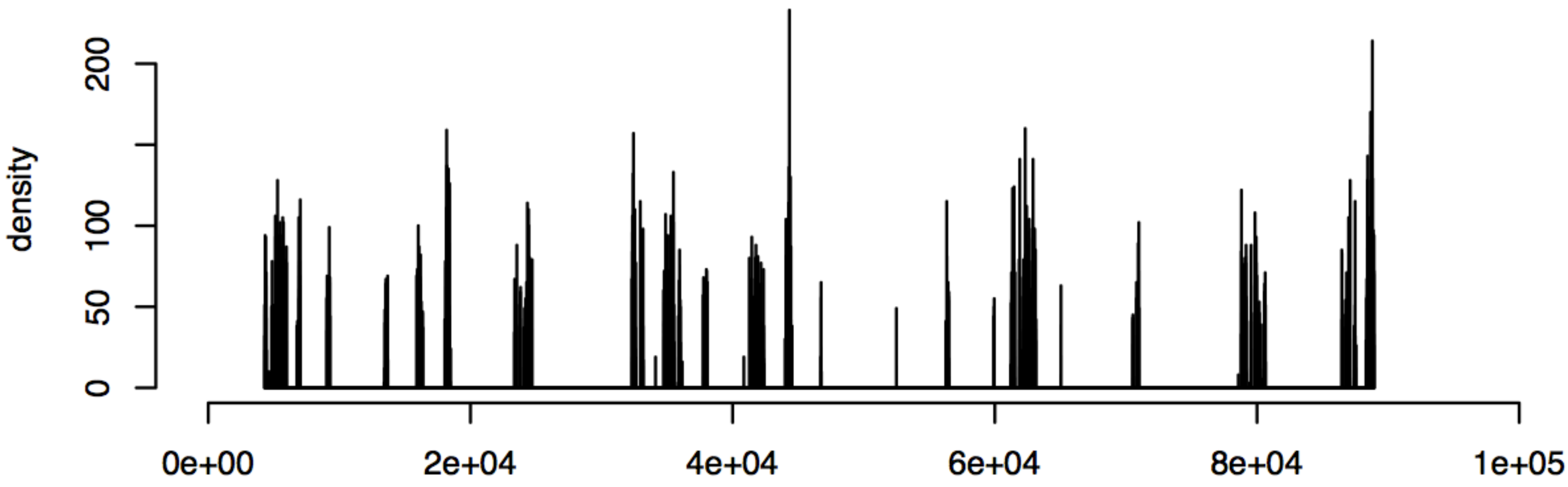


Self-similarity (1D)



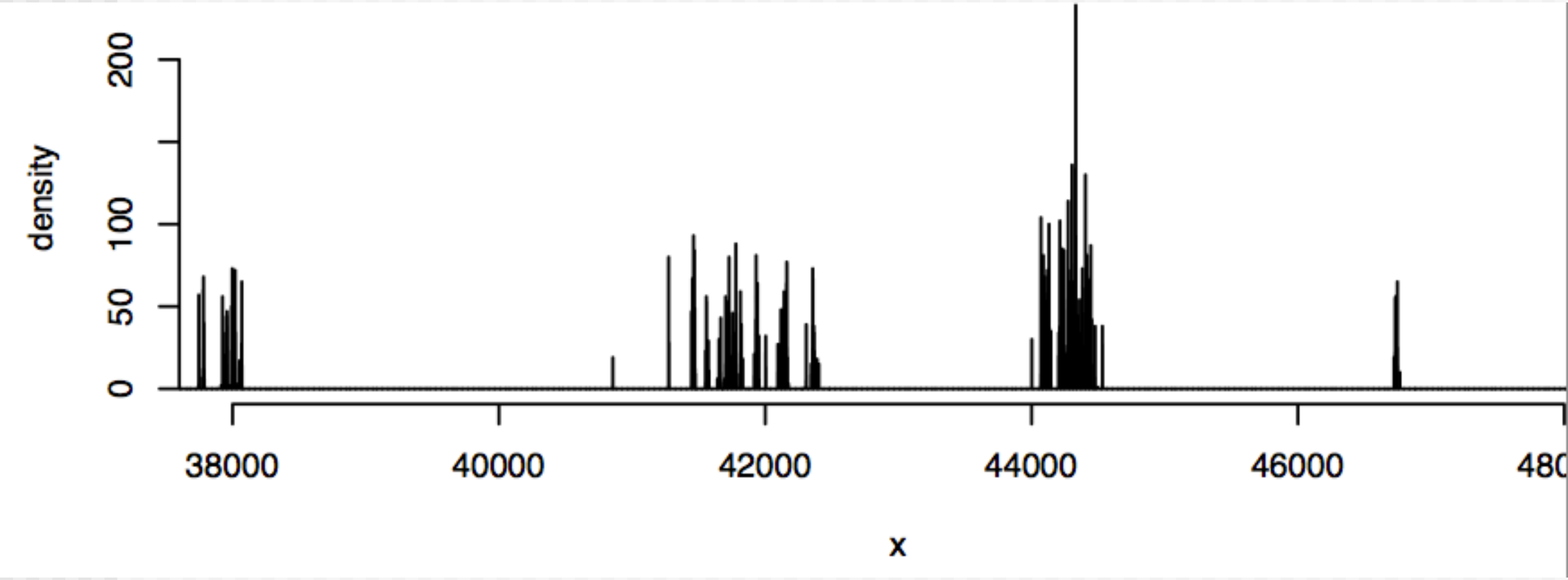
Non-linear clustering in 1D

Whole system (N=10^5 particles)



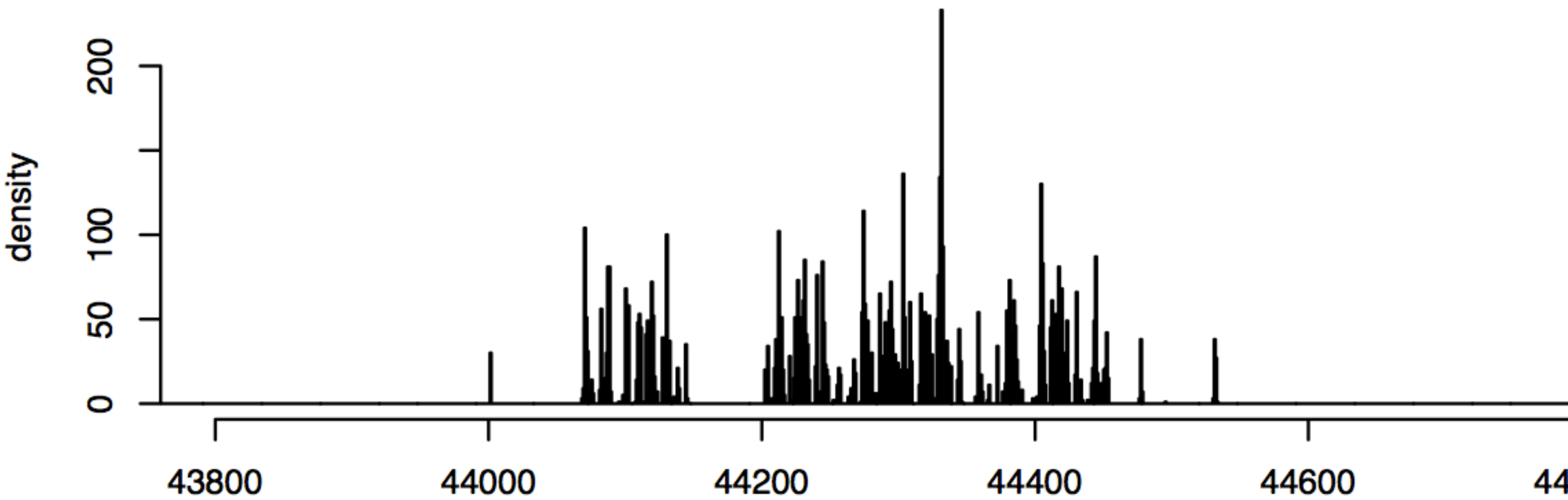
Non-linear clustering in 1D

1/10 th of system



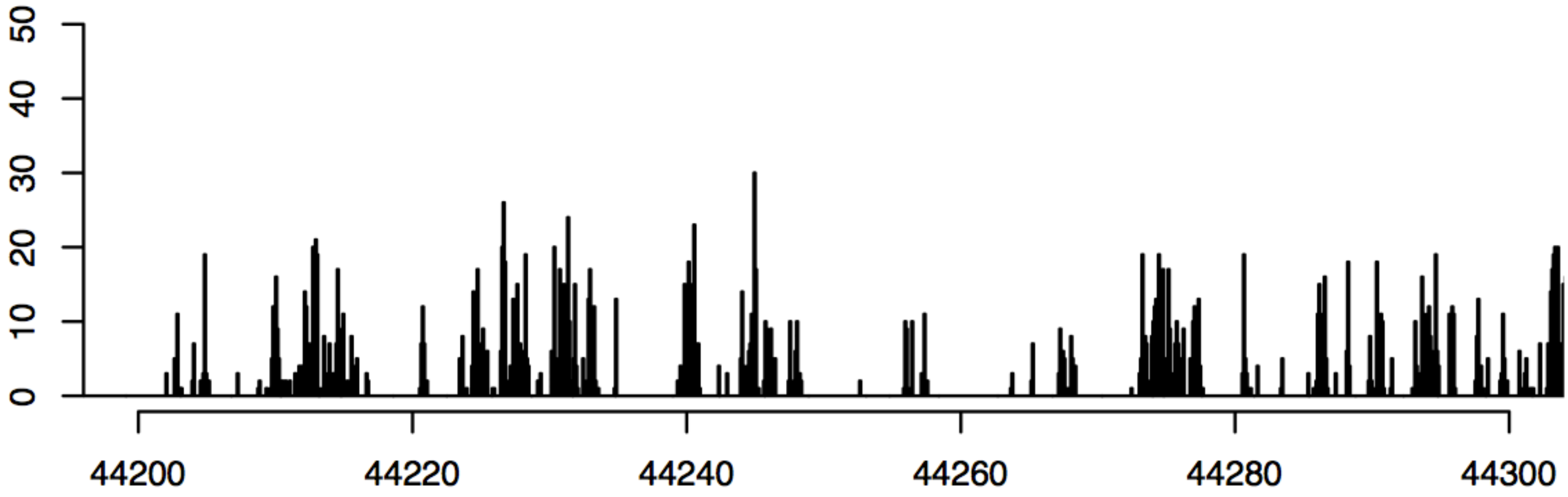
Non-linear clustering in 1D

1/10^2 of system



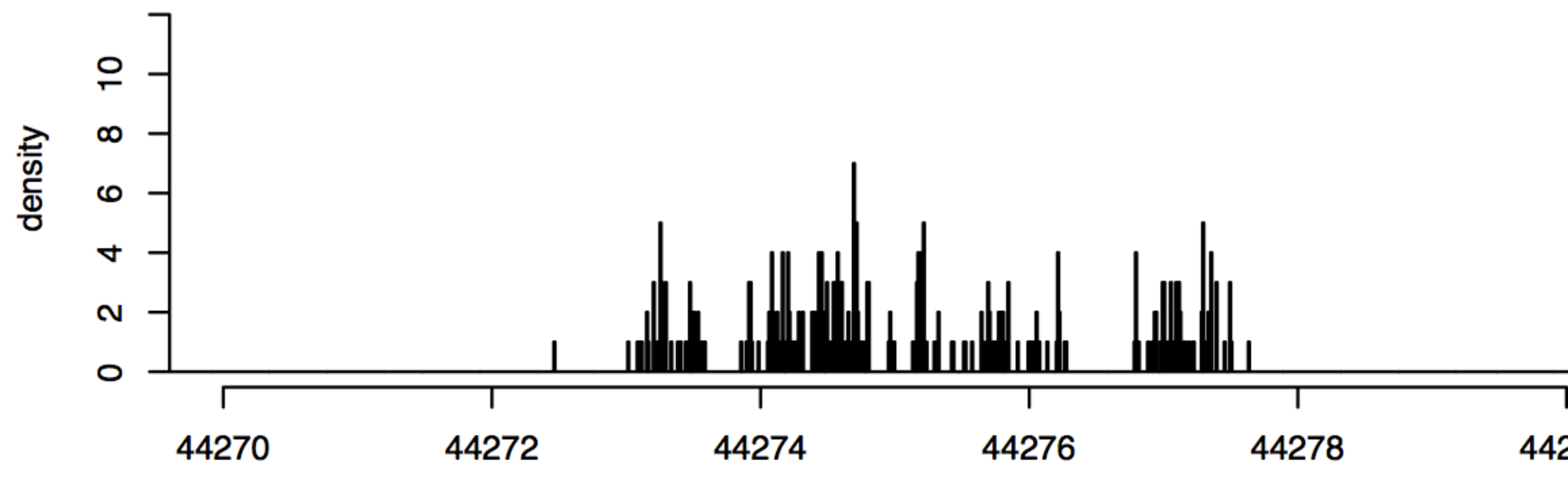
Non-linear clustering in 1D

1/10^3 of system



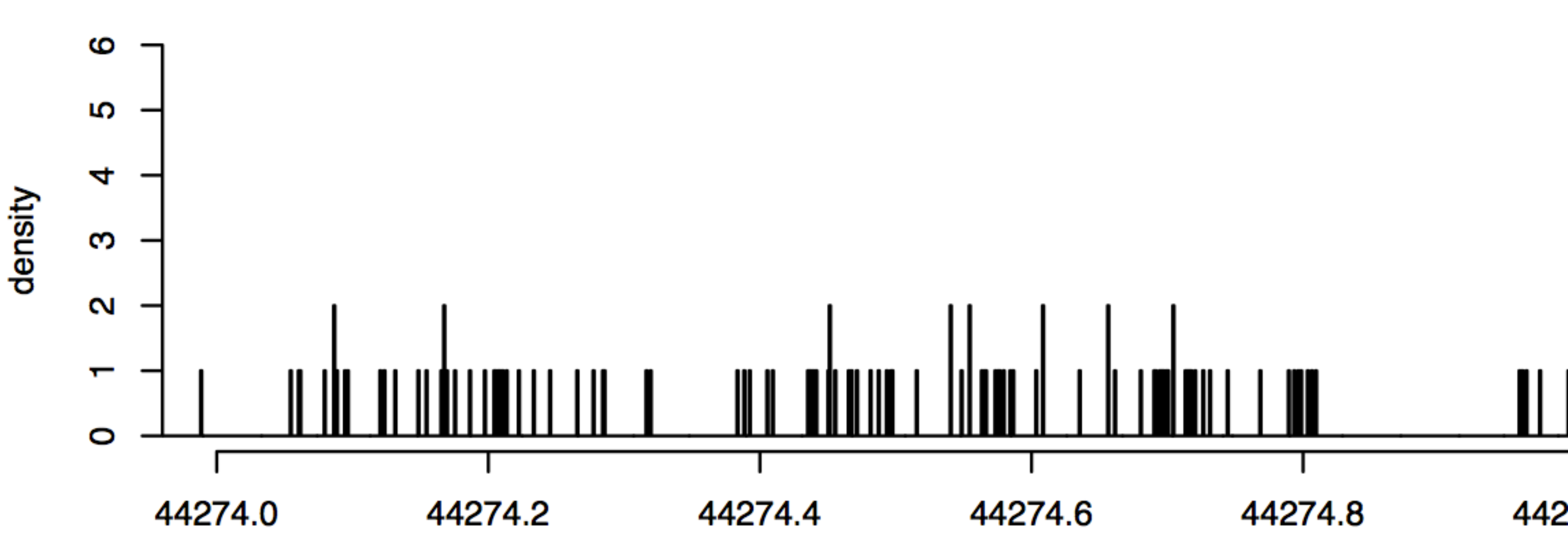
Non-linear clustering in 1D

1/10^4 of system



Non-linear clustering in 1D

1/10^5 of system



Results: scale-invariance in 1D?

(MJ, F. Sicard MNRAS 2011)

Power law behaviour in spatial correlations, over 3-4 orders of magnitudes in expanding models

Appears to extend over an arbitrarily large range of scale, asymptotically apparently without limit..

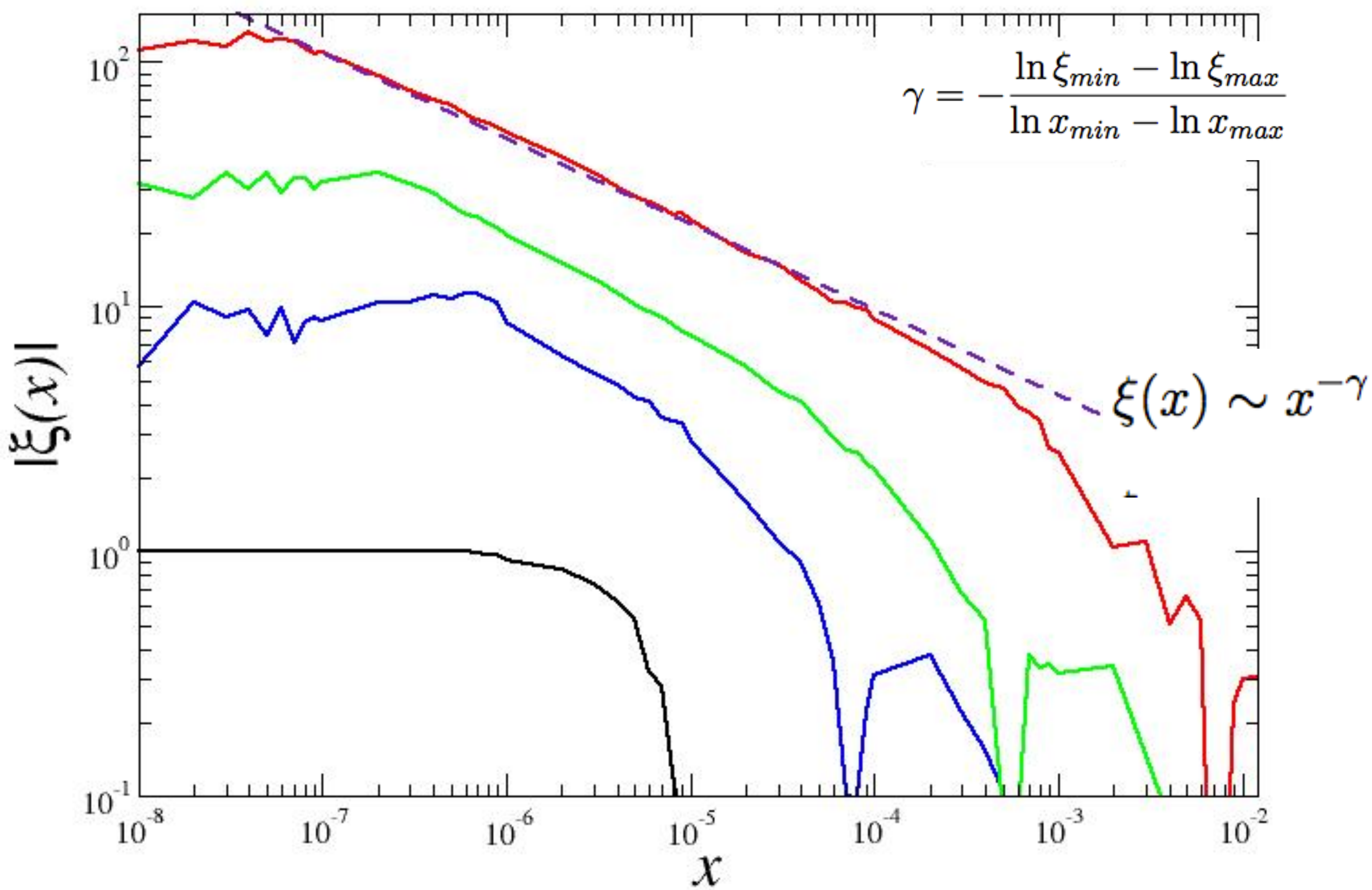
Is it associated with an underlying scale invariance?

Study (multi-)fractal exponents using standard box-counting technique

Confirm findings of [**Miller et al., Phy. Rev. E. (2007)** and refs therein]

strong evidence for fractal structure/scale-invariance

Determination of correlation dimension (1D)



Origin of the exponents?

(MJ, F. Sicard MNRAS 2012)

(
Measured exponents clearly depend both on IC (n) and "cosmology"(ϰ)

"Stable clustering hypothesis" (Peebles 1974 for 3D)

But what does this hypothesis mean in the 1D model?
What is "stability"?

“Physical coordinates” in 1D

Unlike 3D we **do not** derive equations from physical coordinates…nevertheless there are (almost) equivalent coordinates:

For particles in a **finite subsystem S** the eom can be written

$$\frac{d^2 y_i}{d\tau^2} + \Gamma \frac{dy_i}{d\tau} = -g \sum_{j\neq i, j\in S} \mathrm{sgn}(y_i - y_j) + 2gn_0 y_i$$

where y_i= position *relative to CM of S* (Note: no tidal forces!)
Taking

$$\tilde{t} = \frac{3}{\Gamma} e^{\Gamma\tau/3}, \quad r_i = e^{2\Gamma\tau/3} y_i$$

gives

$$\frac{d^2 r_i}{d\tilde{t}^2} = -g \sum_{j\neq i, j\in S} \mathrm{sgn}(r_i - r_j) + \frac{2}{\tilde{t}^2}[1 + \frac{9gn_0}{\Gamma^2}] r_i \,.$$

Second term on right becomes negligible at long times → r_i “physical coordinate“

Correlation dimension in the "stable clustering" hypothesis

MJ, F. Sicard MNRAS 2011

Assume strongly non-linear structures behave as isolated virialized objects

→ Clustering frozen in "physical coordinates"
→ Temporal evolution of **lower cut-off** to power-law

Using "self-similarity" to determine behaviour of **upper cut-off,**
Predict

$$\xi(x) \propto x^{-\gamma_{sc}}$$

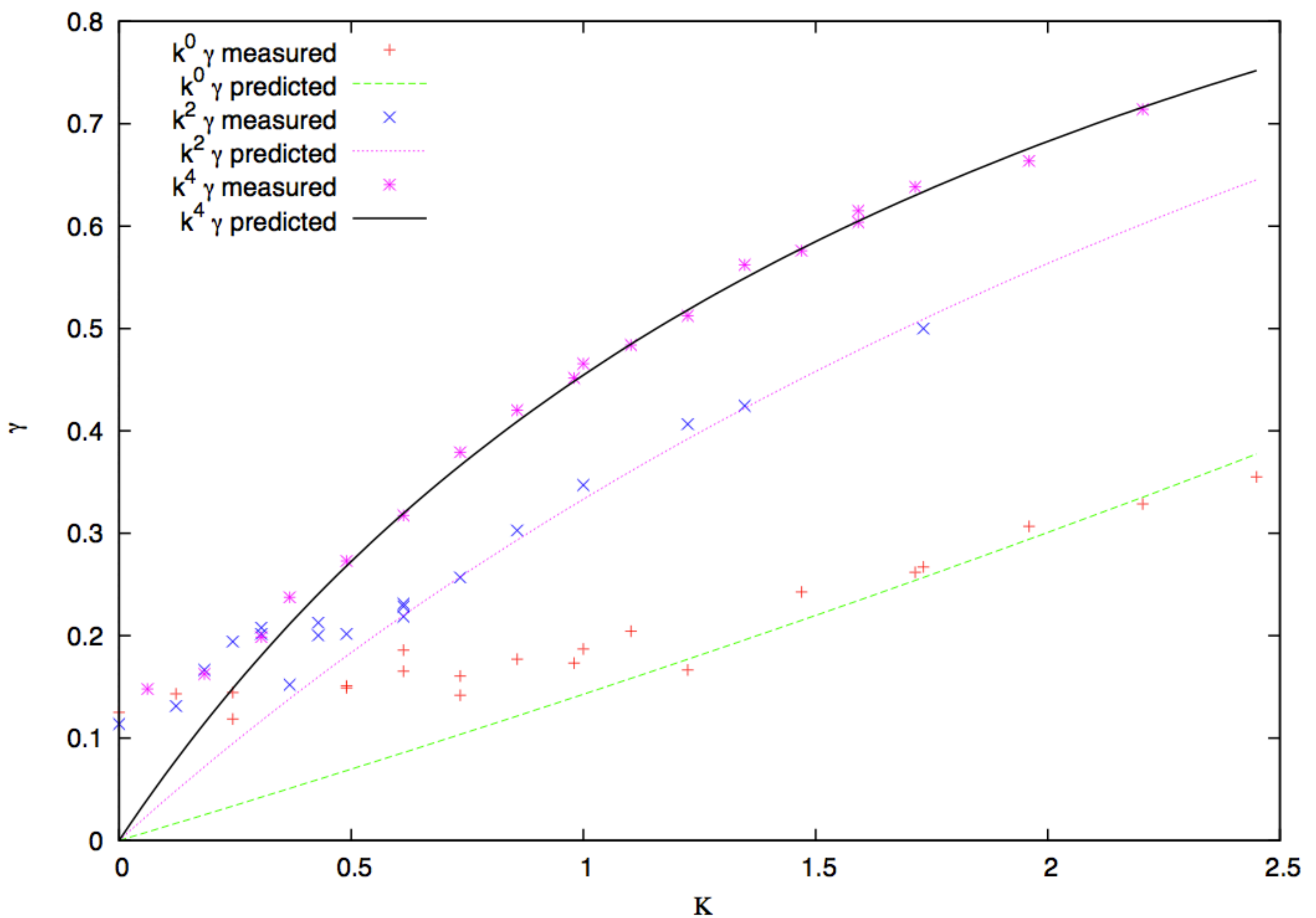
$$\gamma_{sc}(n,\kappa) = \frac{2\kappa(n+1)}{\kappa(2n-1)+3\sqrt{\kappa^2+24}}$$

where

$$\Gamma = \kappa\sqrt{2\pi G\rho_0/3}$$

Exponents of non-linear clustering in 1D models: measurement from simulations

D. Benhaiem and MJ (2012)



Exponents in 1D models: from stable clustering to universality

D. Benhaiem and MJ (2013)

Excellent agreement with stable clustering when $\gamma_{sc}(n, \kappa) \gtrsim 0.2$

Otherwise exponent which is ~ independent of both expansion and IC

→ "universal" non-linear clustering

Why a critical value for validity of stable clustering?

Can show that

$$\left(\frac{L_2}{L_1}\right) = \left(\frac{L_2^0}{L_1^0}\right)^{-\frac{\gamma_{sc}}{1-\gamma_{sc}}} \left(\frac{L_2^0}{L_1^0}\right)$$

where $\left(\frac{L_2}{L_1}\right)$ is ratio of size of two structures when the larger one virializes,

while $\left(\frac{L_2^0}{L_1^0}\right)$ is the ratio of their initial sizes

Thus large exponent → expect substructure to persist (because highly bound)

Open questions about the "non-linear regime"

- **How is non-linear clustering properly characterized ?**
- **How does it depend on initial conditions and cosmology?**

1D suggests the space of cold IC and cosmologies breaks into two regions:

- fractal "virialized hierarchy", non-universal
- fractal "virialized hierarchy" (**or smooth, not so clear..**), universal

Back to cosmology in 3D…

Power-law scaling in galaxy clustering

Observations:

Power law behaviours do characterize galaxy correlations **in some range**

Standard model: power law correlations are an accident….
(cf. Masjedi et al, Astrophy. J. 2008)

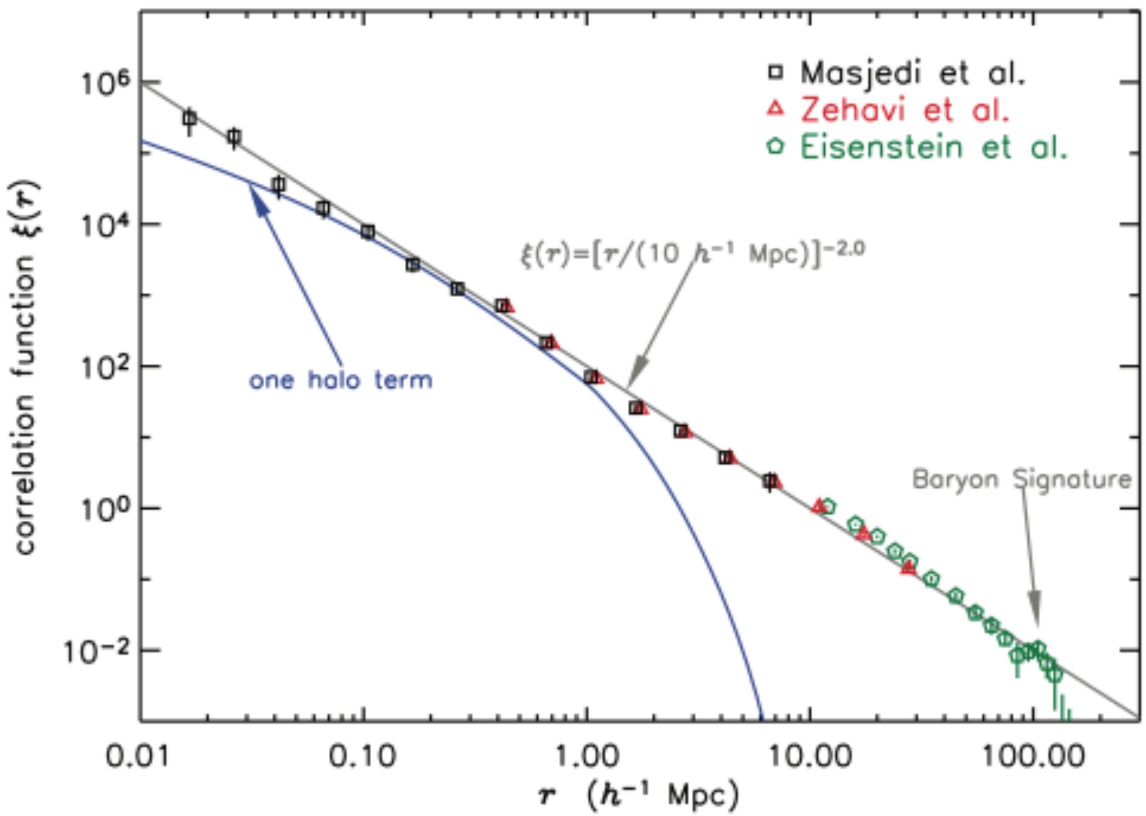


FIG. 4.—Real-space correlation function $\xi(r)$ for the LRG sample ($-23.2 < M_g < -21.2$ and $0.16 < z < 0.36$) calculated as described in the text on small scales, combined with real-space correlation function on intermediate scales

Power-law scaling in galaxy clustering

Observations:

Power law behaviours characterize galaxy correlations **in some range**

Is such power law clustering in galaxies indicative of scale-invariant phenomena?

If yes, is the purely gravitational dynamics giving rise to it?

Current standard model answer**: no, these power-laws are an "accidental"**

Or perhaps resolution of 3D simulations too poor to resolve it?

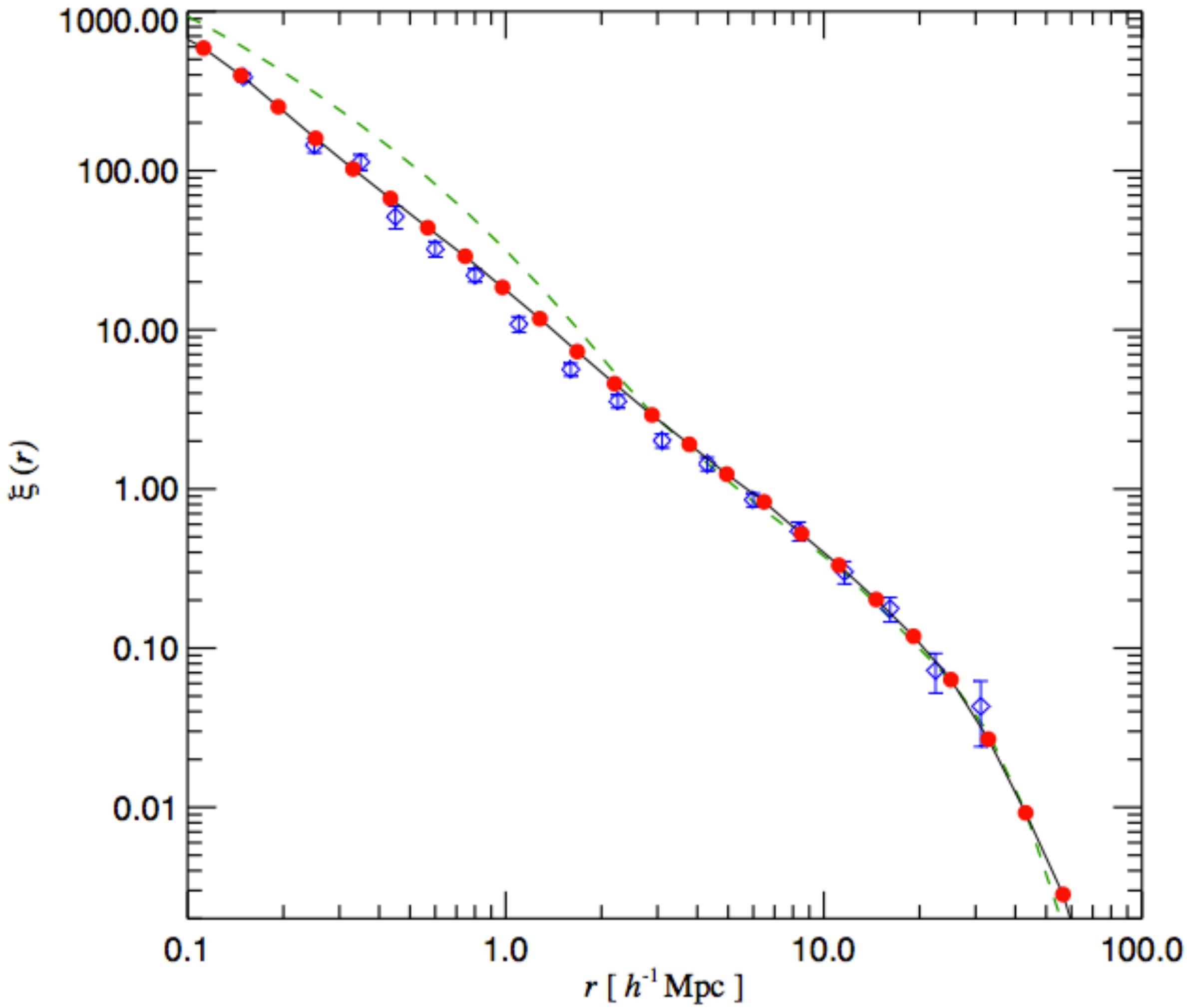


Figure 4: Galaxy 2-point correlation function at the present epoch. Red symbols (with vanishingly small Poisson error-bars) show measurements for model galaxies brighter than $M_K = -23$. Data for the large spectroscopic redshift survey 2dFGRS[28] are shown as blue diamonds. The SDSS[34] and APM[31] surveys give similar results. Both, for the observational data and for the simulated galaxies, the correlation function is very close to a power-law for $r \leq 20\,h^{-1}$Mpc. By contrast the correlation function for the dark matter (dashed line) deviates strongly from a power-law.

(V. Springel et al., Nature 2005)

Stable clustering/resolution in 3D revisited

(D. Benhaiem, MJ and B. Marcos, 2013 +work in progress)

Study of "Gamma cosmology" in 3D..

$$\frac{d^2\mathbf{x}_i}{d\tau^2} + \Gamma\frac{d\mathbf{x}_i}{d\tau} = \mathbf{F}_i^{\mathrm{REG}}$$

$$\Gamma = \kappa\sqrt{2\pi G\rho_0/3}$$

→

Generalisation of stable clustering prediction of Peebles:

$$\gamma_{sc}\left(n,\kappa\right) = \frac{6(3+n)}{5+\sqrt{1+\frac{24}{\kappa^2}+2n}}$$

Stable clustering/resolution in 3D revisited

(D. Benhaiem, MJ and B. Marcos, 2013 +work in progress)

Results: stable clustering prediction works very well in range of scale we can resolve..

Recent work (D. Benhaiem & MJ, 2016), larger simulations:

Breakdown of stable clustering correlated to breakdown of self-similarity

→
Non-linear regime dominated by interaction and merging of structures is strongly affected by UV (i.e. discreteness) effects!

Suggests that a large part of N-body simulation 3D results may be incorrect…

Perhaps VP simulations may help to resolve the issue definitively..?
[See Yoshikawa K. et al., MNRAS (2013), Colombi et al., MNRAS (2015)]

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F. Sicard (University College, London)
T. Worrakitpoonpon (Nonthaburi, Thailand)