Anisotropic Long Range Spin Systems

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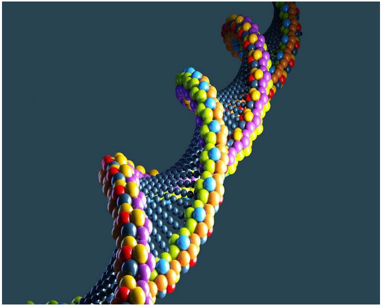
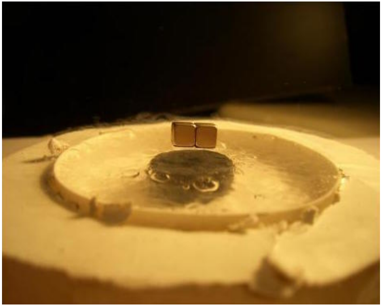
Overview

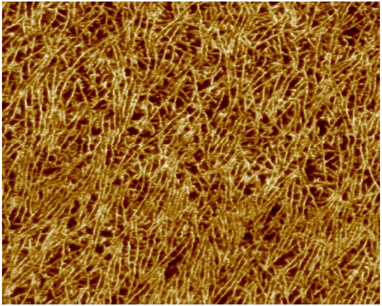
1. Long Range Interactions
   - Spin systems
2. LR Spin Systems
   - Traditional results
   - Controversy
   - Effective Dimension
3. The Anisotropic case
   - Dimensional Analysis
   - Three regimes
   - Effective dimension
   - Critical exponents

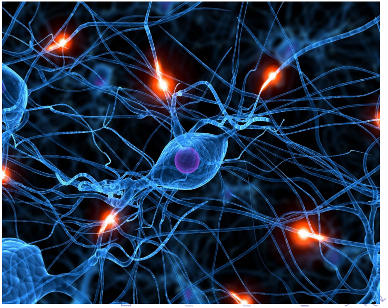
Long range interacting systems.





$$\frac{1}{r^{d+\sigma}}$$





Spin Systems.

Why spin systems

- spin systems are the testbed of statistical mechanics.
- Various Monte Carlo (MC) and perturbative results available.
- Diverse interesting physical problems in a single formalism.

Issues:

- Phase diagram for diverse interaction shapes.
- Description of different symmetry groups.
- Description of high order critical points.

Spin Systems

Lattice Hamiltonian

$$H = -\frac{J}{2}\sum_{ij}\frac{1}{|i-j|^{d+\sigma}}\boldsymbol{S}_i\boldsymbol{S}_j$$

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Mean Field Propagator

$$G(q)^{-1} = J(q) = \int d^d x\, J(i-j)e^{iq\cdot(i-j)}$$

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Leading momentum term

$$\lim_{q\to 0} G^{-1}(q) \propto q^{\sigma} \quad \text{if} \quad \sigma \leq 2$$

$$\lim_{q\to 0} G^{-1}(q) \propto q^{2} \quad \text{if} \quad \sigma > 2$$

Long range interactions in d dimensions

Traditional Results

Three regimes:

- $\mathbf{0} < \sigma < \mathbf{d/2}$ **Mean field exponents** ($\eta = 2 - \sigma$ and $\nu = \sigma^{-1}$).
- $\mathbf{d/2} < \sigma < \mathbf{2}$ **Long range exponents** ($\eta \equiv \eta(\sigma)$ and $\nu \equiv \nu(\sigma)$).
- $\sigma > \mathbf{2}$ **Short range exponents** ($\eta = \eta_{SR}$ and $\nu = \nu_{SR}$).[a]

[a] M.E. Fisher et al. PRL 29,14

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- **$0 < \sigma <$ d/2 Mean field exponents** ($\eta = 2 - \sigma$ and $\nu = \sigma^{-1}$).
- **d/2 $< \sigma <$ 2 Long range exponents** ($\eta \equiv \eta(\sigma)$ and $\nu \equiv \nu(\sigma)$).
- **$\sigma >$ 2 Short range exponents** ($\eta = \eta_{SR}$ and $\nu = \nu_{SR}$).[a]

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Peculiar Long Range Behavior

Using ϵ-expansion technique with $\epsilon = 2\sigma - d$ or $1/N$ expansion is possible to calculate the critical exponent η.

$$\eta = 2 - \sigma + O(\epsilon^3)$$

Exact at any order in ϵ. $\eta = 2 - \sigma$ for all $\sigma < 2$. **Discontinuity** in $\sigma = 2$.

Removal of the discontinuity

Sak's Results

The anomalous dimension cannot be less than $\eta_{\rm SR}$,

$$\eta = 2 - \sigma \quad \sigma < \sigma^*$$
$$\eta = \eta_{\rm SR} \quad \sigma > \sigma^*$$

where $\sigma* = 2 - \eta_{\rm SR}$. **No discontinuity** is present.

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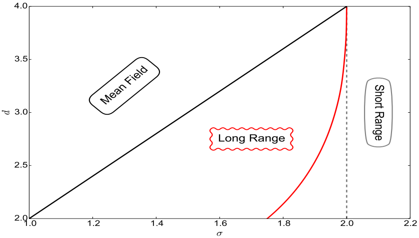
System regimes



Monte Carlo Results: Controversy

Luijte and Blote[a] results (2002) seemed to confirm Sak results, but new, more complete, results (2013)[b] question on Sak validity

[a] E. Luijte & H.W. Blote PRL 89, 025703
[b] M. Picco, arXiv:1207.1018

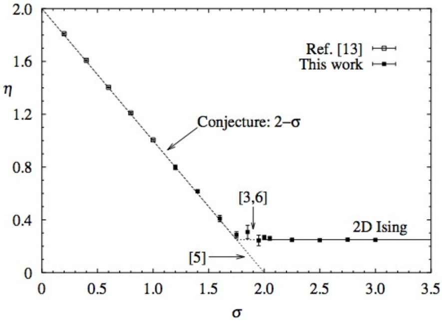


Figure: MC 2002

VS

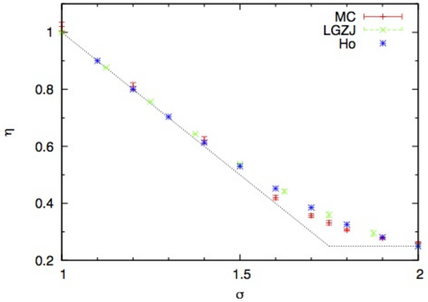


Figure: MC 2013

Effective Dimension

Ginzburg-Landau Free Energy

$$\Phi_{\mathrm{SR}} = \int d^{d_{\mathrm{SR}}}x \left\{-Z_k \psi \Delta \psi + \mu \psi^2 + g\psi^4\right\} + \cdots$$

$$\Phi_{\mathrm{LR}} = \int d^{d_{\mathrm{LR}}}x \left\{-Z_k \psi \Delta^{\frac{\sigma}{2}} \psi - Z_{2,k} \psi \Delta \psi + \mu \psi^2 + g\psi^4\right\} + \cdots$$

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Effective dimension results

$$Z_k = Z_{2,k} = 1 \rightarrow d_{\mathrm{SR}} = \frac{2d_{\mathrm{LR}}}{\sigma}$$

$$Z_{2,k} = 1 \rightarrow d_{\mathrm{LR}} = \frac{(2-\eta_{\mathrm{SR}})d_{\mathrm{LR}}}{\sigma}$$

Qualitative Description

I Approximation Level: No anomalous dimension

$\boldsymbol{d_{\mathrm{SR}} = \frac{2d_{\mathrm{LR}}}{\sigma}}$: Exact $N \to \infty$, Correct σ ranges, $\boldsymbol{\sigma}^* = 2$

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$\boldsymbol{d_{\mathrm{SR}} = \frac{(2-\eta_{\mathrm{SR}})d_{\mathrm{LR}}}{\sigma}}$: Exact $N \to \infty$, Correct σ ranges, $\boldsymbol{\sigma}^* = 2 - \eta_{\mathrm{SR}}$

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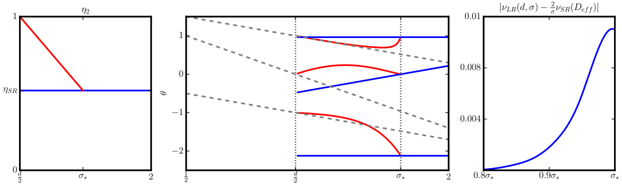
III Approximation Level: Mixed theory space

Competition between Short and Long range fixed points: ~~$\boldsymbol{d_{\mathrm{SR}}}$~~

Fixed Points Solutions and Stability

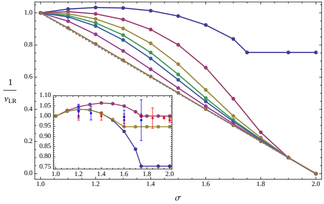


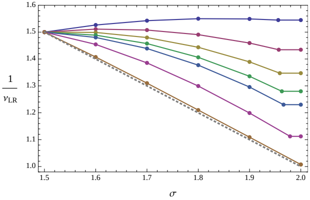
Quantitative Results

Short Range Corrections

Short Range corrections spoil dimensional equivalence. Small everywhere but at $\sigma \simeq \sigma_*$.

Correlation Length Exponent





Anisotropic $O(N)$ models.

Lattice Hamiltonian

$$H = -\sum_{i\neq j} \frac{J_{\parallel}}{2} \frac{\boldsymbol{S}_i \boldsymbol{S}_j}{r_{\parallel,ij}^{d_1+\sigma}} \delta(\boldsymbol{r}_{\perp,ij}) - \sum_{i\neq j} \frac{J_{\perp}}{2} \frac{\boldsymbol{S}_i \boldsymbol{S}_j}{r_{\perp,ij}^{d_2+\tau}} \delta(\boldsymbol{r}_{\parallel,ij}).$$

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Mean Field Propagator

$$\lim_{q \to 0} G(q)^{-1} = \lim_{q \to 0} J(q) = Z_{\parallel} q_{\parallel}^{\sigma} + Z_{\perp} q_{\perp}^{\tau} + \mu + O(q^2)$$

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Effective field theory

$$\int d^d x \left\{ -\frac{Z_{\parallel}}{2} \phi(x) \Delta^{\sigma/2} \phi(x) - \frac{Z_{\parallel}}{2} \phi(x) \Delta^{\tau/2} \phi(x) + \ldots + U(\phi(x)) \right\}$$

Physical Realizations.

Quantum Lattice Hamiltonian

$$H = -\frac{J}{2}\sum_{i\neq j}\frac{\boldsymbol{\sigma}_i^{(z)}\boldsymbol{\sigma}_j^{(z)}}{|i-j|^{d+\sigma}} - h\sum_i \sigma_i^{(x)} \, ,$$

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Mapping between Quantum LR and Anisotropic LR

$$\boldsymbol{\sigma}_i \to S_i$$
$$d_1 = d \qquad d_2 = z$$
$$\sigma = \sigma \qquad \tau = 2.$$

The Quantum LR Ising is obtained for $z = 1$

Critical Behavior

Asymptotic propagators

$$G(q_1, q_1) \approx q_1^{-\sigma+\eta_\sigma} G(1, q_2 q_1^{-\theta}) \approx q_2^{-\tau+\eta_\tau} G(q_1 q_2^{-\frac{1}{\theta}}, 1)$$

Correlation Lengths

$$\xi_\parallel \approx |T - T_c|^{-\nu_1} \qquad \xi_\perp \approx |T - T_c|^{-\nu_2},$$

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Anisotropy index

$$\frac{\sigma - \eta_\sigma}{\tau - \eta_\tau} = \frac{\nu_2}{\nu_1} = \theta.$$

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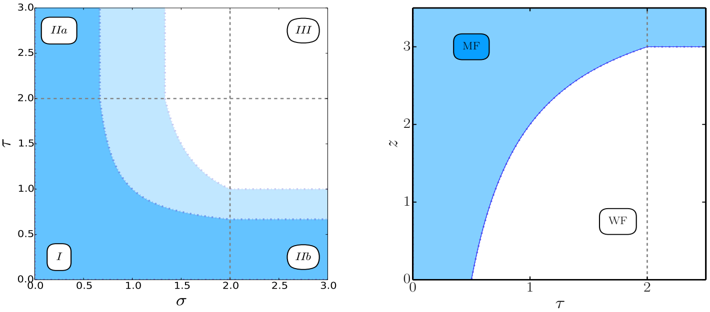
Mean field Results

$$\eta_\sigma=\eta_\tau=0,\quad \nu_1=\sigma^{-1},\quad \nu_2=\tau^{-1}.$$

Mean Field Regions



Three regimes

- Region *I*: Anisotropic pure LR system ($\sigma < \sigma_*$ and $\tau < \tau_*$).
- Region $II_{a\backslash b}$: Anisotropic S-LR system ($\sigma < \sigma_*$ and $\tau > \tau_*$).
- Region *III*: Isotropic SR system ($\sigma > \sigma_*$ and $\tau > \tau_*$).

Effective Dimension

In region I the system is equivalent to an isotropic LR one

$$D = d_1 + \theta d_2, \quad \theta = \frac{\sigma}{\tau}.$$

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Exact in the spherical limit

$$\nu_1 = \frac{\tau}{\tau d_1 + \sigma d_2 - \tau\sigma},$$
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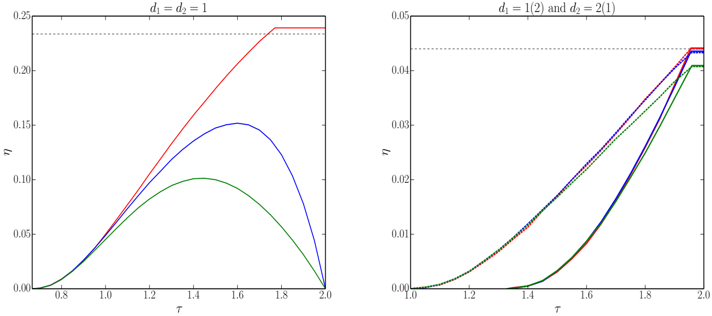
Independent confirmation: the spherical ANNNI model

$$\nu_1 = \frac{L}{(d_1 - 2)L + d_2},$$
$$\nu_2 = \frac{1}{(d_1 - 2)L + d_2}.$$

Anomalous dimension in the three regimes

- Region I: $\eta_1 = 2 - \sigma$ and $\eta_2 = 2 - \tau$ ($\eta_\sigma = \eta_\tau = 0$).
- Region $II_{a\backslash b}$: $\eta_1 = \eta(\tau)$ and $\eta_2 = 2 - \tau$
- Region III: $\eta_1 = \eta_2 = \eta_{SR}$

Anomalous dimension of the SR term in region II



Correlation length exponent

Region I: Anisotropic pure LR system.

$\boldsymbol{D_{eff}} = d_1 + \theta d_2$: Exact for $N \to \infty$. $\sigma^* = 2 - \eta(\tau)$ and $\tau^* = 2 - \eta(\sigma)$

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Region II: Anisotropic mixed S-LR region.

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Region III: Isotropic SR case.

Universality class of an isotropic SR system in dimension $d = d_1 + d_2$.

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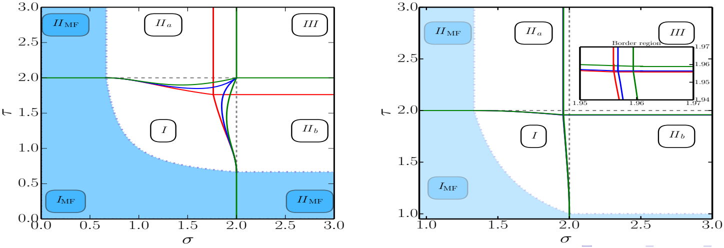

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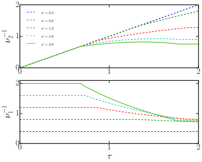
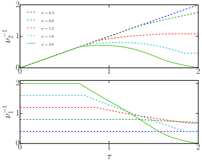
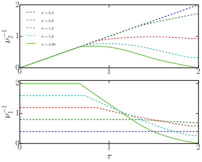


(c) $N = 1$



(d) $N = 2$



(e) $N = 3$

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Thank You