Topological Approach to Microcanonical Thermodynamics and Phase Transition of Interacting Classical Spins

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(Dated: April 19, 2016)

We propose a topological approach suitable to establish a connection between thermodynamics and topology in the microcanonical ensemble. Indeed, we report on results that point to the possibility of describing *interacting classical spin systems* in the thermodynamic limit, including the occurrence of a phase transition, using topology arguments only. Our approach relies on Morse theory, through the determination of the critical points of the potential energy, which is the proper Morse function. Our main finding is to show that, in the context of the studied classical models, the Euler characteristic $\chi(E)$ embeds the necessary features for a correct description of several magnetic thermodynamic quantities of the systems, such as the magnetization, correlation function, susceptibility, and critical temperature. Despite the classical nature of the studied models, such quantities are those that do not violate the laws of thermodynamics [with the proviso that Van der Waals loop states are mean field (MF) artifacts]. We also discuss the subtle connection between our approach using the Euler entropy, defined by the logarithm of the modulus of $\chi(E)$ per site, and that using the *Boltzmann* microcanonical entropy. The approaches based on entropies associated with the total number of microscopic states, or the total number of critical points of the potential energy, are also considered in our analysis. Moreover, the results suggest that the loss of regularity in the Morse function is associated with the occurrence of unstable and metastable thermodynamic solutions in the MF case. The reliability of our approach is tested in two exactly soluble systems: the infinite-range and the short-range XY models in the presence of a magnetic field. Further studies are very desirable in order to clarify the extension of the validity of our proposal.

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