Lectures on the Cosmic Microwave Background

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Lecture III

- Primary Anisotropies (continued)
- Beyond primary anisotropies
- Measurement of angular power spectrum
- Parameter constraints

Equations of motion

To derive the equations of motion for photons, recall our toy model of perturbations in a thermal gas of free massless particles in flat space.

The phase space density of the gas satisfies the collisionless Boltzmann equation

$$\frac{\partial n}{\partial t} = -\hat{p} \cdot \nabla n$$

If we have a detector that registers particles of all energies, it is natural to define

$$\Delta_T(\vec{x}, \hat{p}) = \frac{1}{\bar{\bar{I}}} \int \frac{p^3 dp}{(2\pi)^3} \delta n(\vec{x}, p\, \hat{p})$$

Equations of motion

Toy example:

It satisfies

$$\frac{\partial \Delta_T(\vec{x},\hat{p},t)}{\partial t} + \hat{p}\cdot\nabla\Delta_T(\vec{x},\hat{p},t) = 0$$

Translational invariance makes it convenient to look for solutions

$$\Delta_T(\vec{x},\hat{p},t) = \int \frac{d^3q}{(2\pi)^3}\alpha(\vec{q})\Delta_T(q,\mu,t)e^{i\vec{q}\cdot\vec{x}}$$

$$\hat{q}\cdot\hat{p}$$

$$\frac{\partial \Delta_T(q,\mu,t)}{\partial t} + iq\mu\Delta_T(q,\mu,t) = 0$$

Equations of motion

Toy example:

If we are interested in multipole coefficients $a_{T,\ell\, m}$ and angular power spectra, it is convenient to decompose them as

$$\Delta_T(q,\mu,t) = \sum_\ell (-i)^\ell (2\ell+1) P_\ell(\mu) \Delta_{T,\ell}(q,t)$$

$$a_{T,\ell\, m} = \pi i^\ell \int \frac{d^3q}{(2\pi)^3} \alpha(\vec{q}) Y^*_{\ell\, m}(\hat{q}) \Delta_{T,\ell}(q,t_0)$$

Equations of motion

Toy example:

Then

$$C_{TT\,\ell} = \pi^2 \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \left|\Delta_{T,\ell}(q,t_0)\right|^2$$

Where $\Delta_{T,\ell}(q,t)$ satisfy

$$\dot{\Delta}_{T,\ell}(q,t) + \frac{q}{2\ell+1}\left[(\ell+1)\Delta_{T,\ell+1}(q,t) - \ell\Delta_{T,\ell-1}(q,t)\right] = 0$$

Analogous equations can be derived for the polarization anisotropy.

Equations of motion

Beyond the toy example

For interacting particles one finds

$$\frac{\partial \Delta_T(q,\mu,t)}{\partial t} + iq\mu\Delta_T(q,\mu,t) = -\omega\Delta_T(q,\mu,t) + \omega F\left[\Delta_{T,0}(q,t),\Delta_{T,2}(q,t),t\right]$$

with formal solution

$$\Delta_T(q,\mu,t) = \Delta_T(q,\mu,t_i)e^{-iq\mu(t-t_i)}e^{-\omega(t-t_i)}$$
$$+\omega\int_{t_i}^{t} dt e^{-iq\mu(t-t')}e^{-\omega(t-t')}F[\Delta_{T,0}(q,t),\Delta_{T,2}(q,t),t]$$

Since only low multipoles appear in the collision terms, one can solve a truncation of the hierarchy and obtain the higher multipoles through this "line-of-sight integration"

Equations of motion

Beyond the toy example

The same derivation generalizes to a general spacetime

In this case define the phase space density

$$n(x^i, p_i, t) \equiv \sum_r \delta(x^i - x_r^i(t))\delta(p_i - p_{i\,r}(t))$$

The definition of momentum and the geodesic equation imply

$$\frac{dx^i}{dt} = \frac{p^i}{p^0} \qquad\qquad \frac{dp_i}{dt} = \frac{p^k p^l}{2p^0}\frac{\partial g_{kl}}{\partial x^i}$$

and
$$\frac{\partial n}{\partial t} + \frac{p^k}{p^0}\frac{\partial n}{\partial x^k} + \frac{1}{2}\frac{p^k p^l}{p^0}\frac{\partial g^{kl}}{\partial x^m}\frac{\partial n}{\partial p_m} = C$$

Derivation of the Boltzmann hierarchy as before but more tedious.

Equations of motion

Photons

$$\dot{\Delta}^{(S)}_{T,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{T,\ell+1}(q,t) - \ell\Delta^{(S)}_{T,\ell-1}(q,t)\right]$$
$$= -\omega_c(t)\Delta^{(S)}_{T,\ell}(q,t) - 2\dot{A}_q\delta_{\ell,0} + 2q^2\dot{B}_q\left(\frac{1}{3}\delta_{\ell,0} - \frac{2}{15}\delta_{\ell,2}\right)$$
$$+\omega_c\Delta^{(S)}_{T,0}\delta_{\ell,0} + \frac{1}{10}\omega_c\Pi\delta_{\ell,2} - \frac{4}{3}\frac{q}{a}\omega_c\delta u_{b\,q}\delta_{\ell,1}$$

$$\dot{\Delta}^{(S)}_{P,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{P,\ell+1}(q,t) - \ell\Delta^{(S)}_{P,\ell-1}(q,t)\right]$$
$$= -\omega_c(t)\Delta^{(S)}_{P,\ell}(q,t) + \frac{1}{2}\omega_c(t)\Pi(q,t)\left(\delta_{\ell,0} + \frac{1}{5}\delta_{\ell,2}\right)$$

with source function

$$\Pi = \Delta^{(S)}_{P,0} + \Delta^{(S)}_{T,2} + \Delta^{(S)}_{P,2}$$

Equations of motion

Photons

$$\dot{\Delta}^{(S)}_{T,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{T,\ell+1}(q,t) - \ell\Delta^{(S)}_{T,\ell-1}(q,t)\right]$$
$$= -\omega_c(t)\Delta^{(S)}_{T,\ell}(q,t) - 2\dot{A}_q\delta_{\ell,0} + 2q^2\dot{B}_q\left(\frac{1}{3}\delta_{\ell,0} - \frac{2}{15}\delta_{\ell,2}\right)$$
$$+\omega_c\Delta^{(S)}_{T,0}\delta_{\ell,0} + \frac{1}{10}\omega_c\Pi\delta_{\ell,2} - \frac{4}{3}\frac{q}{a}\omega_c\delta u_{b\,q}\delta_{\ell,1}$$

$$\dot{\Delta}^{(S)}_{P,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{P,\ell+1}(q,t) - \ell\Delta^{(S)}_{P,\ell-1}(q,t)\right]$$
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with source function

$$\Pi = \Delta^{(S)}_{P,0} + \Delta^{(S)}_{T,2} + \Delta^{(S)}_{P,2}$$

Polarization sourced by temperature quadrupole

Equations of motion

Photons

The components of the stress tensor can be written as

$$
\begin{aligned}
\delta\rho_{\gamma\, q} &= \bar{\rho}_\gamma \Delta^{(S)}_{T,0}\,, \\
\delta p_{\gamma\, q} &= \frac{\bar{\rho}_\gamma}{3}\left(\Delta^{(S)}_{T,0} + \Delta^{(S)}_{T,2}\right)\,, \\
\delta u_{\gamma\, q} &= -\frac{3}{4}\frac{a}{q}\Delta^{(S)}_{T,1}\,, \\
q^2 \pi^S_{\gamma\, q} &= \bar{\rho}_\gamma \Delta^{(S)}_{T,2}\,.
\end{aligned}
$$

At early times when Compton scattering is efficient

$$\Delta_{T,\ell} \to 0 \quad \text{for} \quad \ell \geq 2$$

$$\Delta_{P,\ell} \to 0$$

The Boltzmann hierarchy reduces to the equations of hydrodynamics

Equations of motion

(Massless) Neutrinos

$$\dot{\Delta}^{(S)}_{\nu,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{\nu,\ell+1}(q,t) - \ell\Delta^{(S)}_{\nu,\ell-1}(q,t)\right] = - 2\dot{A}_q\delta_{\ell,0} + 2q^2\dot{B}_q\left(\frac{1}{3}\delta_{\ell,0} - \frac{2}{15}\delta_{\ell,2}\right)$$

Baryons

Energy conservation

$$\delta\dot{\rho}_{b\,q} + \frac{3\dot{a}}{a}\delta\rho_{b\,q} - \frac{q^2}{a^2}\overline{\rho}_b\delta u_{b\,q} + \frac{1}{2}\overline{\rho}_b\left(3\dot{A}_q - q^2\dot{B}_q\right) = 0$$

Momentum conservation

$$\delta\dot{u}_{b\,q} + \frac{4}{3}\frac{\overline{\rho}_\gamma}{\overline{\rho}_b}\omega_c(t)\left(\delta u_{b\,q} + \frac{3}{4}\frac{a}{q}\Delta^{(S)}_{T,1}(q,t)\right) = 0$$

Equations of motion

Dark Matter

$$\delta\dot{\rho}_{c\,q} + \frac{3\dot{a}}{a}\delta\rho_{c\,q} + \frac{1}{2}\overline{\rho}_{c\,q}\left(3\dot{A}_q - q^2\dot{B}_q\right) = 0$$

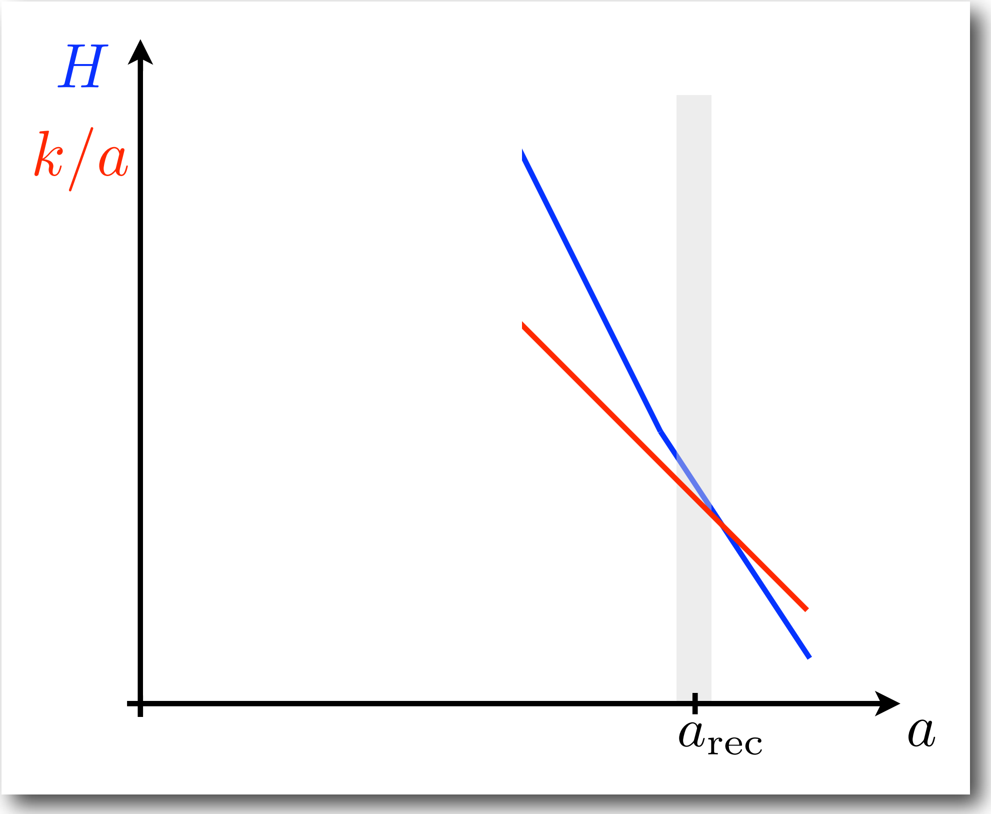
Scalar metric perturbations

$$\frac{q^2}{a^2}A_q + \frac{\dot{a}}{a}\left(3\dot{A}_q - q^2\dot{B}_q\right) = 8\pi G\left(\delta\rho_{q\,b} + \delta\rho_{q\,c} + \overline{\rho}_\gamma\Delta^{(S)}_{T,0} + \overline{\rho}_\nu\Delta^{(S)}_{\nu,0}\right)$$

$$\dot{A}_q = 8\pi G\left(\overline{\rho}_b\delta u_{b\,q} - \frac{a}{q}\overline{\rho}_\gamma\Delta^{(S)}_{T,1}(q,t) - \frac{a}{q}\overline{\rho}_\nu\Delta^{(S)}_{\nu,1}(q,t)\right)$$

Initial Conditions

What remains is the choice of initial conditions



All modes are "outside the horizon" at early times.

$$\frac{q}{a} \ll H$$

Initial Conditions

At early times the Boltzmann hierarchy for photons reduces to the equations of hydrodynamics and we can look for solutions of the form

$$\Delta_{T,0}^{(S)} = \Delta_{\nu,0}^{(S)} = \frac{4}{3}\frac{\delta\rho_c}{\bar{\rho}_c} = \frac{4}{3}\frac{\delta\rho_b}{\bar{\rho}_b} \equiv \Delta_0^{(S)}$$

$$\Delta_{\nu,1}^{(S)} \propto \Delta_{T,1}^{(S)} = -\frac{4}{3}\frac{q}{a}\delta u_{b\,q} \equiv \Delta_1^{(S)}$$

$$\Delta_{T,\ell} \to 0 \quad \text{for} \quad \ell \geq 2$$

$$\Delta_{P,\ell} \to 0$$

These are adiabatic initial conditions

Initial Conditions

In this limit $\mathcal{R}_q = \frac{A_q}{2} + H\delta u_q$ becomes a constant

and we can normalize our solution such that $\mathcal{R}_q \to \mathcal{R}_q^o$

Then

$$\Delta_0^{(S)}(q,t) = \frac{4}{3}\frac{q^2 t^2}{a^2(t)}\mathcal{R}_q^o\,,$$

$$\Delta_1^{(S)}(q,t) = \frac{8}{27}\frac{q^3 t^3}{a^3(t)}\mathcal{R}_q^o\,,$$

$$\Delta_{\nu,2}^{(S)}(q,t) = -\frac{16}{3(15+4f_\nu)}\frac{q^2 t^2}{a^2(t)}\mathcal{R}_q^o\,,$$

$$A_q(t) = \left(2 - \frac{2}{3}\frac{5+4f_\nu}{15+4f_\nu}\frac{q^2 t^2}{a^2(t)}\right)\mathcal{R}_q^o\,,$$

$$q^2 \dot{B}_q(t) = \frac{20}{15+4f_\nu}\frac{q^2 t}{a^2(t)}\mathcal{R}_q^o\,,$$

$$\Delta_{\nu,1}^{(S)}(q,t) = \frac{23+4f_\nu}{15+4f_\nu}\Delta_1^{(S)}(q,t)$$

Initial Conditions

These are the equations and initial conditions used by the Boltzmann codes such as CAMB or CLASS.

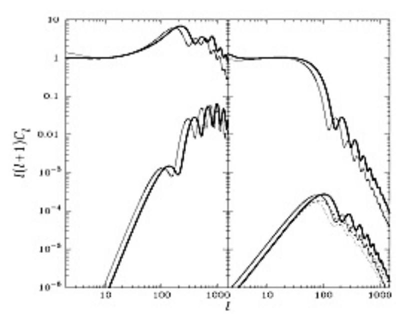
With the solution at hand, one computes

$$a^{(S)}_{T,\ell m} = \pi T_0 i^\ell \int d^3q\ \alpha(\mathbf{q}) Y^{m*}_\ell(\hat{q}) \Delta^{(S)}_{T,\ell}(q, t_0)$$

or directly

$$C^{(S)}_{TT,\ell} = \pi^2 T_0^2 \int q^2 dq\ \left| \Delta^{(S)}_{T,\ell}(q, t_0) \right|^2$$

similarly for polarization and tensor contribution



Code for Anisotropies in the Microwave Background

by Antony Lewis and Anthony Challinor

CLASS

the Cosmic Linear Anisotropy Solving System

From eV to Inflation

$$C^{(S)}_{XX,\ell} = 4\pi T_0^2 \int \frac{dk}{k} \Delta^2_{\mathcal{R}}(k) \left| \int_0^{\tau_0} d\tau S^{(S)}_X(k,\tau) j_\ell(k(\tau_0-\tau)) \right|^2$$

From eV to Inflation

Initial Conditions

Late time evolution

$$C^{(S)}_{XX,\ell} = 4\pi T_0^2 \int \frac{dk}{k}\Delta^2_{\mathcal{R}}(k) \left| \int_0^{\tau_0} d\tau\, S_X^{(S)}(k,\tau)\, j_\ell(k(\tau_0-\tau)) \right|^2$$

Physics of Recombination

Geometry

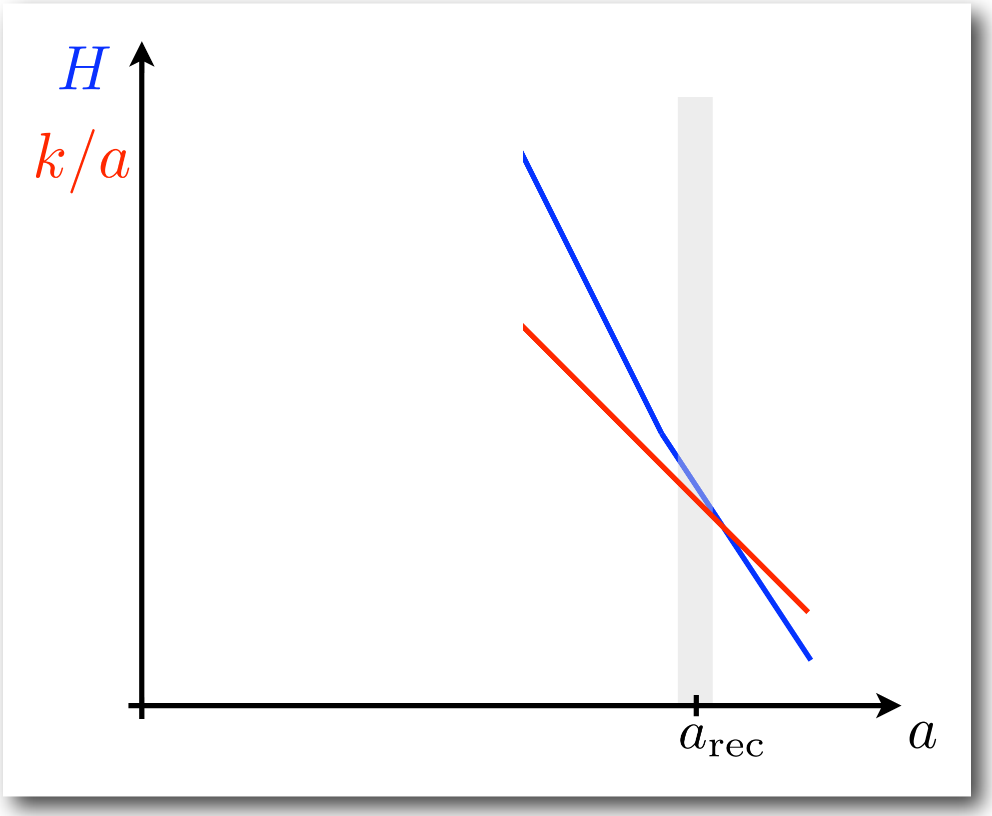
From eV to Inflation

So far, these are initial conditions for the system of equations that governs the evolution of the universe from around few keV to the present

In this limit, the system has 5 solutions that do not decay, one "adiabatic" solution and 4 "isocurvature" solutions. **(Bucher et al. 1999)**

Experimentally, only the adiabatic solution seems excited for which $\mathcal{R}$ is constant.

From eV to Inflation



We can extrapolate backwards very easily at least until the temperatures become high enough for new degrees of freedom to appear.

From eV to Inflation

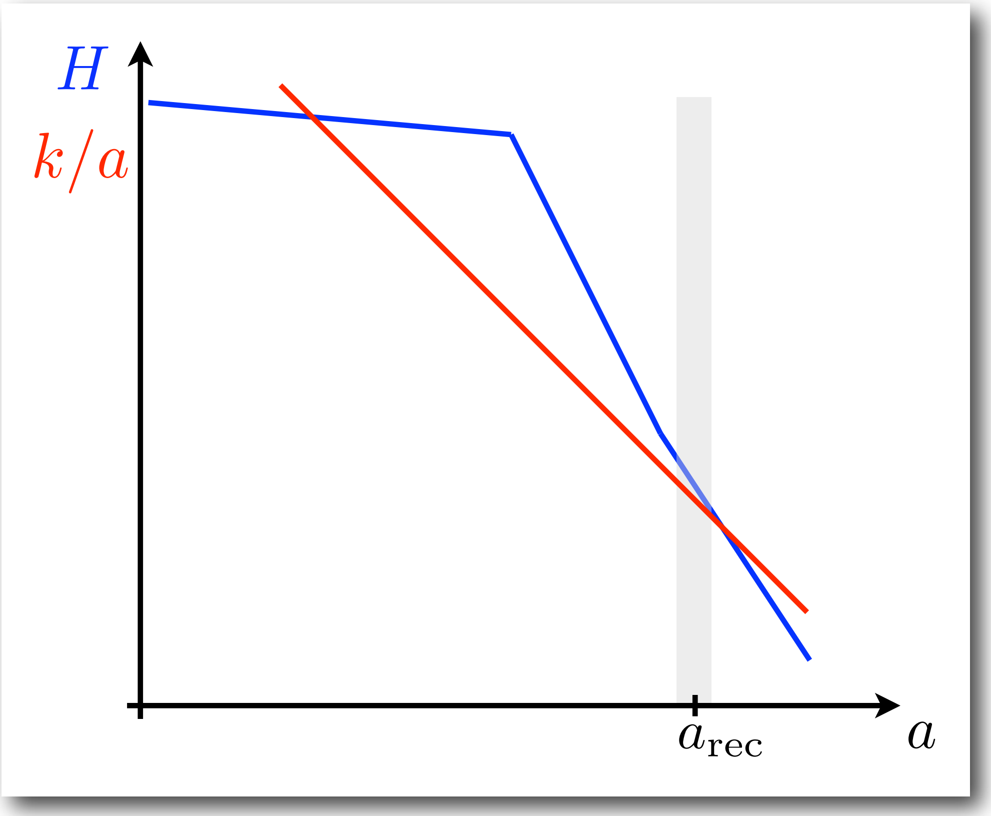
Outside the horizon, this adiabatic solution with constant $\mathcal{R}$ exists not only for the matter content present below a few keV, but for a general matter content. **(Weinberg 2009)**

To generate the perturbations causally, they cannot have been outside the horizon very early on.
This requires a phase with

$$\frac{d}{dt}\left(\frac{q}{a|H|}\right) < 0$$

(e.g. inflation or bounce)

From eV to Inflation



The perturbations are generated as quantum fluctuations while inside the horizon, and then exit the horizon.

From eV to Inflation

There are two cases in which the solution with constant $\mathcal{R}$ is known to be an attractor:

- Single field inflation
- Phase of thermal equilibrium without conserved charges.

In single field inflation, the anisotropies in the CMB directly tell us about the inflationary dynamics!

From eV to Inflation

For standard single field slow-roll inflation, the primordial spectrum of scalar perturbations is

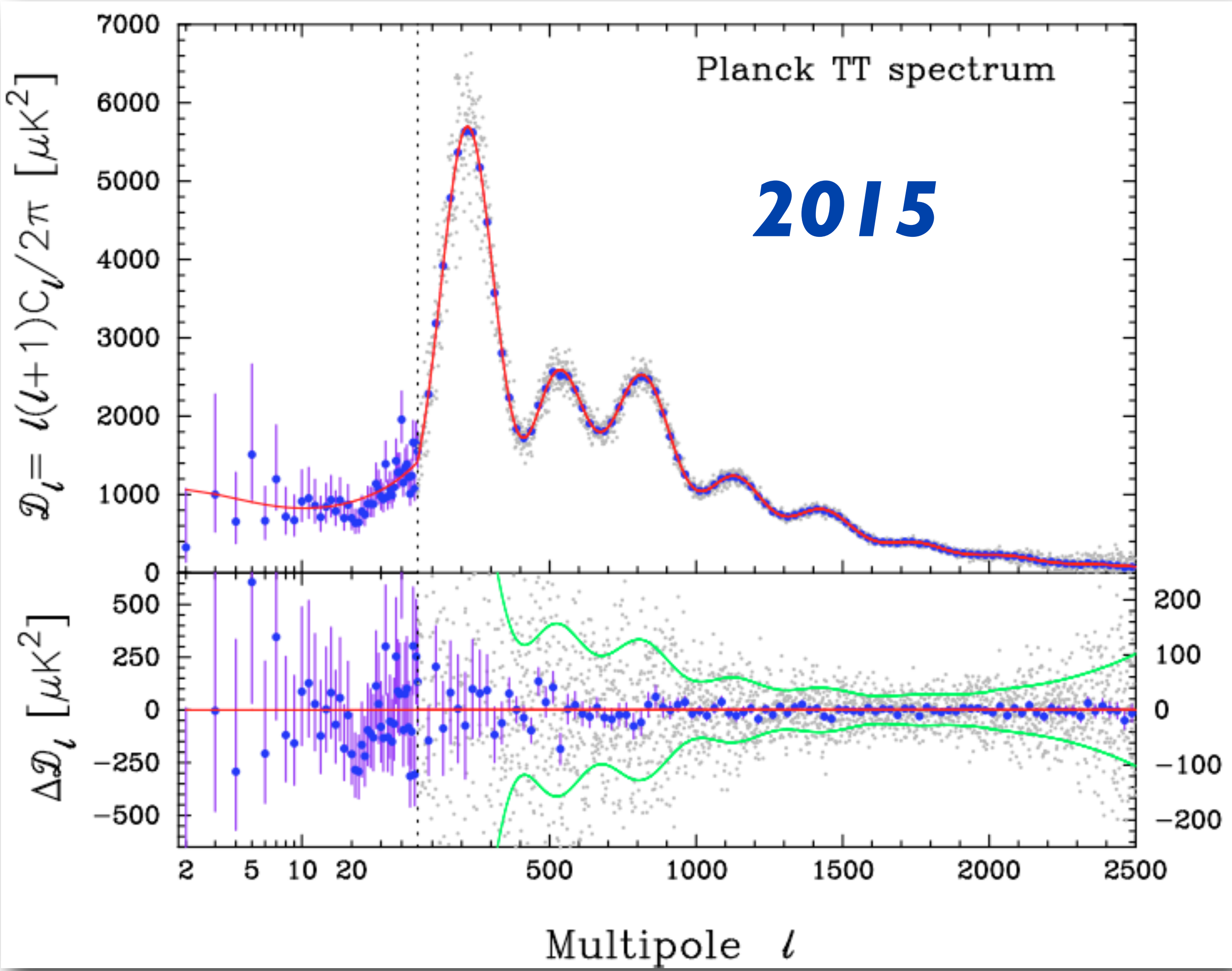
$$\Delta^2_{\mathcal{R}}(q) = \frac{H^2(t_q)}{8\pi^2\epsilon(t_q)} \approx \Delta^2_{\mathcal{R}}\left(\frac{q}{q_*}\right)^{n_s-1}$$

with $$n_s = 1 - 4\epsilon_* - 2\delta_*$$

and $$\epsilon = -\frac{\dot{H}}{H^2} \qquad \delta = \frac{\ddot{H}}{2H\dot{H}}$$

and the 3-pt function too small to be observed.

Power spectrum measurement

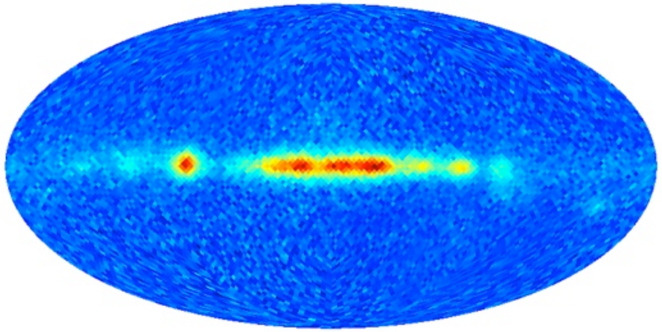


We know how to compute the theory prediction, now we need to understand the data points.

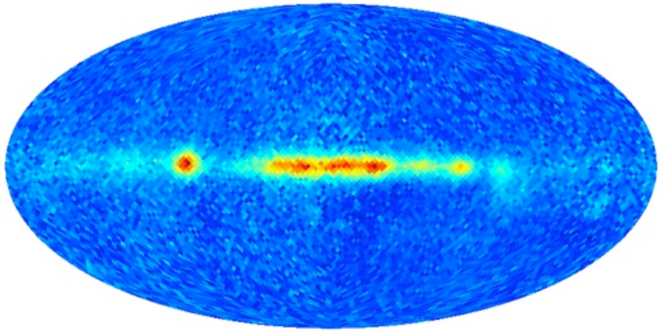
Beyond Primary Anisotropies

CMB data consists of sky maps at different microwave frequencies

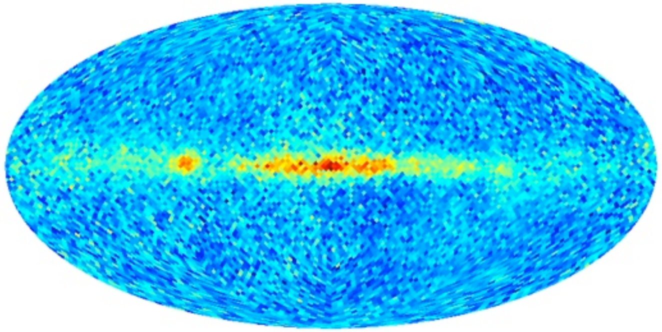
COBE (DMR)
(1989-93)



31 GHz



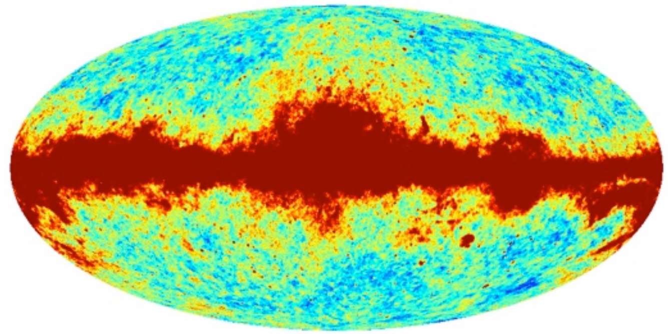
53 GHz



91 GHz

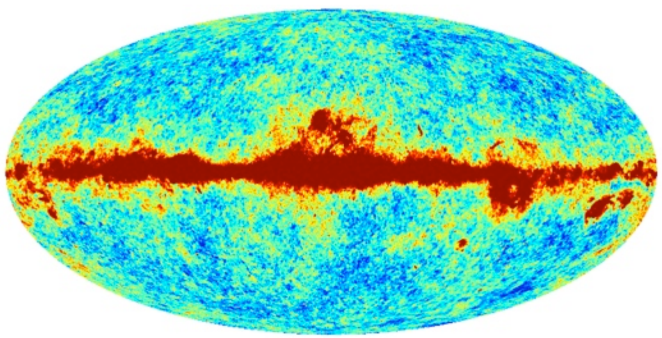
Beyond Primary Anisotropies

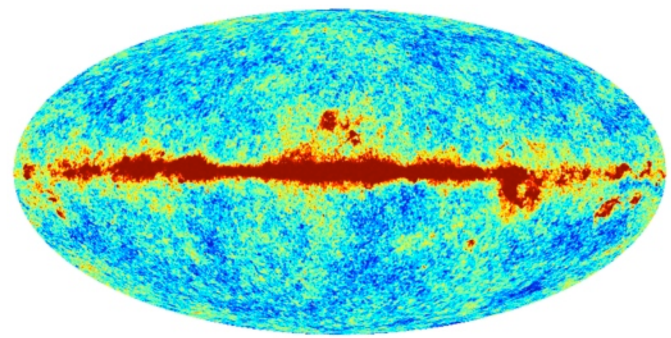
WMAP
(2001-10)



23 GHz

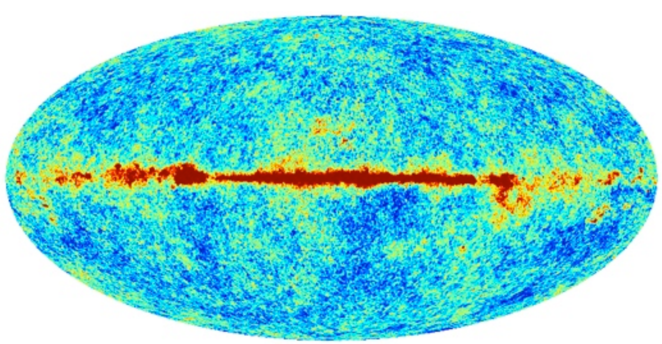
33 GHz

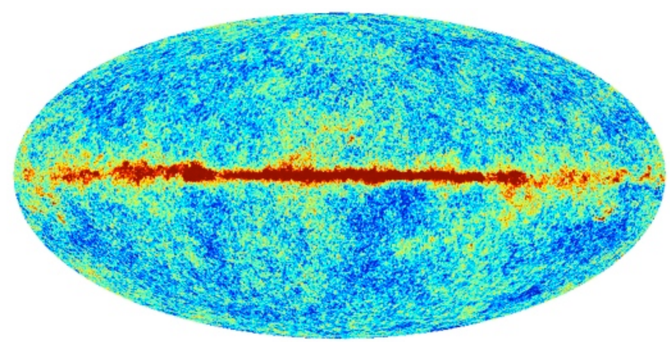




41 GHz

61 GHz

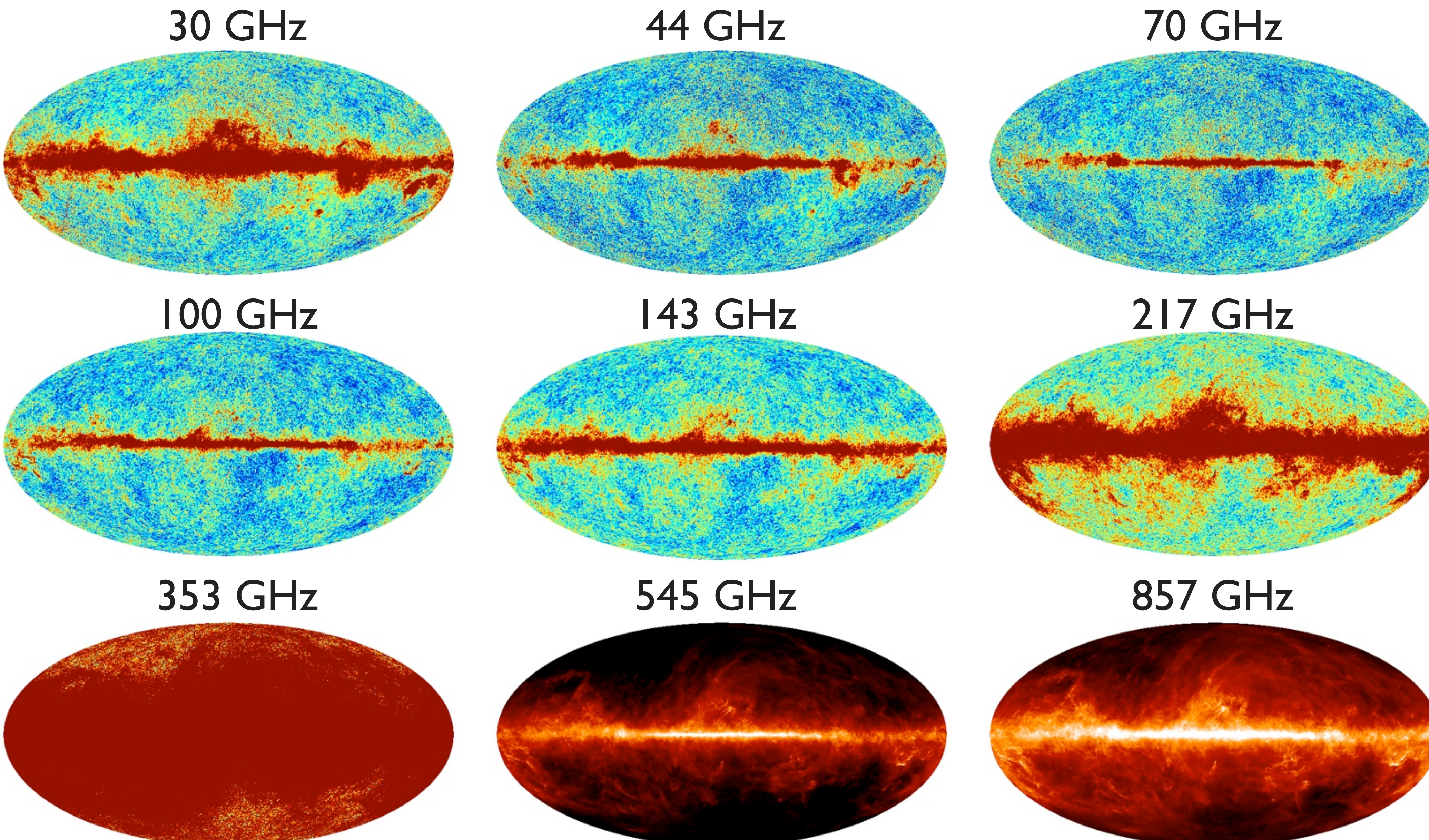




94 GHz

Beyond Primary Anisotropies

Planck



Beyond Primary Anisotropies

To learn about the CMB this means we must understand

- Dust
- Synchrotron
- ...

We have additional ways to probe cosmology

- Reionization
- Thermal SZ effect
- Kinetic SZ effect
- Lensing of the CMB
- ...

Thermal SZ effect

The change in temperature is set by

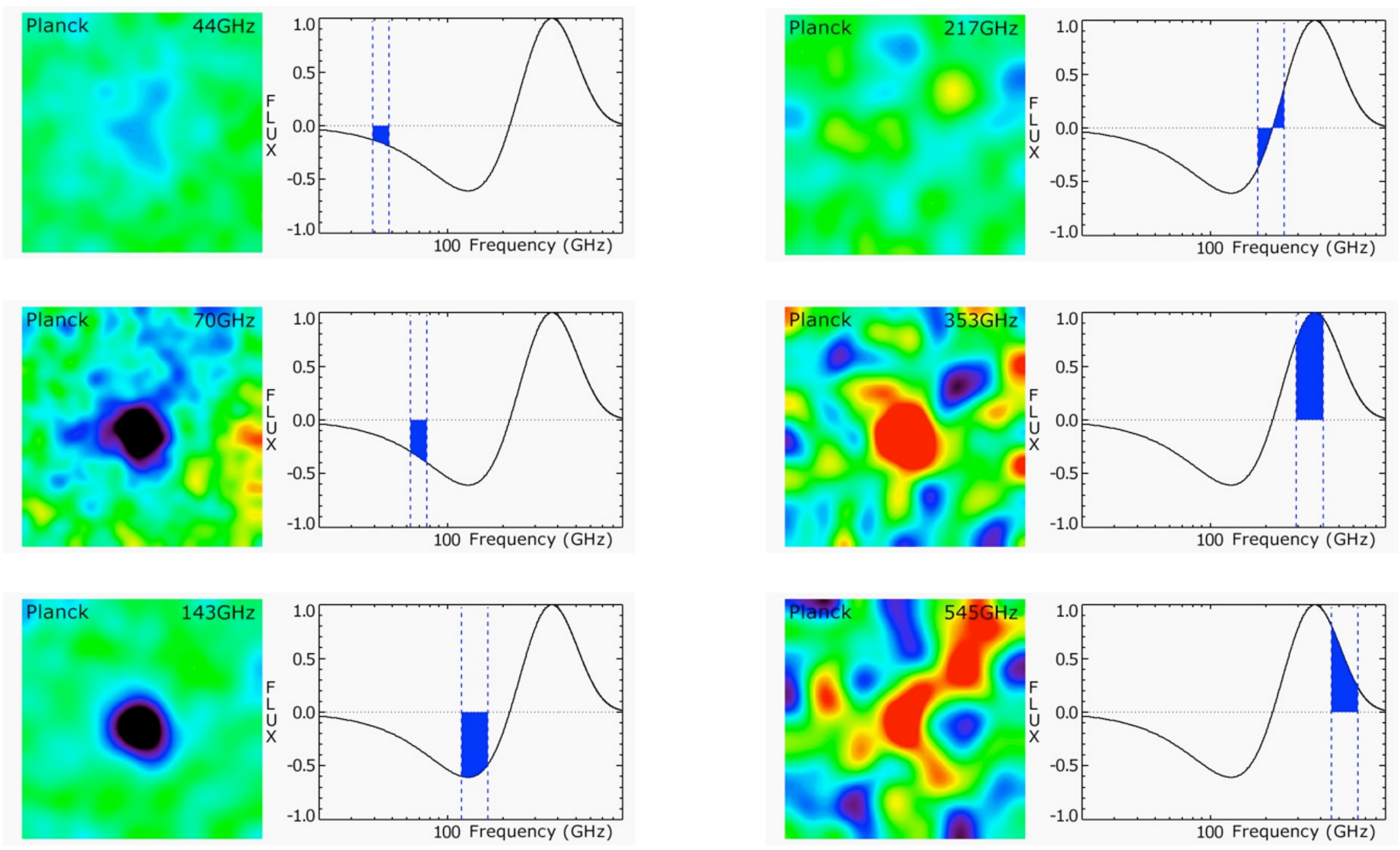
$$\Delta T(\hat{n}) = y(\hat{n})\,(x \coth(x/2) - 4)\,T_0$$

$$x = \frac{h\nu}{kT} \qquad y(\hat{n}) = \int dl\, n_e \sigma_T \frac{kT_e}{m_e}$$

A map of the Compton parameter y is a measure of hot gas in the universe between us and the surface of last scattering.

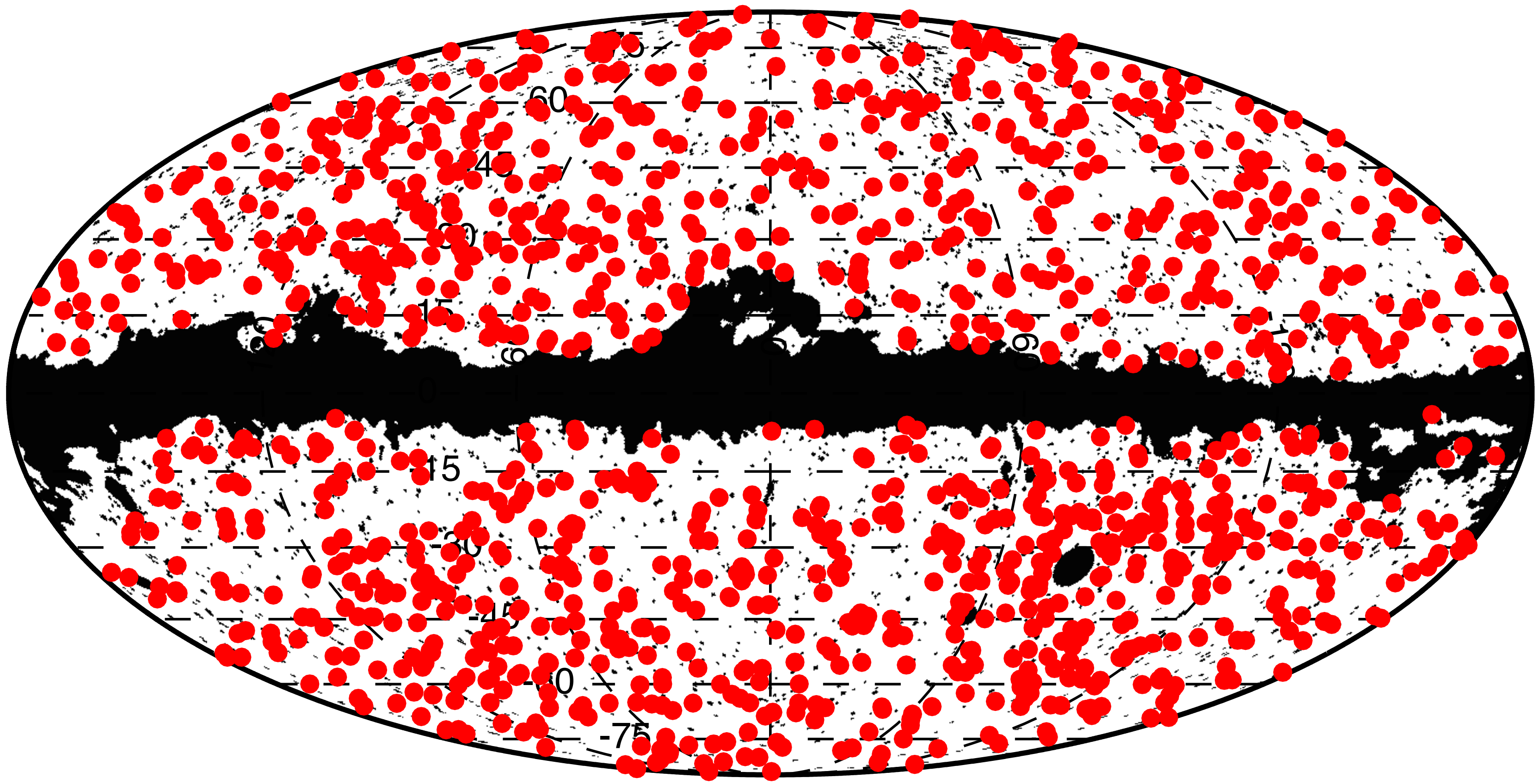
Thermal SZ effect

SZ view of Abell 2319 with Planck



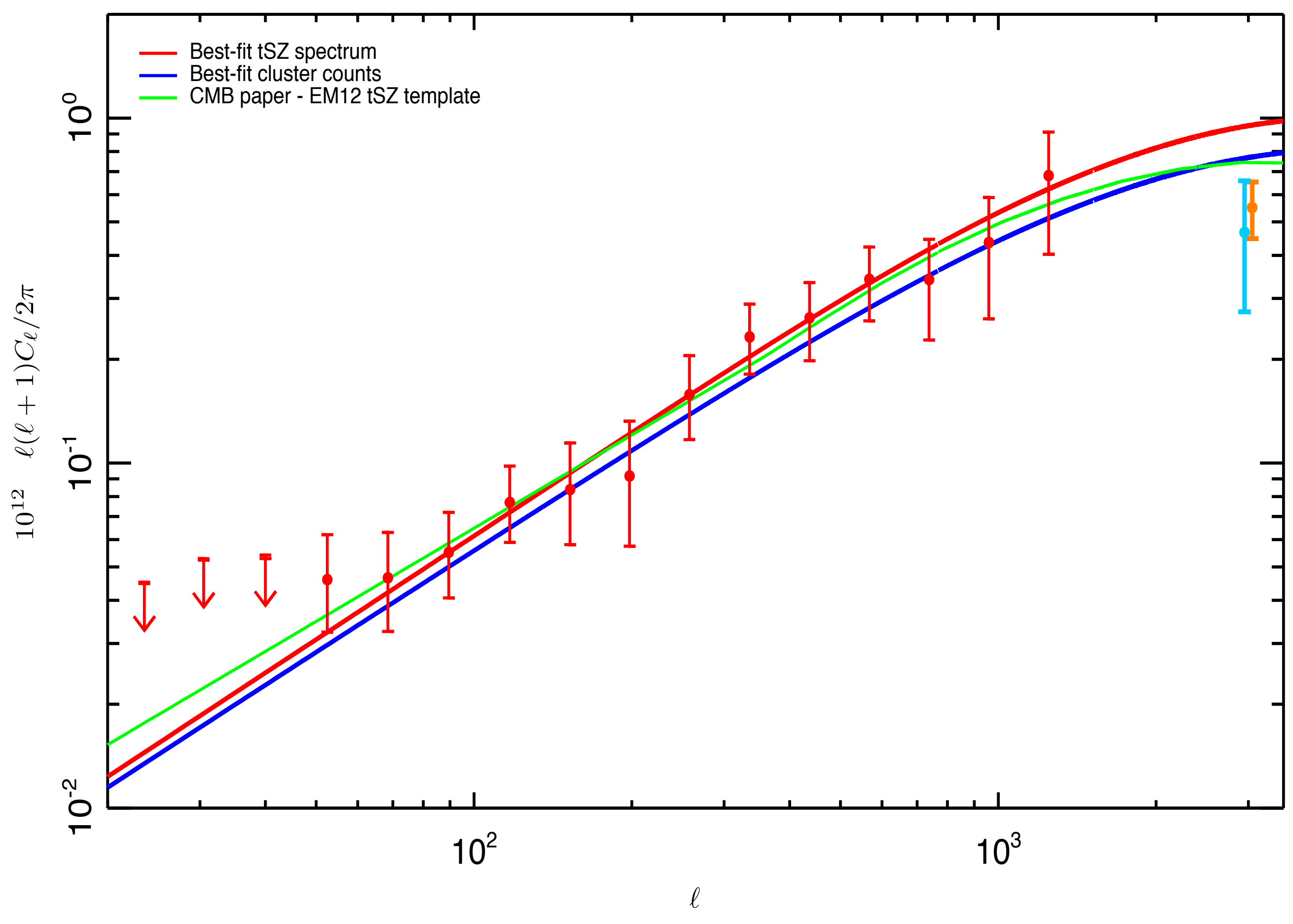
Thermal SZ effect

Planck SZ clusters



Thermal SZ effect

Planck thermal SZ power spectrum



Lensing

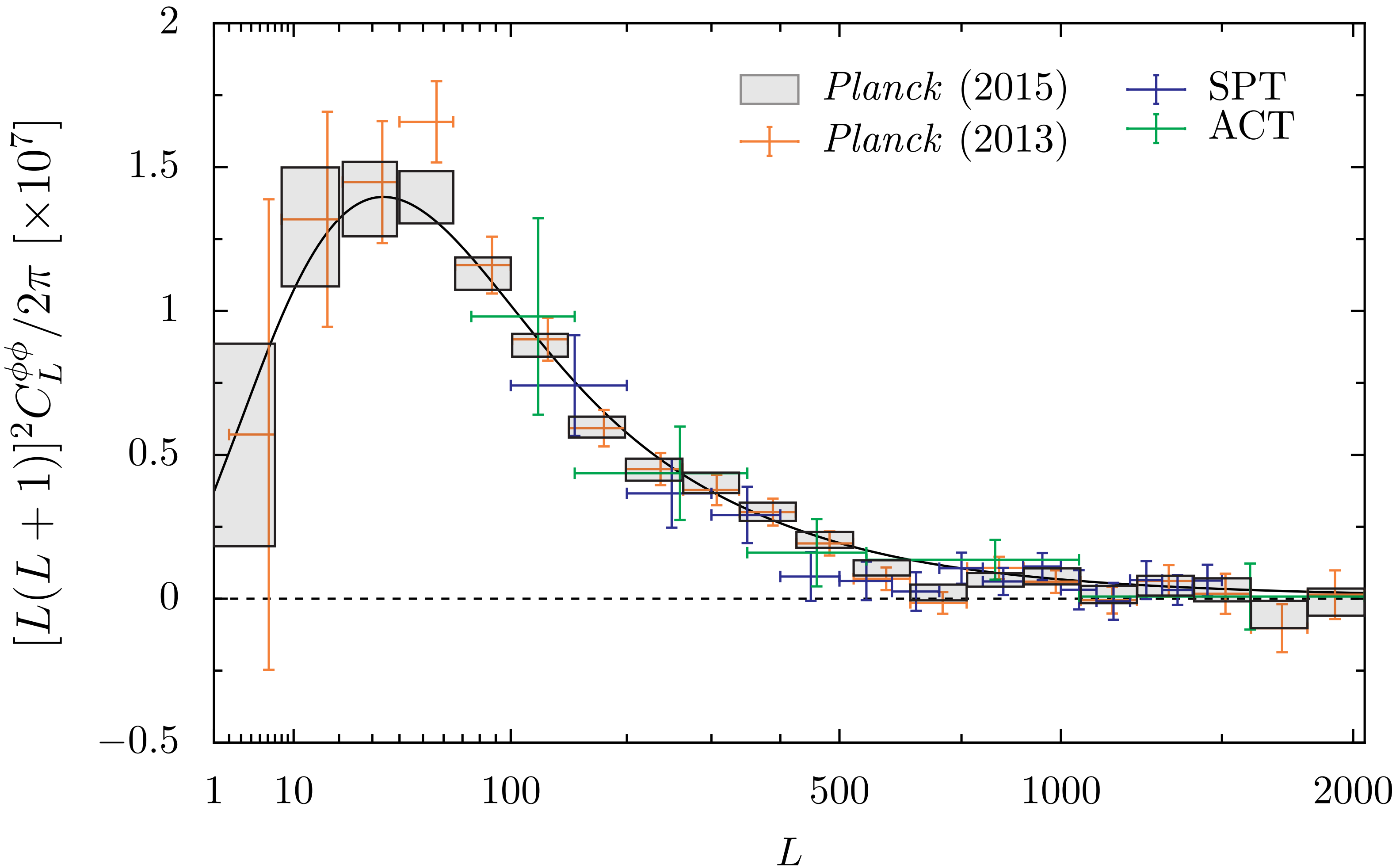
$$T(\hat{n}) = T^{\mathrm{unlensed}}\left(\hat{n} + \nabla\phi(\hat{n})\right)$$

- Washes out acoustic peaks in the power spectrum (this effect is included in all the analyses)
- leads to temperature three-point correlations because of correlations between ISW and lensing
- leads to temperature four-point correlations proportional to power spectrum of lensing field

$$\mathbb{T}^{\ell_1\ell_2}_{\ell_3\ell_4}(L) \approx C_L^{\phi\phi} C_{\ell_2}^{TT} C_{\ell_4}^{TT} F_{\ell_1 L \ell_2} F_{\ell_3 L \ell_4}$$

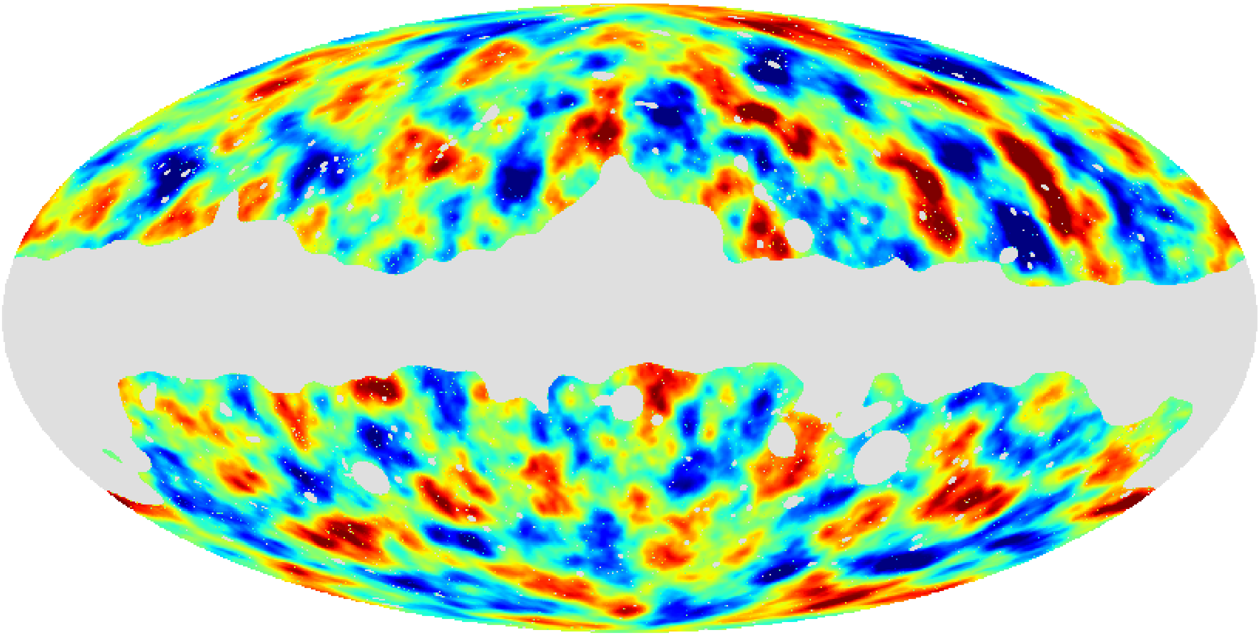
Lensing

Detected at high significance (40σ)



Lensing

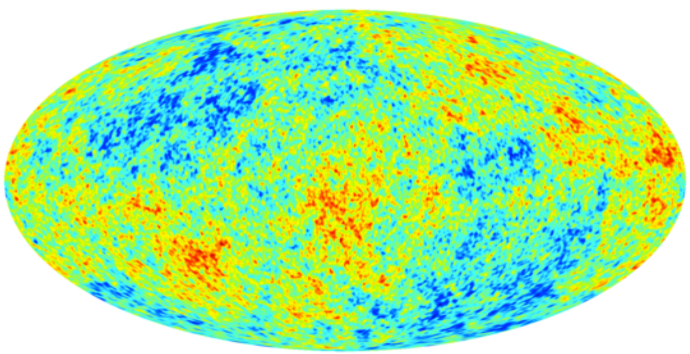
The lensing potential itself can also be reconstructed



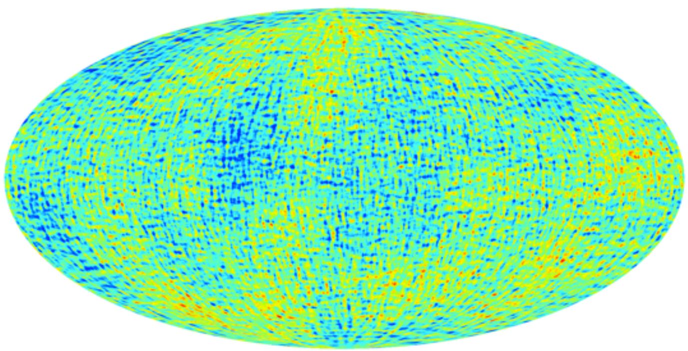
and provides a map (albeit a noisy one) of (the projection of) all matter between us and the surface of last scattering!

Ideal measurement

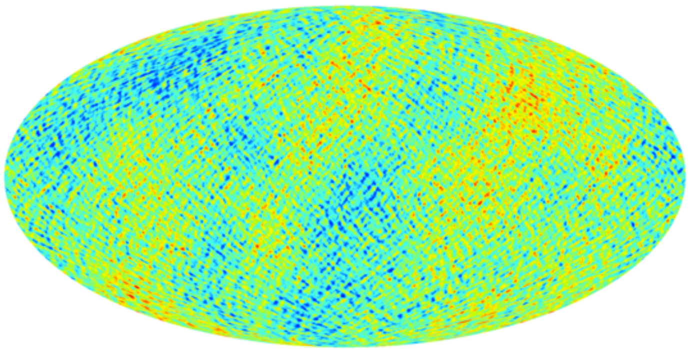
ΔT



Q



U



$$a_{T,\ell m} = \int d^2\hat{n}\, Y_\ell^{m*}(\hat{n}) \Delta T(\hat{n})$$

$$a_{E,\ell\, m}, a_{B,\ell\, m}$$

$$C^{\rm obs}_{TT,\ell} \equiv \frac{1}{2\ell+1} \sum_m \left| a^{\rm obs}_{T,\ell\, m} \right|^2$$

$$C^{\rm obs}_{TE,\ell}, C^{\rm obs}_{EE,\ell}, C^{\rm obs}_{BB,\ell}$$

Ideal measurement

How do we estimate the cosmological parameters of our favorite model?

Denote the parameters by $\vec{\theta}$ and the data by D where D could be $a_{\ell m}^{\rm obs}, C_\ell^{\rm obs}$

We would like to know $P(\vec{\theta}|D)$

We cannot compute it directly, but can use Bayes' theorem

$$P(\vec{\theta}|D) = \frac{P(D|\vec{\theta})P(\vec{\theta})}{P(D)}$$

"prior"

Ideal measurement

How do we estimate the cosmological parameters of our favorite model?

Denote the parameters by $\vec{\theta}$ and the data by D where D could be $a_{\ell m}^{\mathrm{obs}}, C_\ell^{\mathrm{obs}}$

We would like to know $P(\vec{\theta}|D)$

We cannot compute it directly, but can use Bayes' theorem

$$P(\vec{\theta}|D) = \frac{P(D|\vec{\theta})P(\vec{\theta})}{P(D)}$$

"prior"

This suggests to define a likelihood for our experiment

$$\mathcal{L}(\vec{\theta}) = P(D|\vec{\theta})$$

which can be computed for any given theory

Ideal measurement

Warm up: Measurement of temperature anisotropies

For Gaussian perturbations

$$\langle a_{\ell\, m} a^*_{\ell'\, m'} \rangle = C_\ell \delta_{\ell\ell'} \delta_{mm'}$$

and

$$P(a_{\ell\, m}) = \frac{1}{(2\pi C_\ell)^{\frac{2\ell+1}{2}}} \exp\left(-\sum_m \frac{|a_{\ell\, m}|^2}{2C_\ell} \right)$$

Ideal measurement

Warm up: Measurement of temperature anisotropies

For Gaussian perturbations

$$\langle a_{\ell\, m} a^*_{\ell'\, m'} \rangle = C_\ell \delta_{\ell\ell'} \delta_{mm'}$$

and

$$P(a_{\ell\, m}) = \frac{1}{(2\pi C_\ell)^{\frac{2\ell+1}{2}}} \exp\left(- \sum_m \frac{|a_{\ell\, m}|^2}{2C_\ell} \right)$$

So the exact likelihood is

$$\mathcal{L}(\theta) = \prod_\ell \frac{1}{(2\pi C_\ell(\theta))^{\frac{2\ell+1}{2}}} \exp\left(- \sum_m \frac{|a^{\rm obs}_{\ell\, m}|^2}{2C_\ell(\theta)} \right)$$

or for $C_\ell^{\rm obs}$

$$\mathcal{L}(\theta) \propto \prod_\ell \exp\left(- \frac{2\ell+1}{2} \left[\frac{C_\ell^{\rm obs}}{C_\ell(\theta)} + \ln C_\ell(\theta) - \frac{2\ell-1}{2\ell+1} \ln C_\ell^{\rm obs} \right] \right)$$

Ideal measurement

For a measurement including polarization

Define $\mathbf{a}_{\ell\, m} = (a_{T,\ell\, m}, a_{E,\ell\, m}, a_{B,\ell\, m})$

Then $\langle \mathbf{a}_{\ell\, m} \mathbf{a}^{\dagger}_{\ell'\, m'} \rangle = \mathbf{C}_\ell \delta_{\ell\ell'} \delta_{mm'}$

with $\mathbf{C}_\ell = \begin{pmatrix} C_{TT,\ell} & C_{TE,\ell} & 0 \\ C_{TE,\ell} & C_{EE,\ell} & 0 \\ 0 & 0 & C_{BB,\ell} \end{pmatrix}$

Then the exact likelihood is

$$\mathcal{L}(\theta) = \prod_\ell \frac{1}{(2\pi \det \mathbf{C}_\ell(\theta))^{\frac{2\ell+1}{2}}} \exp\left(-\frac{1}{2} \sum_m \mathbf{a}^{\dagger\,\mathrm{obs}}_{\ell\, m} \mathbf{C}^{-1}_\ell(\theta) \mathbf{a}^{\mathrm{obs}}_{\ell\, m} \right)$$

or

$$\mathcal{L}(\theta) \propto \prod_\ell \frac{(\det \mathbf{C}^{\mathrm{obs}}_\ell)^{\frac{2\ell-n}{2}}}{(\det \mathbf{C}_\ell(\theta))^{\frac{2\ell+1}{2}}} \exp\left(-\frac{2\ell+1}{2} \mathrm{tr} \mathbf{C}^{\mathrm{obs}}_\ell \mathbf{C}^{-1}_\ell \right)$$

Adding Real World Effects

In realistic measurements, we have to incorporate

- Noise of the experiment
- Finite resolution of the experiment
- Pixelization of maps
- Masks
- ...

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Notice that these likelihoods are Gaussian in terms of $\mathbf{a}_{\ell\, m}^{\mathrm{obs}}$ but not in terms of $\mathrm{C}_{\ell}^{\mathrm{obs}}$

Incorporating these effects is thus easy in map space where the likelihoods are Gaussian

Adding Real World Effects

Pixel space likelihood

$$\langle \Delta T_i \Delta T_j \rangle = \mathbf{C}_{ij} + \mathbf{N}_{ij}$$

observed pixels

Noise covariance matrix

Pixel covariance for signal

The probability distribution for the ΔT_i is

$$P(\Delta T_i) = \frac{1}{(2\pi)^{N_{\rm pix}/2}\sqrt{\det(\mathbf{C}+\mathbf{N})}} \exp\left(-\frac{1}{2} \sum_{ij} \Delta T_i (\mathbf{C}+\mathbf{N})^{-1}_{ij} \Delta T_j \right)$$

Adding Real World Effects

Pixel space likelihood

So the exact likelihood in pixel space is

$$\mathcal{L}(\theta) = \frac{1}{(2\pi)^{N_{\rm pix}/2}\sqrt{\det(\mathbf{C}(\theta)+\mathbf{N})}} \exp\left(-\frac{1}{2}\sum_{ij} \Delta T_i^{\rm obs} (\mathbf{C}(\theta)+\mathbf{N})^{-1}_{ij} \Delta T_j^{\rm obs}\right)$$

This easily extends to polarization

Adding Real World Effects

Pixel space likelihood

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This easily extends to polarization

Unfortunately evaluating such likelihoods is prohibitively expensive for high resolution full sky experiments such as WMAP or Planck.

To make progress, one uses approximations for the likelihoods based on the $\mathbf{C}_\ell^{\rm obs}$.

Adding Real World Effects

Pseudo-C_ℓ likelihood

One (of many) approximations is a fiducial Gaussian approximation

$$\mathcal{L}(\theta) \propto \frac{1}{\sqrt{\det(\mathcal{C}_{\rm fid})}} \exp\left[-\frac{1}{2}(\mathbf{C}^{\rm obs} - \mathbf{C}(\theta))^t \mathcal{C}_{\rm fid}^{-1} (\mathbf{C}^{\rm obs} - \mathbf{C}(\theta))\right]$$

with covariance matrix $\mathcal{C}_{\rm fid} = \langle \mathbf{C}\mathbf{C}^t \rangle$ evaluated for some fiducial cosmology close to the true cosmology.

The covariance matrix can be computed analytically even for masked maps and in the presence of noise

Adding Real World Effects

Spectra and covariance for pseudo-C_ℓ likelihood

For masked sky maps

$$\Delta\tilde{T}_i^a = W_i^a(\Delta T_i^a + N_i^a)$$

we have multipole coefficients

$$\tilde{a}_{\ell m}^a = \sum_i \Omega_i \Delta\tilde{T}_i^a Y_{\ell m}^*(\hat{n}_i)$$

and pseudo-spectra

$$\tilde{C}_\ell^{ab} \equiv \frac{1}{2\ell+1}\sum_m \tilde{a}_{\ell m}^a \tilde{a}_{\ell m}^{b*}$$

These are related to the underlying power spectra by

$$\langle \tilde{C}_\ell^{ab} \rangle = \sum_{\ell'} M_{\ell\ell'}^{ab}(p_{\ell'} b_{\ell'}^{ab})^2 \langle \hat{C}_{\ell'}^{ab} \rangle + \tilde{N}_\ell^{ab}$$

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beam

mode coupling matrix

pixel window function

Adding Real World Effects

Spectra and covariance for pseudo-C_ℓ likelihood

Their covariance matrix is

$$
\begin{aligned}
\langle \Delta\tilde{C}_\ell^{ab}\Delta\tilde{C}_{\ell'}^{cd}\rangle &= \sqrt{C_\ell^{ac}C_\ell^{bd}C_{\ell'}^{ac}C_{\ell'}^{bd}}\,\Xi(\ell,\ell',W^{(ac)(bd)}) + \sqrt{C_\ell^{ad}C_\ell^{bc}C_{\ell'}^{ad}C_{\ell'}^{bc}}\,\Xi(\ell,\ell',W^{(ad)(bc)}) \\
&+ \sqrt{C_\ell^{ac}C_{\ell'}^{ac}}\,\Xi(\ell,\ell',W_\sigma^{(ac)(bd)}) + \sqrt{C_\ell^{ad}C_{\ell'}^{ad}}\,\Xi(\ell,\ell',W_\sigma^{(ad)(bc)}) \\
&+ \sqrt{C_\ell^{bd}C_{\ell'}^{bd}}\,\Xi(\ell,\ell',W_\sigma^{(bd)(ac)}) + \sqrt{C_\ell^{bc}C_{\ell'}^{bc}}\,\Xi(\ell,\ell',W_\sigma^{(bc)(ad)}) \\
&+ \Xi(\ell,\ell',W_{\sigma\sigma}^{(ac)(bd)}) + \Xi(\ell,\ell',W_{\sigma\sigma}^{(ad)(bc)})
\end{aligned}
$$

with

$$
\begin{aligned}
W_\ell^{(ac)(bd)} &= \frac{1}{2\ell+1}\sum_m w_{\ell m}^{ac} w_{\ell m}^{bd*}\,, \\
W_{\sigma\,\ell}^{(ac)(bd)} &= \frac{1}{2\ell+1}\sum_m w_{\ell m}^{ac} w_{\sigma\,\ell m}^{bd*}\,, \\
W_{\sigma\sigma\,\ell}^{(ac)(bd)} &= \frac{1}{2\ell+1}\sum_m w_{\sigma\,\ell m}^{ac} w_{\sigma\,\ell m}^{bd*}\,,
\end{aligned}
$$

$$
\begin{aligned}
w_{\ell m}^{ab} &= \sum_i \Omega_i W_i^a W_i^b Y_{\ell m}^*(\hat{n}_i)\,, \\
w_{\sigma\,\ell m}^{ab} &= \sum_i \Omega_i^2 W_i^{a2} \sigma_i^{a2} \delta^{ab} Y_{\ell m}^*(\hat{n}_i)\,.
\end{aligned}
$$

Adding Real World Effects

Hybrid likelihoods

Pixel based likelihoods are exact but prohibitively expensive for full sky, high resolution experiments

Pseudo-C_ℓ likelihood only accurate for high enough multipoles as the C_ℓ obey a χ-square distribution with $2\ell+1$ degrees of freedom

This suggests using a hybrid likelihood consisting of a pixel based likelihood on large scales and a pseudo-C_ℓ likelihood on small scales

Parameter estimation

To find the likelihood as function of our parameters, we could evaluate it on a grid.

Since the likelihoods are typically costly to evaluate and especially for higher dimensional parameter spaces this is too time consuming.

We sample them using Markov Chain Monte Carlo methods instead.

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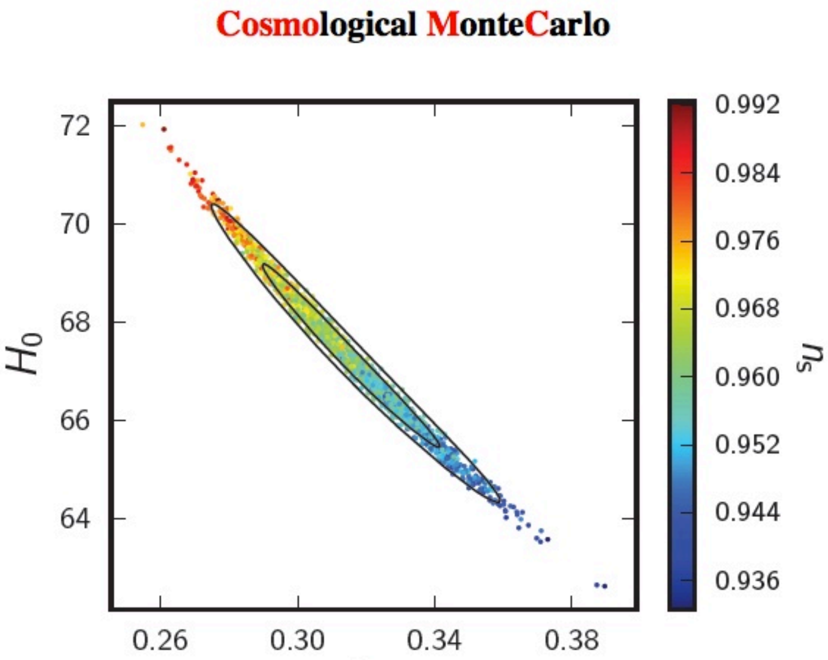
Typically with CosmoMC but other tools exist



Parameter estimation

Metropolis-Hastings

- Choose a starting point in parameter space and compute $\mathcal{L}(\theta_0)$
- Pick a randomly chosen second point and compute $\epsilon = \mathcal{L}(\theta_1)/\mathcal{L}(\theta_0)$
- If $\epsilon > 1$ keep the point, if $\epsilon < 1$ keep with probability ϵ
- Repeat

With some additional work this will generate random points drawn from $\mathcal{L}(\theta)$, which can be used to find best-fits, means, error bars...