Scalar field critical collapse in 2+1 dimensions

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joint work with Carsten Gundlach and Tadeusz Chmaj

1. Introduction
   - Critical collapse
   - 2+1 dimensions
   - Previous work

2. Results
   - Evolution of generic initial data
   - The critical solution
   - Derivation of γ and δ

Choptuik's discovery of the critical phenomena in gravitational collapse:

- Consider a one-parameter family of initial data with parameter p
  - $p > p_*$ (supercritical) - black hole formation
  - $p < p_*$ (subcritical) - dispersion to infinity
- Near the black hole threshold

$$M_{BH} \sim (p - p_*)^{\delta}$$

and

$$|R|_{max} \sim (p_* - p)^{-2\gamma},$$

where from dimensional analysis $\delta = \gamma(d - 2)$

- In the region of a large curvature or just before the apparent horizon formation, the solution approaches the universal *critical solution*

The critical solution has a few defining properties:

- universality
- regularity
- scale-invariance (CSS) or scale-periodicity (DSS)
- has one unstable mode in the perturbation theory
- the unstable mode is also unique and its growth factor is associated with the power in the scaling law
- in the adapted coordinates (T, x):

$$g_{\mu\nu}(T, x) = e^{-2T} \bar{g}_{\mu\nu}(x)$$

In DSS case $\bar{g}$ is periodic in T with period Δ

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However, in the presence of $\Lambda < 0$ black holes exist

$$ds^2 = -\left(\frac{\bar{r}^2}{\ell^2} - M\right) d\bar{t}^2 + \left(\frac{\bar{r}^2}{\ell^2} - M\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\theta^2,$$

where $\Lambda = -1/\ell^2$

- $M = -1$ - AdS spacetime
- $-1 < M < 0$ - spacetimes with conical singularity
- $M > 0$ - black hole solutions

The first numerical simulation of critical collapse of a scalar field in 2+1 dimensions was done by Pretorius and Choptuik and by Husain and Olivier:

- P&C:
  - the critical solution is CSS
  - Ricci scaling at 0: $|R|_{max} \sim (p_* - p)^{-2\gamma}$, where $\gamma = 1.15 - 1.25$
  - they concluded that $\delta = 2\gamma$ (the formula from dimensional analysis is obviously not applicable here)
- H&O: mass scaling exponent $\delta = 0.81$
- Results are not conclusive

The Garfinkle solution for $\Lambda = 0$

The spherically symmetric metric in double null coordinates:

$$ds^2 = -e^{2\mathcal{A}}dudv + \bar{r}^2 d\theta^2.$$

Introduce similarity coordinates:

$$x = \frac{v}{u}, \qquad T = -\ln\left(-\frac{u}{\ell}\right),$$

and assume the ansatz

$$\phi = c(T + f(x)), \qquad \mathcal{A} = \mathcal{A}(x), \qquad \bar{r} = e^{-T}R(x).$$

- The Garfinkle solution is regular at the center at $x = 1$
- The Garfinkle solution is generically singular at $x = 0$ (the lightcone)
- However, for given values of $\bar{c}$:

$$\bar{c}^2 = 1 - \frac{1}{2n},$$

where $\bar{c}^2 = 8\pi G c^2$, it is also analytic at $x = 0$ if $n = 1, 2...$

Stability analysis (Garfinkle & Gundlach) shows n unstable modes with $k = \frac{m}{2n}$, where either $m = 2n - 1$ or $1 < m < n$

The next step is to incorporate Λ into the field equations.

We follow Cavagliá, Clément and Fabbri and expand in powers of e^{-2T} $(-u^2/\ell^2)$

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_0(x) + \sum_{n=1}^{\infty} e^{-2nT} \mathcal{A}_n(x), \\
R &= R_0(x) + \sum_{n=1}^{\infty} e^{-2nT} R_n(x), \\
\phi &= c[T + f_0(x) + \sum_{n=1}^{\infty} e^{-2nT} f_n(x)]
\end{aligned}$$

We chose the numerical method of Pretorius and Choptuik. The metric ansatz is

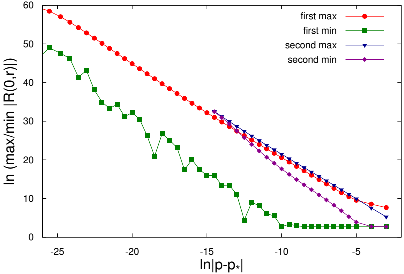
$$ds^2 = \cos^{-2}\left(\frac{r}{\ell}\right) e^{2A}(-dt^2 + dr^2) + \ell^2 \tan^2\left(\frac{r}{\ell}\right) e^{2B}\, d\theta^2,$$

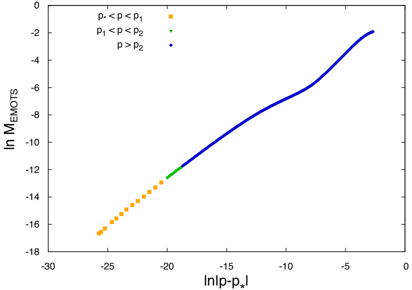
so that compared to the previous double null form we have

$$\mathcal{A} = A - \ln[\cos(r/\ell)], \qquad \bar{r} = \ell \tan(r/\ell) \exp B.$$

- Timelike infinity of asymptotically AdS spacetime is brought to $r = \pi\ell/2$
- The metric effectively represents the metric in double null coordinates $u = t - r$ and $v = t + r$, but is evolved on the grid in t and r
- We use free evolution scheme, so metric functions A and B are evolved from initial data at $t = 0$
- We choose units such that $G = 2$, $\ell = 2/\pi$

Ricci and mass scaling





$$|R|_{max} \sim (p - p_*)^{-2\gamma},$$

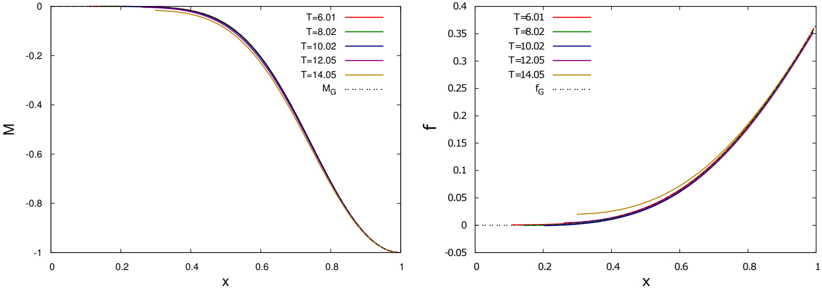
where $\gamma = 1.23$

$$M_{EMOTS} \sim (p_* - p)^{\delta},$$

where $\delta = 0.68$

Definitely $\delta \neq 2\gamma$!

Using similarity coordinates (x, T) we compare the numerical results with the Garfinkle solution



There is a good agreement with the Garfinkle solution with $n = 4$ inside the lightcone $0 < x \leq 1$.

Numerical results seem to coincide with the Garfinkle solution inside the lightcone.

- One possible extension of the Garfinkle solution is the analytic continuation - consider $-1 < x \leq 1$
- Another possible continuation is the null continuation for which

$$\phi = c[T + 2\ln 2], \qquad e^{2\mathcal{A}} = 2^{-4\bar{c}^2}, \qquad R = \frac{1}{2}$$

- or in terms of u

$$\phi = -2c\ln\left(\frac{(-u/\ell)^2}{2}\right), \quad e^{2\mathcal{A}} = 2n^2\left(\frac{(-u/\ell)^2}{2}\right)^{4\bar{c}^2},$$

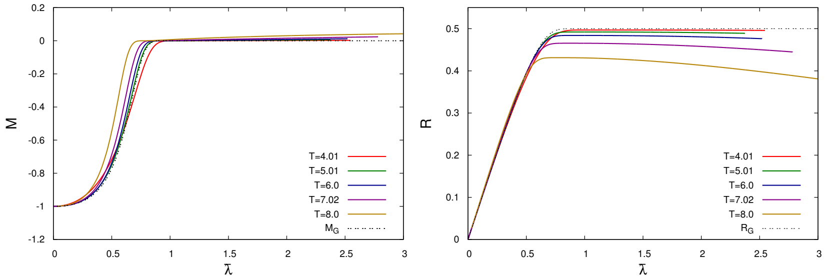
$$R = e^T \ell \frac{(-u/\ell)^{2n}}{2}$$

- The null continuation can be matched to the Garfinkle solution
- Matching is C^{n-1}
- Similarily, perturbations and Λ-corrections also can be matched

To go beyond the lightcone we introduce a new coordinate

$$\lambda = -\frac{s}{u},$$

where s is the affine parameter along outgoing null geodesics.



The analytic continuation (beyond $\bar{\lambda} = 1$) is ruled out and the null continuation seems to be a good approximation.

Assume that the true critical solution has exactly one unstable mode with the exponent λ_0.

- In the near critical region

$$\phi \simeq \phi_* + c(p)e^{\lambda_0 T}v(x) \simeq \phi_* + c'(p_*)|p - p_*|e^{\lambda_0 T}v(x)$$

- The first max of Ricci is reached when the solution moves away from the critical solution. When this happens the growing mode reached $O(1)$

$$|p - p_*|e^{\lambda_0 T} \sim 1.$$

- Ricci has the dimension $1/L^2$ and the scale is given by e^{-T}

$$R_{max} \propto e^{2T} \propto |p - p_*|^{-2\gamma},$$

  where $\gamma = \frac{1}{\lambda_0}$

- For the top mode of $n = 4$ Garfinkle solution $\gamma = \frac{8}{7} \approx 1.14$

- Clearly, the formula obtained from dimensional analysis is not applicable to $d = 2$
- Λ plays a role in black hole formation leading to the anomalous scaling exponent
- Assumption: the true critical solution is well approximated by null-extended and Λ-corrected $n = 4$ Garfinkle solution and has one unstable mode
- Assuming that the AH forms in the null continuation just beyond the lightcone gives

  $$M_{AH} \propto (p_* - p)^{\delta},$$

  where $\delta = \frac{2}{2+\lambda_0} = \frac{2\gamma}{2\gamma+1} = \frac{16}{23}$
- The other unstable modes will not contribute anyway
- There is no evidence for unstable modes with $m = 2$ and $m = 3$ in numerical simulations

Conclusions

- In higher dimensions the presence of negative Λ is negligible in the regions of large curvature
- In 2+1 the effect of a negative Λ cannot be perturbative: its presence is crucial for a black hole formation
- Our simulations give evidence that the critical solution is well approximated by null-extended and Λ-corrected $n = 4$ Garfinkle solution
- Numerically, we do not have evidence for three growing modes
- Assuming that the critical solution has one unstable mode with $k = 7/8$ we calculated $\gamma = 8/7$ and $\delta = 16/23$ which are in good agreement with numerical results