Symplectic symmetries of AdS_3 phase space

Ali Seraj
Institute for Research in Fundamental Sciences (IPM)

In collaboration with

G.Compère, M.M. Sheikh-Jabbari and P. Mao

International Center for Theoretical Physics (ICTP)
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Motivation

Dynamics in 3d Einstein gravity

- No local dynamics in the bulk
- Chern-Simons formulation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \Leftrightarrow F = 0 = \bar{F}$$

- Many interesting solutions
- Boundary dynamics

Surface Dynamics in 3d Einstein gravity

- Asymptotic symmetries
- Conserved charges uniquely label the solutions

Extension into the bulk

- Boundary data $\xrightarrow{\text{Einstein equations}}$ asymptotically AdS_3 solutions
- Asymptotic symmetries $\xrightarrow{\text{gauge fixing}}$ symmetries everywhere in the bulk
- *symplectic symmetry*

Symplectic geometry

- *Definition*:

$$\textit{phase space} \quad = \quad \text{a } \textit{manifold } \mathbf{\Gamma} \quad + \quad \text{a } \textit{symplectic form } \Omega$$

- Properties of symplectic form Ω

1. 2-form $\quad \Omega_{ab} = -\Omega_{ba}$
2. Closed $\quad d\Omega = 0$
3. Nondegenerate $\quad \det \Omega \neq 0$

- Symmetries → charges

$$X^a = \Omega^{ab}\, \partial_b\, H_X$$

- Poisson bracket → Algebra of charges

$$\{f, g\} = \Omega^{ab}\, \partial_a f\, \partial_b g$$

Asymptotically AdS_3 Geometries

Asymptotically AdS_3 geometries

- **Gauge Fixing.** Coordinates (z, x^+, x^-) in Fefferman-Graham gauge

$$ds^2 = \frac{1}{z^2}\left[dz^2 + \gamma_{ij}\, dx^i\, dx^j\right]$$
$$\gamma_{ij} = g^{(0)}_{ij}(x^+, x^-) + g^{sub}_{ij}(z, x^+, x^-)$$

- **boundary conditions.** Brown-Henneaux

$$g^{(0)} = \eta = -dx^+ dx^- \qquad \varphi \sim \varphi + 2\pi$$

Asymptotically AdS_3 geometries

- Asymptotically AdS_3 geometries (*off-shell*)

$$z = 1/r \quad \rightarrow \quad \boxed{ds^2 = \frac{dr^2}{r^2} + \left(r^2\, \eta_{ij} + g^{(sub)}_{ij}(r, x^\pm)\right) dx^i\, dx^j}$$

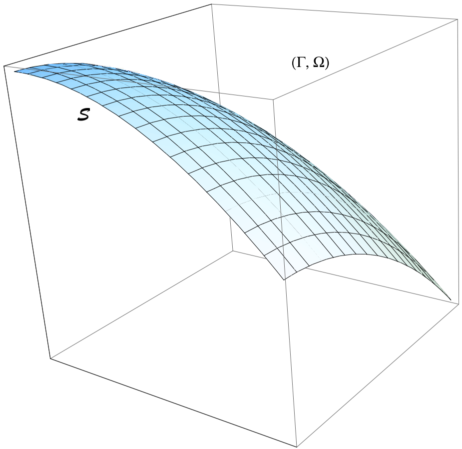
- Asymptotically AdS_3 geometries (*on-shell*)[*Bañados(1998)*]

$$\boxed{ds^2 = \ell^2 \frac{dr^2}{r^2} - \left(r dx^+ - \ell^2 \frac{L_-(x^-) dx^-}{r}\right)\left(r dx^- - \ell^2 \frac{L_+(x^+) dx^+}{r}\right)}$$

Phase Space of asymptotically AdS_3 Geometries

Schematic picture of the phase space

- **Γ**: The set of all asymptotically AdS_3 geometries (off-shell)
- $\mathcal{S}$: The set of all asymptotically AdS_3 geometries (on-shell)



Symplectic structure

[*Lee, Wald(1990), Barnich, Brandt(2001)*]

Covariant phase space formulation

- **Symplectic structure**

$$\Omega(\delta_1\phi, \delta_2\phi) = \int_\Sigma \boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi)$$

Σ an arbitrary spacelike hypersurface

- The *symplectic current* $\boldsymbol{\omega}$ is obtained by the second variation of the Lagrangian (*Lee-Wald 1990*)

$$\delta\mathbf{L}[\Phi] = \mathbf{E}[\Phi]\delta\Phi + d\boldsymbol{\Theta}[\delta\Phi, \Phi]$$
$$\boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi, \Phi] = \delta_1\boldsymbol{\Theta}[\delta_2\Phi, \Phi] - \delta_2\boldsymbol{\Theta}[\delta_1\Phi, \Phi]$$

- Alternative definition (*Barnich-Brandt 2001*)

A strange property

[*G.Compère, P.Mao, A.S and M.M.Sheikh-Jabbari, JHEP* **1601**, *080 (2016).*]

Theorem

- Projection of symplectic form on the solution submanifold ***vanishes***

$$\boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi, \Phi]\Big|_{\mathcal{S}} = 0, \qquad \forall\Phi \in \mathcal{S}, \quad \forall\delta_1\Phi, \delta_2\Phi \in T\mathcal{S}$$

- $\mathcal{S}$ is a ***null*** surface in $\boldsymbol{\Gamma}$ (or an *isotropic* submanifold)
- This also happens in classical mechanics
- If γ is a first class constraint,

$$X^a = \Omega^{ab}\partial_b\gamma$$

is a null direction on $\gamma = const$ surface

Symmetries of AdS_3 phase space

Symplectic symmetries of AdS$_3$ phase space

[*G. Barnich, C. Troessaert (2010)*]

[*G. Barnich, A. Gomberoff and H. A. Gonzalez, (2012)*]

[*G.Compère, P.Mao, A.S and M.M.Sheikh-Jabbari, (2015).*]

- Diffeomorphism is the gauge symmetry of Einstein theory

$$g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\chi g_{\mu\nu}, \qquad \forall \chi$$

- However, we have fixed the gauge $\quad g_{rr} = \frac{1}{r^2}, \quad g_{ri} = 0$
- Moreover, we have fixed a boundary condition $\quad g^{(0)} = -dx^+ \, dx^-$

Residual gauge symmetries over solutions

$$\boxed{\chi = -\frac{r}{2}(\epsilon'_+ + \epsilon'_-)\partial_r + \Big(\epsilon_+ + \frac{\ell^2 r^2 \epsilon''_- + \ell^4 L_- \epsilon''_+}{2(r^4 - \ell^4 L_+ L_-)}\Big)\partial_+ + \Big(\epsilon_- + \frac{\ell^2 r^2 \epsilon''_+ + \ell^4 L_+ \epsilon''_-}{2(r^4 - \ell^4 L_+ L_-)}\Big)\partial_-}$$

- *Symplectic symmetry* $\qquad \boldsymbol{\omega}[\delta\Phi, \delta_\chi \Phi, \Phi]\Big|_{\mathcal{S}} = 0$

Symmetries of the phase space

$$\chi = -\frac{r}{2}(\epsilon'_+ + \epsilon'_-)\partial_r + \Big(\epsilon_+ + \frac{\ell^2 r^2 \epsilon''_- + \ell^4 L_- \epsilon''_+}{2(r^4 - \ell^4 L_+ L_-)}\Big)\partial_+ + \Big(\epsilon_- + \frac{\ell^2 r^2 \epsilon''_+ + \ell^4 L_+ \epsilon''_-}{2(r^4 - \ell^4 L_+ L_-)}\Big)\partial_-$$

- Extention of Brown-Henneaux asymptotic symmetries into the bulk

$$\chi^{(BH)} = -\frac{r}{2}(\epsilon'_+ + \epsilon'_-)\partial_r + \epsilon_+\partial_+ + \epsilon_-\partial_-$$

- Transform solutions to solutions everywhere

$$g_{\mu\nu}(L) + \mathcal{L}_\chi g_{\mu\nu}(L) = g_{\mu\nu}(L + \delta_\epsilon L),$$

$$\delta_\epsilon L = \epsilon\partial L + 2L\partial\epsilon - \frac{1}{2}\partial^3\epsilon$$

- $[\chi(\epsilon_1; L), \chi(\epsilon_2; L)]$ does not close !

Algebra of residual symmetries

Adjusted bracket *[Barnich and C. Troessaert (2010)]*

The adjusted (modified) bracket $[\,\cdot\,,\,\cdot\,]_*$

$$\Big[\chi(\epsilon_1;L),\chi(\epsilon_2;L)\Big]_* \equiv \Big[\chi(\epsilon_1;L),\chi(\epsilon_2;L)\Big]_{L.B} - \Big(\delta^L_{\epsilon_1}\chi(\epsilon_2;L) - \delta^L_{\epsilon_2}\chi(\epsilon_1;L)\Big),$$

- Fourier expansion of $\epsilon_+ = e^{inx^+} \to \chi_n, \qquad \epsilon_- = e^{inx^-} \to \bar{\chi}_n$

$$\Big[\chi_m,\chi_n\Big]_* = (m-n)\;\chi_{m+n},$$
$$\Big[\bar{\chi}_m,\bar{\chi}_n\Big]_* = (m-n)\;\bar{\chi}_{m+n}$$

- 2 copies of Witt algebra

Symmetries and conserved charges

Theorem [*Wald (1993), Barnich, Brandt(2001), Compère (2007)*]

Symplectic form associate a ***charge*** to a symmetry

$$\boldsymbol{\omega}[\delta\Phi, \delta_\chi\Phi, \Phi] = d\,\boldsymbol{k}_\chi[\delta\Phi, \Phi]$$

- Charge variation

$$\delta Q_\chi \equiv \oint_{\partial\Sigma} \boldsymbol{k}_\chi[\delta\Phi, \Phi]\,.$$

- Definition of charge (assuming ***integrability***)

$$Q_\chi \equiv \int_{\phi_0}^{\phi} \delta Q_\chi$$

Charges and their algebra

[*G. Compre, L. Donnay, P. H. Lambert and W. Schulgin, (2015)*]

[*G.Compère, P.Mao, A.S and M.M.Sheikh-Jabbari, (2016).*]

Conserved charges everywhere

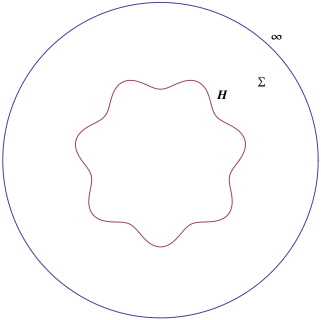
- Charges corresponding to *symplectic symmetries* can be computed at any closed surface

$$\boldsymbol{\omega}[\delta\Phi, \delta_\chi\Phi, \Phi] = d\,\boldsymbol{k}_\chi[\delta\Phi, \Phi]$$

$$\Downarrow$$

$$\delta Q_\chi\Big|_\infty - \delta Q_\chi\Big|_H = \int_\Sigma \boldsymbol{\omega}(\delta\Phi, \delta_\chi\Phi, \Phi)$$

$$= 0$$



Algebra of charges

Theorem

- Algebra of charges is a central extension of the algebra of symplectic symmetries

$$\boxed{Q_\chi[g] = \frac{\ell}{8\pi G}\int_0^{2\pi} d\phi\left(\epsilon_+(x^+)L_+(x^+) + \epsilon_-(x^-)L_-(x^-)\right)}$$

- Fourier expansion

$$\{Q_m, Q_n\} = (m-n)Q_{m+n} + \frac{c}{12}m^3\delta_{m+n,0},$$
$$\{\bar{Q}_m, \bar{Q}_n\} = (m-n)\bar{Q}_{m+n} + \frac{c}{12}m^3\delta_{m+n,0},$$

where $c = \frac{3\ell}{2G}$. This is the symmetry algebra of a 2 dimensional CFT

Symmetries at the horizon

Symplectic symmetries *induce* a symmetry algebra near the *horizon* of black hole.

- Extremal black holes in 3 dimensions
  [*G.Compère, P.Mao, A.S and M.M.Sheikh-Jabbari, (2016).*]

- Non-extremal black holes
  [*G. Compère and J. Long, (2016)*]
  [*H. Afshar, S. Detournay, D. Grumiller and B. Oblak, (2015)*]
  [*L. Donnay, G. Giribet, H. A. Gonzalez and M. Pino, (2015)*]

Results

- Asymptotically AdS_3 geometries form a phase space.
- The set of solutions form a null surface in the phase space
- Symplectic symmetries are the extension of asymptotic symmetries inside the bulk
- Charges can be computed over *any* closed surface which is a smooth deformation of the circle at infinity

Thank you for your attention