Scrambling of locally perturbed thermal states

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Outline

1. Motivating & quantifying scrambling
2. 2d CFT discussion
   - Set-up
   - Large c 2d CFTs at finite T & thermo field double
3. Brief holographic discussion
4. Quantum chaos & butterfly effect vs scrambling

Motivation

Question

If we input some localised information in a quantum system, such as a perturbation, does it remain localised or does it spread over the entire system ?

- $\exists$ delocalisation $\sim$ scrambling

Measures of scrambling

1. Any arbitrary subsystem up to half of the state's dof is *nearly maximally entangled* : Page scrambling
2. If $\exists$ scrambling $\Rightarrow$ information about input can not be deduced by local output measurements $\Rightarrow$ mutual information may quantify this

Quantifying scrambling

Any unitary operator

$$U(t) = \sum_{i,j=0}^{2^n-1} u_{ij} |i\rangle \langle j|$$

can be mapped into a 2n-qubit state treating input and output legs equally

$$|U(t)\rangle = \frac{1}{2^{n/2}} \sum_{i,j=0}^{2^n-1} u_{ij} |i\rangle_{\text{in}} \otimes |i\rangle_{\text{out}}$$

This is an example of the channel-state duality in QI.

Quantifying scrambling

Given some local disturbance in A,

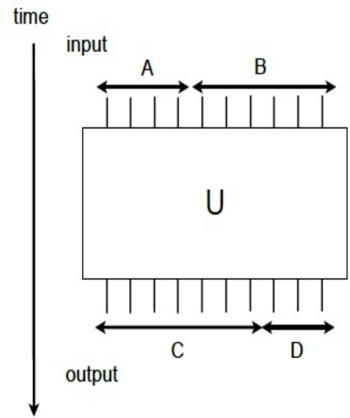
$\exists$ scrambling $\Rightarrow$

not measurable in C

$\Downarrow$

$$I(A:C) = S_A + S_C - S_{A \cup C}$$

small



Quantifying scrambling

Same argument and conclusion for local region $D \Rightarrow I(A:D)$ small
Amount of information that is non-locally hidden in CD by computing

$$I(A:CD) - I(A:C) - I(A:D)$$

In QI, this is captured by the tripartite information

$$I_3(A:C:D) = I(A:C) + I(A:D) - I(A:CD)$$

being very negative (Hosur, Qi, Roberts & Yoshida)

- Today, we will focus on $I(A:C) \approx 0$

BHs : scrambling vs quantum cloning

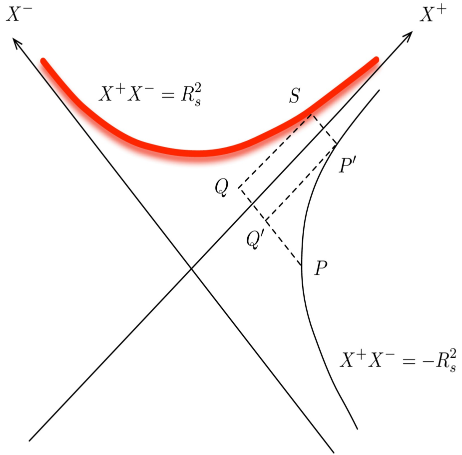
BH physics suggest speed at which thermality is regained is faster than in diffusive systems (scrambling) (Susskind-Sekino)

1. scrambling time from causality bound preventing quantum cloning

   $\tau_{\text{scrambling}} \sim \beta \log S$

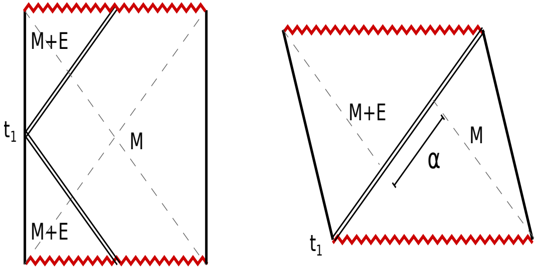
2. Faster than diffusion

   $\tau_{\text{diff}} \sim S^{2/d} \gg \log S$



Perturbing eternal BH (Shenker & Stanford)

- Perturbation turned on at time t_1 on the left boundary
- Backreaction can be non-trivial, no matter how light the perturbation is, depending on the t_1 scale
- Shock-wave description



Small perturbations get blue shifted near horizon (Shenker-Stanford)

$$t^\star \sim \beta \log m_p \beta$$

CFT calculation

Consider a 2d CFT at finite temperature

$$\rho_\beta$$

Perturb the thermal state by a local primary operator

$$\mathcal{O}_w(x_0,0)\,\rho_\beta\,\mathcal{O}_w^\dagger(x_0,0)$$

Evolve the system unitarily

$$e^{-iHt}\,\left(\mathcal{O}_w(x_0,0)\,\rho_\beta\,\mathcal{O}_w^\dagger(x_0,0)\right)\,e^{iHt}$$

Question

Time scale $t^\star$ at which

$$I(A:B)(t^\star)=0$$

CFT : overall strategy

Non-compact 2d CFT $\Rightarrow$ no Poincaré recurrences

Logic :

1. Describe density operator and its regularisation (on top of the UV cut-off)
2. Use replica trick to compute entanglement entropy $\Rightarrow$ correlators in 2d CFT
3. Use large c limit to compute these correlators analytically
4. Solve for the scrambling time scale $t^\star$

Remark : this main set-up was also considered in Roberts & Stanford

CFT set-up

Consider a perturbation, generated at time $t = 0$ at $x = -\ell$, created by a primary operator $\mathcal{O}$ acting on the vacuum of the 2d CFT :

$$|\Psi_{\mathcal{O}}(t)\rangle = \sqrt{\mathcal{N}}\, e^{-iHt}\, e^{-\epsilon H}\mathcal{O}(0, -\ell)|0\rangle$$

- $\mathcal{O}$ is inserted at $t = 0$ and $x = -\ell$ and dynamically evolved afterwards
- ϵ is a small parameter smearing the UV behaviour of the local operator (separation in euclidean time)

Density matrix :

$$\rho(t) = \mathcal{N}\, e^{-iHt}\, e^{-\epsilon H}\mathcal{O}(0, -\ell)|0\rangle\langle 0|\mathcal{O}^{\dagger}(0, -\ell)\, e^{iHt}\, e^{-\epsilon H}$$
$$= \mathcal{N}\, \mathcal{O}(\omega_2, \bar{\omega}_2)|0\rangle\langle 0|\mathcal{O}^{\dagger}(\omega_1, \bar{\omega}_1)$$

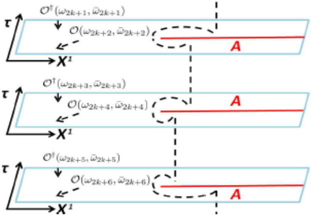
where $\omega_1 = -\ell + i(\epsilon - it)$, $\omega_2 = -\ell - i(\epsilon + it)$ $(\bar{\omega}_1 = -\ell - i(\epsilon - it))$

Entanglement entropy : replica trick

- Method 1 : uniformization

$$\mathrm{Tr}\rho_A^n \sim \langle \mathcal{O}(\omega_1,\bar{\omega}_1)\mathcal{O}^\dagger(\omega_2,\bar{\omega}_2)...\mathcal{O}^\dagger(\omega_{2n},\bar{\omega}_{2n})\rangle_{\Sigma_n}$$

  - Σ_n Riemann surface
  - $\omega_{2k+1} = e^{2\pi ik}\omega_1$
  - $\omega_{2k+2} = e^{2\pi ik}\omega_2$



- Method 2 : Twist operators

$$\mathrm{Tr}\rho_A^n \sim \langle\psi|\sigma(\omega_1,\bar{\omega}_1)\tilde{\sigma}(\omega_2,\bar{\omega}_2)|\psi\rangle$$

  - Calculation done in n-copies of the original CFT
  - Twist operators emerge because of the existence of some internal symmetry when swapping these copies

Perturbations at finite temperature

Same set-up as before, but now

1. we perturb a thermal state at $t = -t_\omega$:

$$\rho(t) \equiv \mathcal{N}\mathcal{O}(\omega_2, \bar{\omega}_2)\, e^{-\beta H}\, \mathcal{O}^\dagger(\omega_1, \bar{\omega}_1)$$

   with

$$\omega_1 = x_0 + t + t_\omega + i\epsilon \qquad \bar{\omega}_1 = x_0 - t - t_\omega - i\epsilon$$
$$\omega_2 = x_0 + t + t_\omega - i\epsilon \qquad \bar{\omega}_2 = x_0 - t - t_\omega + i\epsilon\,.$$

2. A pair of operators will be inserted on a cylinder, separated $2i\epsilon$

Thermofield double set-up

Consider two non-interacting 2d CFTs, say CFT_L and CFT_R, with isomorphic Hilbert spaces $\mathcal{H}_{L,R}$

Thermofield double state :

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2}E_n} |n\rangle_L |n\rangle_R$$

- $|n\rangle_L$ is an eigenstate of the hamiltonian H_L acting on $\mathcal{H}_L$ with eigenvalue E_n (and similarly for $|n\rangle_R$).
- $|n\rangle_L$ is the CPT conjugate of the state $|n\rangle_R$
- Notation : $|n\rangle_L \otimes |n\rangle_R$ as $|n\rangle_L |n\rangle_R$.
- Thermal reduced density

$$\rho_R(\beta) = \text{tr}_{\mathcal{H}_L}\left(|\Psi_\beta\rangle \langle\Psi_\beta|\right) = \frac{1}{Z(\beta)} \sum_{n\in\mathcal{H}_R} e^{-\beta E_n} |n\rangle_R \langle n|_R \ ,$$

Thermofield double : observables

- Single sided correlators are thermal

$$\langle\Psi_\beta|\,\mathcal{O}_R(x_1,t_1)\ldots\mathcal{O}_R(x_n,t_n)\,|\Psi_\beta\rangle = \mathrm{tr}_{\mathcal{H}_R}\left(\rho_R(\beta)\mathcal{O}_R(x_1,t_1)\ldots\mathcal{O}_R(x_n,t_n)\right)\,.$$

- Two sided correlators : by analytic continuation

$$\langle\Psi_\beta|\,\mathcal{O}_L(x_1,-t)\ldots\mathcal{O}_R(x_n',t_n')\,|\Psi_\beta\rangle = \mathrm{tr}_{\mathcal{H}_R}\left(\rho_R(\beta)\mathcal{O}_R(x_1,t-i\beta/2)\ldots\mathcal{O}_R(x_n',t_n')\right)\,.$$

Will use this observation when computing Renyi entropies

CFT considerations

As discussed by Morrison & Roberts (see also Hartman & Maldacena) :

- *single* sided thermal correlation functions are computed on a *single* cylinder with periodicity $\tau \sim \tau + \beta$
- *two-sided* correlators involve a path integral over a cylinder with the same periodicity $\tau \sim \tau + \beta$, where *all* operators $\mathcal{O}_R$ are inserted at $\tau = i\beta/2$, whereas $\mathcal{O}_L$ are inserted at $\tau = 0$

Set-up : Consider thermofield double state

- two finite intervals: $A = [y, y + L]$ in the left CFT_L and $B = [y, y + L]$ in the right CFT_R
- perturb the TFD by the insertion of a local primary operator $\mathcal{O}_L$ acting on CFT_L at $x = 0\,,\, t_- = -t_\omega$

Calculation of S_A

$$S_A = -\lim_{n\to 1} \frac{1}{n-1} \log\left(\mathrm{Tr}\, \rho_A^n(t)\right)$$

where

$$\mathrm{Tr}\, \rho_A^n(t) = \frac{\langle \psi(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\tilde{\sigma}(x_3,\bar{x}_3)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle \psi(x,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

with the insertion points

$$\begin{aligned} x_1 &= -i\epsilon\,, \quad x_2 = y - t_\omega - t_-, \quad x_3 = y + L - t_\omega - t_-\,, \quad x_4 = +i\epsilon \\ \bar{x}_1 &= +i\epsilon\,, \quad \bar{x}_2 = y + t_\omega + t_-, \quad \bar{x}_3 = y + L + t_\omega + t_-\,, \quad \bar{x}_4 = -i\epsilon \end{aligned}$$

with conformal dimensions

$$H_\psi = n h_\psi\,, \quad H_\sigma = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

Conformal maps

1. From the cylinder to the plane

$$\omega(x) = e^{2\pi x/\beta}$$

2. Standard map : $\omega_1 \to 0$, $\omega_2 \to z$, $\omega_3 \to 1$ and $\omega_4 \to \infty$

$$z(\omega) = \frac{(\omega_1 - \omega)\omega_{34}}{\omega_{13}(\omega - \omega_4)}$$

where the cross-ratio satisfies

$$z = \frac{\omega_{12}\omega_{34}}{\omega_{13}\omega_{24}}$$

Result

$$S_A^{(n)} = \frac{c(n+1)}{6}\log\left(\frac{\beta}{\pi\epsilon_{UV}}\sinh\frac{\pi L}{\beta}\right) + \frac{1}{n-1}\log\left(|1-z|^{4H_\sigma}G(z,\bar{z})\right)$$

where

$$G(z,\bar{z}) = \langle\psi|\,\sigma(z,\bar{z})\tilde{\sigma}(1,1)\,|\psi\rangle$$

Using the large c results derived by Fitzpatrick, Kaplan & Walters in the limit $n \to 1$

$$\Delta S_A = \frac{c}{6}\log\left(\frac{z^{\frac{1}{2}(1-\alpha_\psi)}\bar{z}^{\frac{1}{2}(1-\bar{\alpha}_\psi)}(1-z^{\alpha_\psi})(1-\bar{z}^{\bar{\alpha}_\psi})}{\alpha_\psi\bar{\alpha}_\psi(1-z)(1-\bar{z})}\right)$$

where $\alpha_\psi = \sqrt{1-\frac{24h_\psi}{c}}$.

Final result

Analysing the imaginary parts, we reach the conclusions :

- $(z, \bar{z}) \to (1, 1)$ for $t + t_\omega < y$ and $t + t_\omega > y + L$
- $(z, \bar{z}) \to (e^{2\pi i}, 1)$ for $y < t + t_\omega < y + L$

The importance of this monodromy has been emphasized by several groups including Asplund, Bernamonti, Galli & Hartman and Roberts & Stanford

$$\boxed{\Delta S_A = 0\,, \quad t_- + t_\omega < y \text{ and } t_- + t_\omega > y + L}$$

$$\boxed{\begin{array}{l} \Delta S_A = \dfrac{c}{6} \log \left[\dfrac{\beta}{\pi \epsilon} \dfrac{\sin \pi \alpha_\psi}{\alpha_\psi} \dfrac{\sinh\left(\frac{\pi(y+L-t_- - t_\omega)}{\beta}\right) \sinh\left(\frac{\pi(t_- + t_\omega - y)}{\beta}\right)}{\sinh\left(\frac{\pi L}{\beta}\right)} \right] \\ \qquad y < t_- + t_\omega < y + L \end{array}}$$

Calculation of S_B

Very similar, but with different insertion points :

$$\mathrm{Tr}\,\rho_A^n(t) = \frac{\langle \psi(x_1,\bar{x}_1)\sigma(x_5,\bar{x}_5)\tilde{\sigma}(x_6,\bar{x}_6)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle \psi(x,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

with the insertion points

$$x_5 = y + L + i\frac{\beta}{2} - t_+ - t_\omega, \qquad x_6 = y + i\frac{\beta}{2} - t_+ - t_\omega$$
$$\bar{x}_5 = y + L - i\frac{\beta}{2} + t_+ + t_\omega, \qquad \bar{x}_6 = y - i\frac{\beta}{2} + t_+ + t_\omega$$

We always obtain the expected thermal answer at all times

$$\boxed{S_B = \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon_{\mathrm{UV}}}\sinh\frac{\pi L}{\beta}\right)}$$

Calculation of $S_{A\cup B}$

Very similar, but with different insertion points :

$$\mathrm{Tr}\,\rho^n_{A\cup B}(t) = \frac{\langle\psi(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\tilde{\sigma}(x_3,\bar{x}_3)\sigma(x_5,\bar{x}_5)\tilde{\sigma}(x_6,\bar{x}_6)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle\psi(x,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

with the insertion points

$$\begin{aligned}
x_1 &= -i\epsilon, \quad x_2 = y - t_- - t_\omega, \quad x_3 = y + L - t_- - t_\omega, \quad x_4 = +i\epsilon \\
\bar{x}_1 &= +i\epsilon, \quad \bar{x}_2 = y + t_- + t_\omega, \quad \bar{x}_3 = y + L + t_- + t_\omega, \quad \bar{x}_4 = -i\epsilon \\
x_5 &= y + L + i\frac{\beta}{2} - t_+ - t_\omega, \quad x_6 = y + i\frac{\beta}{2} - t_+ - t_\omega, \\
\bar{x}_5 &= y + L - i\frac{\beta}{2} + t_+ + t_\omega, \quad \bar{x}_6 = y - i\frac{\beta}{2} + t_+ + t_\omega\,.
\end{aligned}$$

Strategy

Using conformal maps

$$\mathrm{Tr}\,\rho^n_{A\cup B} = \left|\frac{\beta}{\pi\epsilon_{UV}}\sinh\left(\frac{\pi L}{\beta}\right)\right|^{-8H_\sigma} |1-z|^{4H_\sigma}\,|z_{56}|^{4H_\sigma}$$
$$\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$

where all cross-ratios z, z_i are analytically known.

- $\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$ expected 6-pt function

S-channel (I)

Let us introduce a resolution of the identity

$$\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle = \sum_\alpha \langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\,|\alpha\rangle\,\langle\alpha|\,\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$

- $(z,\bar{z}) \to (1,1)$ for $\frac{\epsilon}{\beta} \ll 1 \Rightarrow$ use OPE !!
- $\sigma(z,\bar{z})\tilde{\sigma}(1,1) \sim \mathbb{I} +$ corrections in $(z-1)^r\ \mathcal{O}_r$
- Orthogonality of 2-pt functions $\Rightarrow |\alpha\rangle = |\psi\rangle$ dominant

Thus,

$$\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle \simeq \langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\,|\psi\rangle\ \langle\psi|\,\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$

S-channel (II)

Using conformal maps

$$\langle\psi|\,\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle = |1-\tilde{z}_5|^{4H_\sigma}\,|z_{56}|^{-4H_\sigma}\,\langle\psi|\,\sigma(\tilde{z}_5,\bar{\tilde{z}}_5)\tilde{\sigma}(1,1)|\psi\rangle\,,$$

we obtain

$$\mathrm{Tr}\,\rho^n_{A\cup B} \simeq \left|\frac{\beta}{\pi\epsilon_{UV}}\sinh\left(\frac{\pi L}{\beta}\right)\right|^{-8H_\sigma} |1-z|^{4H_\sigma}\,|1-\tilde{z}_5|^{4H_\sigma}\,G(z,\bar{z})G(\tilde{z}_5,\bar{\tilde{z}}_5)+\ldots$$

Since $\tilde{z}_5 = z_5$, the cross-ratio determining S_B, we derive

$$S_{A\cup B} = S_A + S_B, \qquad \text{and} \qquad I(A:B) = 0$$

This resembles the bulk calculation from two geodesics joining pairs of points in the same boundary !!

T-channel (I)

We could introduce the resolution of the identity as follows

$$\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle = \sum_\alpha \langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(z_6,\bar{z}_6)\,|\alpha\rangle\,\langle\alpha|\,\sigma(z_5,\bar{z}_5)\tilde{\sigma}(1,1)|\psi\rangle\,.$$

- $(z_5,\bar{z}_5)\to(1,1)$ for $\frac{\epsilon}{\beta}\ll 1 \Rightarrow$ use OPE !!
- As before, $|\alpha\rangle = |\psi\rangle$ dominant contribution !!

T-channel (II)

In this case,

$$\mathrm{Tr}\,\rho^n_{A\cup B} \simeq \left|\frac{\beta}{\pi\epsilon_{UV}}\sinh\left(\frac{\pi L}{\beta}\right)\right|^{-8H_\sigma}\left|\frac{x}{1-x}\right|^{4H_\sigma}|1-z_5|^{4H_\sigma}|1-\tilde{z}_2|^{4H_\sigma}$$
$$G(\tilde{z}_2,\bar{\tilde{z}}_2)G(z_5,\bar{z}_5)+\ldots$$

where $(x,\bar{x})$ are the cross-ratios computed out of the insertion points of the four twist operators

$$x=\frac{z_{23}z_{56}}{z_{25}z_{36}}=\frac{w_{23}w_{56}}{w_{25}w_{36}}=\frac{2\sinh^2\frac{\pi L}{\beta}}{\cosh\frac{2\pi L}{\beta}+\cosh\frac{2\pi(t_- - t_+)}{\beta}}=\bar{x}\,,$$

T-channel (III)

For $t_- + t_\omega > y + L$, we derive

$$\boxed{\begin{aligned} S_{A\cup B} \simeq{}& \frac{c}{6}\log\left(\frac{\sinh\frac{\pi(t_-+t_\omega-y)}{\beta}\cosh\frac{\pi(t_++t_\omega-y)}{\beta}}{\cosh\frac{\pi\Delta t}{\beta}}\right) \\ &+\frac{c}{6}\log\left(\frac{\sinh\frac{\pi(t_-+t_\omega-y-L)}{\beta}\cosh\frac{\pi(t_++t_\omega-y-L)}{\beta}}{\cosh\frac{\pi\Delta t}{\beta}}\right) \\ &+\frac{2c}{3}\log\left|\frac{\beta}{\pi\varepsilon_{UV}}\cosh\left(\frac{\pi\Delta t}{\beta}\right)\right| + \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi}\right)\end{aligned}}$$

where $\Delta t = t_- - t_+$

- To derive this result we used Fitzpatrick, Kaplan & Walters
- Assumption : conformal block of the identity is dominant

Mutual information

In the regime $t_\mp + t_\omega > y + L > y$,

$$
\boxed{\begin{aligned}
I(A:B) \simeq\; & \frac{2c}{3}\log\left(\frac{\beta}{\pi\varepsilon_{UV}}\sinh\frac{\pi L}{\beta}\right) - \frac{2c}{3}\log\left|\frac{\beta}{\pi\varepsilon_{UV}}\cosh\left(\frac{\pi\Delta t}{\beta}\right)\right| \\
& - \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi}\right) \\
& - \frac{c}{6}\log\left(\frac{\sinh\frac{\pi(t_- + t_\omega - y)}{\beta}\cosh\frac{\pi(t_+ + t_\omega - y)}{\beta}}{\cosh\frac{\pi\Delta t}{\beta}}\right) \\
& - \frac{c}{6}\log\left(\frac{\sinh\frac{\pi(t_- + t_\omega - y - L)}{\beta}\cosh\frac{\pi(t_+ + t_\omega - y - L)}{\beta}}{\cosh\frac{\pi\Delta t}{\beta}}\right)
\end{aligned}}
$$

- take $t_- = t_+ = 0$ and look for $t^\star_\omega$ satisfying

$$I(A:B)(t^\star_\omega) = 0$$

Scrambling time

Assuming $t^\star_\omega/\beta \gg 1$, we obtain

$$\boxed{t^\star_\omega = y + \frac{L}{2} - \frac{\beta}{2\pi}\log\left(\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi}\right) + \frac{\beta}{\pi}\log\left(2\sinh\frac{\pi L}{\beta}\right)}$$

- if $h_\psi \ll c$, then

$$t^\star_\omega = y + \frac{L}{2} + \frac{\beta}{2\pi}\log\left(\frac{\pi S_{density}}{4E_\psi}\right) + \frac{\beta}{\pi}\log\left(2\sinh\frac{\pi L}{\beta}\right)$$

where we used

$$\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi} \sim \frac{\pi E_\psi}{S_{\text{density}}}$$

with $S_{\text{density}} = \frac{\pi c}{3\beta}$ and $E_\psi = \frac{\pi h_\psi}{\epsilon}$

Holographic considerations

Main idea & strategy :

- Static point particle at $r = 0$ in global AdS_3

$$ds^2 = -\left(r^2 + R^2 - \mu\right) d\tau^2 + \frac{R^2 dr^2}{r^2 + R^2 - \mu} + r^2 d\varphi^2 \,,$$

- Holographic entanglement entropy known

$$S_A = \frac{c}{6}\left[\log\left(\frac{r_\infty^{(1)} \cdot r_\infty^{(2)}}{R^2}\right) + \log \frac{2\cos\left(|\Delta\tau_\infty|\alpha_\mu\right) - 2\cos\left(|\Delta\varphi_\infty|\alpha_\mu\right)}{\alpha_\mu^2}\right]$$

- Map metric to Kruskal coordinates, while boosting the particle, to describe a free falling particle in eternal BTZ
  - Use an initial condition ensuring the particle carries the right energy, from CFT and stress tensor perspective
- Map endpoints & compute entanglement entropy

Holographic comments

Calculations involve many explicit technical details, leading to

1. Exact matching of dominant CFT contributions with the holographic model geodesic calculations
   - S-channel and T-channel contributions precisely match the two dominant geodesics computing $S_{A\cup B}$
2. In the limit of large t_ω :
   - free falling particle becomes almost null with energy localised at the horizon
   - matches the schock-wave descriptions proposed/used by Shenker, Stanford, Roberts, Susskind, ...

Relation to quantum chaos & butterfly effect

Kitaev, and later Maldacena, Shenker & Stanford, suggested the use of a large commutator

$$-\langle [W(t),\ V(0)]^2 \rangle_\beta$$

to diagnose the butterfly effect. Using the regularised quantity,

$$A = -\mathrm{tr}\left(y^2\ [W(t),\ V(0)]\ y^2\ [W(t),\ V(0)] \right) \quad \text{where} \quad y^4 = \frac{e^{-\beta H}}{Z(\beta)}$$

it follows

$$\begin{aligned} A &= \mathrm{tr}\left(y^2\, W(t) V(0)\, y^2 V(0) W(t) \right) + \mathrm{tr}\left(y^2\ V(0) W(t)\, y^2 W(t) V(0) \right) \\ &- F(t + i\frac{\beta}{4}) - F(t - i\frac{\beta}{4}) \\ F(t) &= \mathrm{tr}\left(y V(0) y W(t) y V(0) y W(t) \right) . \end{aligned}$$

Notice $F(t)$ involves out of time ordered (OTO) correlators.

Relation to quantum chaos & butterfly effect

Using the thermo-field double state :

- First two terms are order 1, for all t, since they are norms of states
- Growth of the regularised commutator, requires $F(t \pm i\beta/4)$ to become small

$$F(t) = \langle\Psi|V_L V_R|\Psi\rangle$$

with $|\Psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_{m,n} e^{-\beta(E_m+E_n)/4} W(t)_{nm}|m\rangle_L|n\rangle_R$

- small t, $F(t)$ is order one due to its nearly maximally entangled nature
- as t increases, correlations can get destroyed and $F(t)$ decreases

$\Rightarrow$ quantum chaos $\sim$ destruction of these correlations.

Holographic considerations(IMPROVE)

- Time ordered correlators :

$$\langle V(0)V(0)W(t)W(t)\rangle \sim \langle VV\rangle\langle WW\rangle + \mathcal{O}\left(e^{-t/t_d}\right)$$

where $t_d \sim \beta$ is controlled by BH quasi-normal modes.

- Holographic calculations determine OTOs to behave like (Shenker, Stanford, Roberts, Susskind)

$$\langle W(t)V(0)W(t)V(0)\rangle_\beta = f_0 - \frac{f_1}{N^2}\,e^{2\pi t/\beta} + \mathcal{O}(N^{-4})$$

OTO decays at time scales $t^\star \sim \frac{\beta}{2\pi}\log N^2$.
Characteristic feature for holographic theories.

Question : Any CFT evidence for this behaviour ?

Large c 2d CFT considerations

Euclidean correlators

$$\frac{\langle V^\dagger(z_1,\bar z_1)V(z_2,\bar z_2)W^\dagger(z_3,\bar z_3)W(z_4,\bar z_4)\rangle}{\langle V^\dagger(z_1,\bar z_1)V(z_2,\bar z_2)\rangle\langle W^\dagger(z_3,\bar z_3)W(z_4,\bar z_4)\rangle} = \mathcal{A}(z,\bar z)$$

have branch cuts from $z \in (1,\infty)$ & OTOs defined by analytic continuation

$$\begin{aligned} z_1 &= e^{\frac{2\pi}{\beta}i\varepsilon_1}\,, \quad \bar z_1 = e^{-\frac{2\pi}{\beta}i\varepsilon_1} \\ z_2 &= e^{\frac{2\pi}{\beta}i\varepsilon_2}\,, \quad \bar z_2 = e^{-\frac{2\pi}{\beta}i\varepsilon_2} \\ z_3 &= e^{\frac{2\pi}{\beta}(t+i\varepsilon_3-x)}\,, \quad \bar z_3 = e^{\frac{2\pi}{\beta}(-t-i\varepsilon_3-x)} \\ z_4 &= e^{\frac{2\pi}{\beta}(t+i\varepsilon_4-x)}\,, \quad \bar z_4 = e^{\frac{2\pi}{\beta}(-t-i\varepsilon_4-x)} \end{aligned}$$

Chaos explored in the regime $t - x \gg \beta$

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}} \approx -e^{-\frac{2\pi}{\beta}(t-x)}\,\varepsilon_{12}^\star\varepsilon_{34} \to 0\,, \qquad \frac{\bar z}{z}\ \text{fixed}$$

Large c 2d CFT considerations

Assuming

- large c,
- sparse spectrum of light operators
- absence of light single-trace operators of spin $s > 2$

Perlmutter, building on Roberts & Stanford

$$\frac{\langle VW(t)VW(t)\rangle_\beta}{\langle VV\rangle_\beta\langle W(t)W(t)\rangle_\beta} \sim 1 - \frac{i}{c\,\varepsilon_{12}^\star\varepsilon_{34}}\, e^{-\frac{2\pi(t-x)}{\beta}}\, f(e^{-\frac{4\pi}{\beta}x}) + \dots$$

Scrambling time $\sim$ scale at which expansion breaks down

$$t_\star \sim \frac{\beta}{2\pi}\log c$$

Butterfly effect $\Rightarrow$ scrambling (in QM)

Hosur, Qi, Roberts & Yoshida showed that averaging OTOs over a complete basis of operators in the subsystems A and D

$$|\langle \mathcal{O}_D(t)\, \mathcal{O}_A(0)\, \mathcal{O}_D(t)\, \mathcal{O}_A(0)\rangle| \sim 2^{-S^{(2)}_{A\cup C}}$$

Since $S_{A\cup C} \geq S^{(2)}_{A\cup C}$

$$I(A:C) \leq 2a - \log_2 \epsilon \quad \text{with} \quad \epsilon \sim f_0 - \frac{f_1}{N^2}\, e^{\lambda_L t}$$

where we already assumed the system is quantum chaotic.
Introducing $N^2 \sim e^{\lambda_L t^\star}$, then we find

$$I(A:C) \leq 2a - \natural\, e^{\lambda_L (t-t^\star)} + \dots$$

- If system is chaotic, QI magnitudes relevant to scrambling approach their Haar-scrambled values.

Scrambling $\Rightarrow$ quantum chaos ?

Question : what about integrable 2d CFTs ?
Intuition : OTO should remain of order one (lack of quantum chaos)

Caputa, Numawara & Veliz-Osorio (see also Qi & Gu) find this behaviour for 2d rational CFTs

- $\exists$ scrambling in $SU(N)_k$ WZW models in the large c limit
- order of limits concerning $c \to \infty$??