Dynamical Boundary Diffeomorphisms in 2+1 Dimensions

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The idea in a nutshell

- Action on manifold with boundary has two pieces:

$$I = I_{\mathit{bulk}} + I_{\mathit{bdry}}$$

  Boundary piece needed
  - Classically: to allow extrema
  - Quantum mechanically: to ensure proper "sewing" of path integrals
- Gauge symmetries of I_{bulk} will typically be broken by I_{bdry}
  - Formerly nonphysical degrees of freedom become dynamical at boundary
- Action for new degrees of freedom is induced from I_{bdry}
- Results for (2+1)-dimensional gravity:
  - Asymptotically AdS: Liouvulle action
  - Asymptotically flat: action related to Liouville, Hill's equation
    (work in progress . . .)

$(2+1)$-dimensional gravity

Two tactics

- Start with Chern-Simons formulation
  - simple decomposition $A = g^{-1}dg + g^{-1}\bar{A}g$
  - standard reduction to WZNW model at boundary (plus constraints)
  - boundary term known to be right for "sewing"
  - ***but*** doesn't generalize to higher dimensions

- Use standard metric or vielbein formulation
  - no simple decomposition into gauge-fixed fields + diffeos
  - boundary theory may not be local
  - but presumably more widely applicable

The AdS case

Fefferman-Graham expansion of metric:

$$ds^2 = -\ell^2 d\rho^2 + g_{ij}dx^i dx^j, \quad \text{with } g_{ij} = e^{2\rho}\overset{(0)}{g}_{ij}(x) + \overset{(2)}{g}_{ij}(x) + \dots$$

Field equations:

$$\overset{(2)}{g}{}^i{}_i = -\frac{\ell^2}{2}\overset{(0)}{R}, \qquad \overset{(0)}{\nabla}_i \overset{(2)}{g}_{jk} - \overset{(0)}{\nabla}_j \overset{(2)}{g}_{ik} = 0$$

Diffeomorphism:

$$\rho \to \rho + \frac{1}{2}\varphi(x) + e^{-2\rho}\overset{(2)}{f}(x) + \dots$$

$$x^i \to x^i + e^{-2\rho}\overset{(2)}{h}{}^i(x) + \dots$$

Form invariance of metric:

$$\overset{(2)}{h}{}^i = -\frac{\ell^2}{4}e^{-\varphi}\overset{(0)}{g}{}^{ij}\partial_i\varphi, \qquad \overset{(2)}{f} = -\frac{\ell^2}{16}e^{-\varphi}\overset{(0)}{g}{}^{ij}\partial_i\varphi\partial_j\varphi.$$

New spatial metric:

$$g_{ij} = e^{2\rho} e^{\varphi} \overset{(0)}{g}_{ij} + 8\pi G \ell T_{ij} + \left(\overset{(2)}{g}_{ij} - \frac{\ell^2}{2} \overset{(0)}{g}_{ij} \overset{(0)}{\Delta} \varphi - \frac{\ell^2}{4} \lambda \overset{(0)}{g}_{ij} e^{\varphi} \right) + \dots$$

where

$$T_{ij} = \frac{\ell}{32\pi G} \Big[\partial_i \varphi \partial_j \varphi - \frac{1}{2} \overset{(0)}{g}_{ij} \overset{(0)}{g}{}^{kl} \partial_k \varphi \partial_l \varphi - 2 \overset{(0)}{\nabla}_i \overset{(0)}{\nabla}_j \varphi + 2 \overset{(0)}{g}_{ij} \overset{(0)}{\Delta} \varphi + \lambda \overset{(0)}{g}_{ij} e^{\varphi} \Big]$$

(Liouville stress-energy tensor)

Action

$$I_{grav} = \frac{1}{16\pi G}\int_M d^3x\sqrt{^{(3)}g}\left({}^{(3)}R + \frac{2}{\ell^2}\right) + \frac{1}{8\pi G}\int_{\partial M} d^2x\sqrt{\gamma}K - \frac{1}{8\pi G\ell}\int_{\partial M} d^2x\sqrt{\gamma}$$

Original boundary at $\rho = \bar{\rho}$; new boundary at

$$\rho = \bar{\rho} + \frac{1}{2}\varphi(x) + e^{-2\bar{\rho}}\overset{(2)}{f}(x) = F(x)$$

Compute new normal, extrinsic curvature, induced metric: find

$$I_{grav} = -\frac{\ell}{16\pi G}\int_{\partial M} d^2x\sqrt{\overset{(0)}{g}}\left(\overset{(0)}{g}{}^{ij}\partial_i F\partial_j F - F\overset{(0)}{R}\right)$$

(Liouville action in limit $\lambda \to 0$)

What does this mean?

- CFT with right central charge to match Brown-Henneaux:
  Cardy formula gives correct entropy

- Coupling "classical" source at boundary gives right Hawking radiation (Emparan & Sachs)

- Minimal: "effective description" of black hole states

- Maximal: Liouville theory "really" describes black hole states

Liouville theory has two sectors:

– "normalizable states": $\Delta \geq \frac{c-1}{24}, \quad c_{\mathit{eff}} = 1$
  $\Leftrightarrow$ BTZ black holes

– "nonnormalizable states": fill in gap inf $\Delta, \quad c_{\mathit{eff}} = c$
  $\Leftrightarrow$ point particles/conical defects

The asymptotically flat case (work in progress*...)

Again partially gauge-fix metric: Bondi coordinates

$$ds^2 = -2dudr + sdu^2 + 2tdud\phi + r^2 e^{2\varphi} d\phi^2$$

First problem: need right boundary terms

$$\begin{aligned}\delta I_{grav} &= \textit{bulk piece} \\ &+ \frac{1}{16\pi G}\int_{\partial M} d^2x\,[-\partial_r(re^{\varphi}\delta s) - 2(\partial_u + s\partial_r)(re^{\varphi}\delta\varphi)] + \mathcal{O}(1/r) \\ &= \cdots + \frac{1}{16\pi G}\,\delta\int_{\partial M} d^2x\,[-2se^{\varphi} - 2re^{\varphi}\partial_u\varphi + e^{\varphi}(s - r\partial_r s)] \\ &- \frac{1}{16\pi G}\int_{\partial M} d^2x\, e^{\varphi}(s - r\partial_r s)\delta\varphi + \mathcal{O}(1/r)\end{aligned}$$

**Warning: do not believe all factors of 2*

From field equations (Barnich & Troessaert)

$$s = -2r\partial_u\varphi + e^{-2\varphi}\left[-(\partial_\phi\varphi)^2 + 2\partial_\phi^2\varphi + \Theta\right] \quad \text{with } \partial_u\Theta = 0$$

In this form, can integrate $e^{\varphi}(s - r\partial_r s)\delta\varphi$; find

$$I_{bdry} = \frac{1}{8\pi G}\int_{\partial M} d^2x \left[e^{-\varphi}(\partial_\phi\varphi)^2 + e^{-\varphi}\Theta\right]$$

Not quite Liouville action, but has interesting properties. . .

Let $\chi = e^{-\varphi/2}$

Then equations of motion are $\quad \partial_\phi^2\chi - \frac{\Theta}{4}\chi = 0$

(Hill's equation)

Action for diffeomorphisms: start with flat base metric

$$ds^2 = -2d\bar{u}d\bar{r} + d\bar{u}^2 + \bar{r}^2 d\bar{\phi}^2$$

Diffeomorphism

$$\bar{u} = u_0 + \frac{u_1}{r} + \dots, \qquad \bar{\phi} = \phi_0 + \frac{\phi_1}{r} + \dots, \qquad \bar{r} = ar + ab_0 + \frac{ab_1}{r} + \dots$$

Form invariance of metric $\Rightarrow$

$$\partial_u u_0 = \frac{1}{a}$$

$$\partial_\phi \phi_0 = \frac{1}{\beta} \qquad \text{with } \partial_u \beta = 0$$

$$b_0 = \frac{\beta}{a}\partial_\phi\left(\beta\partial_\phi u_0\right) + \frac{1}{2}\frac{\beta^2}{a^2}A$$

$$\text{where } g_{\phi\phi} = \frac{a^2}{\beta^2}r^2 + Ar$$

Confirm form

$$s = -2r\partial_u\varphi + e^{-2\varphi}\left[-(\partial_\phi\varphi)^2 + 2\partial_\phi^2\varphi + \Theta\right]$$

with $\varphi = \ln(a/\beta) = \ln(\partial_\phi\phi_0/\partial_u u_0)$ and

$$\Theta = 2\partial_\phi\left(\frac{\partial_\phi\beta}{\beta}\right) + \left(\frac{\partial_\phi\beta}{\beta}\right)^2 = -2\{\phi_0;\phi\}$$

Schwarzian derivative $\{f;z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$

Boundary action is then

$$I_{bdry} = \frac{1}{\pi G}\int_{\partial M} d^2x\left[\frac{1}{2}(\partial_\phi\chi)^2 - \frac{1}{4}\{\phi_0;\phi\}\chi^2\right]$$

with $\chi = e^{-\varphi/2}$

What can we say about this action?

• No u derivatives (why?)

• Equations of motion are Hill's equation

$$\partial_\phi^2 \chi + \frac{1}{2}\{\phi_0; \phi\}\chi = 0$$

Schwarzian derivative form of potential ⇒ periodicity of solutions

• Consider the auxiliary two-dimensional metric

$$d\tilde{s}^2 = \frac{1}{\beta}\left(du\,d\phi + e^{-\varphi}du^2\right)$$

If $\partial_u \varphi = 0$, then

$$I_{bdry} = \frac{1}{16\pi G}\int d^2x\sqrt{-\tilde{g}}\,\tilde{R}\,\Box^{-1}\tilde{R}$$

(Polyakov action with $c = 6/G$)

⇒ connections with CFT, but to be worked out ...