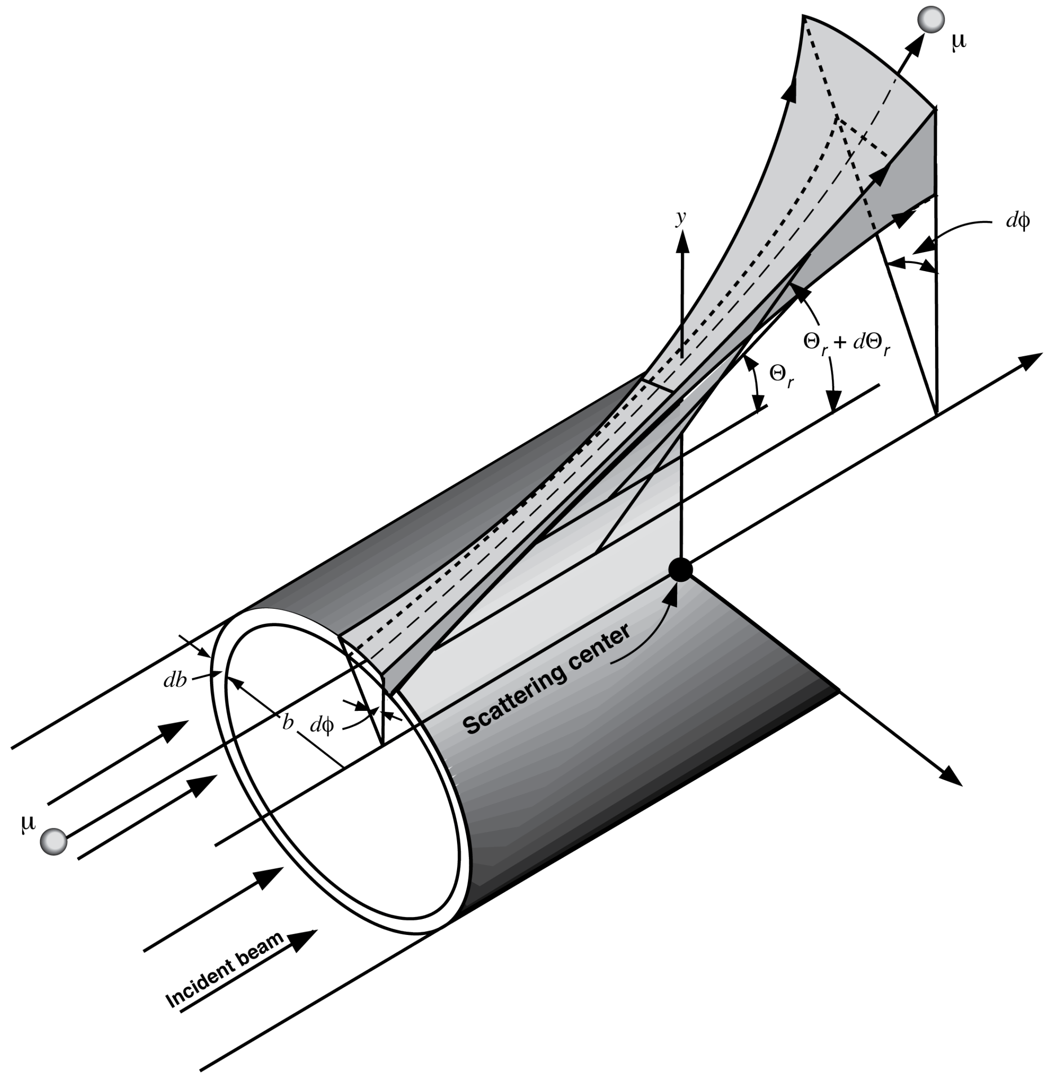
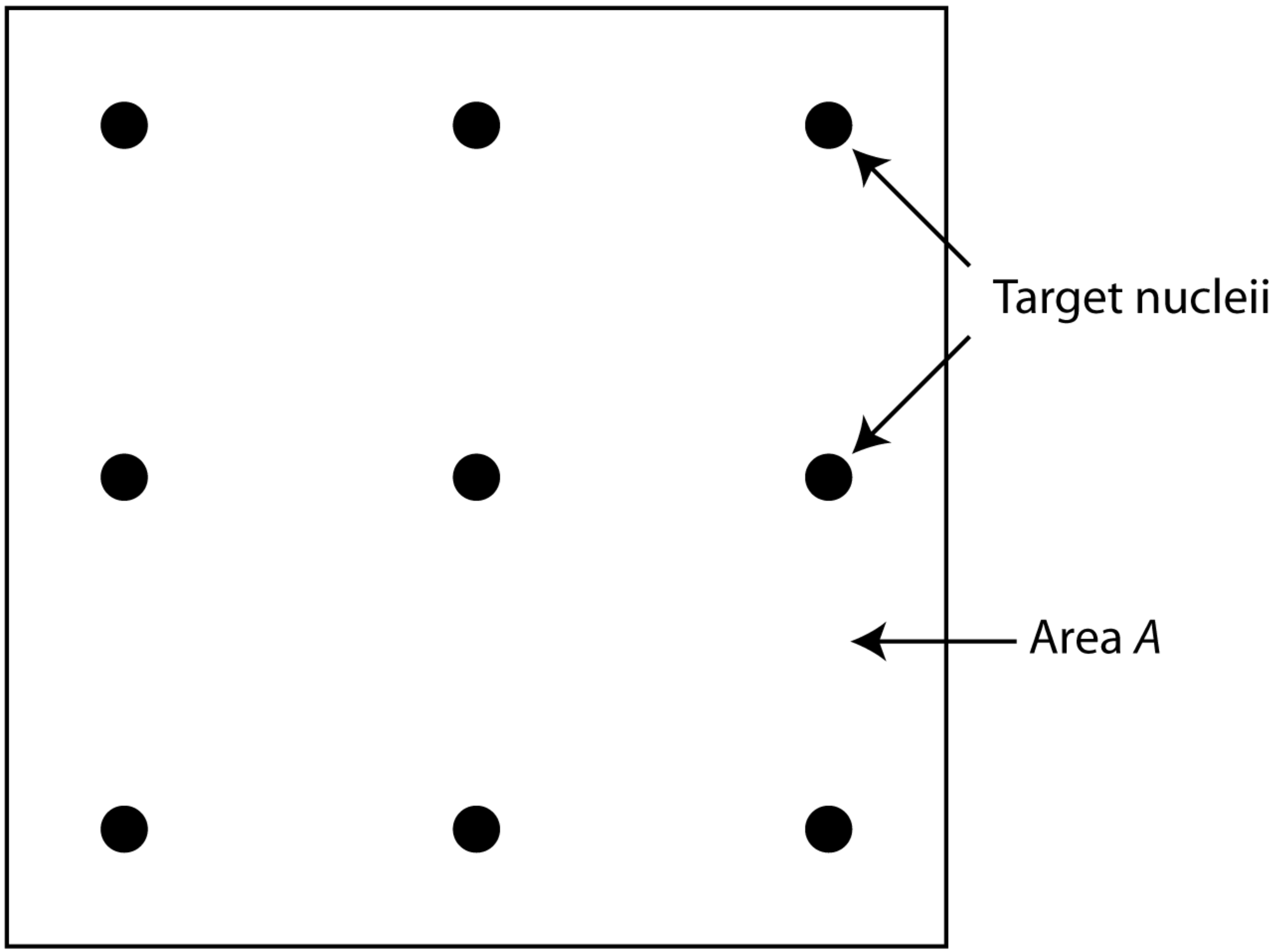
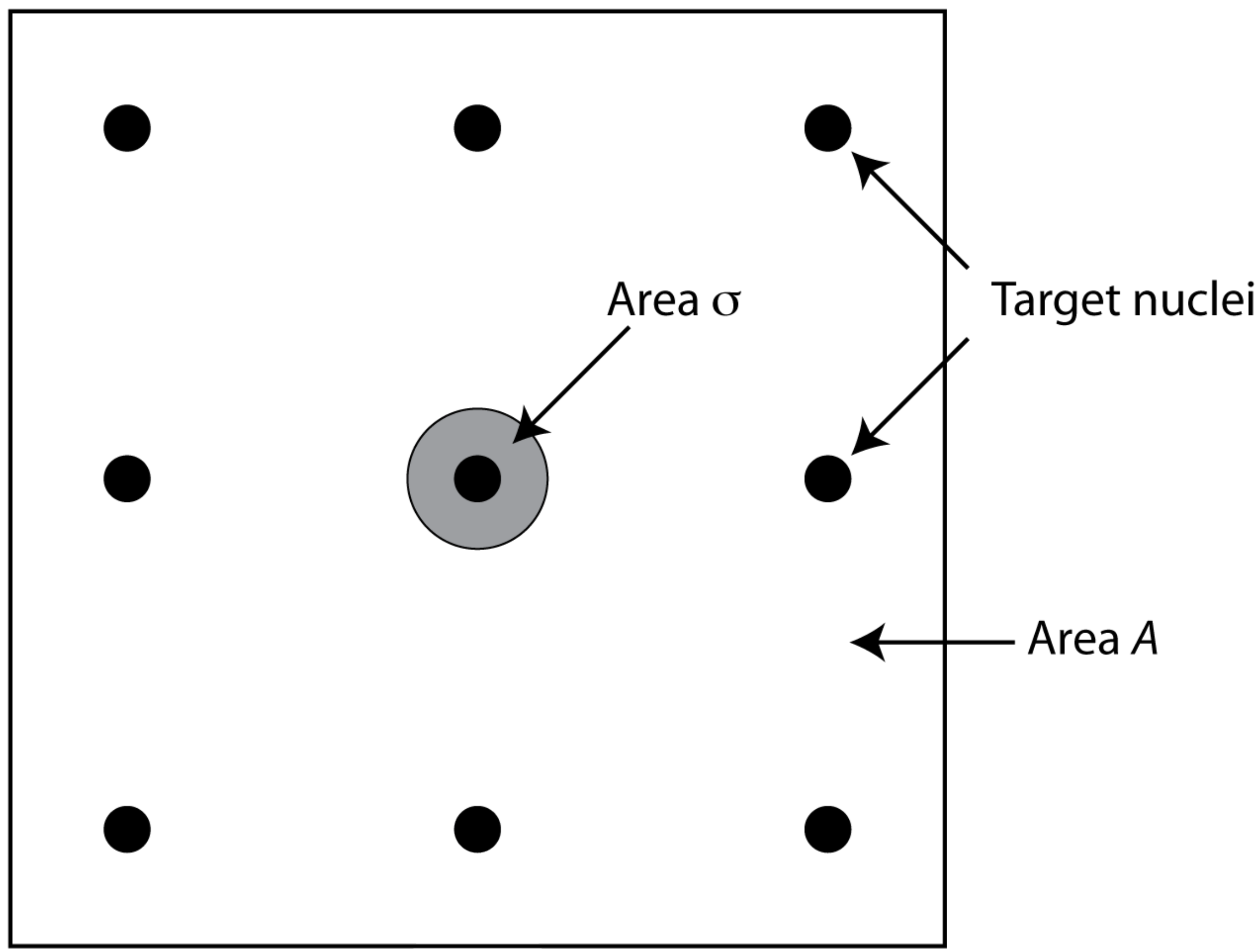
Part II
Introduction to Reaction and Scattering Cross-Sections



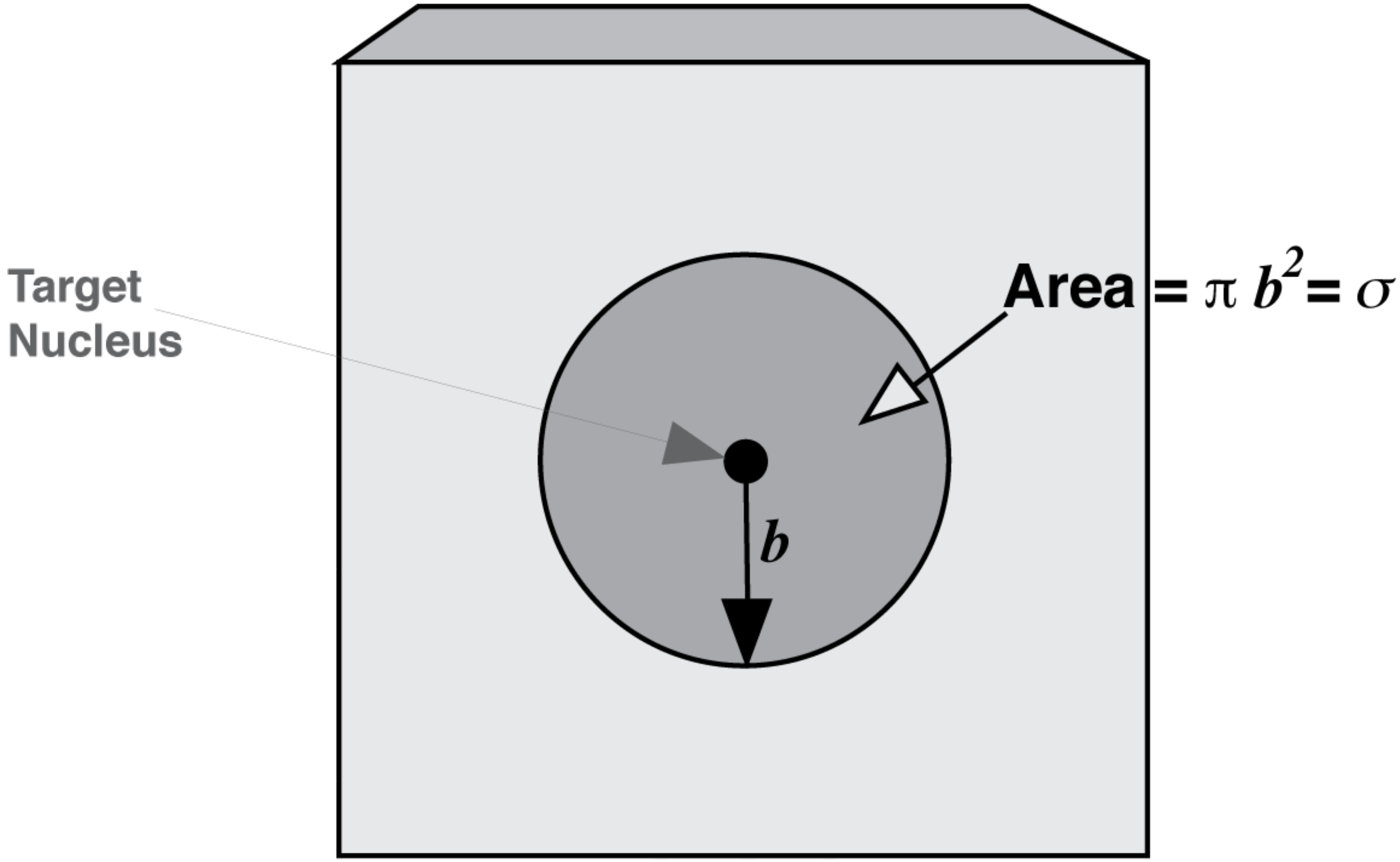
1. Reaction Cross-Section Introduction



The cross-section, σ, is the area that one atom presents to a beam of incident particles for initiating a specific "interaction"



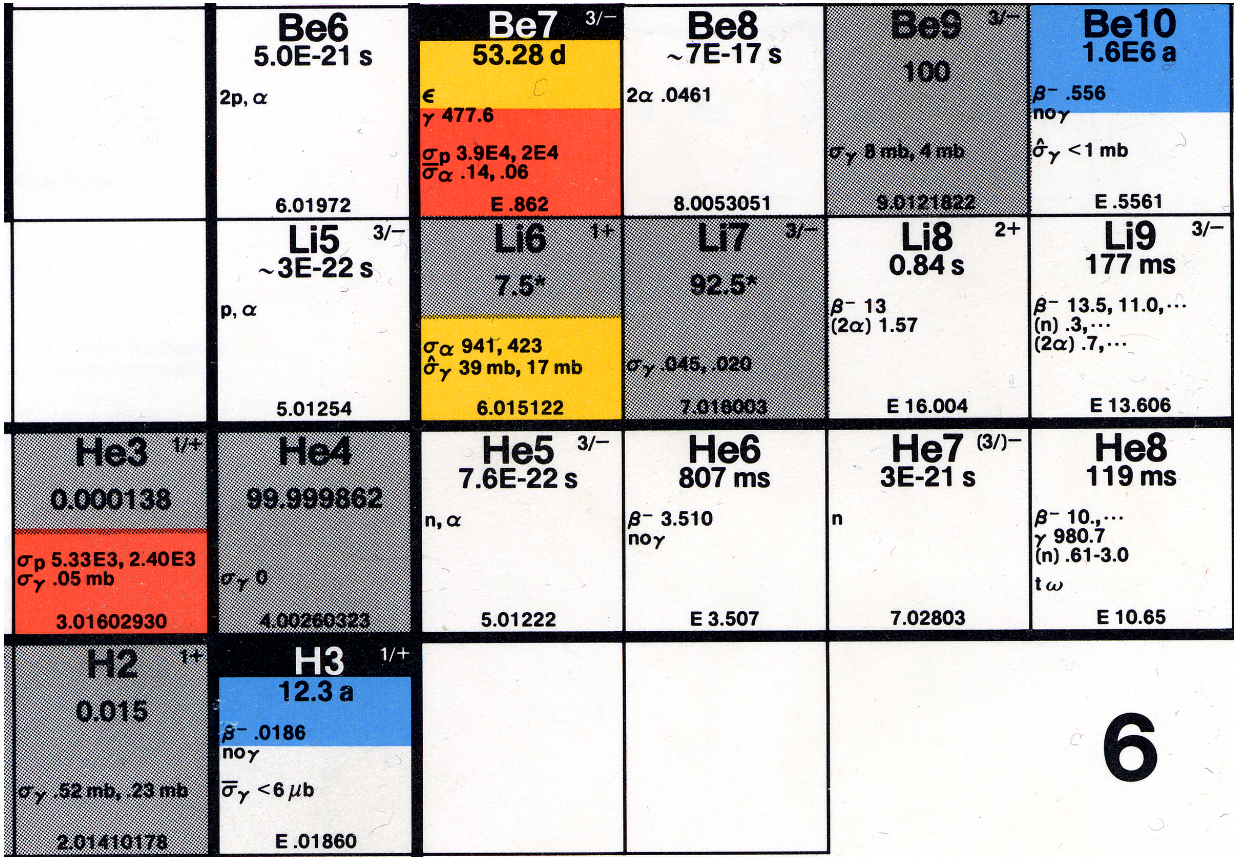
The impact parameter, *b*



$$\sigma\left[\mathrm{cm}^2\right]=\pi\,(b\left[\mathrm{cm}\right])^2$$

$$\sigma=\pi\,b^2$$

Transmutation reactions



$$^{6}_{3}\mathrm{Li} + {}^{1}_{0}n \rightarrow {}^{4}_{2}\mathrm{He} + {}^{3}_{1}\mathrm{H}$$

$$^{6}_{3}\mathrm{Li}\left(n,\alpha\right){}^{3}_{1}\mathrm{H}$$

Li transmutation via thermal *n* capture

$$\sigma_\alpha = 941 \text{ [barns]} = 941 \cdot 10^{-24} \text{ [cm}^2\text{]}$$

$$b_\alpha = \sqrt{\frac{\sigma_\alpha}{\pi}} = 1.73 \cdot 10^{-11} \text{ [cm]} = 1.73 \cdot 10^{-13} \text{ [m]} = 173 \text{ [fm]}$$

How large is 173 fm?

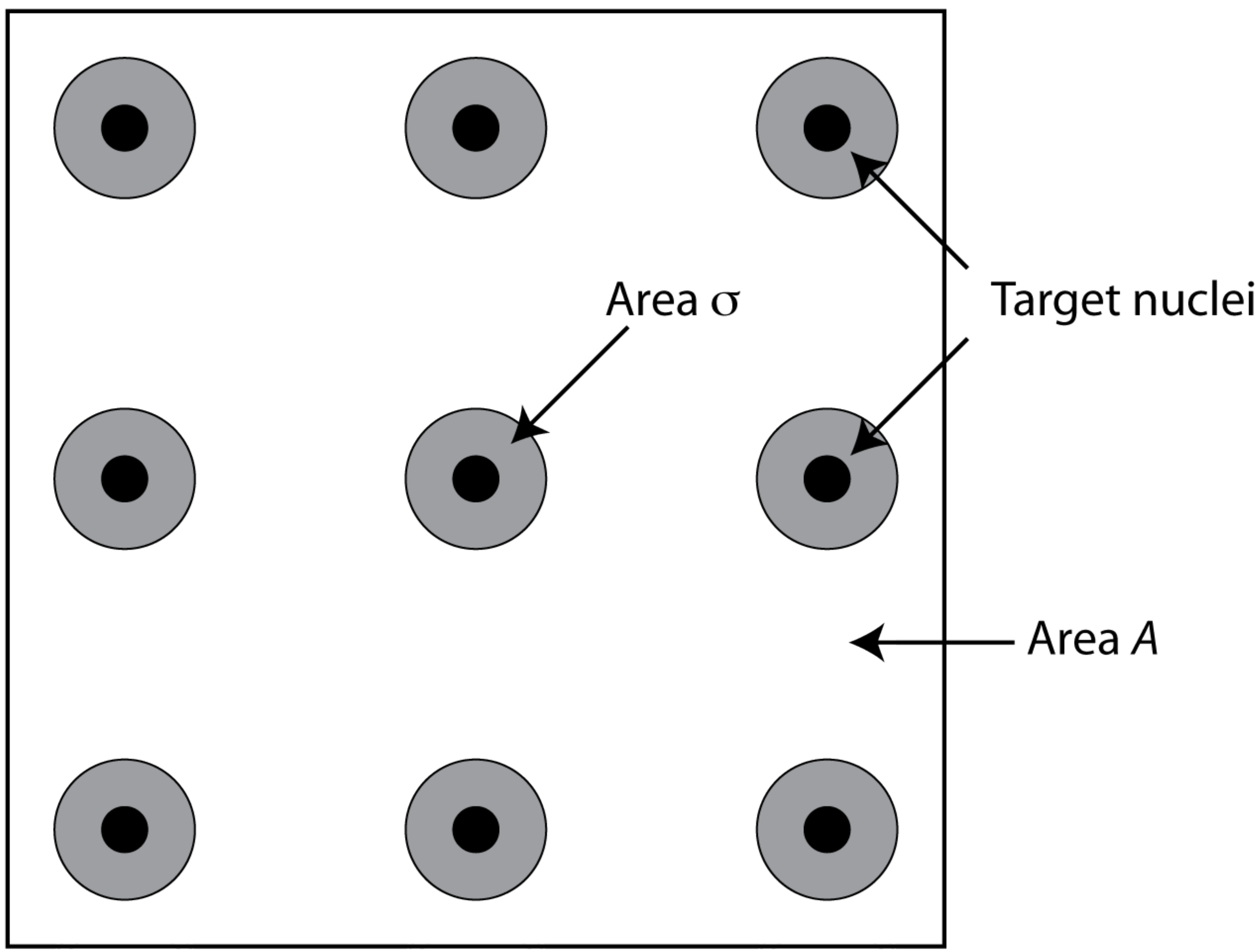
$$R = 1.25 \cdot 10^{-5}\ M_T^{1/3}\ [\text{Å}]$$

$$M_T\left({}^{6}\text{Li}\right) = 6.01522\ [\text{amu}]$$

$$R = 2.27332 \cdot 10^{-5}\ [\text{Å}]$$

$$= 2.27332 \cdot 10^{-5}\ [\text{Å}] = 2.27\ [\text{fm}]$$

Reaction Cross-Section(s). Relationship to the Probability of Interaction



Fluence

$$\Phi\left[\frac{\text{interactions}}{\text{cm}^2}\right] = \Phi_0\left[\frac{\text{projectile particles}}{\text{cm}^2}\right]\hat{\rho}\left[\frac{\text{target atoms}}{\text{cm}^2}\right] \quad \sigma\left[\frac{\text{cm}^2}{\text{target atom}}\right]$$

$$\Phi = \Phi_0\ \hat{\rho}\ \sigma$$

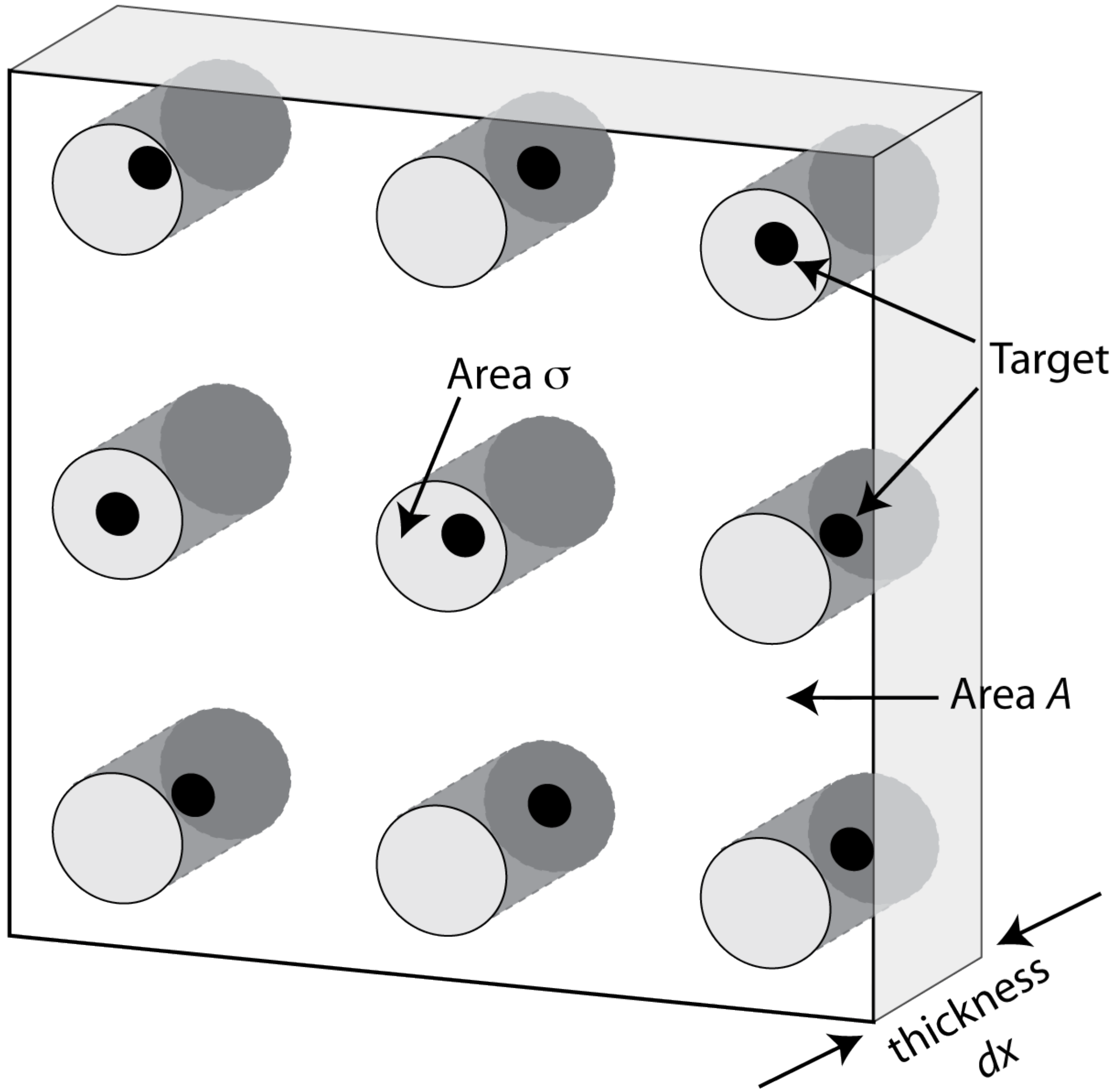
Probability vs. Cross-Section

$$Probabilty\ of\ Interaction = \frac{\Phi \left[\frac{\text{interactions}}{\text{cm}^2} \right]}{\Phi_0 \left[\frac{\text{projectile particles}}{\text{cm}^2} \right]}$$

$$= \hat{\rho} \left[\frac{\text{target atoms}}{\text{cm}^2} \right] \quad \sigma \left[\frac{\text{cm}^2}{\text{target atom}} \right]$$

$$P = \frac{\Phi}{\Phi_0} = \hat{\rho}\, \sigma$$

3-D



$$\Phi \left[\frac{\text{interactions}}{\text{cm}^2} \right] = \Phi_0 \left[\frac{\text{projectile particles}}{\text{cm}^2} \right] \rho \left[\frac{\text{target atoms}}{\text{cm}^3} \right] \cdot dx \; [\text{cm}] \; \sigma \left[\frac{\text{cm}^2}{\text{target atom}} \right]$$

$$\Phi = \Phi_0 \left(\rho \cdot dx \right) \sigma$$

3-D

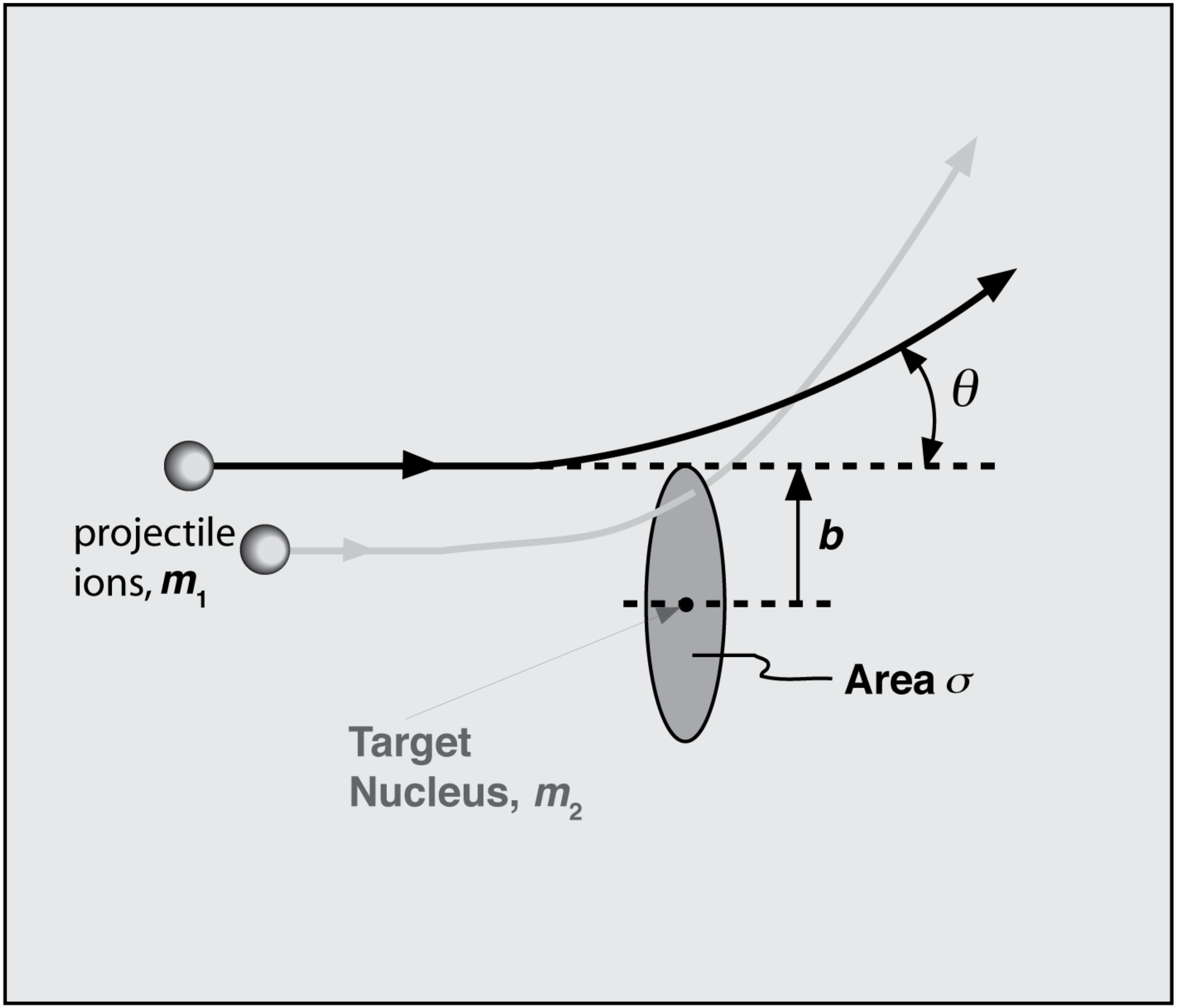
Probabilty of Interaction

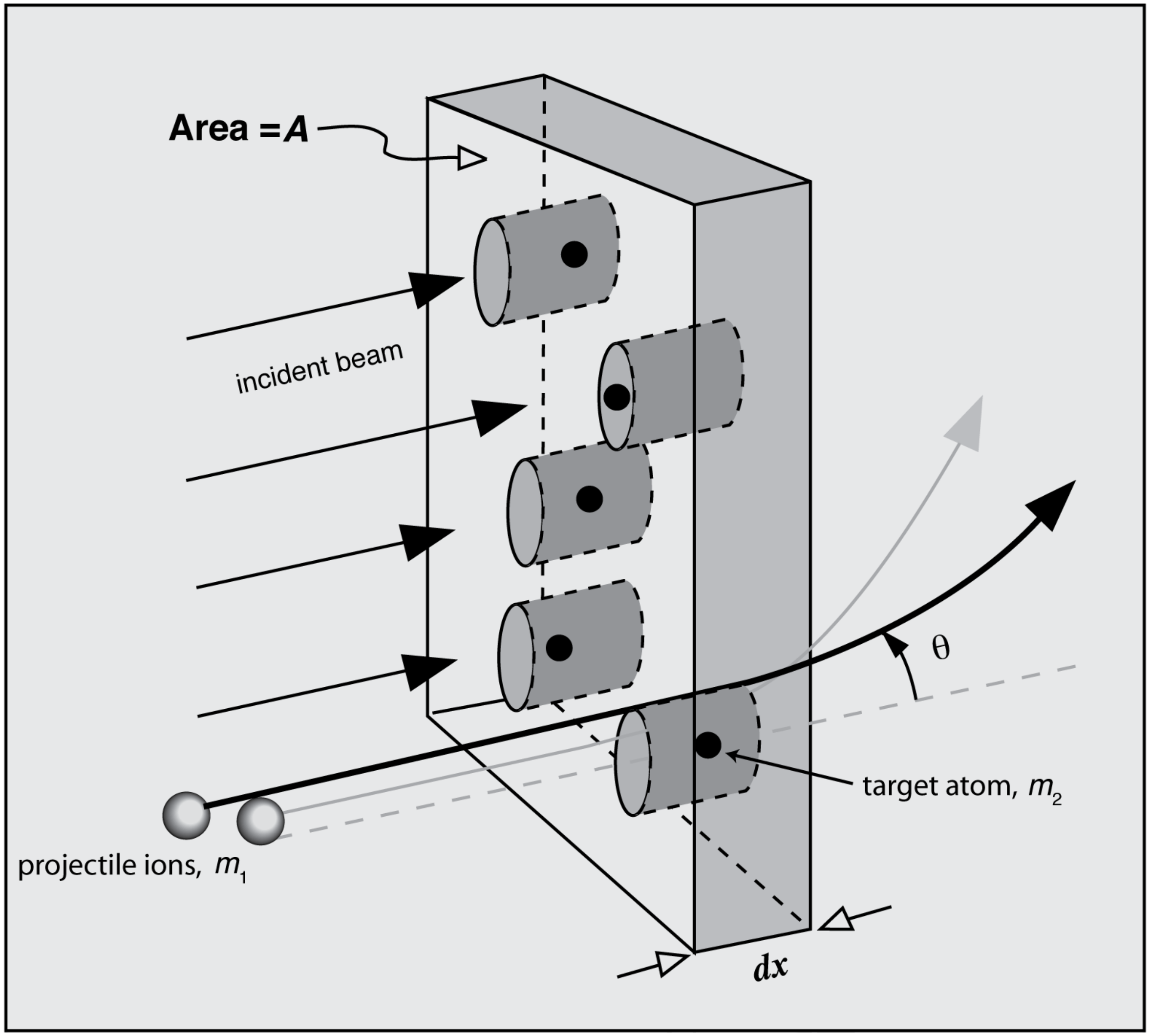
$$= \frac{\Phi\left[\frac{\text{interactions}}{\text{cm}^2}\right]}{\Phi_0\left[\frac{\text{projectile particles}}{\text{cm}^2}\right]}$$

$$= \rho\left[\frac{\text{target atoms}}{\text{cm}^3}\right]\cdot dx\ [\text{cm}] \quad \sigma\left[\frac{\text{cm}^2}{\text{target atom}}\right]$$

$$P = \frac{\Phi}{\Phi_0} = (\rho \cdot dx)\,\sigma$$

The Total Scattering Cross-Section, σ





$$\Phi\left[\frac{\text{projectile ions scattered to } \theta \le \theta \le \pi}{\text{cm}^2}\right] = \Phi_0\left[\frac{\text{projectile ions}}{\text{cm}^2}\right]\rho\left[\frac{\text{target nuclei}}{\text{cm}^3}\right]\cdot dx\ [\text{cm}]\quad \sigma\left[\frac{\text{cm}^2}{\text{target nucleus}}\right]$$

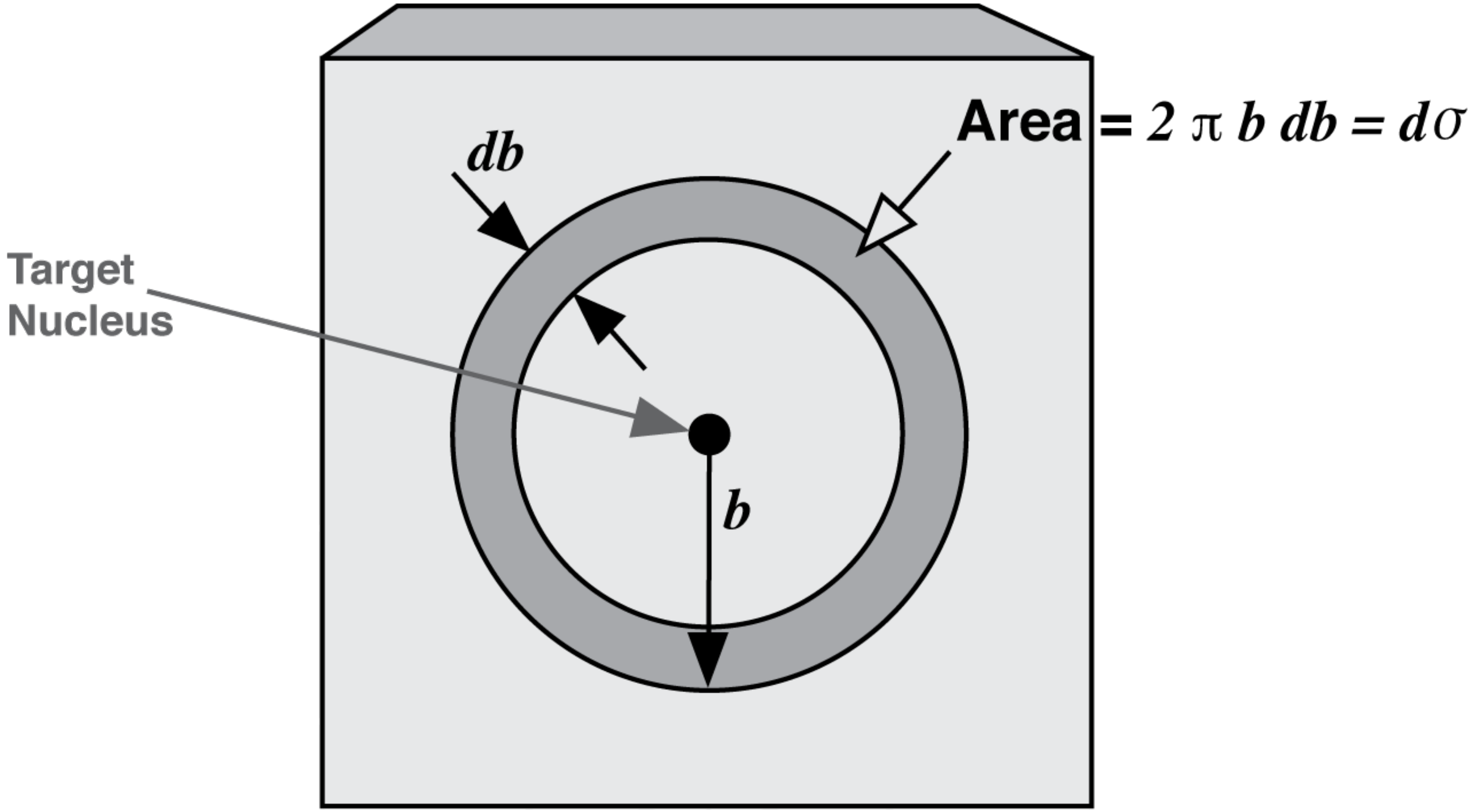
$$\Phi = \Phi_0\,(\rho\cdot dx)\,\sigma$$

Probabilty of Scattering to $\theta \leq \theta \leq \pi$

$$= \frac{\Phi\left[\frac{\text{projectile ions scattered to } \theta \leq \theta \leq \pi}{\text{cm}^2}\right]}{\Phi_0\left[\frac{\text{projectile ions}}{\text{cm}^2}\right]} = \rho\left[\frac{\text{target nuclei}}{\text{cm}^3}\right] \cdot dx\ [\text{cm}] \quad \sigma\left[\frac{\text{cm}^2}{\text{target nucleus}}\right]$$

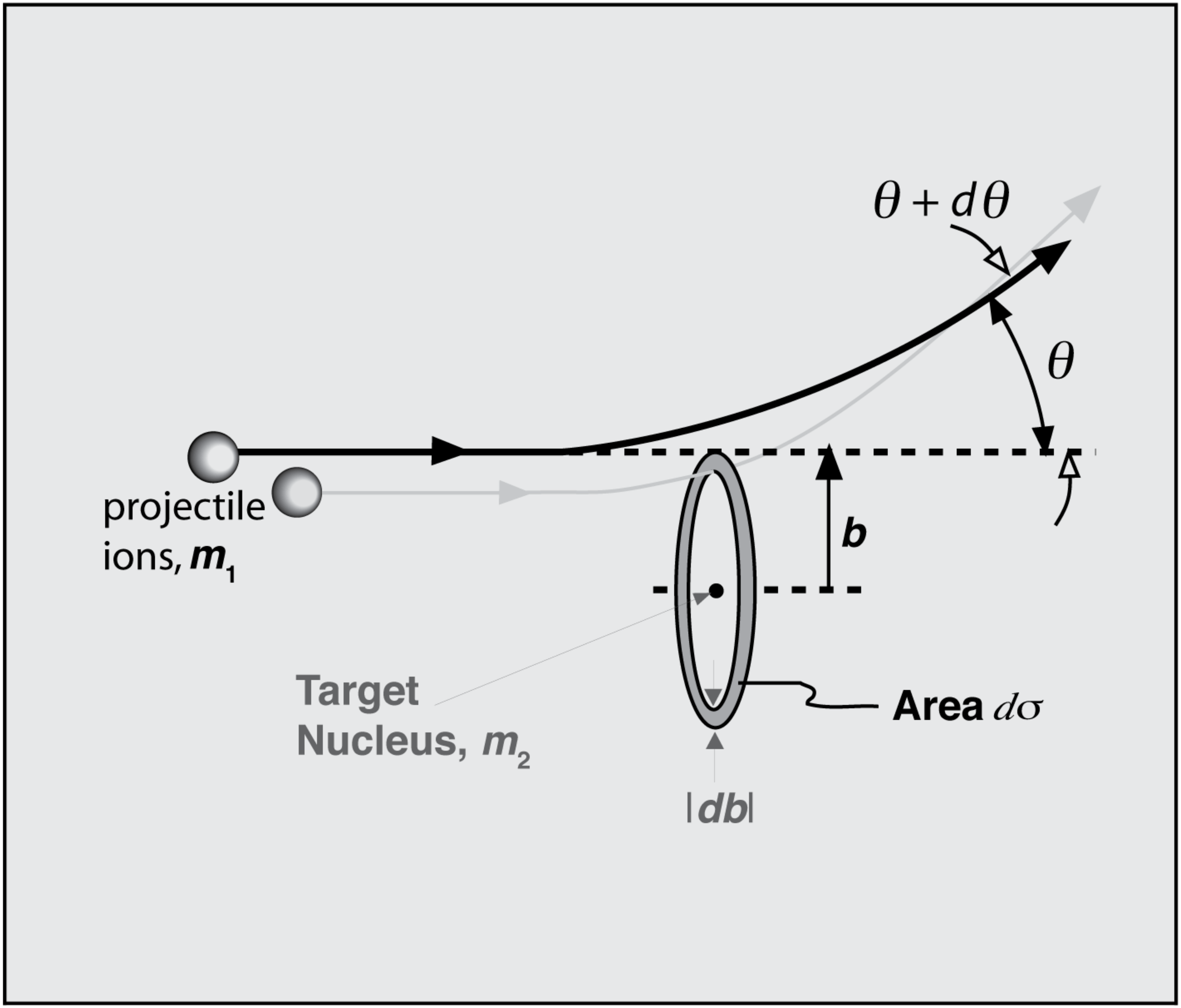
$$P = \frac{\Phi}{\Phi_0} = (\rho \cdot dx)\,\sigma$$

The Differential Scattering Cross-Section, *dσ*/*d*Ω

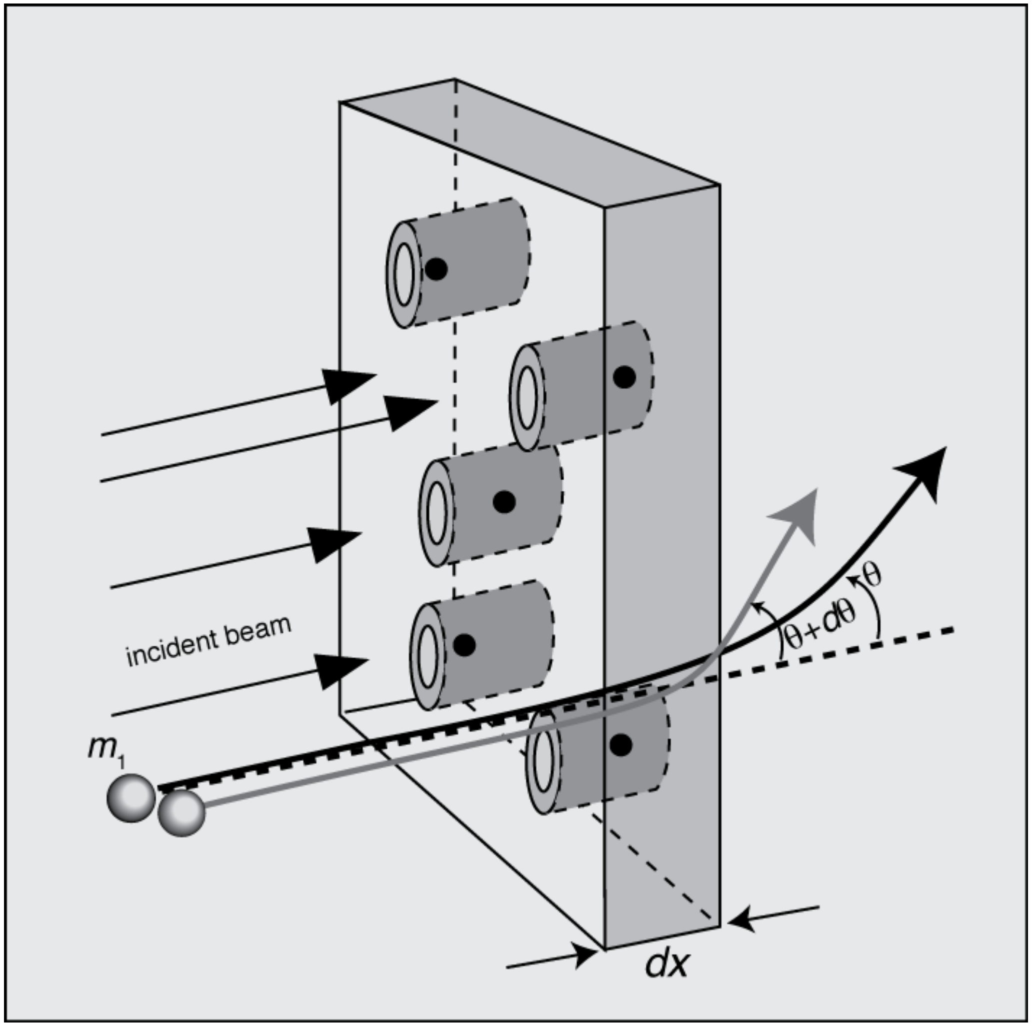


$$d\sigma\left[\mathrm{cm}^2\right] = 2\,\pi\,b\left[\mathrm{cm}\right]db\left[\mathrm{cm}\right]$$

$$d\sigma = 2\,\pi\,b\,db$$



θ + dθ
θ
b
projectile
ions, m1
Target
Nucleus, m2
Area dσ
|db|



$$d\Phi \left[\frac{\text{projectile ions scattered to } \theta \le \theta \le \theta + d\theta}{\text{cm}^2} \right] = \Phi_0 \left[\frac{\text{projectile ions}}{\text{cm}^2} \right] \rho \left[\frac{\text{target nuclei}}{\text{cm}^3} \right] \cdot dx \text{ [cm]} \quad d\sigma \left[\frac{\text{cm}^2}{\text{target nucleus}} \right]$$

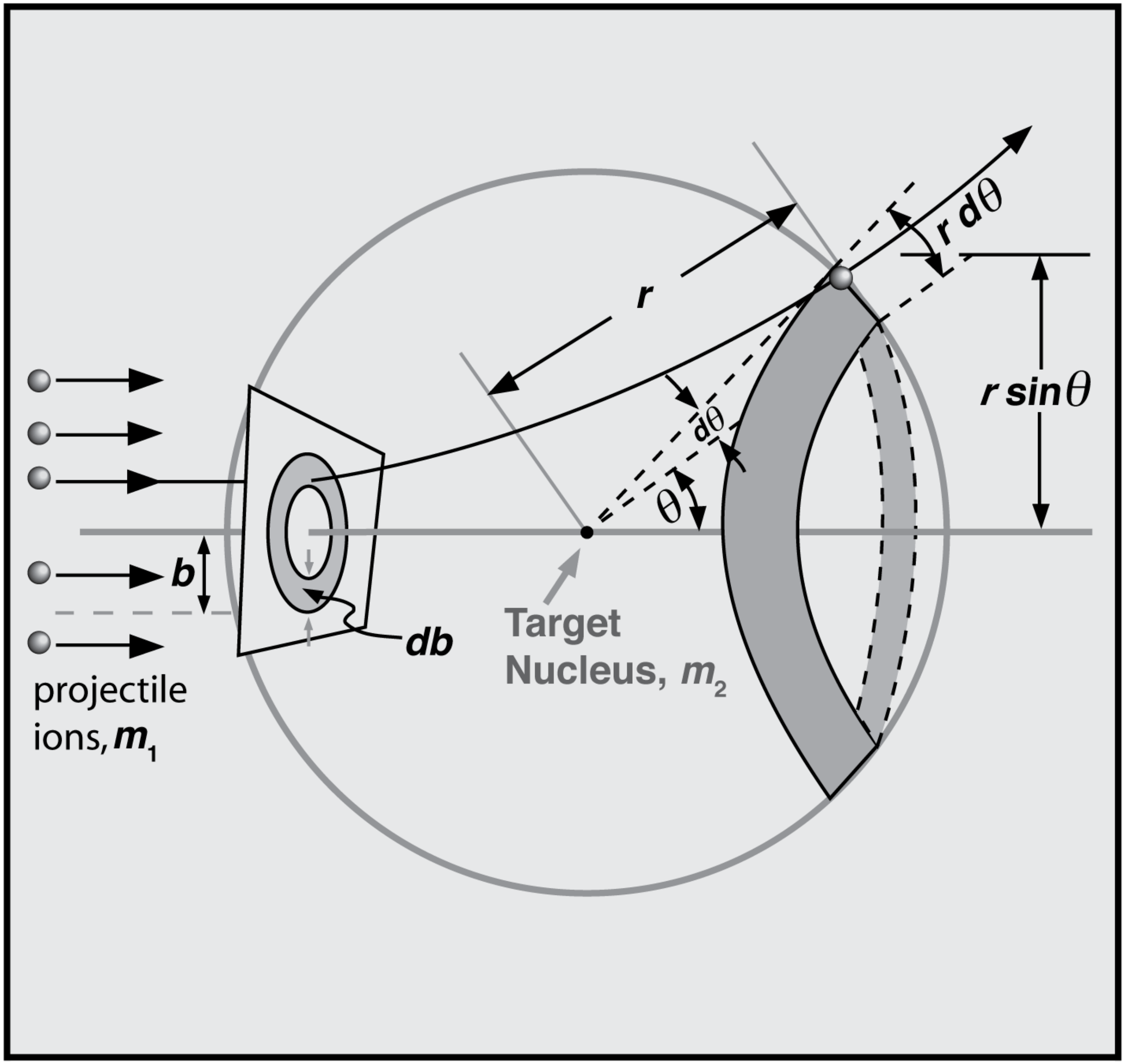
$$d\Phi = \Phi_0 \, (\rho \cdot dx) \, d\sigma$$

Differential Probabilty of Scattering to $\theta \leq \theta \leq \theta + d\theta$

$$= \frac{d\Phi \left[\dfrac{\begin{array}{l}\text{projectile ions} \\ \text{scattered to} \\ \theta \leq \theta \leq \theta + d\theta\end{array}}{\text{cm}^2} \right]}{\Phi_0 \left[\dfrac{\text{projectile ions}}{\text{cm}^2} \right]} = \rho \left[\frac{\text{target nuclei}}{\text{cm}^3} \right] \cdot dx \ [\text{cm}] \quad d\sigma \left[\frac{\text{cm}^2}{\text{target nucleus}} \right]$$

$$P(\theta, \theta + d\theta) = \frac{d\Phi}{\Phi_0} = (\rho \cdot dx)\, d\sigma$$

Solid angle, $d\Omega$, associated with scattering in 3-D, to angles between θ and $\theta + d\theta$



$$\frac{d\Omega}{4\pi} = \frac{2\pi\left(r \sin\theta\right)\left(r\, d\theta\right)}{4\pi r^2}$$

$$d\Omega = 2\pi \sin\theta \, d\theta$$

Fractional ion fluence scattered through angle θ and within solid angle Ω

$$= \frac{d\Phi\left[\dfrac{\begin{array}{l}\text{projectile ions}\\ \text{scattered to}\\ \theta \le \theta \le \theta + d\theta\end{array}}{\text{cm}^2}\right]}{\Phi_0\left[\dfrac{\text{projectile ions}}{\text{cm}^2}\right]} = \rho\left[\frac{\text{target nuclei}}{\text{cm}^3}\right]\cdot dx\ [\text{cm}]\int_\Omega \frac{d\sigma}{d\Omega}\left[\frac{\text{cm}^2}{\text{target nucleus}\cdot\text{steradian}}\right]d\Omega\left[\text{steradians}\right]$$

$$P(\Omega(\theta)) = \frac{d\Phi}{\Phi_0} = (\rho\cdot dx)\int_\Omega \frac{d\sigma}{d\Omega}\,d\Omega$$

Rutherford backscattering (RBS) detector:

Fractional ion fluence scattered through angle θ and within solid angle Ω

$$P(\Omega(\theta)) = \frac{d\Phi}{\Phi_0} = (\rho\cdot dx)\,\frac{d\sigma}{d\Omega}\,\Omega$$

The Differential Energy Transfer Cross-Section, $d\sigma(E,T)/dT$

$$d\sigma(b)/db \rightarrow d\sigma(E,\theta)/d\Omega \rightarrow d\sigma(E,T)/dT$$

State (without proof) an important relationship:

$$\boxed{\frac{d\sigma\left(E,T\right)}{dT} = \frac{4\;\pi}{\Lambda\;E}\,\frac{d\sigma\left(E,\theta\left(T\right)\right)}{d\Omega}}$$

Cross-Sections for a Hard-Sphere Interaction Potential

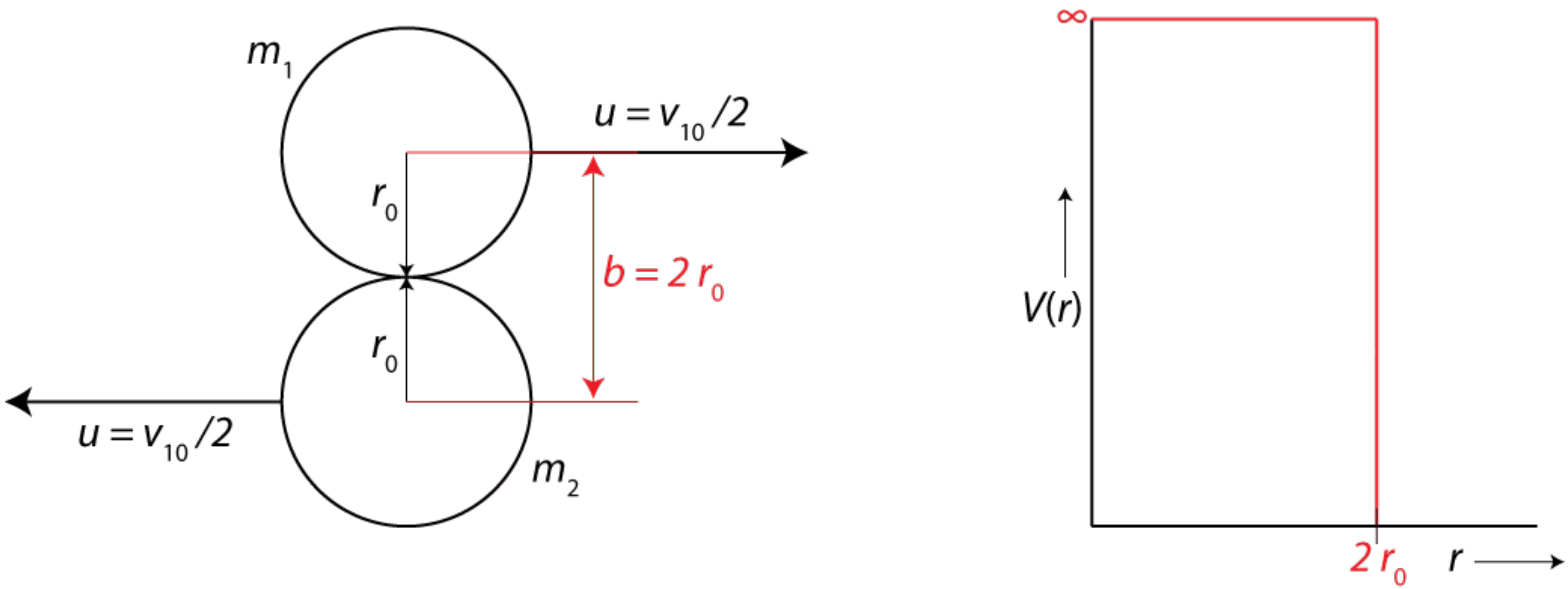


Figure 13.

$$\sigma = \pi\, b^2 \quad \rightarrow \quad d\sigma = 2\,\pi\, b\, db \quad \rightarrow \quad \boxed{\frac{d\sigma(b)}{db} = 2\,\pi\, b}$$

Hard sphere – scattering is isotropic in the *CM* system

$$\boxed{\frac{d\sigma\left(E,\theta\left(T\right)\right)}{\sigma(E)} = \frac{d\Omega(\theta)}{4\,\pi} = \frac{2\,\pi\,\sin\theta\;d\theta}{4\,\pi}}$$

Hard-sphere
Differential energy transfer cross-section

$$\frac{d\sigma\left(E,T\right)}{dT} = \frac{4\,\pi}{\Lambda\,E}\frac{d\sigma\left(E,\theta\left(T\right)\right)}{d\Omega}$$

$$= \frac{4\,\pi}{\Lambda\,E}\frac{\sigma(E)}{4\,\pi}$$

$$\boxed{\frac{d\sigma\left(E,T\right)}{dT} = \frac{\sigma(E)}{\Lambda\,E}}$$

Cross-Sections for a *Rutherford* Interaction Potential

$$V(r) = \frac{K}{r}$$

$$K = (Z_1\ e)(Z_2\ e)$$

$$= Z_1\ Z_2\ e^2$$

Cross-Sections for a *Rutherford* Interaction Potential

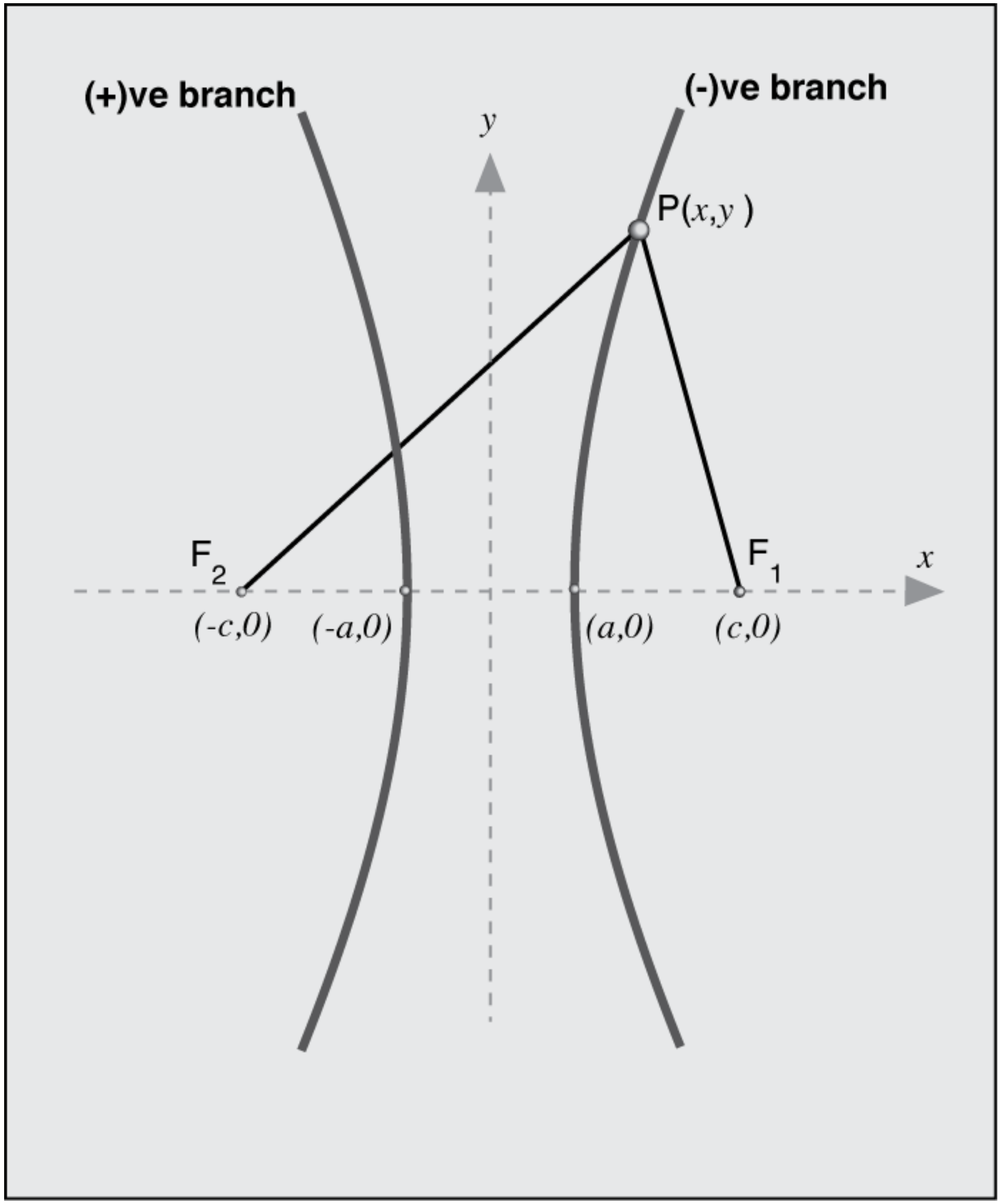
$$d\sigma(b)/db \rightarrow d\sigma(E,\theta)/d\Omega \rightarrow d\sigma(E,T)/dT$$

$$\frac{d\sigma(b)}{db} = 2\,\pi\,b$$

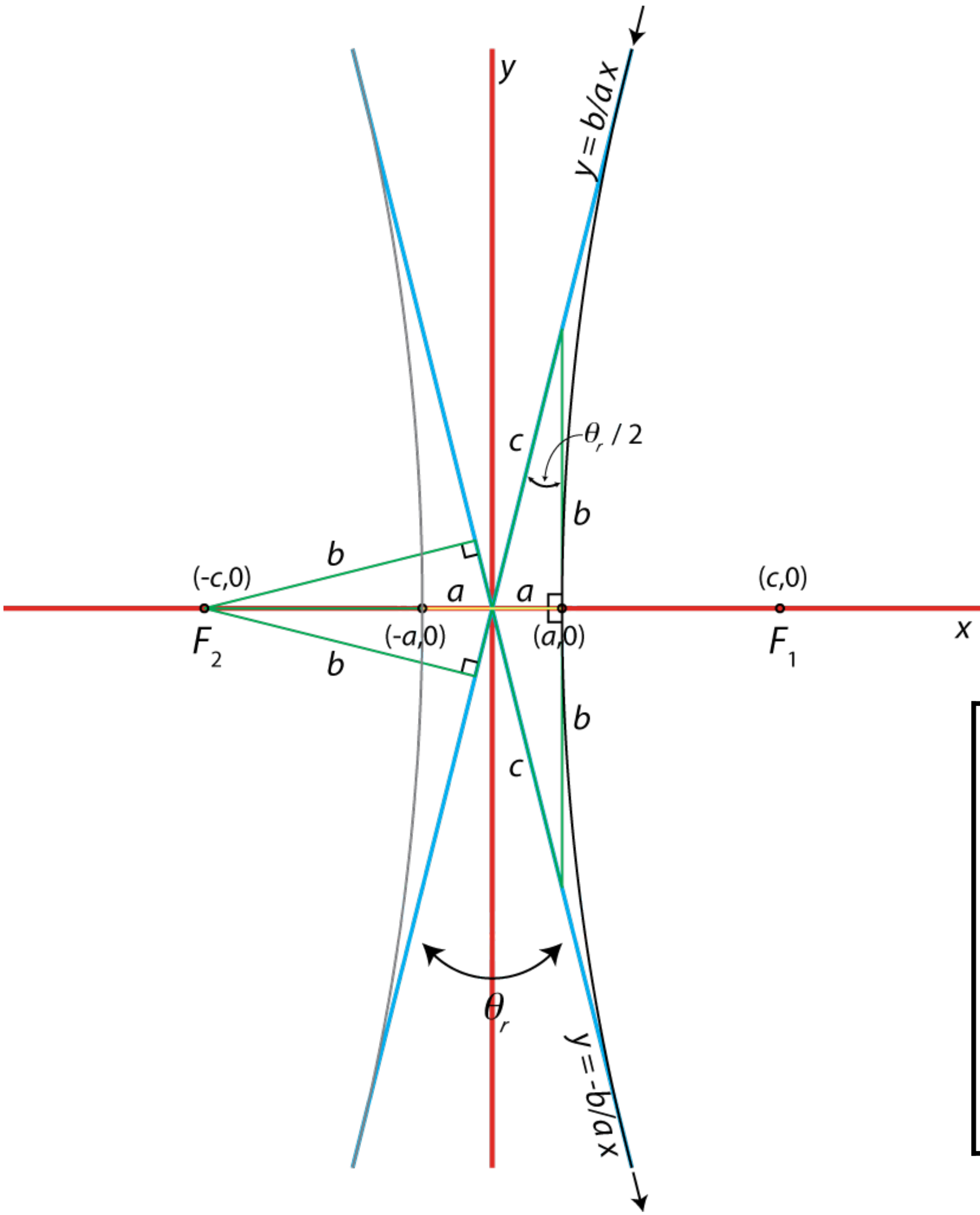
Next, we require an equation that provides a one-to-one mapping of

$$b \rightarrow \theta$$

Hyperbolic solution to the equation of motion



Hyperbolic solution to the equation of motion



$$\frac{b}{a} = \cot\left(\frac{\theta_r}{2}\right)$$

$$a = \frac{K\left(1 + \frac{m_1}{m_2}\right)}{2E}$$

Rutherford cross-section for angular scattering

$$\frac{d\sigma(E,\theta)}{d\Omega} = \left(\frac{K\left(1+\frac{m_1}{m_2}\right)}{4\,E} \right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$

Rutherford cross-section for energy transfer

$$d\sigma(E,\theta)\,/\,d\Omega \rightarrow d\sigma(E,T)\,/\,dT$$

$$\boxed{\frac{d\sigma(E,T)}{dT} = \frac{\pi\, K^2\, x}{E\, T^2}}$$

where $x = m_1 / m_2$ and $K = Z_1\, Z_2\, e^2$

Rutherford scattering: *Differential probability* of energy transfer between *T* and *T*+*dT*

$$P(T, T + dT) = (\rho\, dx)\, \frac{d\sigma(E,T)}{dT}\, dT$$

$$= (\rho\, dx)\, \frac{\pi\, K^2\, x}{E\, T^2}\, dT$$

$$
\begin{aligned}
\sigma(E) &= \int_{T=0}^{T=\Lambda E} d\sigma(E,T) \\
&= \int_{T=0}^{T=\Lambda E} \frac{\pi\ K^2\ x}{E}\ \frac{dT}{T^2} \\
&= \frac{\pi\ K^2\ x}{E} \int_{T=0}^{T=\Lambda E} \frac{dT}{T^2} \\
&= \frac{\pi\ K^2\ x}{E} \int_{T=0}^{T=\Lambda E} \frac{dT}{T^2} \\
&= -\frac{\pi\ K^2\ x}{E}\ \frac{1}{T}\Bigg|_{T=0}^{T=\Lambda E} \\
&= \frac{\pi\ K^2\ x}{E}\left(\frac{1}{0} - \frac{1}{\Lambda\ E}\right)
\end{aligned}
$$

The solution:
The *Screened Coulomb Potential*

$$V(r) = \frac{K}{r} e^{-\frac{r}{\hat{a}}}$$