Nuclear Structure
(II) Collective models

P. Van Isacker, GANIL, France

NSDD Workshop, Trieste, August 2016

Collective nuclear models

(Rigid) rotor model

(Harmonic quadrupole) vibrator model

Liquid-drop model of vibrations and rotations

Interacting boson model

Rotation of a symmetric top

Energy spectrum:

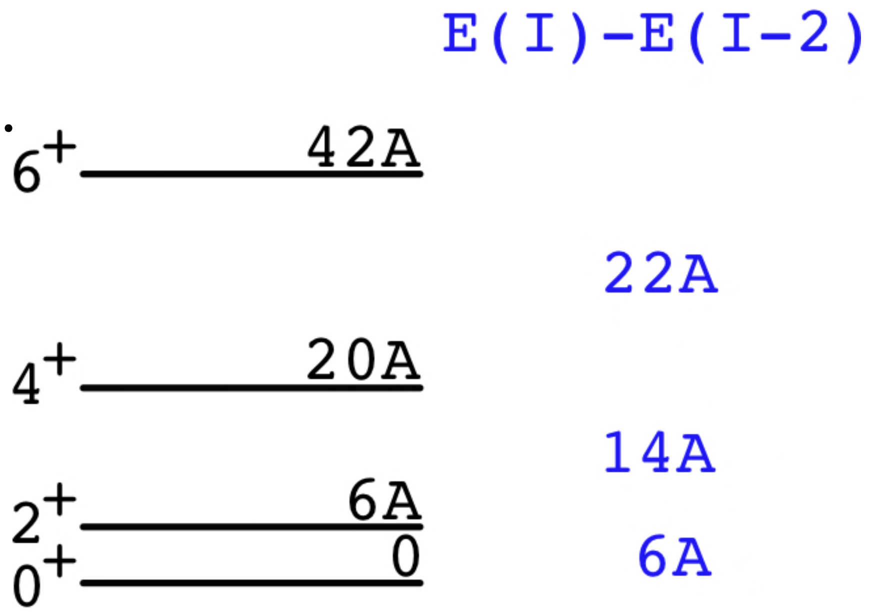
$$E_{\mathrm{rot}}(I) = \frac{\hbar^2}{2\mathfrak{I}} I(I+1)$$

$$\equiv A\, I(I+1), \quad I^\pi = 0^+, 2^+, \ldots$$

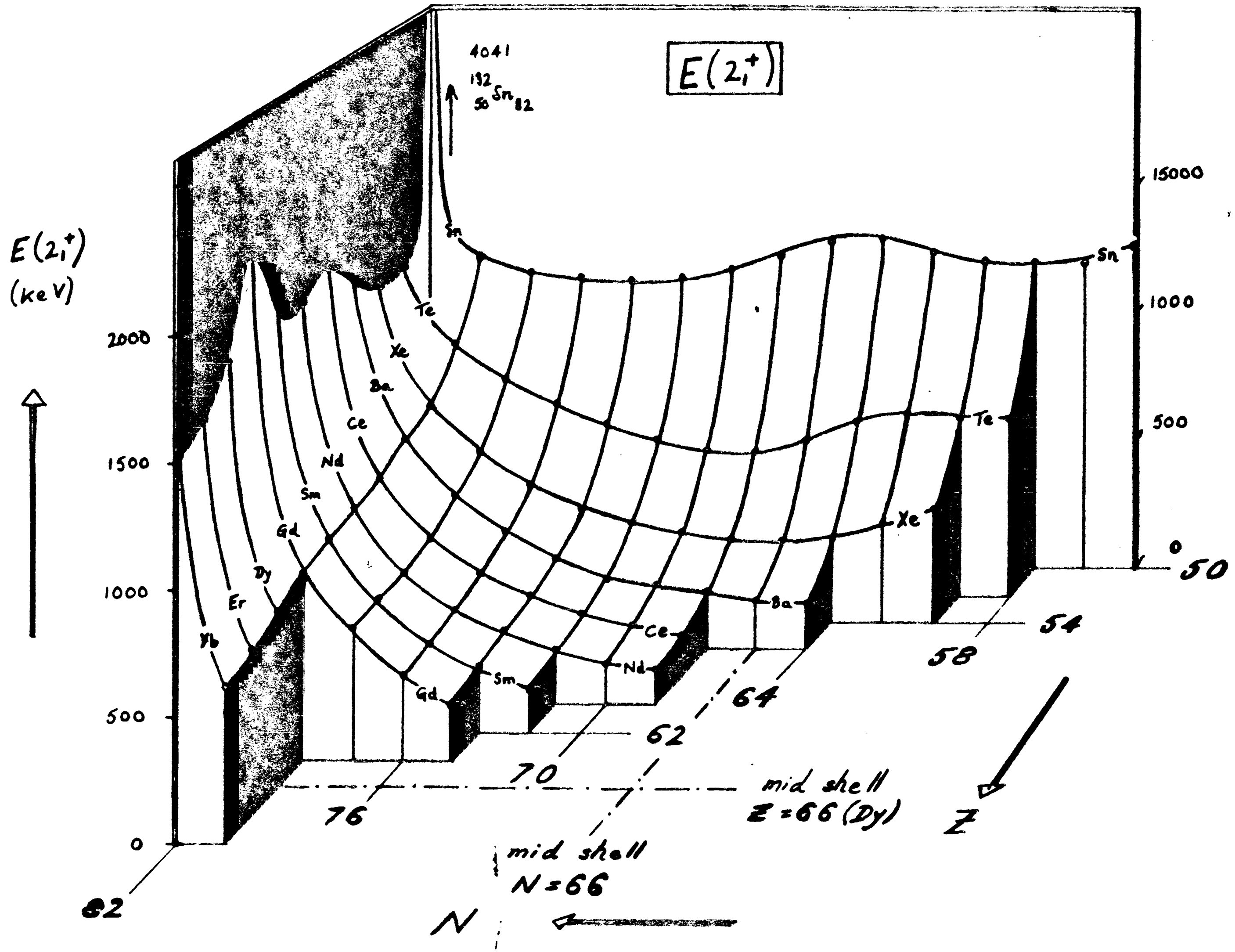
Large deformation $\Rightarrow$ large $\mathfrak{I} \Rightarrow$ low $E_x(2^+)$.

R_{42} energy ratio:

$$E_{\mathrm{rot}}(4^+)/E_{\mathrm{rot}}(2^+) = 3.333\ldots$$

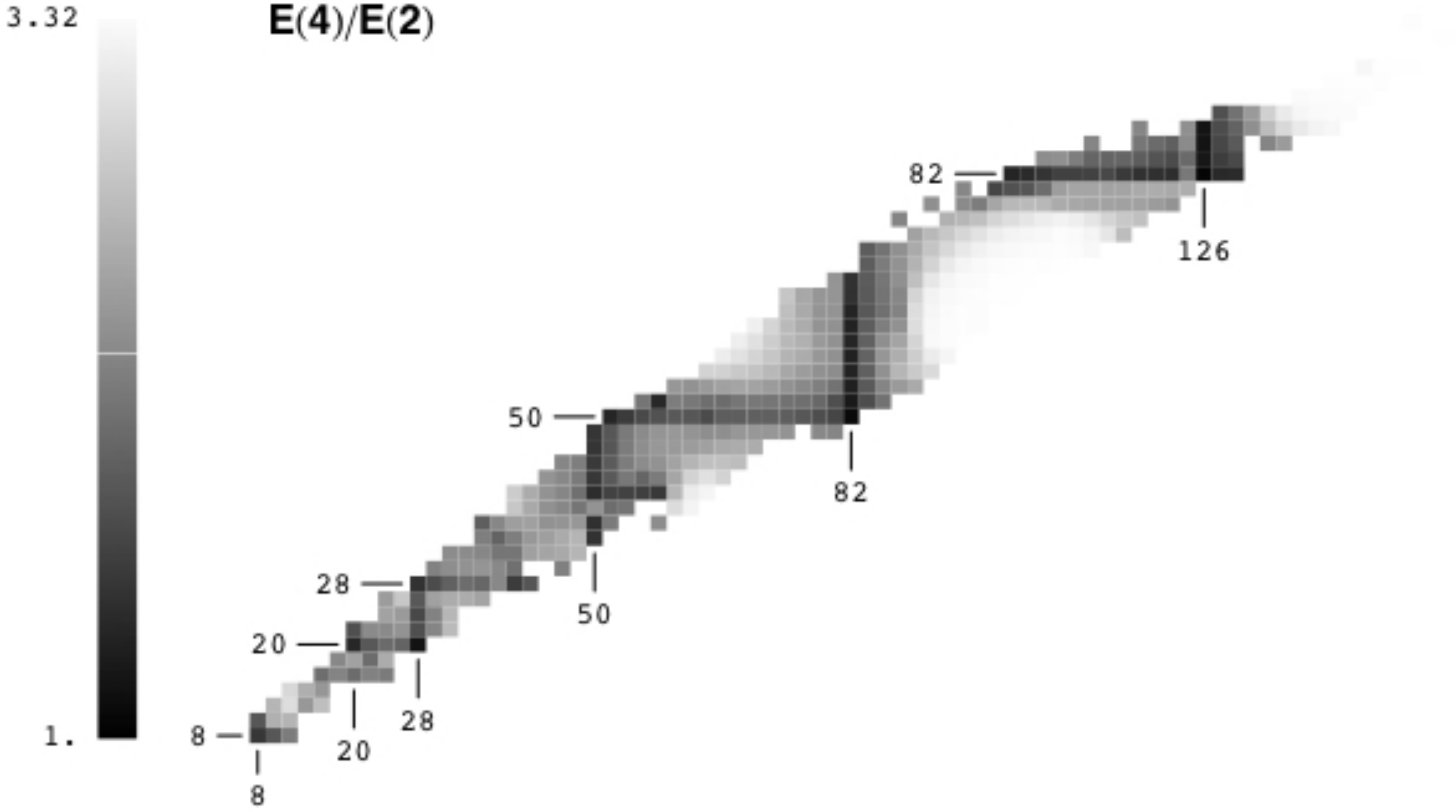


Evolution of $E_x(2^+)$



J.L. Wood, private communication

The ratio R_{42}



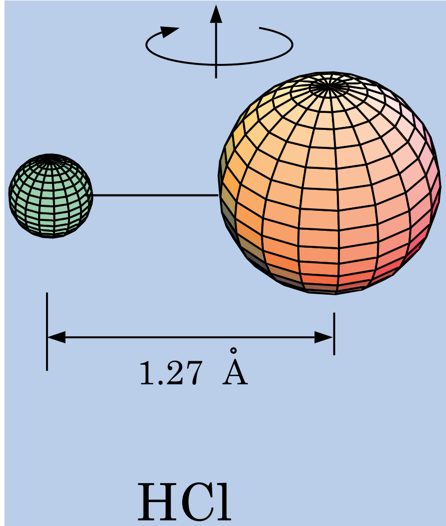
Rotation of an asymmetric top

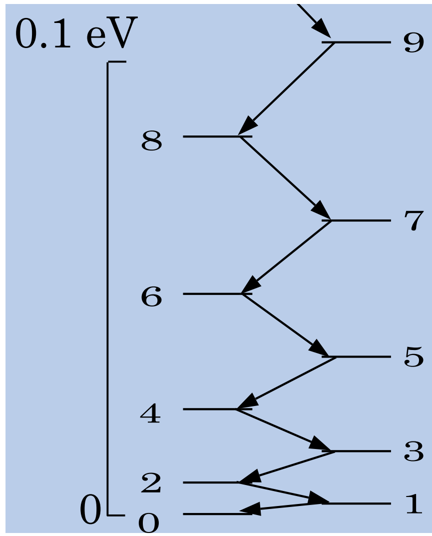
Energy spectrum:

$$E_{\text{rot}}\left(I^{\pi}\right)=\frac{\hbar^2}{2\Im}I\left(I+1\right)$$

$$I^{\pi}=0^+,1^-,2^+,3^-,4^+,\ldots$$

Reflection symmetry only allows even I with positive parity π.





Nuclear shapes

Shapes can be characterized by variables $\alpha_{\lambda\mu}$ in a surface parameterization:

$$R(\theta,\varphi) = R_0\left(1 + \sum_{\lambda}\sum_{\mu=-\lambda}^{+\lambda}\alpha_{\lambda\mu}Y^{*}_{\lambda\mu}(\theta,\varphi)\right)$$

λ=0: compression (high energy)

λ=1: translation (not an intrinsic deformation)

λ=2: quadrupole deformation

λ=3: octupole deformation

Quadrupole shapes

Since the surface $R(\theta,\varphi)$ is real: $\left(\alpha_{\lambda\mu}\right)^* = (-1)^\mu \alpha_{\lambda-\mu}$

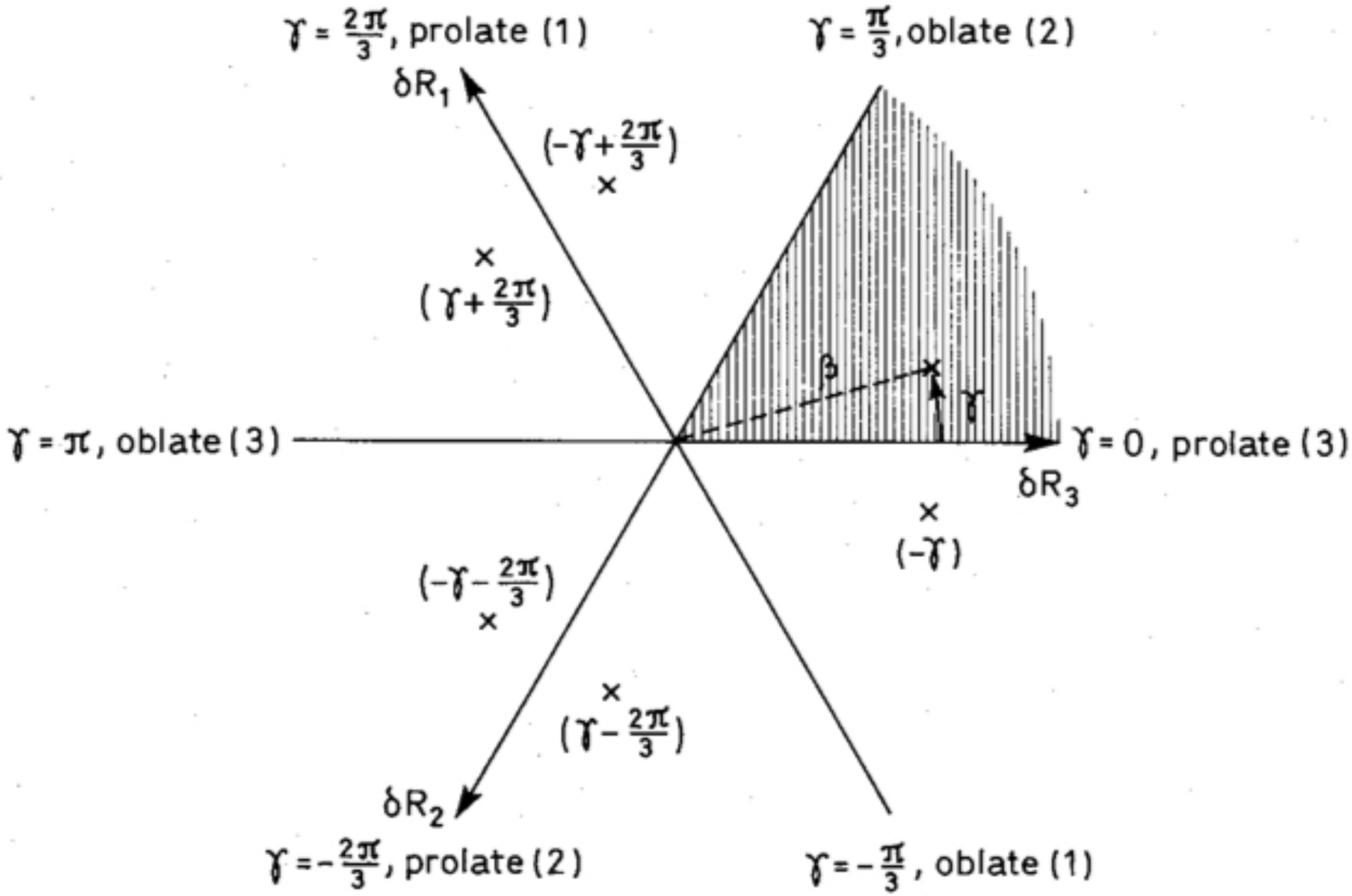
➔ Five independent quadrupole variables (λ=2).

Equivalent to three Euler angles and two intrinsic variables β and γ:

$$\alpha_{2\mu} = \sum_\nu a_{2\nu} D^2_{\mu\nu}(\Omega), \quad a_{21} = a_{2-1} = 0, \quad a_{22} = a_{2-2}$$

$$a_{20} = \beta\cos\gamma, \quad a_{22} = \frac{1}{\sqrt{2}}\beta\sin\gamma$$

The (β,γ) plane



Modes of nuclear vibration

Nucleus is considered as a droplet of nuclear matter with an equilibrium shape. Vibrations are modes of excitation around that shape.

Character of vibrations depends on symmetry of equilibrium shape. Two important cases in nuclei:

Spherical equilibrium shape

Spheroidal equilibrium shape

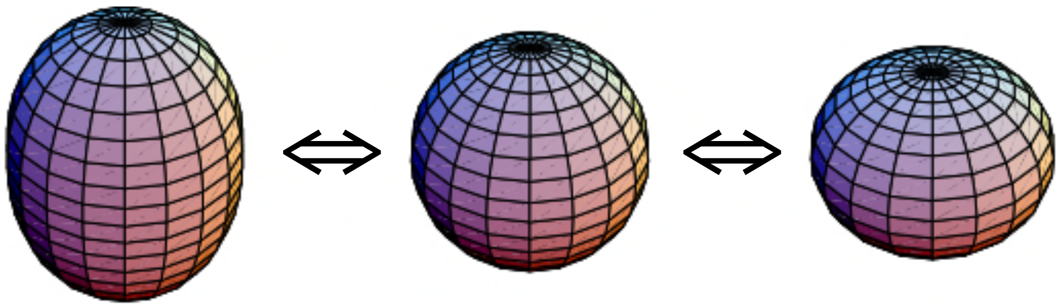
Vibrations about a spherical shape

Vibrations are characterized by λ in the surface parametrization:

λ=0: compression (high energy)

λ=1: translation (not an intrinsic excitation)

λ=2: quadrupole vibration



λ=3: octupole vibration

Spherical quadrupole vibrations

Energy spectrum:

$$E_{\text{vib}}(n) = \left(n + \tfrac{5}{2}\right)\hbar\omega, n = 0,1\ldots$$

***R*$_{42}$ energy ratio:**

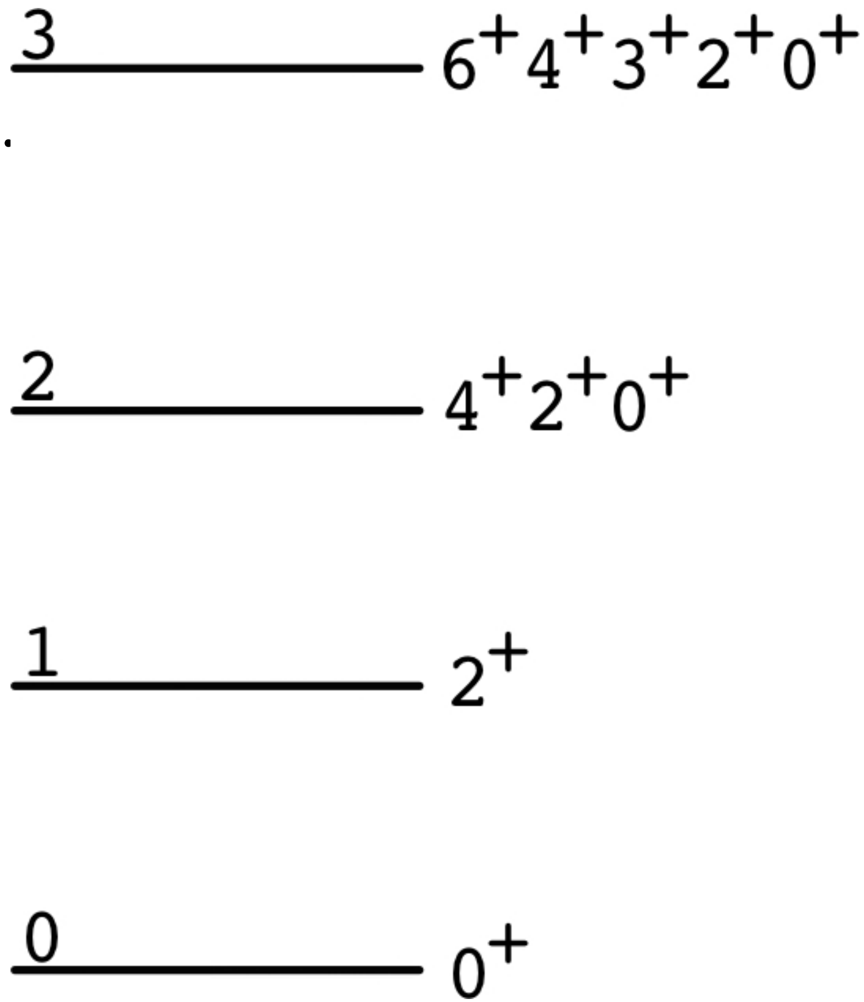
$$E_{\text{vib}}\left(4^+\right)/E_{\text{vib}}\left(2^+\right) = 2$$

E2 transitions:

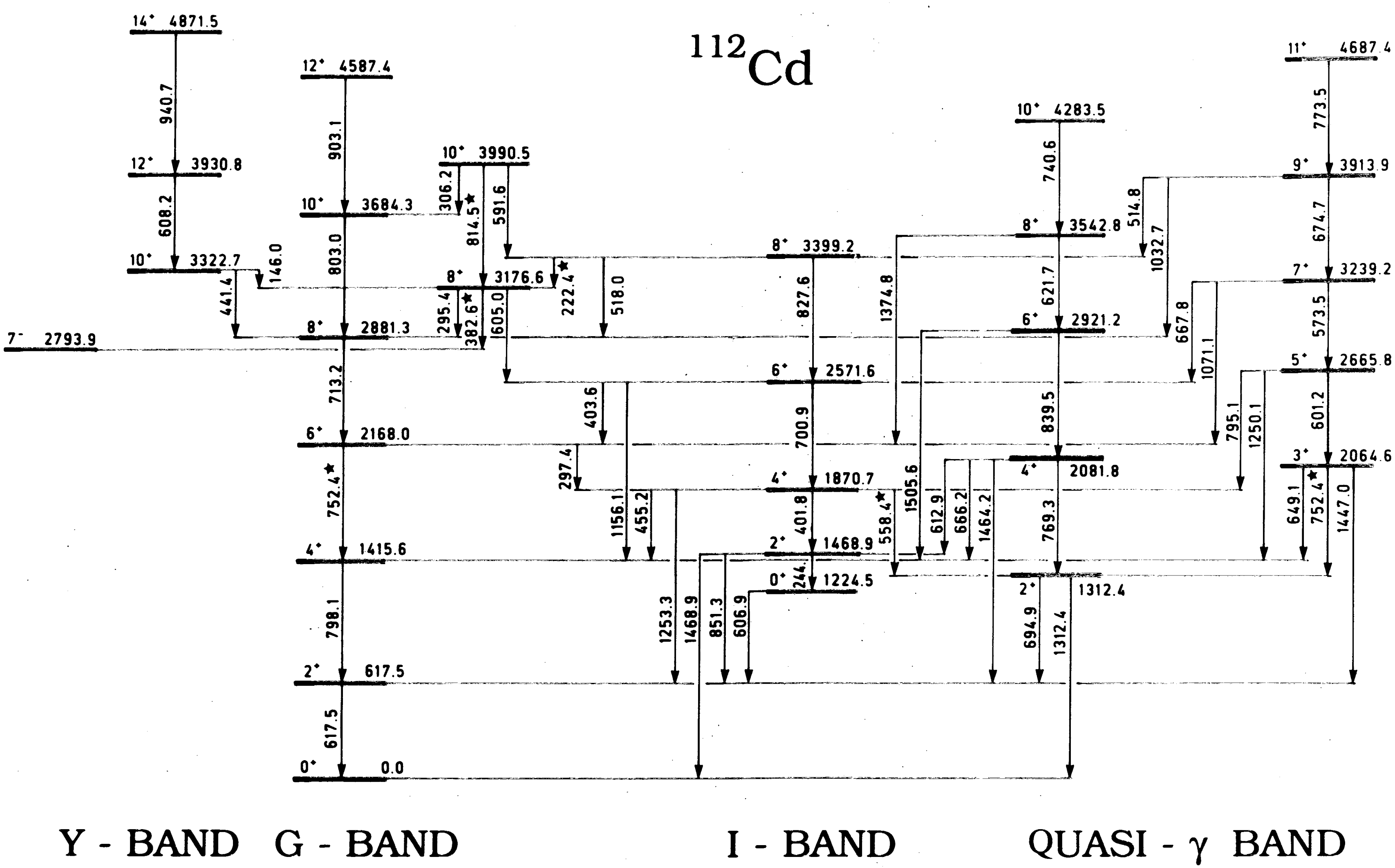
$$B\left(\text{E2}; 2_1^+ \to 0_1^+\right) = \alpha^2$$

$$B\left(\text{E2}; 2_2^+ \to 0_1^+\right) = 0$$

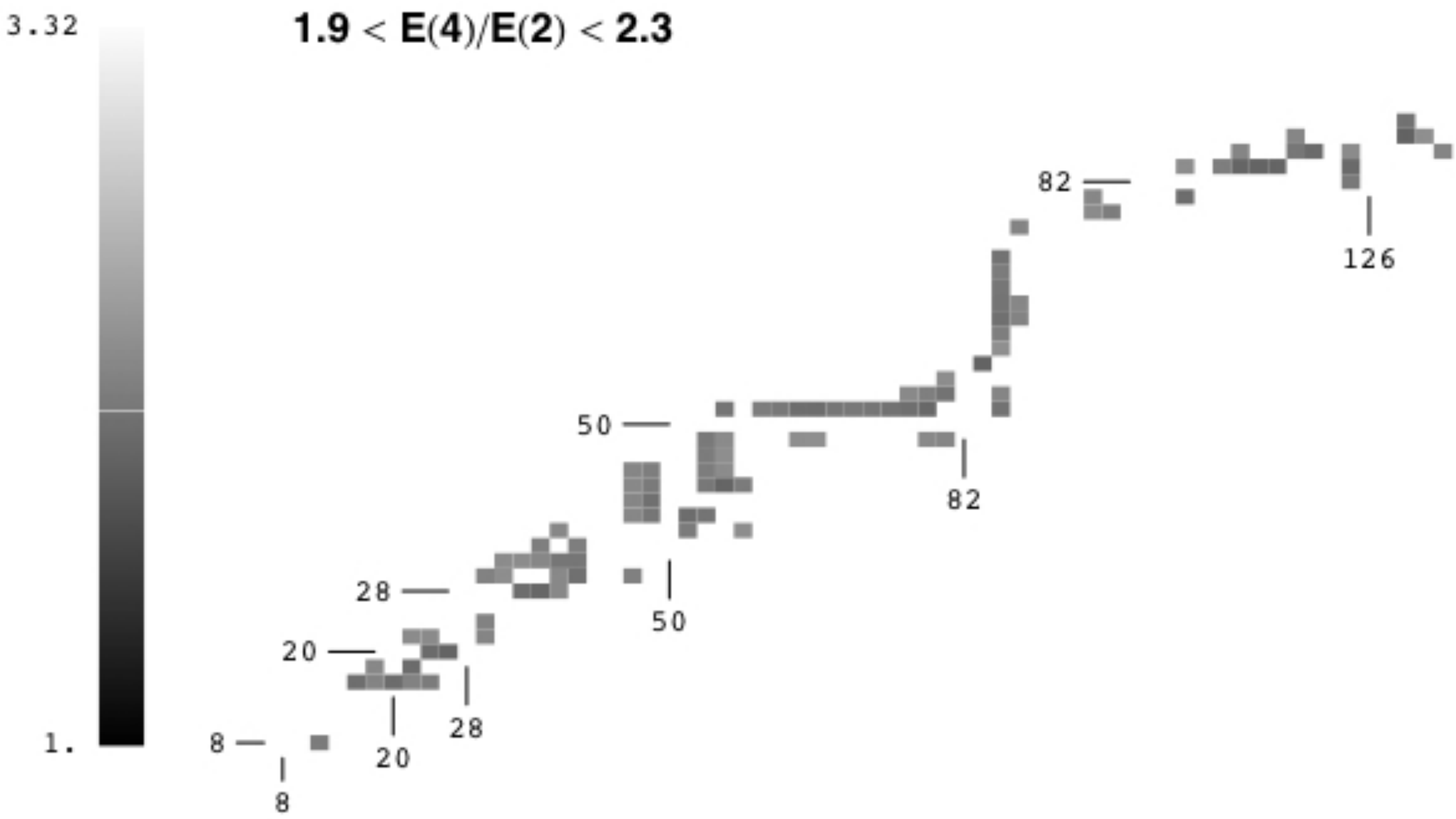
$$B\left(\text{E2}; n = 2 \to n = 1\right) = 2\alpha^2$$



Example of ^{112}Cd



Possible vibrational nuclei from R_{42}

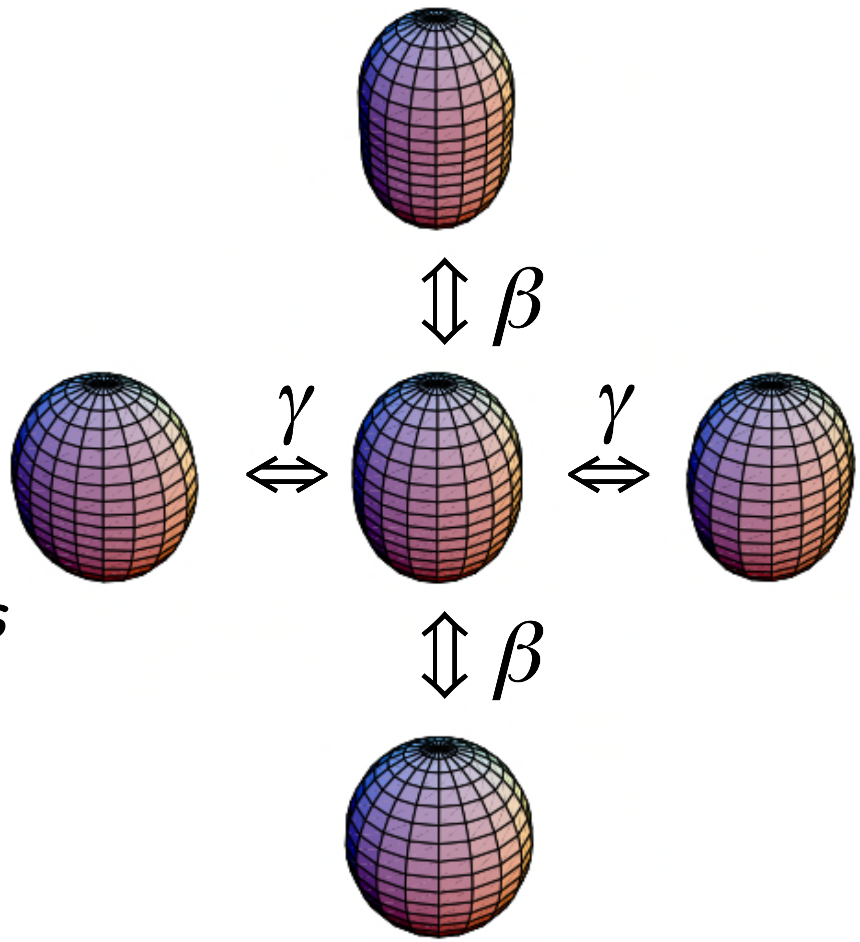


Spheroidal quadrupole vibrations

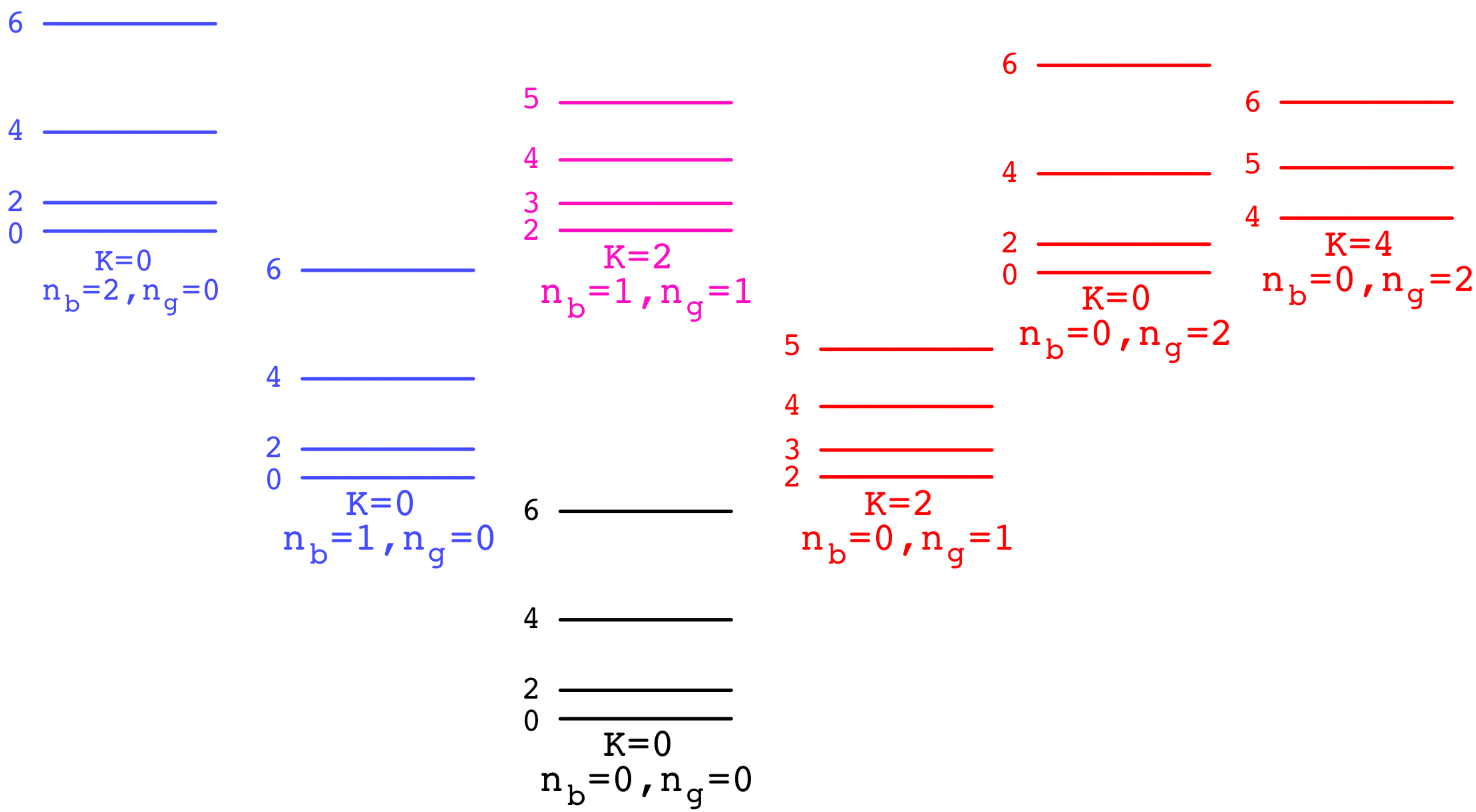
The vibration of a shape with axial symmetry is characterized by $a_{\lambda\nu}$.

Quadrupole oscillations:

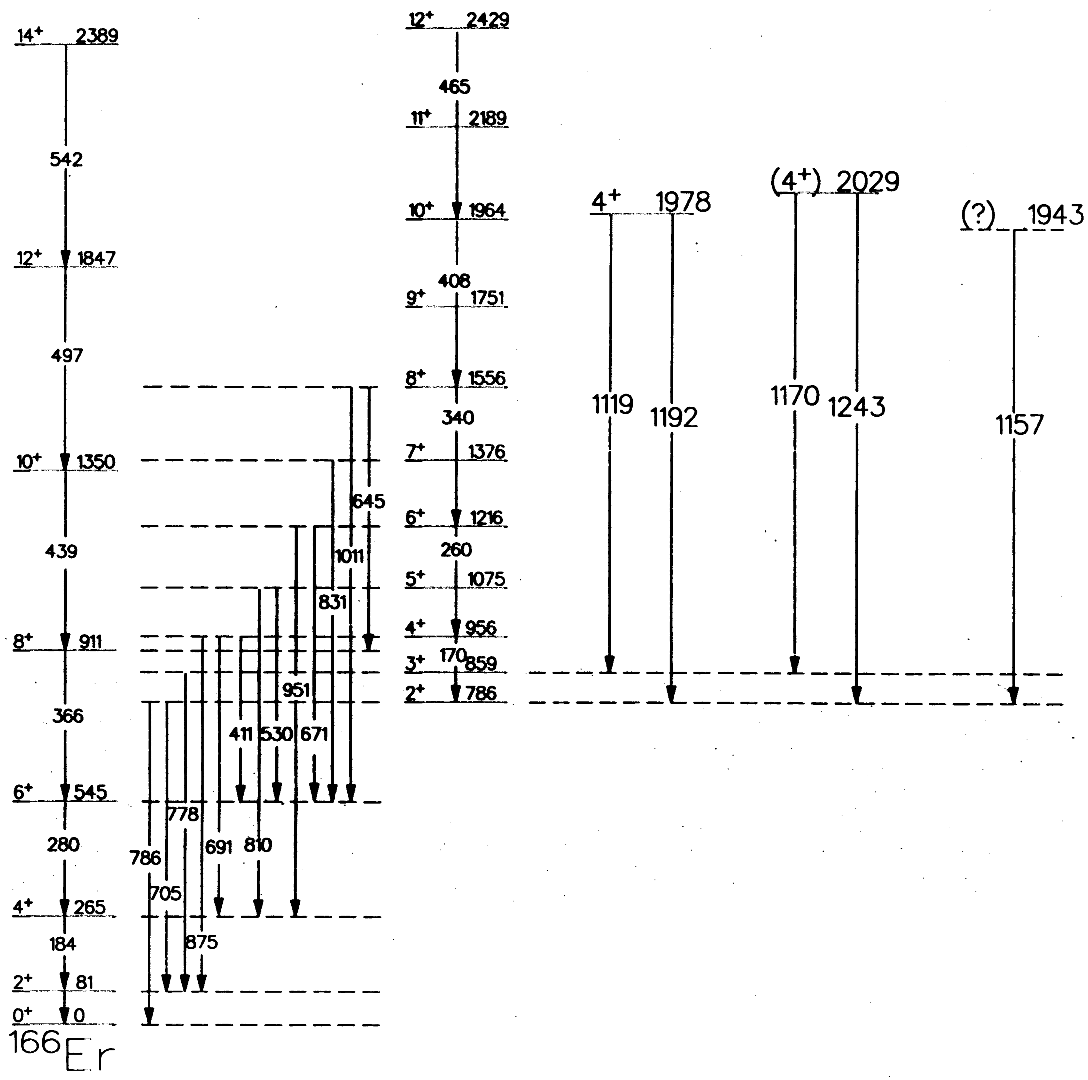
- *ν=0: along the axis of symmetry (β)*
- *ν=±1: spurious rotation*
- *ν=±2: perpendicular to axis of symmetry (γ)*



Spectrum of spheroidal vibrations



Example of ^{166}Er

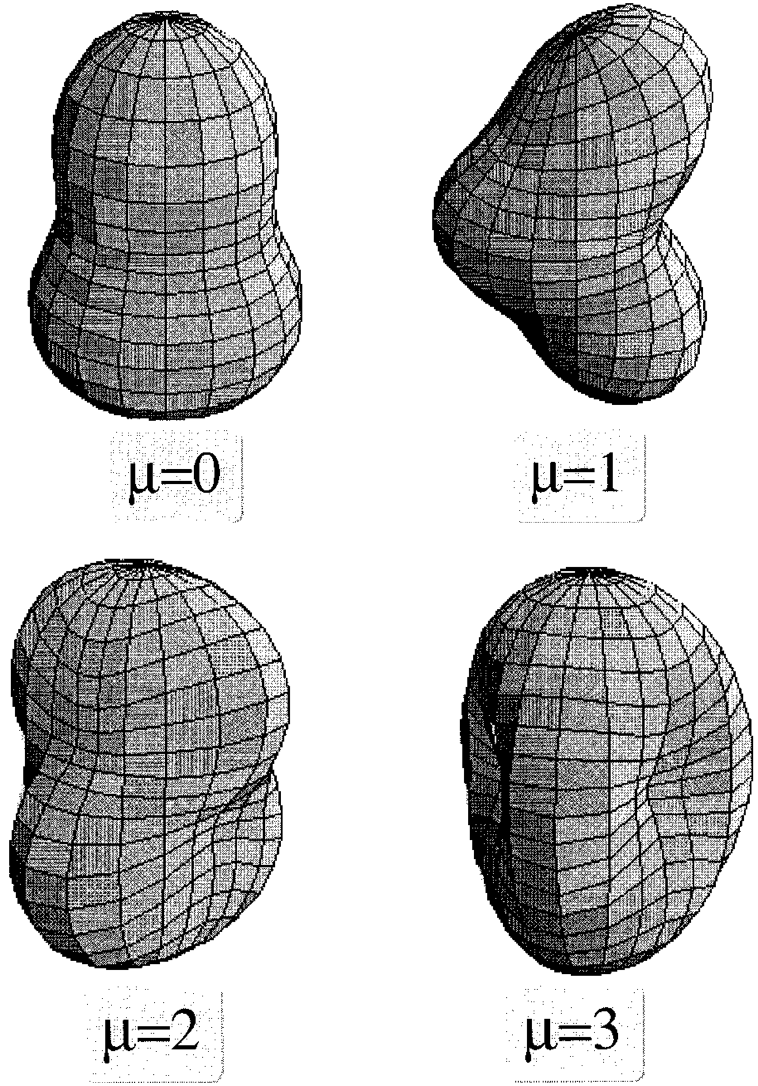


Quadrupole-octupole shapes

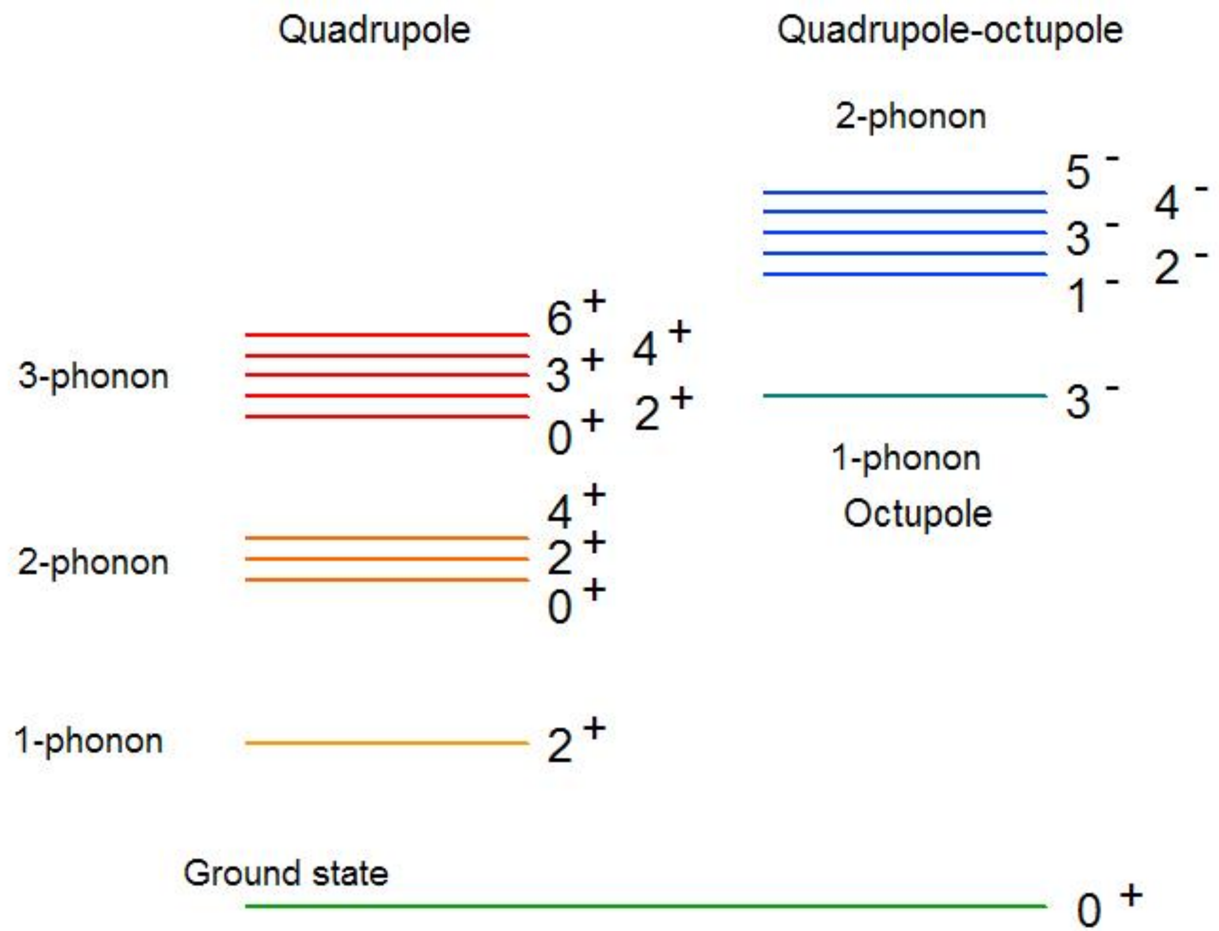
It is difficult to define an intrinsic frame for a pure octupole shape.

Quadrupole-octupole: use quadrupole frame ➔ two quadrupole and seven octupole intrinsic variables $\alpha_{3\mu}$.

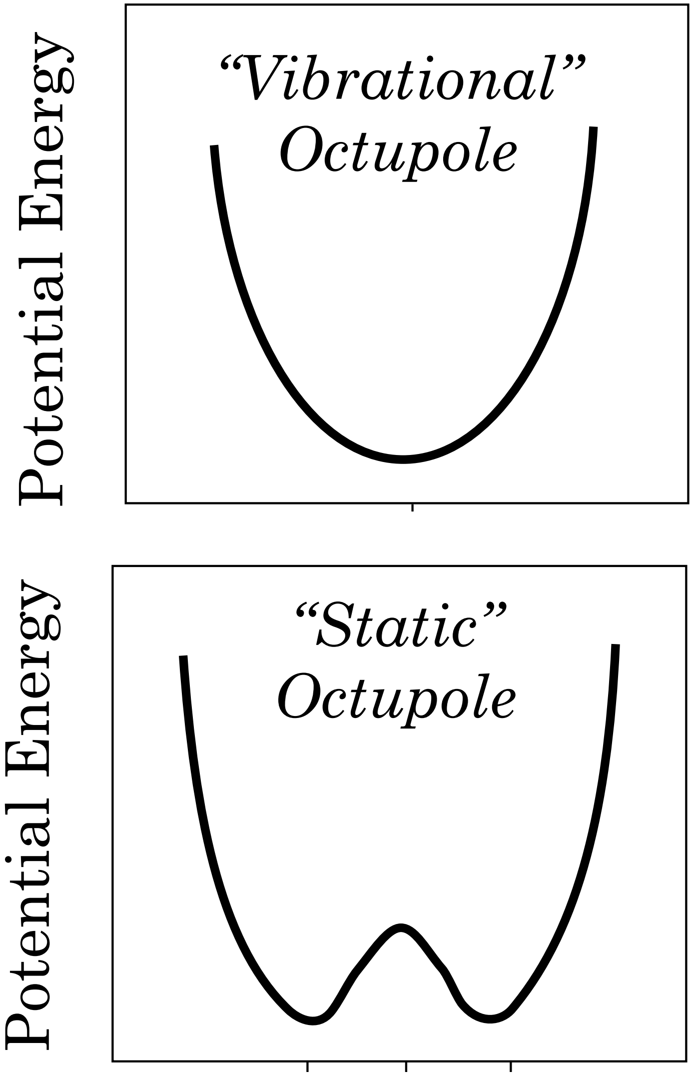
Most important case: β_3=α_{30} (axial symmetry).

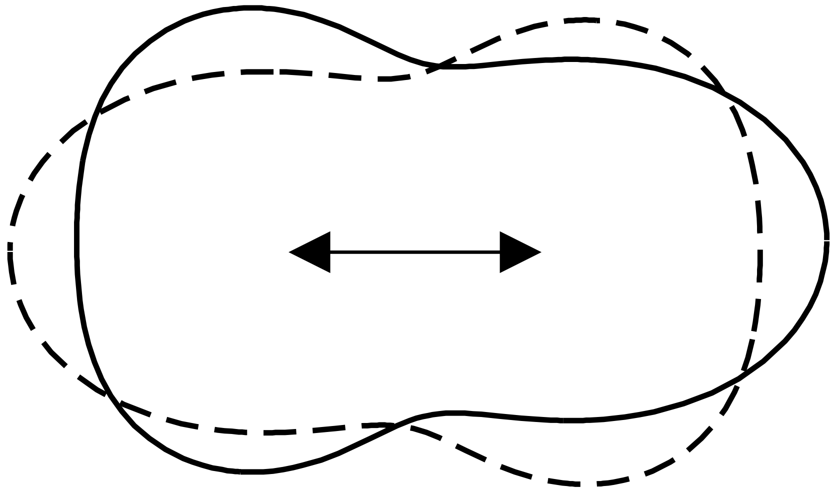


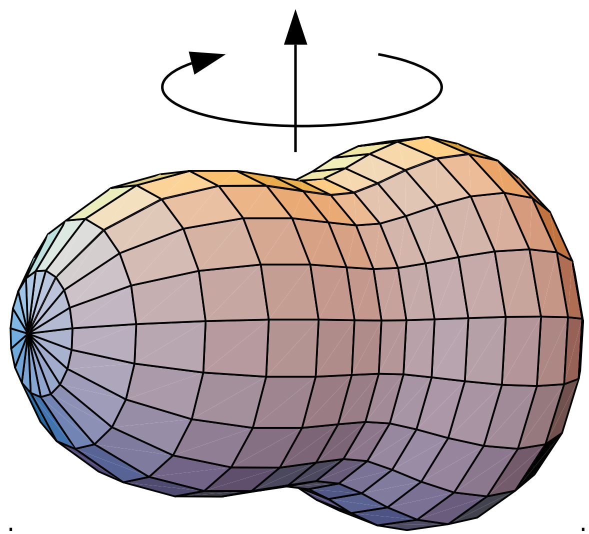
Quadrupole-octupole vibrations



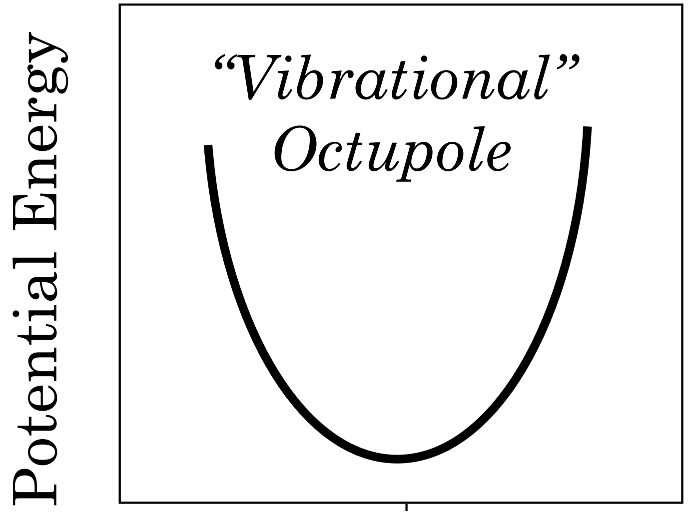
Octupole rotation-vibrations

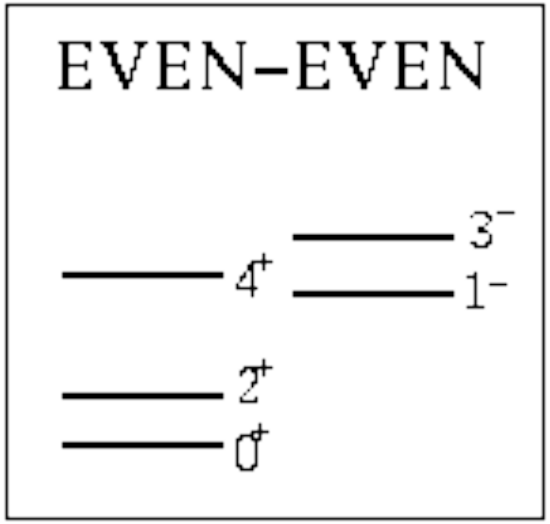


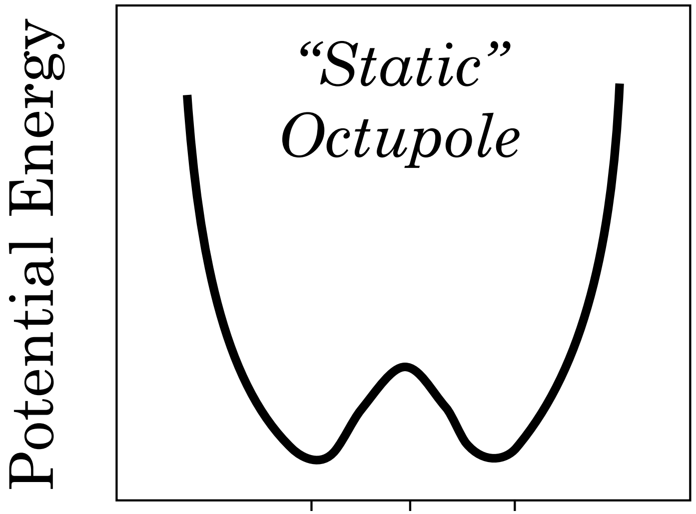


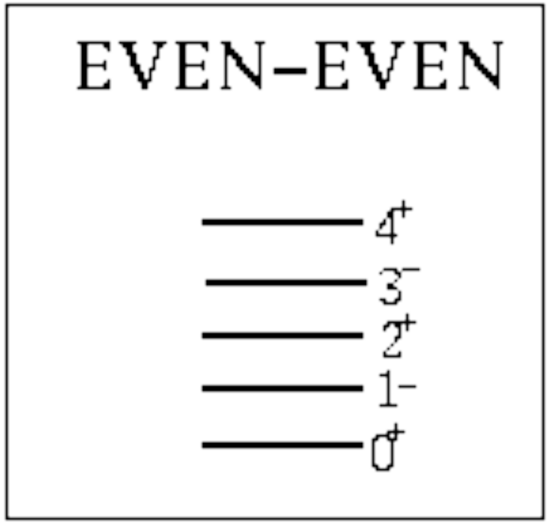


Octupole rotation-vibrations

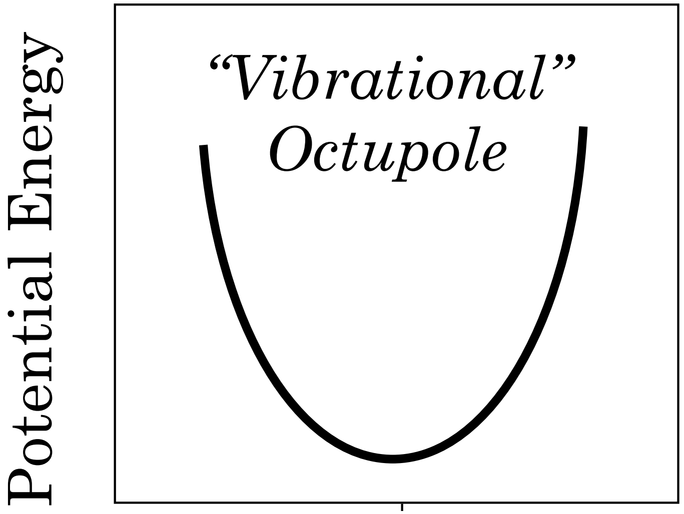


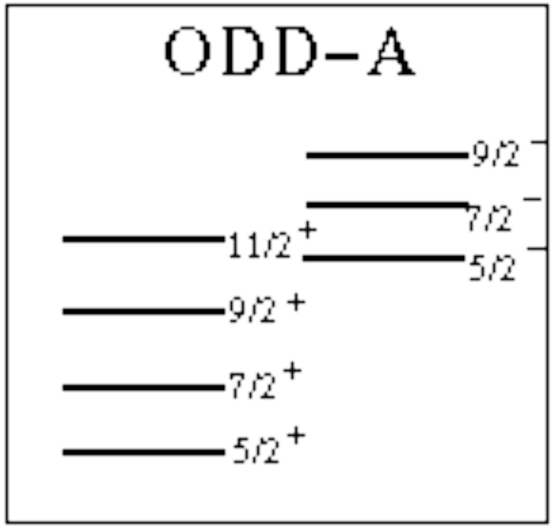


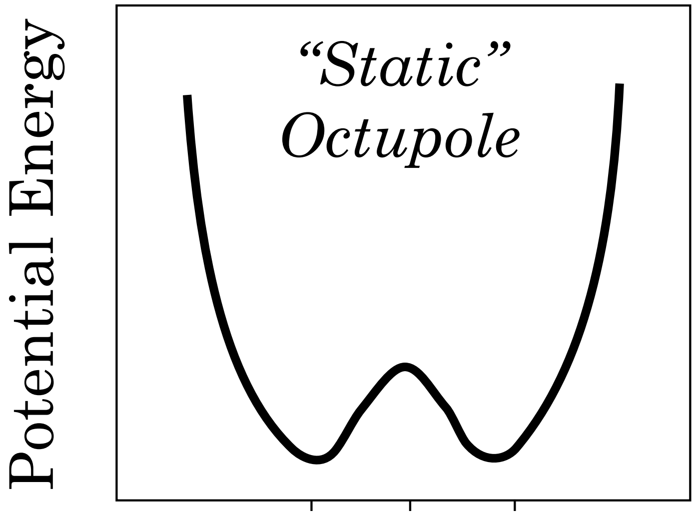


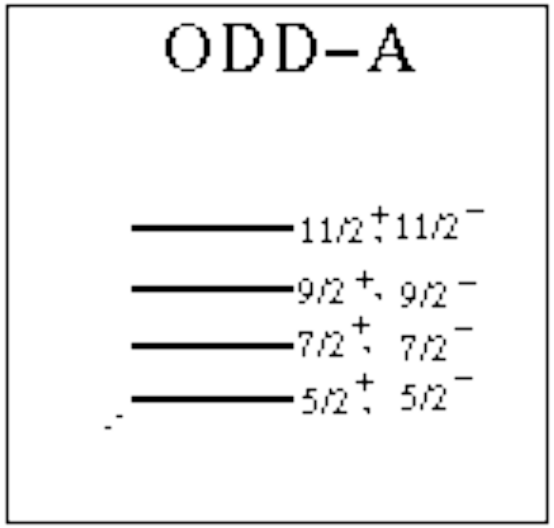


Octupole rotation-vibrations

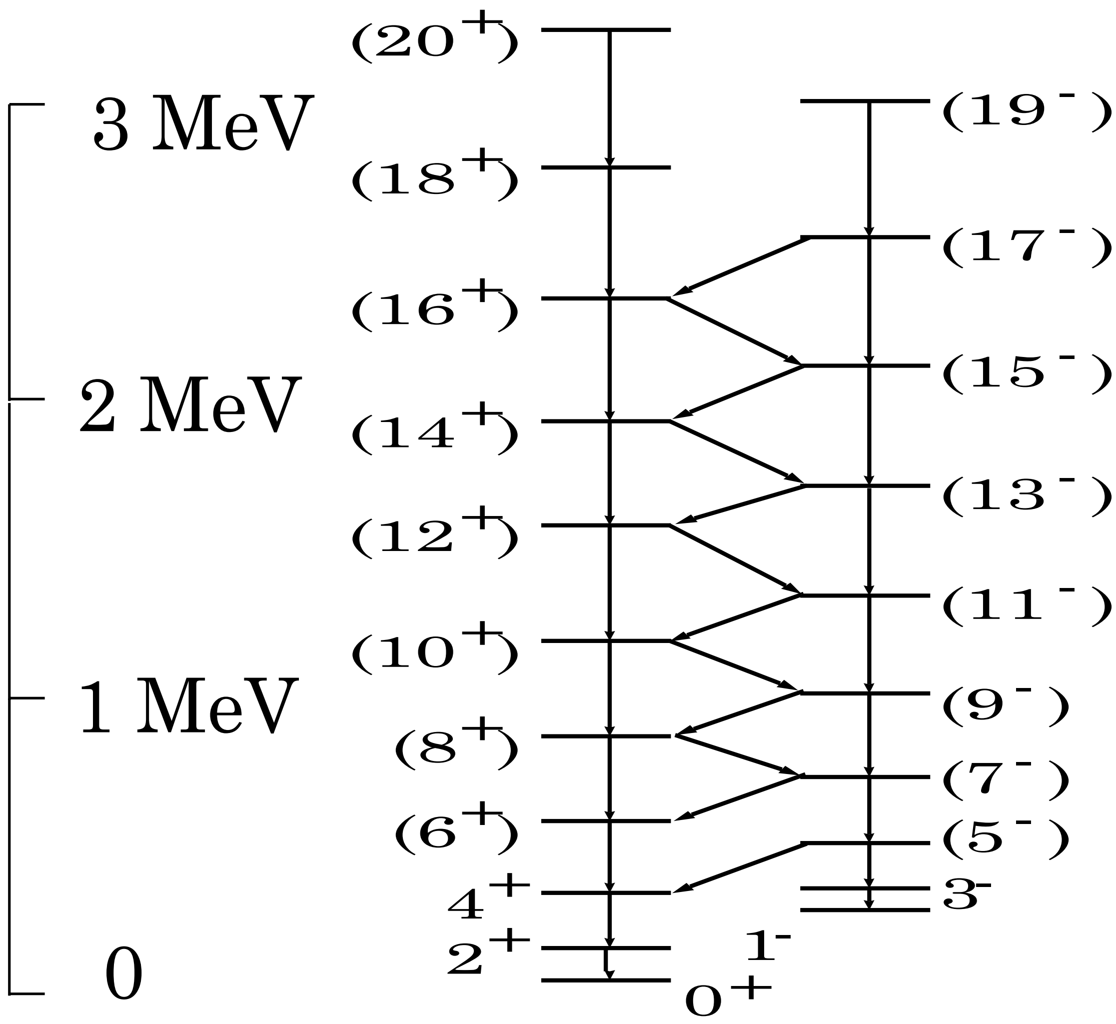








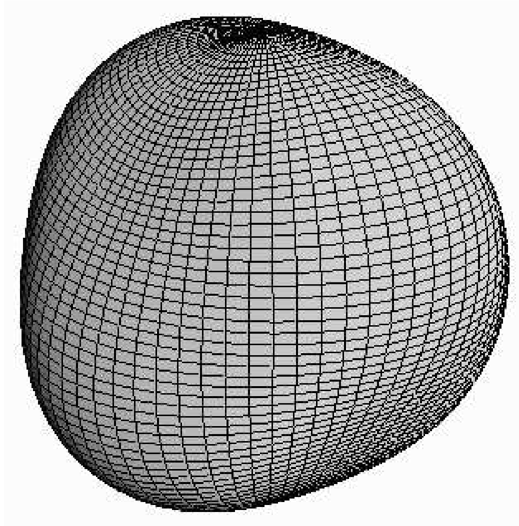
Example: ^{222}Ra



Discrete nuclear symmetries

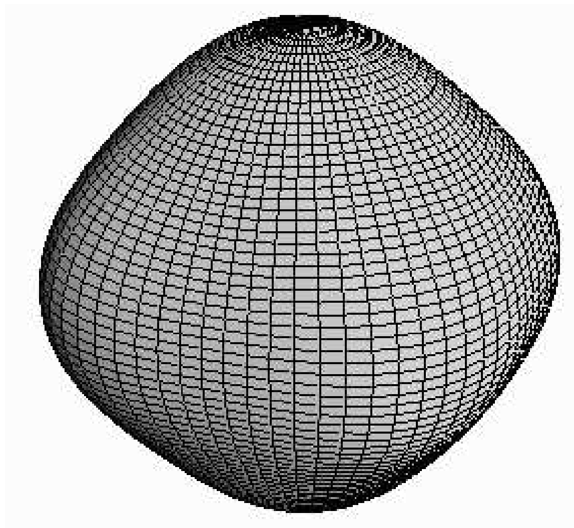
Tetrahedral symmetry:

$$\alpha_{3\pm 2} \neq 0$$



Octahedral symmetry:

$$\alpha_{40} = \sqrt{\frac{14}{5}}\alpha_{4\pm 2} \neq 0$$



Experimental evidence?

J. Dudek *et al.*, Phys. Rev. Lett. **88** (2002) 252502

Rigid rotor model

Hamiltonian of quantum-mechanical rotor in terms of 'rotational' angular momentum ***R***:

$$\hat{H}_{\mathrm{rot}} = \frac{\hbar^2}{2}\left[\frac{R_1^2}{\Im_1} + \frac{R_2^2}{\Im_2} + \frac{R_3^2}{\Im_3}\right] = \frac{\hbar^2}{2}\sum_{i=1}^{3}\frac{R_i^2}{\Im_i}$$

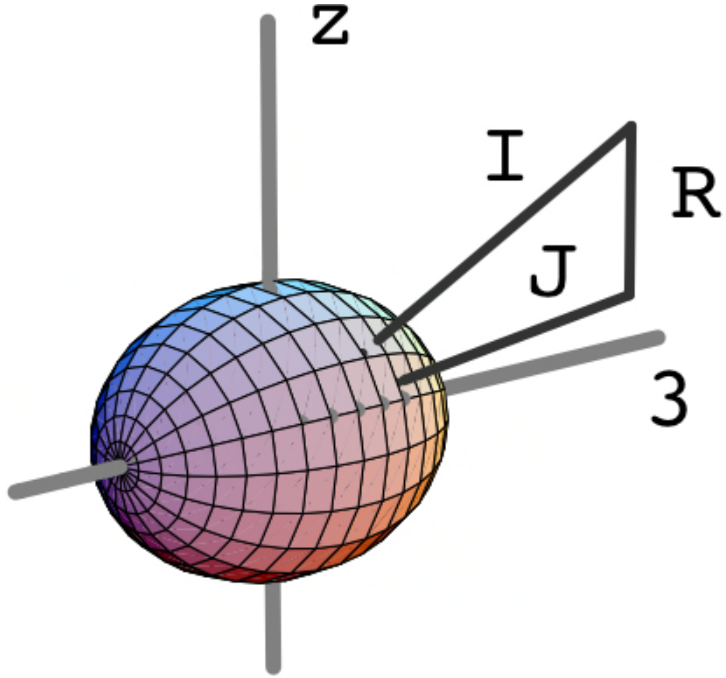
Nuclei have an additional intrinsic part H_{intr} with 'intrinsic' angular momentum ***J***.

The total angular momentum is ***I=R+J***.

Ground-state band of axial rotor

The ground-state spin of even-even nuclei is *I*=0. Hence *K*=0 for ground-state band:

$$E_I = \frac{\hbar^2}{2\Im} I(I+1)$$



E2 properties of rotational nuclei

Intra-band E2 transitions:

$$B(\mathrm{E2}; KI_{\mathrm{i}} \to KI_{\mathrm{f}}) = \frac{5}{16\pi} \langle I_{\mathrm{i}} K \, 20 | I_{\mathrm{f}} K \rangle^2 e^2 Q_0(K)^2$$

E2 moments:

$$Q(KI) = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)} Q_0(K)$$

$Q_0(K)$ is the 'intrinsic' quadrupole moment:

$$e\hat{Q}_0 \equiv \int \rho(\mathbf{r}') r^2 (3\cos^2\theta' - 1) d\mathbf{r}', \quad Q_0(K) = \langle K | \hat{Q}_0 | K \rangle$$

E2 properties of gs bands

For the ground state (*I=K*):

$$Q\left(I=K\right)=\frac{I\left(2I-1\right)}{\left(I+1\right)\left(2I+3\right)}Q_0\left(K\right)$$

For the gsb in even-even nuclei (*K=0*):

$$B\left(\mathrm{E}2;I\rightarrow I-2\right)=\frac{15}{32\pi}\frac{I\left(I-1\right)}{\left(2I-1\right)\left(2I+1\right)}e^2Q_0^2$$

$$Q\left(I\right)=-\frac{I}{2I+3}Q_0$$

$$\Rightarrow \left|eQ\left(2_1^+\right)\right|=\frac{2}{7}\sqrt{16\pi\cdot B\left(\mathrm{E}2;2_1^+\rightarrow 0_1^+\right)}$$

Generalized intensity relations

Mixing of *K* arises from

*Dependence of Q_0 on **I** (stretching)*

Coriolis interaction

Triaxiality

Generalized *intra*- and *inter*-band matrix elements (*eg* E2):

$$\frac{\sqrt{B\left(\mathrm{E2}; K_\mathrm{i} I_\mathrm{i} \rightarrow K_\mathrm{f} I_\mathrm{f}\right)}}{\left|\left\langle I_\mathrm{i} K_\mathrm{i}\; 2 K_\mathrm{f} - K_\mathrm{i} \middle| I_\mathrm{f} K_\mathrm{f} \right\rangle\right|} = M_0 + M_1 \Delta + M_2 \Delta^2 + \cdots$$

with $\Delta = I_\mathrm{f}\left(I_\mathrm{f} + 1\right) - I_\mathrm{i}\left(I_\mathrm{i} + 1\right)$

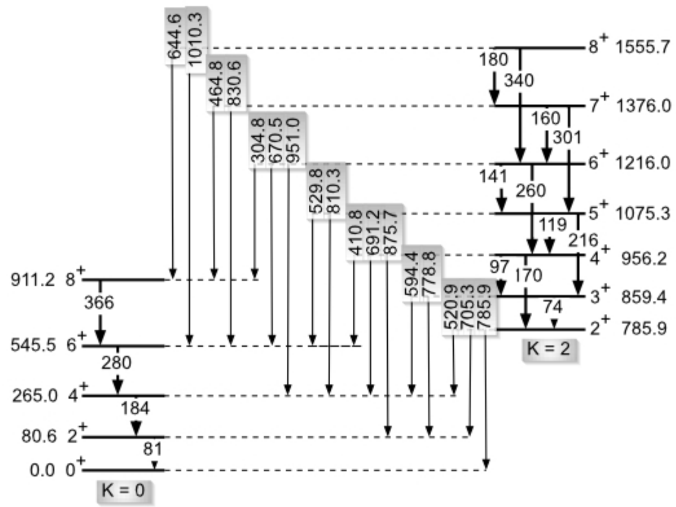
Inter-band E2 transitions

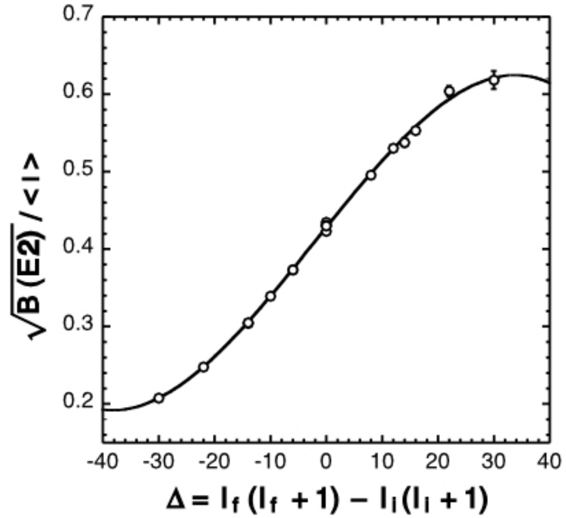
Example of $\gamma \to$g transitions in ^{166}Er:

$$\frac{\sqrt{B\left(\mathrm{E2}; I_\gamma \to I_\mathrm{g}\right)}}{\left|\left\langle I_\gamma 2\, 2 -2 \middle| I_\mathrm{g} 0\right\rangle\right|}$$

$$= M_0 + M_1 \Delta + M_2 \Delta^2 + \cdots$$

$$\Delta = I_\mathrm{g}\left(I_\mathrm{g}+1\right) - I_\gamma\left(I_\gamma+1\right)$$





W.D. Kulp *et al.*, Phys. Rev. C **73** (2006) 014308

Rigid triaxial rotor

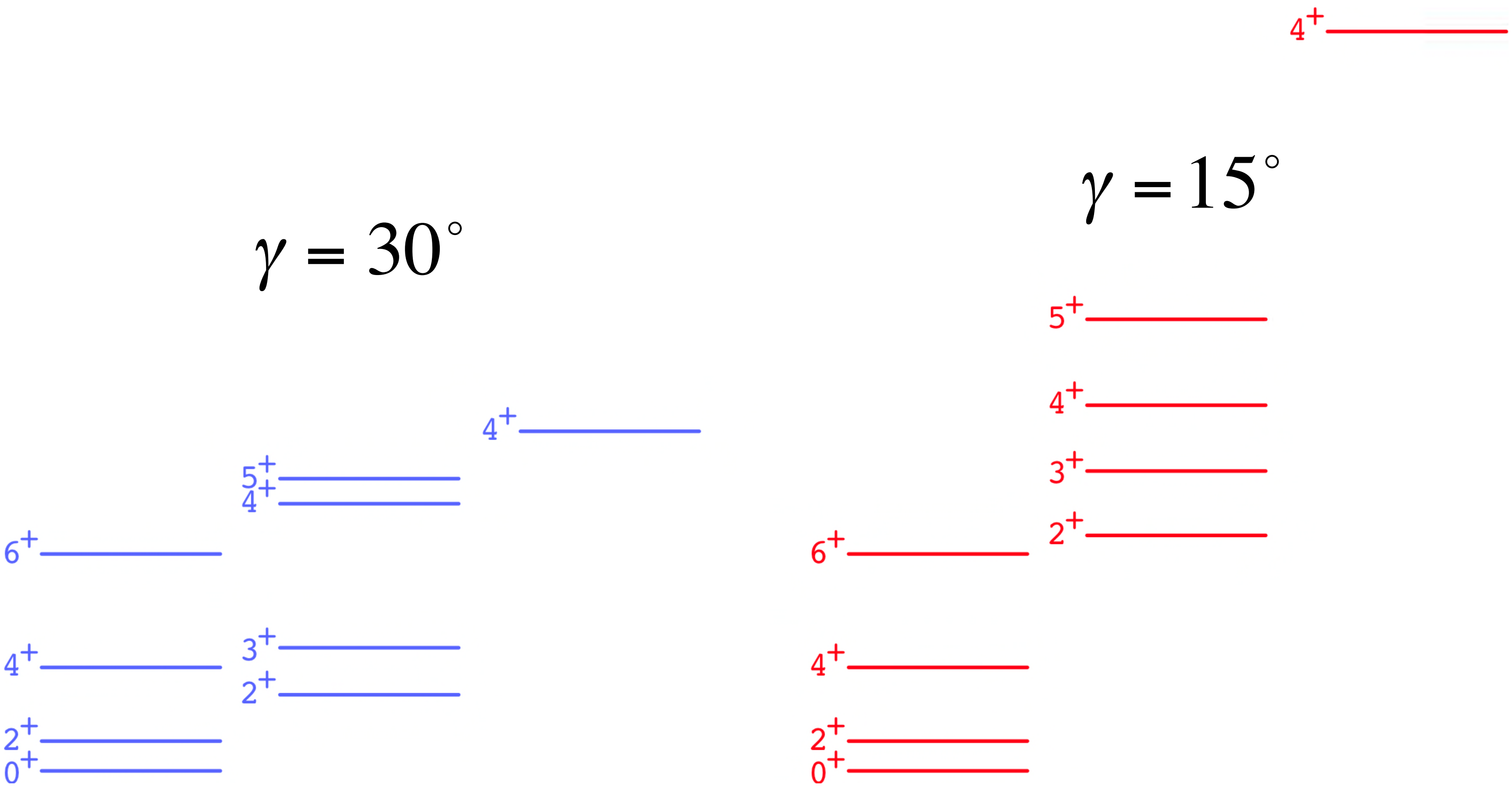
Triaxial rotor hamiltonian $\Im_1 \neq \Im_2 \neq \Im_3$:

$$\hat{H}'_{\mathrm{rot}} = \sum_{i=1}^{3} \frac{\hbar^2}{2\Im_i} I_i^2 = \underbrace{\frac{\hbar^2}{2\Im} I^2 + \frac{\hbar^2}{2\Im_f} I_3^2}_{\hat{H}'_{\mathrm{axial}}} + \underbrace{\frac{\hbar^2}{2\Im_g}\left(I_+^2 + I_-^2\right)}_{\hat{H}'_{\mathrm{mix}}}$$

$$\frac{1}{\Im} = \frac{1}{2}\left(\frac{1}{\Im_1} + \frac{1}{\Im_2}\right), \quad \frac{1}{\Im_f} = \frac{1}{\Im_3} - \frac{1}{\Im}, \quad \frac{1}{\Im_g} = \frac{1}{4}\left(\frac{1}{\Im_1} - \frac{1}{\Im_2}\right)$$

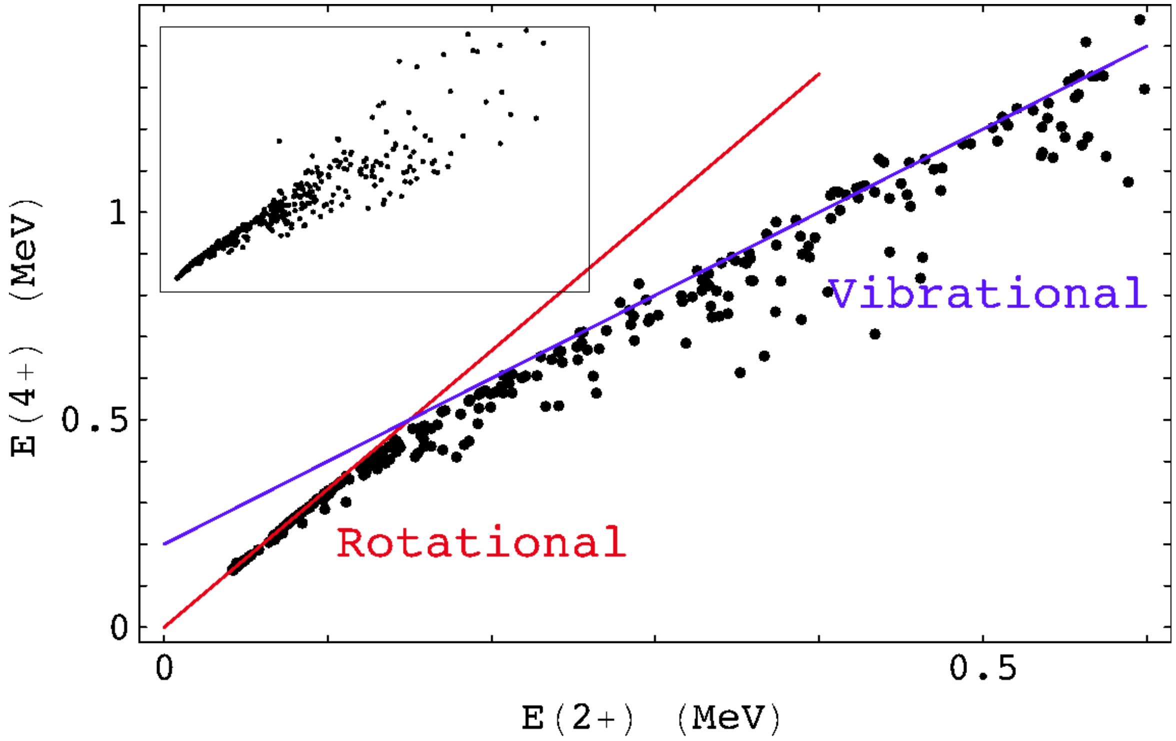
H'_{mix} non-diagonal in axial basis $|KIM\rangle \Rightarrow K$ is *not* a conserved quantum number

Rigid triaxial rotor spectra



Tri-partite classification of nuclei

Empirical evidence for seniority-type, vibrational- and rotational-like nuclei.



N.V. Zamfir *et al.*, Phys. Rev. Lett. **72** (1994) 3480

Interacting boson model

Describe the nucleus as a system of *N* interacting *s* and *d* bosons. Hamiltonian:

$$\hat{H}_{\mathrm{IBM}} = \sum_{i=1}^{6} \varepsilon_i \hat{b}_i^+ \hat{b}_i + \sum_{i_1 i_2 i_3 i_4 = 1}^{6} \upsilon_{i_1 i_2 i_3 i_4} \hat{b}_{i_1}^+ \hat{b}_{i_2}^+ \hat{b}_{i_3} \hat{b}_{i_4}$$

Justification from

Shell model: s and d bosons are associated with S and D fermion (Cooper) pairs.

Geometric model: for large boson number the IBM reduces to a liquid-drop hamiltonian.

Dimensions

Assume Ω available 1-fermion states. Number of *n*-fermion states is $\binom{\Omega}{n} = \frac{\Omega!}{n!(\Omega - n)!}$

Assume Ω available 1-boson states. Number of *n*-boson states is $\binom{\Omega + n - 1}{n} = \frac{(\Omega + n - 1)!}{n!(\Omega - 1)!}$

Example: $^{162}Dy_{96}$ with 14 neutrons (Ω=44) and 16 protons (Ω=32) ($^{132}Sn_{82}$ inert core).

SM dimension: $7 \cdot 10^{19}$

IBM dimension: 15504

Dynamical symmetries

Boson hamiltonian is of the form

$$\hat{H}_{\mathrm{IBM}} = \sum_{i=1}^{6} \varepsilon_i \hat{b}_i^{+} \hat{b}_i + \sum_{i_1 i_2 i_3 i_4 = 1}^{6} \upsilon_{i_1 i_2 i_3 i_4} \hat{b}_{i_1}^{+} \hat{b}_{i_2}^{+} \hat{b}_{i_3} \hat{b}_{i_4}$$

In general not solvable analytically.

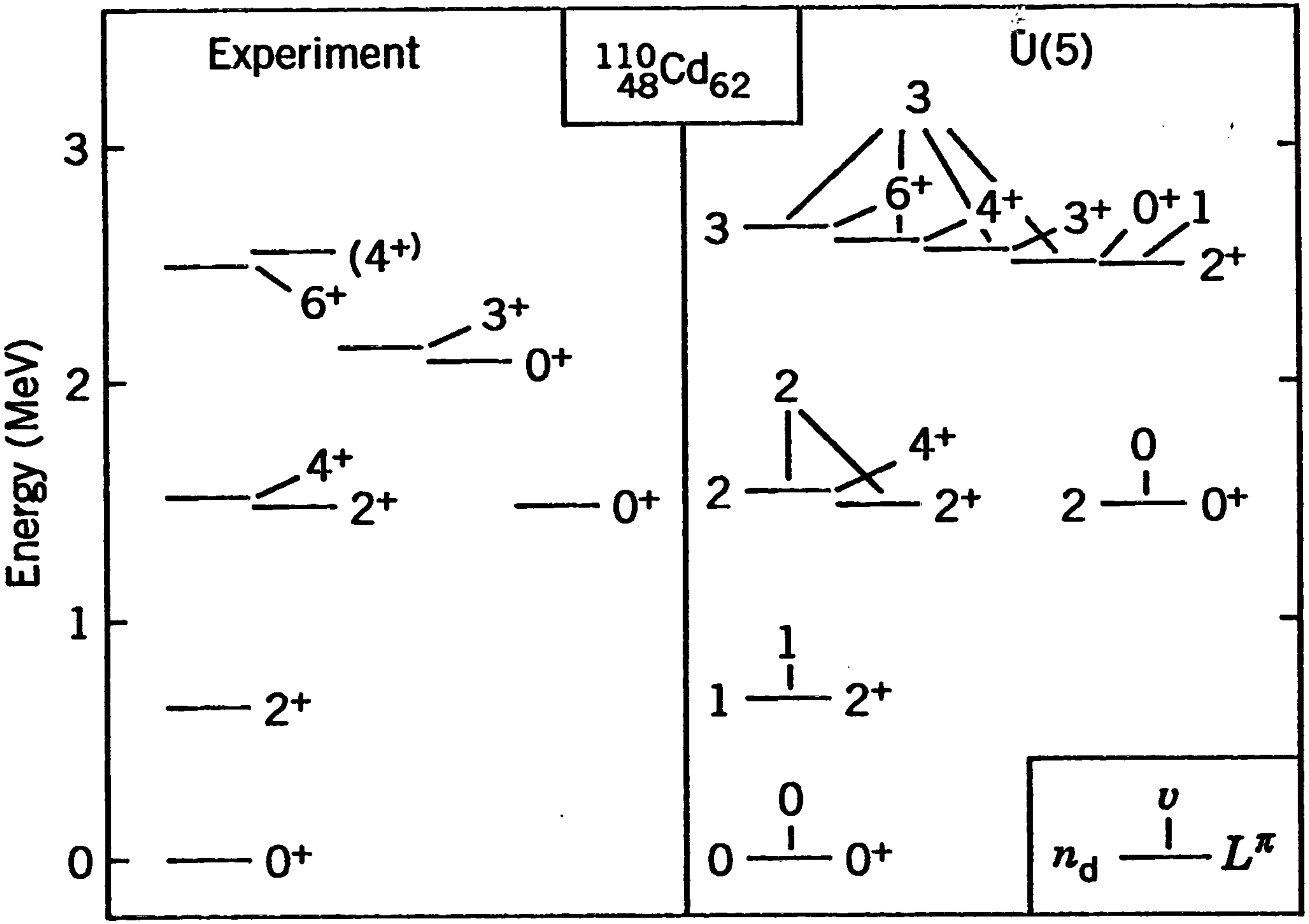
Three solvable cases with SO(3) symmetry:

$$\mathrm{U}(6) \supset \mathrm{U}(5) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3)$$

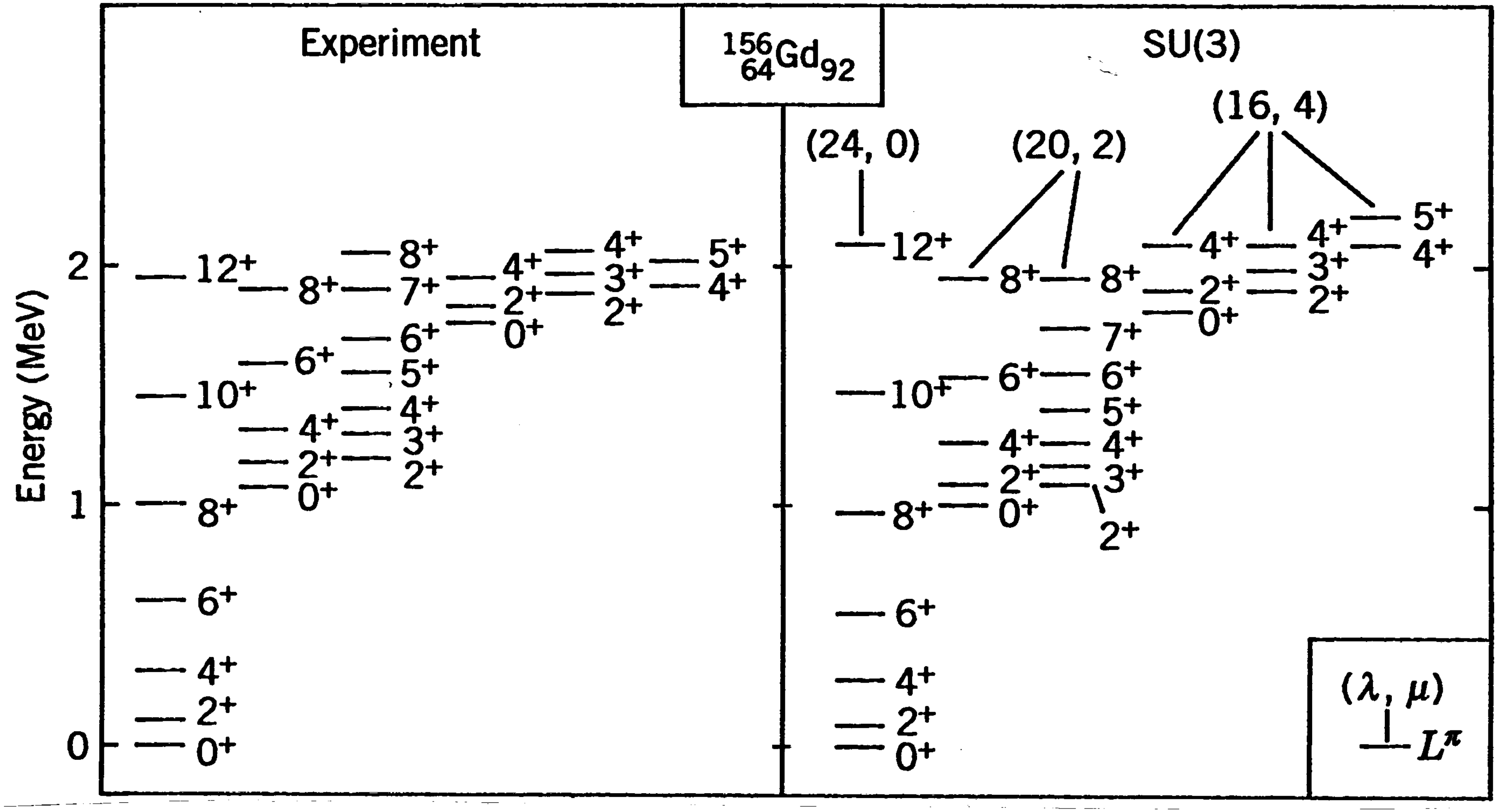
$$\mathrm{U}(6) \supset \mathrm{SU}(3) \supset \mathrm{SO}(3)$$

$$\mathrm{U}(6) \supset \mathrm{SO}(6) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3)$$

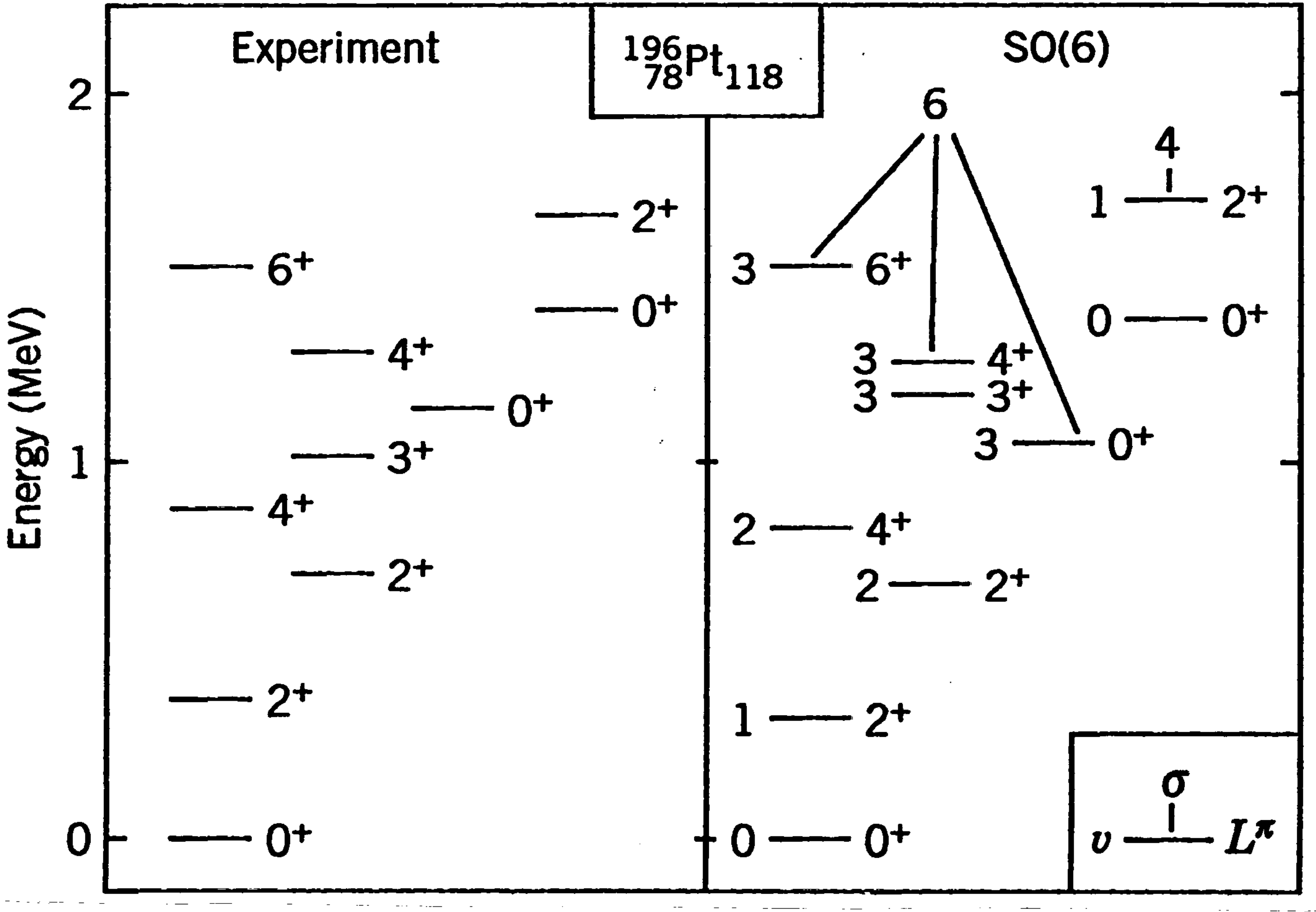
U(5) vibrational limit: $^{110}Cd_{62}$



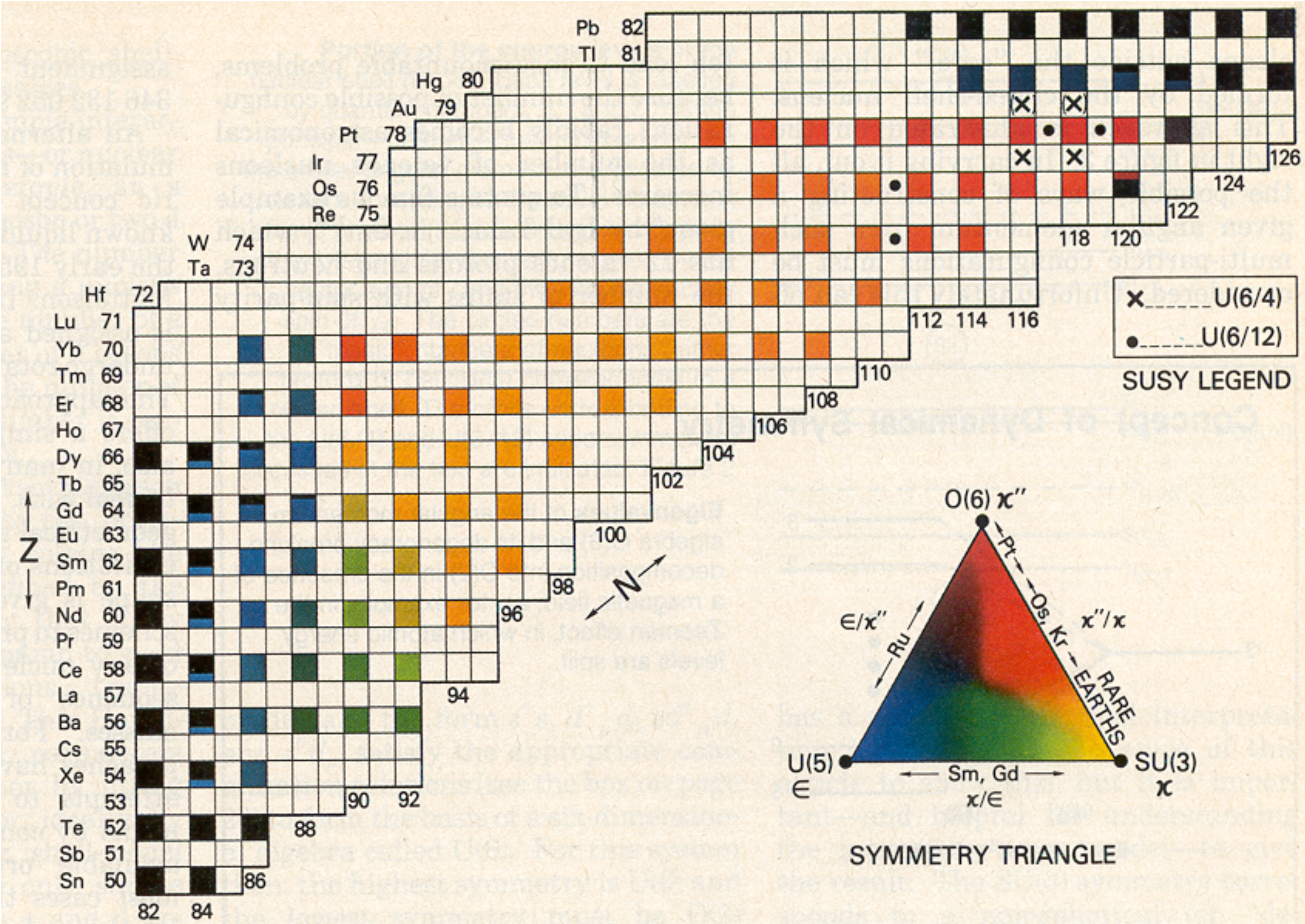
SU(3) rotational limit: $^{156}Gd_{92}$



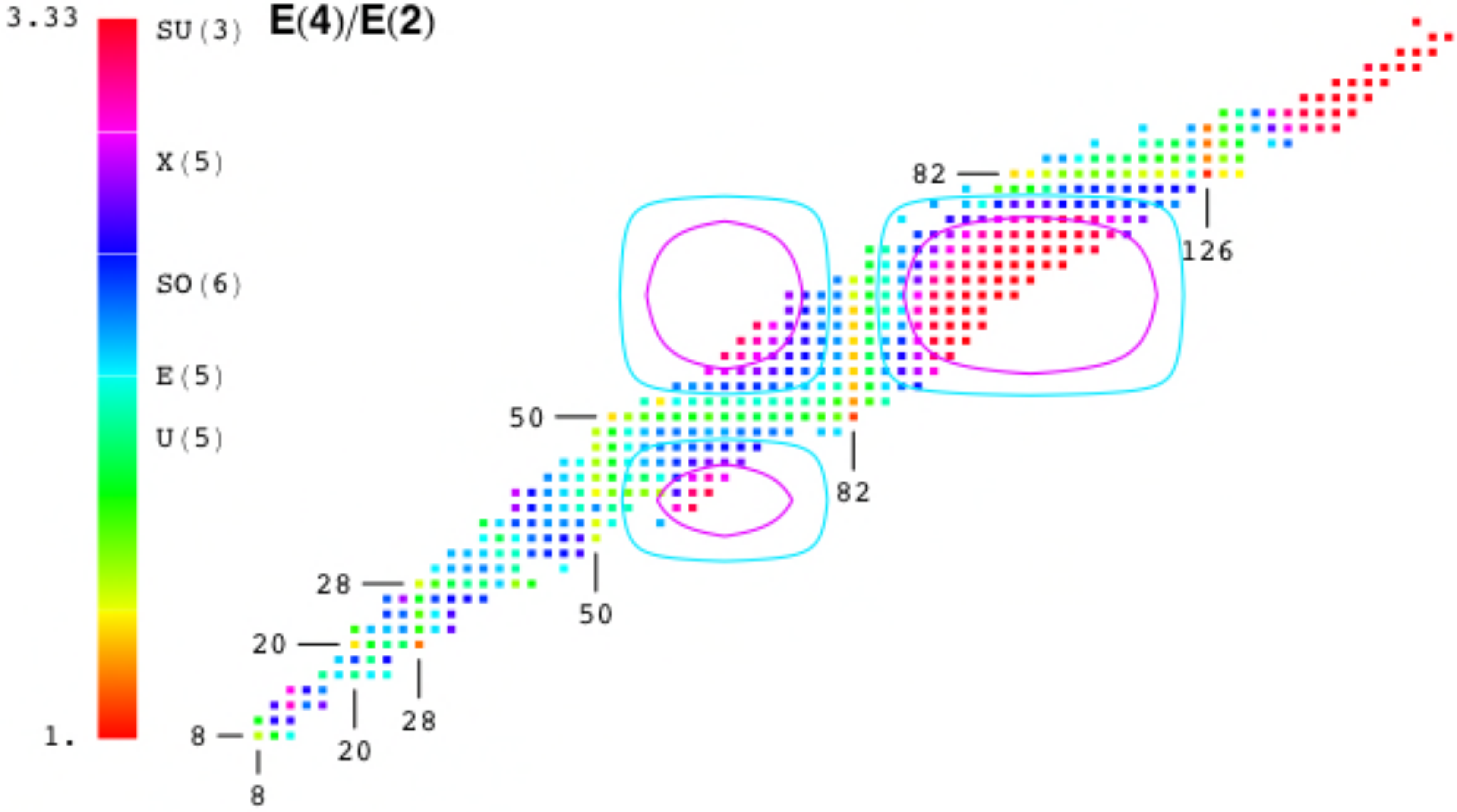
SO(6) γ-unstable limit: $^{196}Pt_{118}$



Applications of IBM



The ratio R_{42}



Bibliography

A. Bohr and B.R. Mottelson, *Nuclear Structure. I Single-Particle Motion* (Benjamin, 1969).

A. Bohr and B.R. Mottelson, *Nuclear Structure. II Nuclear Deformations* (Benjamin, 1975).

R.D. Lawson, *Theory of the Nuclear Shell Model* (Oxford UP, 1980).

K.L.G. Heyde, *The Nuclear Shell Model* (Springer-Verlag, 1990).

I. Talmi, *Simple Models of Complex Nuclei* (Harwood, 1993).

F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge UP, 1987).

Rigid axially symmetric rotor

For $\Im_1 = \Im_2 = \Im \neq \Im_3$ the rotor hamiltonian is

$$\hat{H}_{\text{rot}} = \sum_{i=1}^{3} \frac{\hbar^2}{2\Im_i} I_i^{\,2} = \frac{\hbar^2}{2\Im} \sum_{i=1}^{3} I_i^{\,2} + \frac{\hbar^2}{2}\left(\frac{1}{\Im_3} - \frac{1}{\Im}\right) I_3^{\,2}$$

Eigenvalues of H_{rot}:

$$E_{KI} = \frac{\hbar^2}{2\Im} I(I+1) + \frac{\hbar^2}{2}\left(\frac{1}{\Im_3} - \frac{1}{\Im}\right) K^2$$

Eigenvectors $|KIM\rangle$ of H_{rot} satisfy:

$$I^2 |KIM\rangle = I(I+1) |KIM\rangle,$$

$$I_z |KIM\rangle = M |KIM\rangle, \quad I_3 |KIM\rangle = K |KIM\rangle$$

Electric (quadrupole) properties

Partial γ-ray half-life:

$$T_{1/2}^{\gamma}(\mathrm{E}\lambda) = \ln 2 \left\{ \frac{8\pi}{\hbar} \frac{\lambda+1}{\lambda\left[(2\lambda+1)!!\right]^2} \left(\frac{E_\gamma}{\hbar c}\right)^{2\lambda+1} B(\mathrm{E}\lambda) \right\}^{-1}$$

Electric quadrupole transitions:

$$B(\mathrm{E2}; I_\mathrm{i} \to I_\mathrm{f}) = \frac{1}{2I_\mathrm{i}+1} \sum_{M_\mathrm{i}} \sum_{M_\mathrm{f}\mu} \left| \left\langle I_\mathrm{f} M_\mathrm{f} \left| \sum_{k=1}^{A} e_k r_k^2 Y_{2\mu}(\theta_k, \varphi_k) \right| I_\mathrm{i} M_\mathrm{i} \right\rangle \right|^2$$

Electric quadrupole moments:

$$eQ(I) = \left\langle IM = I \left| \sqrt{\frac{16\pi}{5}} \sum_{k=1}^{A} e_k r_k^2 Y_{20}(\theta_k, \varphi_k) \right| IM = I \right\rangle$$

Magnetic (dipole) properties

Partial γ-ray half-life:

$$T^{\gamma}_{1/2}(\mathrm{M}\lambda) = \ln 2 \left\{ \frac{8\pi}{\hbar} \frac{\lambda+1}{\lambda[(2\lambda+1)!!]^2} \left(\frac{E_\gamma}{\hbar c}\right)^{2\lambda+1} B(\mathrm{M}\lambda) \right\}^{-1}$$

Magnetic dipole transitions:

$$B(\mathrm{M1}; I_\mathrm{i} \rightarrow I_\mathrm{f}) = \frac{1}{2I_\mathrm{i}+1} \sum_{M_\mathrm{i}} \sum_{M_\mathrm{f}\mu} \left| \left\langle I_\mathrm{f} M_\mathrm{f} \right| \sum_{k=1}^{A} \left(g_k^l l_{k,\mu} + g_k^s s_{k,\mu} \right) \left| I_\mathrm{i} M_\mathrm{i} \right\rangle \right|^2$$

Magnetic dipole moments:

$$\mu(I) = \left\langle IM = I \right| \sum_{k=1}^{A} \left(g_k^l l_{k,z} + g_k^s s_{k,z} \right) \left| IM = I \right\rangle$$

Classical limit of IBM

For large boson number *N*, a *coherent* (or *intrinsic*) state is an approximate eigenstate,

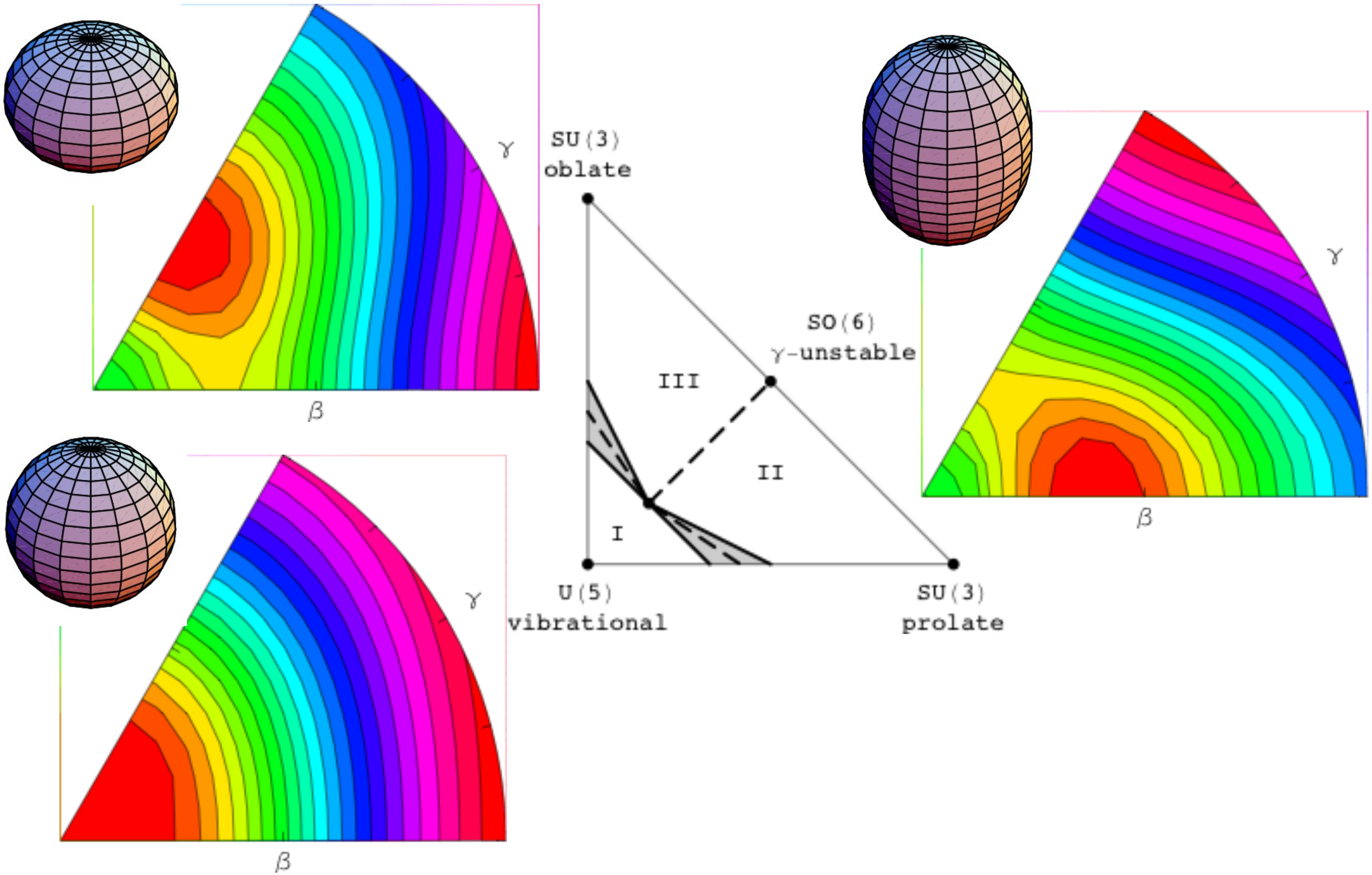
$$\hat{H}_{\mathrm{IBM}}\left|N;\alpha_\mu\right\rangle \approx E\left|N;\alpha_\mu\right\rangle, \qquad \left|N;\alpha_\mu\right\rangle \propto \left(s^+ + \sum_\mu \alpha_\mu d_\mu^+\right)^N \left|\mathrm{o}\right\rangle$$

The real parameters α_μ are related to the three Euler angles and shape variables β and γ.

Any IBM hamiltonian yields energy surface:

$$\left\langle N;\alpha_\mu\right|\hat{H}_{\mathrm{IBM}}\left|N;\alpha_\mu\right\rangle = \left\langle N;\beta\gamma\right|\hat{H}_{\mathrm{IBM}}\left|N;\beta\gamma\right\rangle \equiv V(\beta,\gamma)$$

Phase diagram of IBM



J. Jolie *et al.* , Phys. Rev. Lett. **87** (2001) 162501

Extensions of IBM

Neutron and proton degrees freedom (IBM-2):

F-spin multiplets ($N_\nu+N_\pi$=constant)

Scissors excitations

Fermion degrees of freedom (IBFM):

Odd-mass nuclei

Supersymmetry (doublets & quartets)

Other boson degrees of freedom:

Isospin T=0 & T=1 pairs (IBM-3 & IBM-4)

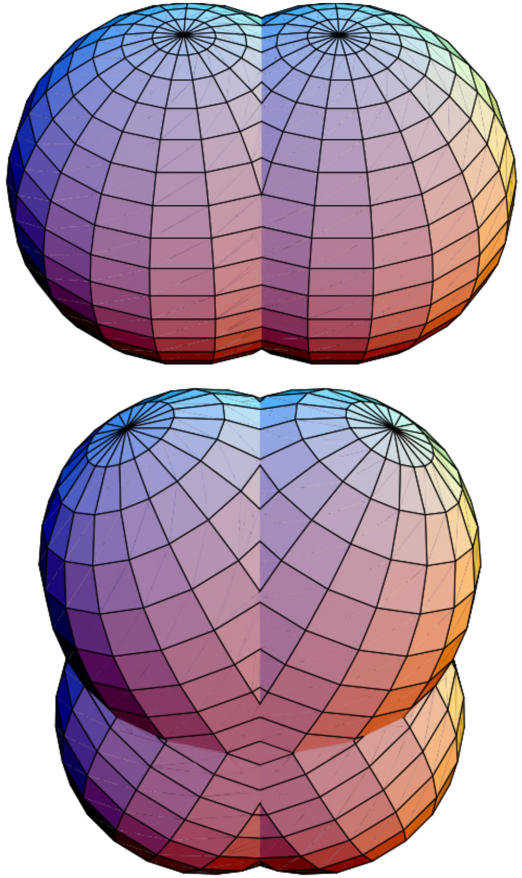
Higher multipole (g,...) pairs

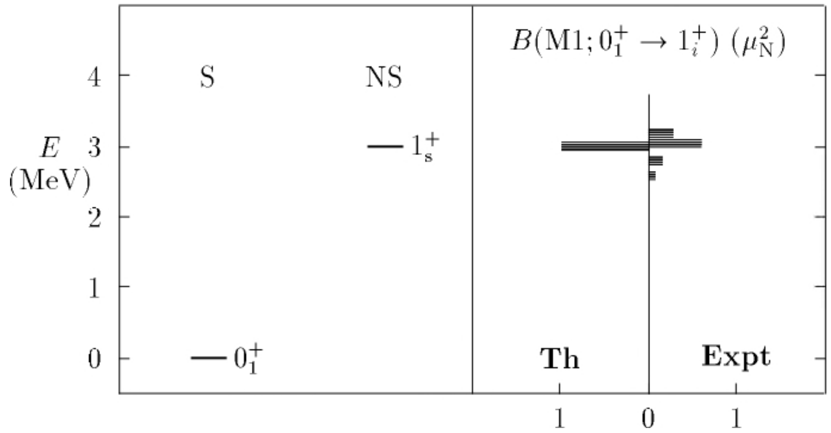
Scissors mode

Collective displacement modes between neutrons and protons:

Linear displacement (giant dipole resonance):
R_ν-R_π $\Rightarrow$ E1 excitation.

Angular displacement (scissors resonance):
L_ν-L_π $\Rightarrow$ M1 excitation.

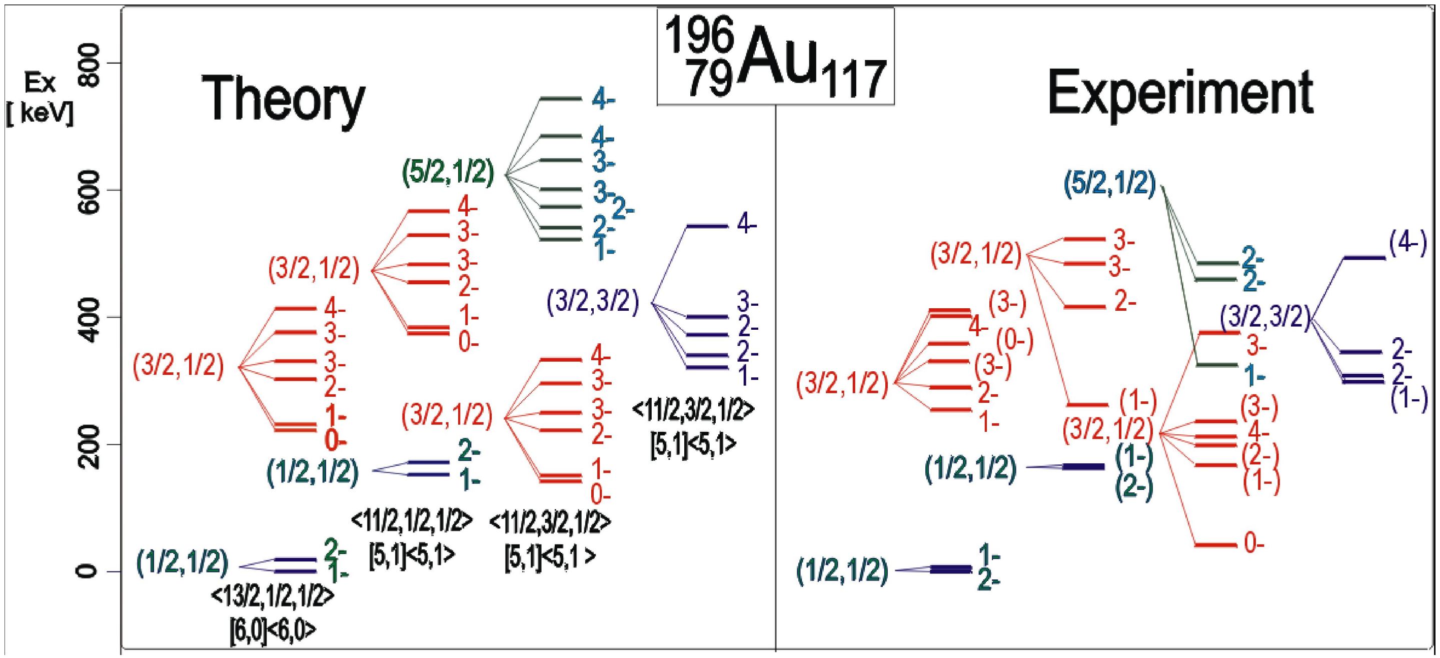




Supersymmetry

A simultaneous description of even- and odd-mass nuclei (doublets) or of even-even, even-odd, odd-even and odd-odd nuclei (quartets).

Example of ^{194}Pt, ^{195}Pt, ^{195}Au & ^{196}Au:



Bosons + fermions

Odd-mass nuclei are fermions.

Describe an odd-mass nucleus as N bosons + 1 fermion mutually interacting. Hamiltonian:

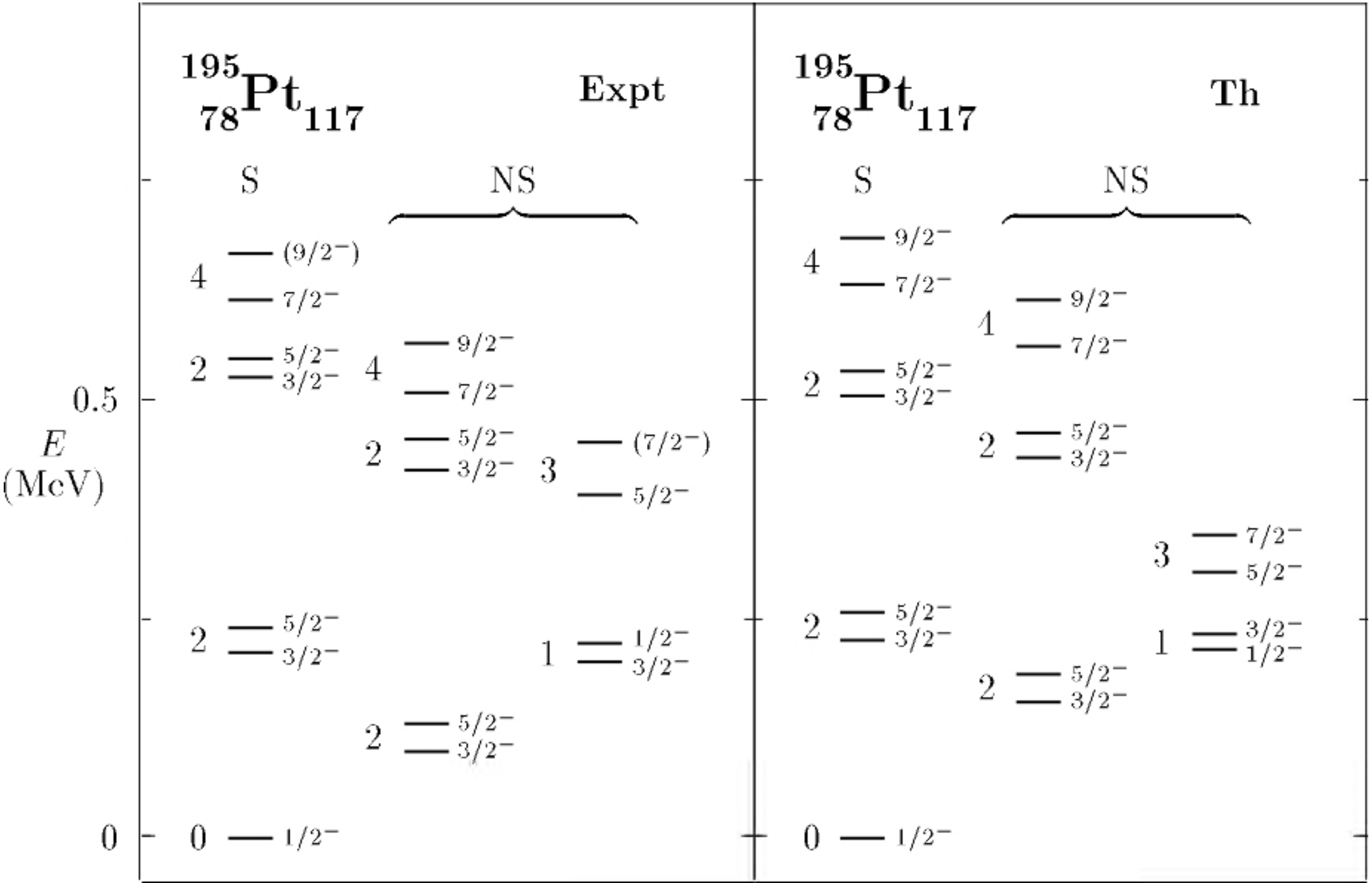
$$\hat{H}_{\mathrm{IBFM}} = \hat{H}_{\mathrm{IBM}} + \sum_{j=1}^{\Omega} \bar{\varepsilon}_j \hat{a}_j^+ \hat{a}_j + \sum_{i_1 i_2 = 1}^{6} \sum_{j_1 j_2 = 1}^{\Omega} \bar{\upsilon}_{i_1 j_1 i_2 j_2} \hat{b}_{i_1}^+ \hat{a}_{j_1}^+ \hat{b}_{i_2} \hat{a}_{j_2}$$

Algebra:

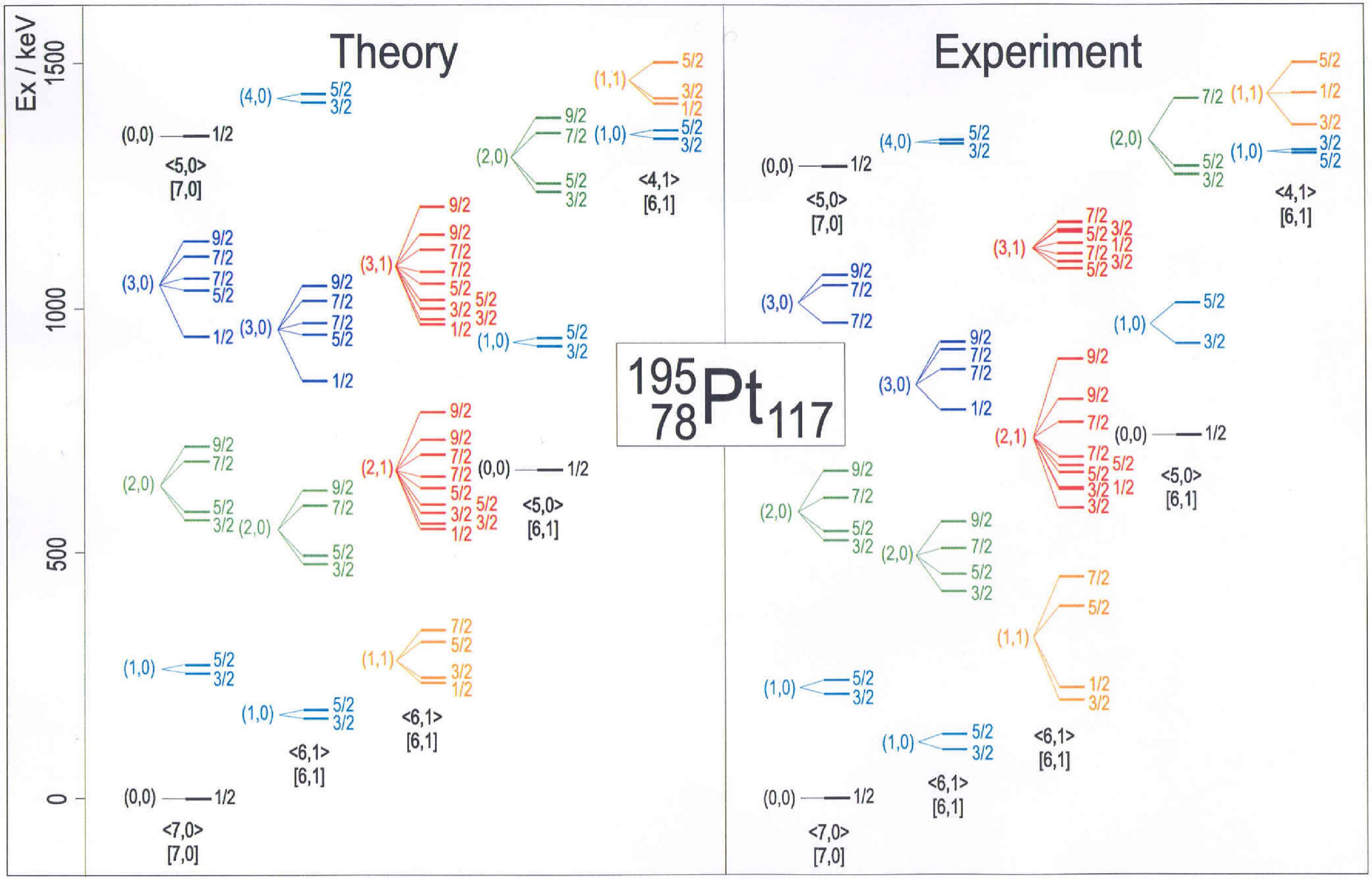
$$\mathrm{U}(6) \oplus \mathrm{U}(\Omega) = \begin{Bmatrix} \hat{b}_{i_1}^+ \hat{b}_{i_2} & \\ & \hat{a}_{j_1}^+ \hat{a}_{j_2} \end{Bmatrix}$$

Many-body problem is solved analytically for certain energies ε and interactions υ.

Example: $^{195}Pt_{117}$



Example: $^{195}Pt_{117}$ (new data)



Nuclear supersymmetry

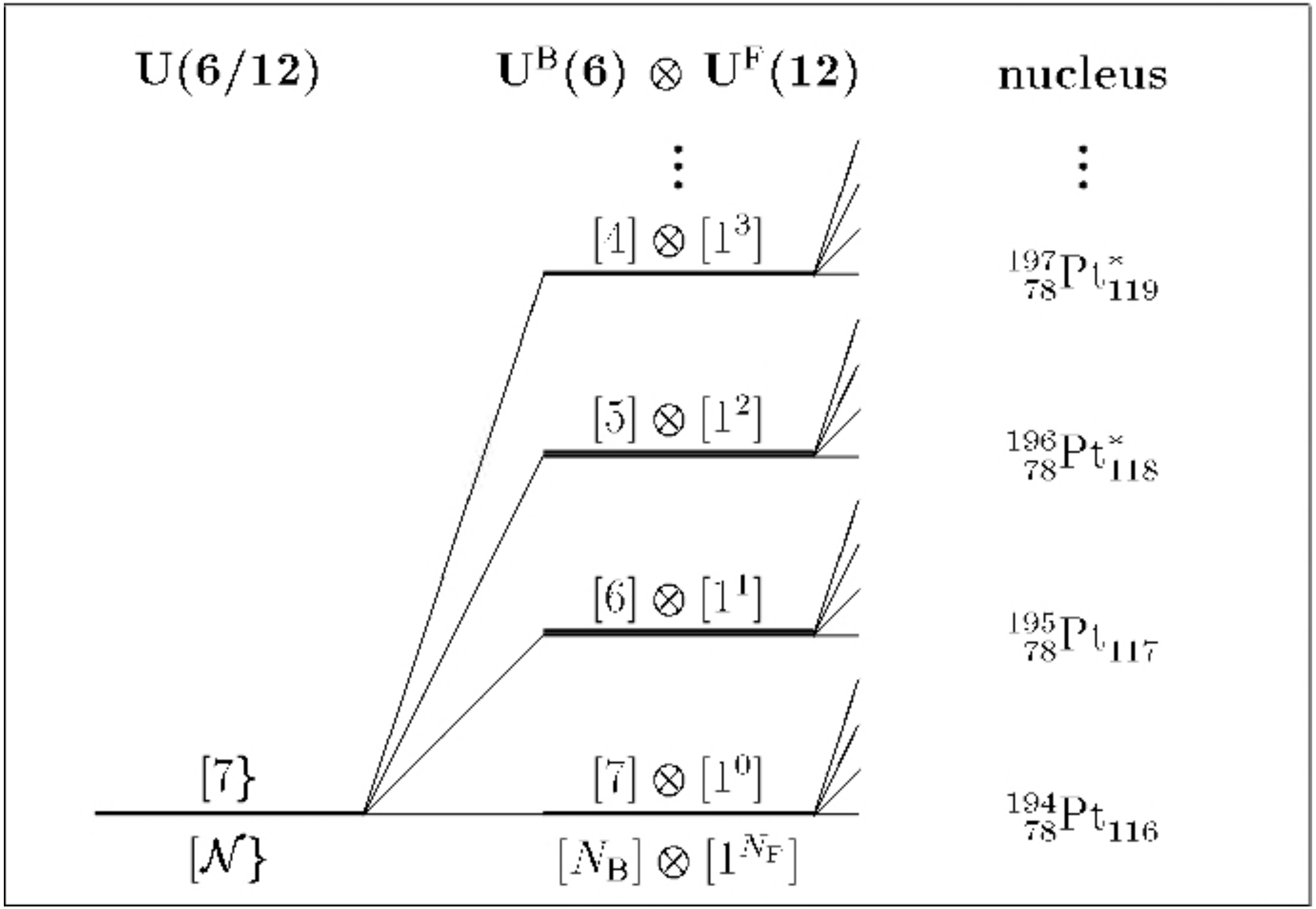
Up to now: separate description of even-even and odd-mass nuclei with the algebra

$$\mathrm{U}(6) \oplus \mathrm{U}(\Omega) = \begin{Bmatrix} \hat{b}^+_{i_1}\hat{b}_{i_2} & \\ & \hat{a}^+_{j_1}\hat{a}_{j_2} \end{Bmatrix}$$

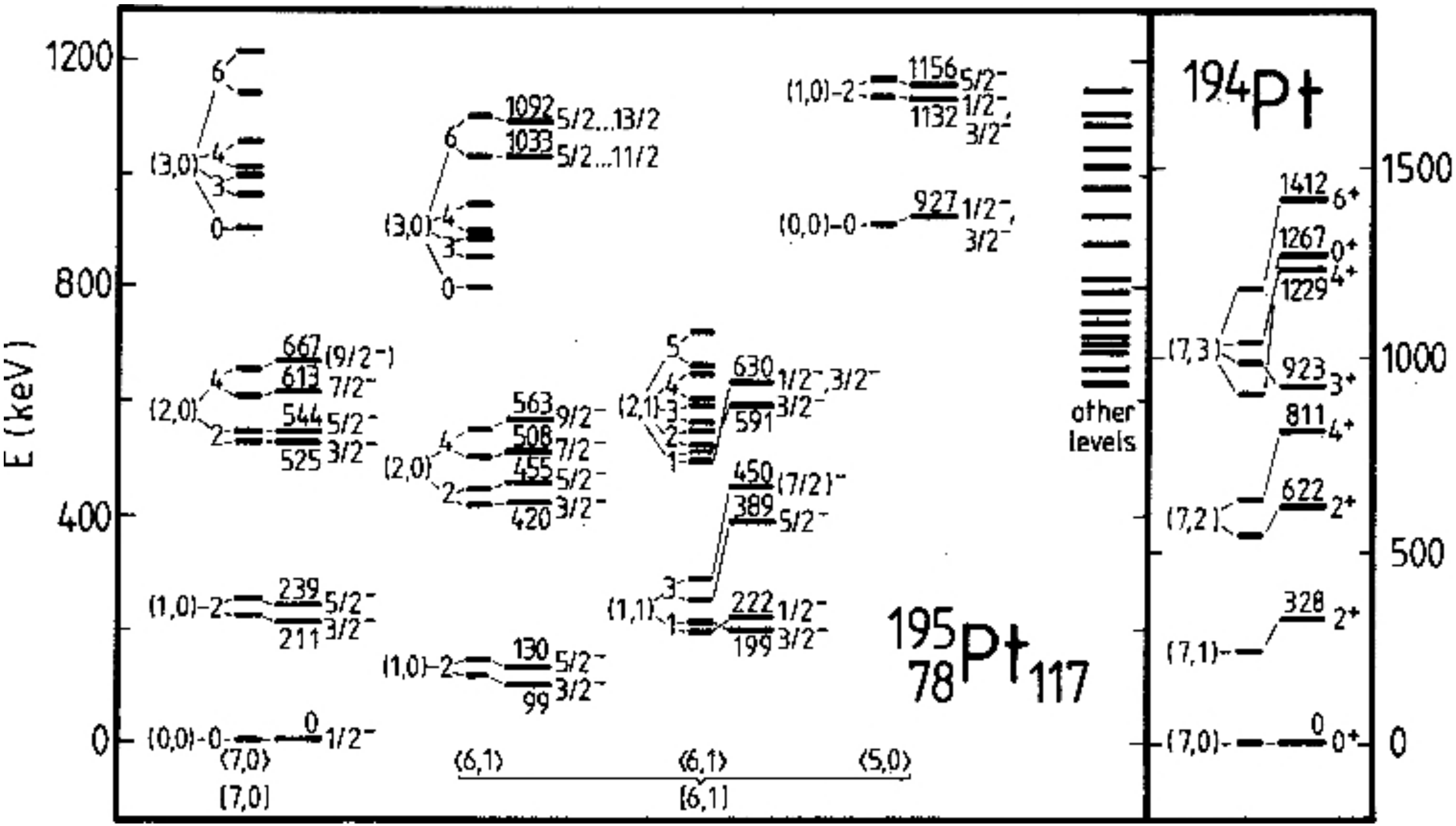
Simultaneous description of even-even and odd-mass nuclei with the superalgebra

$$\mathrm{U}(6/\Omega) = \begin{Bmatrix} \hat{b}^+_{i_1}\hat{b}_{i_2} & \hat{b}^+_{i_1}\hat{a}_{j_2} \\ \hat{a}^+_{j_1}\hat{b}_{i_2} & \hat{a}^+_{j_1}\hat{a}_{j_2} \end{Bmatrix}$$

U(6/12) supermultiplet



Example: $^{194}Pt_{116}$ & $^{195}Pt_{117}$



Example: $^{196}Au_{117}$