Non-Perturbative Topological Strings

Marcos Mariño
University of Geneva

[Alba Grassi, Yasuyuki Hatsuda and M.M., 1410.3382]

[Ricardo Couso-Santamaria, M.M. and Ricardo Schiappa, to appear]

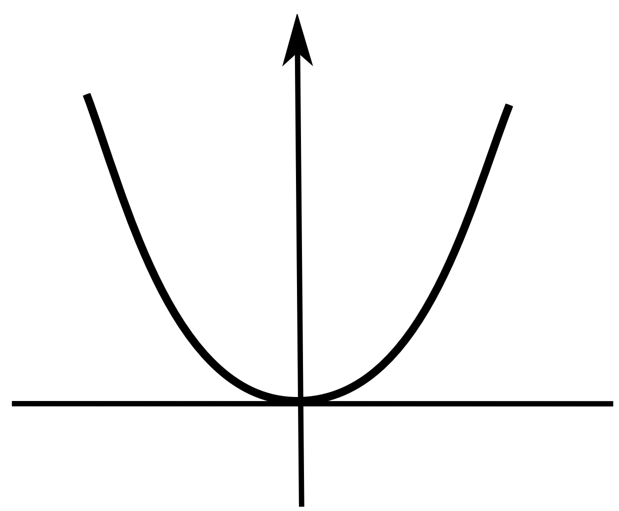
As is well-known, (topological) string theory is defined by a formal, divergent genus expansion

$$F_{\mathrm{TS}}(t, g_s) = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2} \qquad F_g(t) \sim (2g)!$$

Question: is there a *well-defined, computable function* of the moduli *t* and the string coupling constant which has this genus expansion as an asymptotic expansion?

This is the problem of formulating (topological) string theory non-perturbatively.

Since the expression "non-perturbative topological string" has been much (ab)used in recent years, it is useful to go back to basics, i.e. to Quantum Mechanics



$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + \frac{\mathsf{x}^2}{2} + g\mathsf{x}^4$$

quartic oscillator

Standard perturbation theory gives a formal, divergent series for the ground state energy

$$\varphi(g) = \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \cdots$$

The non-perturbative completion of this perturbative series is given by the spectral theory of Schrödinger operators on the Hilbert space $L^2(\mathbb{R})$, and relies on two theorems.

The first theorem says that the operator H^{-1} is compact for all positive g, so we have a discrete, well-defined spectrum, and the ground state energy $E_0(g)$ is a well-defined function of g.

The second theorem says that this function has an asymptotic expansion which agrees with the series obtained in perturbation theory.

$$E_0(g) \sim \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \cdots$$

Note that this function is effectively calculable, by using for example numerical diagonalization

$$E_0(1) = 0.8037706512...$$

A more complicated example of non-perturbative completion: AdS/CFT. Now there are two parameters

$$F_{\text{gauge}}(N, g_s) \sim \sum_{g=0}^{\infty} F_g(\lambda) g_s^{2g-2}$$

gauge theory free energy

genus expansion in superstring theory

't Hooft parameter

$$\lambda = N g_s \propto (L/\ell_s)^4$$

AdS radius

Note that *N* is a positive integer in the non-perturbative definition, while it is a real, continuous variable in superstring theory: the non-perturbative definition seems to cover less than the perturbative series! More on this later.

The definition of non-perturbative definition

A non-perturbative definition of a formal perturbative series is a well-defined function, computable of the relevant parameters (at least in some range), and such that its asymptotic expansion is given by the original perturbative series, *and nothing else*

In the case of topological strings on toric CYs, previous "non-perturbative definitions" in the literature either do not satisfy this criterium, or apply only to particular geometries, or both

Topological strings on toric Calabi-Yau's

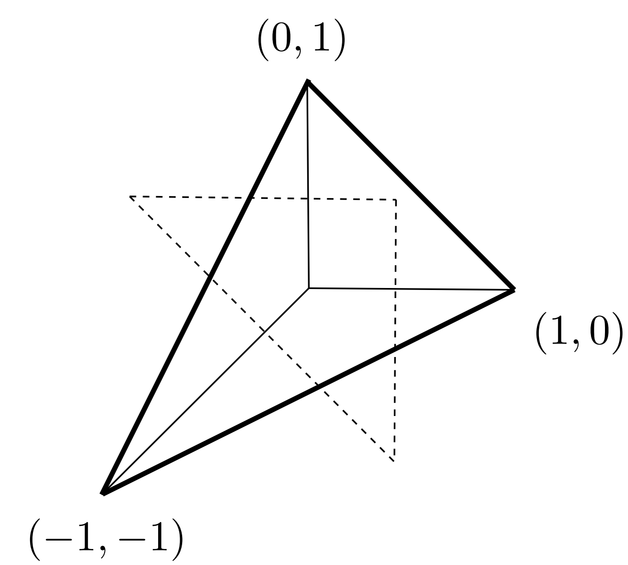
The simplest yet non-trivial CY threefolds are *toric CYs*, which are noncompact. They can be described by *Newton polygons.* Their mirror manifolds reduce to algebraic curves

$$W_X(\mathrm{e}^x, \mathrm{e}^y) = 0$$

given by the Newton polynomial of the polygon

Example: the canonical bundle over $\mathbb{P}^2$

"local $\mathbb{P}^2$"



inner point of the polygon

$$W_X(\mathrm{e}^x, \mathrm{e}^y) = \quad \mathrm{e}^x + \mathrm{e}^y + \mathrm{e}^{-x-y} + \kappa = 0$$

The topological string free energies at genus *g* encode the Gromov-Witten invariants of these threefolds:

$$F_g^{\mathrm{LR}}(t) = \sum_{d \geq 1} N_{g,d}\, \mathrm{e}^{-dt}$$

The free energies can be defined in different "frames", similar to the duality frames in Seiberg-Witten theory. These frames are related by symplectic transformations, and implemented by formal integral transforms [Aganagic-Bouchard-Klemm]. They contain the same perturbative information. The GW invariants correspond to the so-called large radius (LR) or "electric frame".

In this talk I will be mostly interested in the conifold, or "magnetic" frame, where the free energies look like

$$F_g(\lambda) = \frac{B_{2g}}{2g(2g-2)}\lambda^{2-2g} + \sum_{n\geq 1} F_{g,n}\lambda^n$$

vanishing period at the conifold point

They lead to a formal, divergent genus expansion

$$F_{\mathrm{TS}}(\lambda, g_s) = \sum_{g\geq 0} F_g(\lambda) g_s^{2g-2}$$

Thanks to the work of Albrecht Klemm and collaborators, it is possible to calculate the free energies to very high genus (we did it up to *g=114*).

Operators from mirror curves

In order to find the "dual" quantum systems to topological string theories, it was first proposed in [ADKMV] that mirror curves can be "quantized" by promoting *x, y* to canonically conjugate Heisenberg operators

$$[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar \qquad \hbar \in \mathbb{R}_{>0}$$

For simplicity, we will focus on mirror curves of genus one. Weyl quantization of the Newton polynomial produces a *self-adjoint operator* on $L^2(\mathbb{R})$

$$W_X(\mathrm{e}^x, \mathrm{e}^y) \to \mathsf{O}_X$$

local $\mathbb{P}^2$ $\qquad \mathsf{O} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{x}-\mathsf{y}}$

Theorem

[Grassi-Hatsuda-M.M., Kashaev-M.M., Laptev-Schimmer-Takhtajan]

The operator $\rho_X = \mathsf{O}_X^{-1}$ on $L^2(\mathbb{R})$ is positive definite and of trace class

discrete spectrum! $\mathrm{e}^{-E_n}, \qquad n = 0, 1, \cdots$

local $\mathbb{P}^2$

$\hbar = 2\pi$

n	E_n
0	2.56264206862381937
1	3.91821318829983977
2	4.91178982376733606
3	5.73573703542155946
4	6.45535922844299896

similar to confining potentials in Schrödinger theory

Spectral theory

The spectral information of this operator can be collected in various ways. We have the *spectral traces*

$$Z_\ell = \mathrm{Tr}\, \rho_X^\ell = \sum_{n\geq 0} \mathrm{e}^{-\ell E_n}, \qquad \ell = 1, 2, \cdots$$

and the *Fredholm determinant*

$$\Xi_X(\kappa) = \det(1 + \kappa \rho_X) = \exp\left\{ \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell} Z_\ell \kappa^\ell \right\}$$

$$= 1 + \sum_{N=1}^{\infty} Z_X(N, \hbar) \kappa^N$$

"fermionic" spectral traces

$Z_X(N,\hbar)$ are well-defined due to the trace class property of ρ_X, and they can be computed from first principles in the QM model. They are combinations of the standard spectral traces. For example,

$$Z_X(1,\hbar) = \int_{\mathbb{R}} \rho_X(x,x)\mathrm{d}x$$

The integral kernel of the operator ρ_X can be written explicitly, for many geometries, in terms of Faddeev's quantum dilogarithm [Kashaev-M.M.]. This leads to analytic computations of the fermionic spectral traces in many cases - a rare luxury in Quantum Mechanics!

The quantum theory is particularly simple when $\hbar = 2\pi$, and more generally when

$$\hbar = \frac{r}{s}\pi, \qquad \frac{r}{s} \in \mathbb{Q}_{>0}$$

Some sample values for local $\mathbb{P}^2$ [Kashaev-M.M., Okuyama-Zakany]

$$Z(1, 2\pi) = \frac{1}{9}$$

$$Z(2, 4\pi) = \frac{5}{324} - \frac{1}{12\sqrt{3}\pi}$$

A non-perturbative definition

We now claim that

$$F_X(N, \hbar) = \log Z_X(N, \hbar)$$

gives a non-perturbative definition of the topological string free energy in the conifold frame, with the dictionary

$$g_s = \frac{1}{\hbar} \qquad \lambda = \frac{N}{\hbar}$$ 't Hooft parameter

Does this agree with our requirements? We have just seen that the fermionic spectral trace is well-defined when *N* is a positive integer and $\hbar$ positive and real

The asymptotic regime is the standard 't Hooft limit

$$N \to \infty \qquad \qquad \frac{N}{\hbar} = \lambda \quad \text{fixed}$$
$$\hbar \to \infty$$

We conjecture that the asymptotic expansion of $F_X(N, \hbar)$ in this regime is precisely the genus expansion of the topological string, and nothing else

$$F_X(N, \hbar) = \log Z_X(N, \hbar) \sim \sum_{g \geq 0} F_g(\lambda) \hbar^{2-2g}$$

This conjecture has been checked in massive detail. I will give additional evidence for it in this talk.

We now address the issue of the discreteness of *N*. Clearly, not all values of the topological string parameters can be covered if *N* is just a positive integer.

However, we have a stronger conjecture which gives an *exact* formula for $Z_X(N,\hbar)$

$$Z_X(N,\hbar) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \mathrm{e}^{J_X(\mu,\hbar) - N\mu} \mathrm{d}\mu$$

calculable from BPS invariants of X

If this conjecture is true (and it has been checked to amazing precision) the fermionic spectral trace, which was defined for positive, integer *N*, can be extended to an *entire* function on the complex *N* plane [Codesido-Grassi-M.M.]

Borel resummation

There is a traditional way to produce well-defined quantities from factorially divergent series: Borel resummation

$$\varphi(g) = \sum_{n\geq 0} a_n g^n \xrightarrow{\text{Borel transform}} \widehat{\varphi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!}\zeta^n$$

Laplace transform

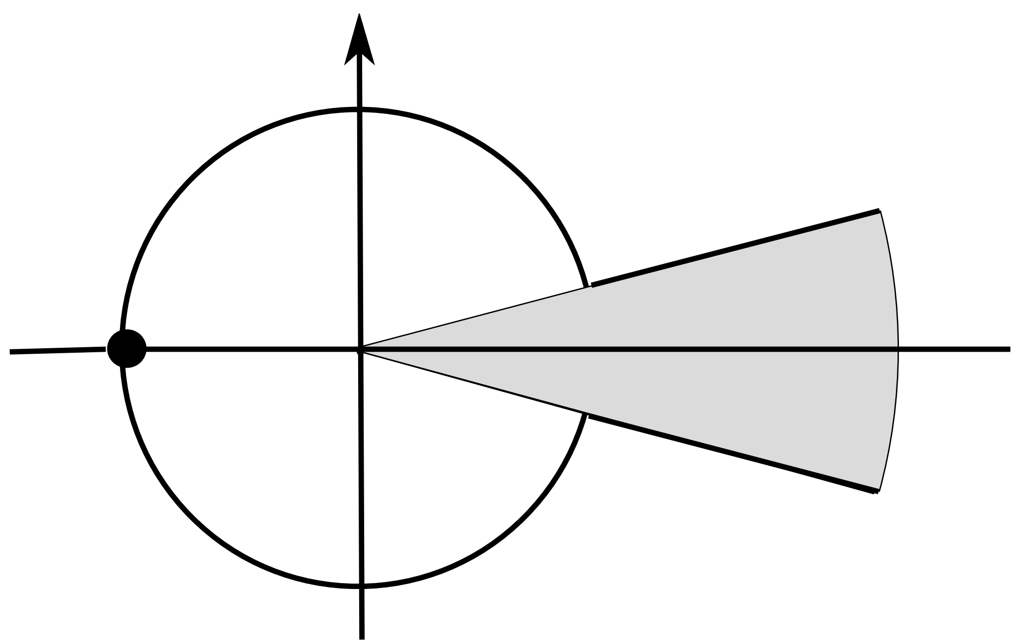
asymptotics

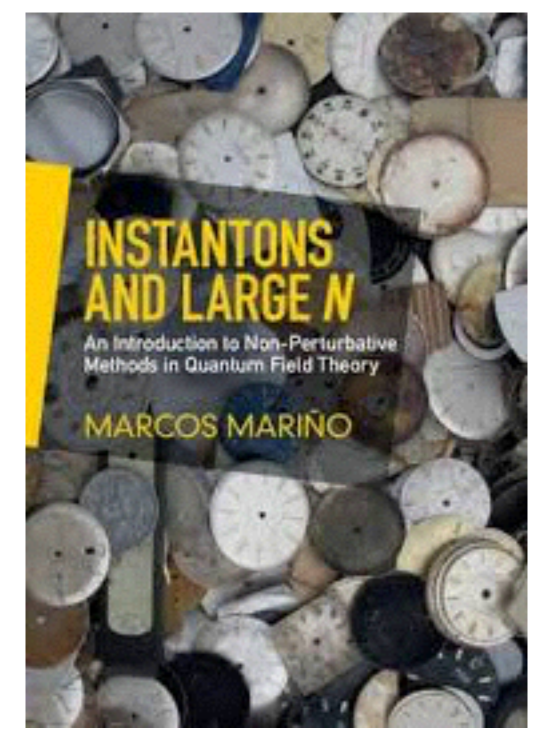
$$s(\phi)(z) = \int_0^\infty \mathrm{e}^{-\zeta}\widehat{\varphi}(z\zeta)\mathrm{d}\zeta$$

Borel resummation

When this procedure makes sense, we say that the series is Borel summable. This requires that the Borel transform has no singularities along the positive real axis.

analytic structure of $\widehat{\varphi}(\zeta)$



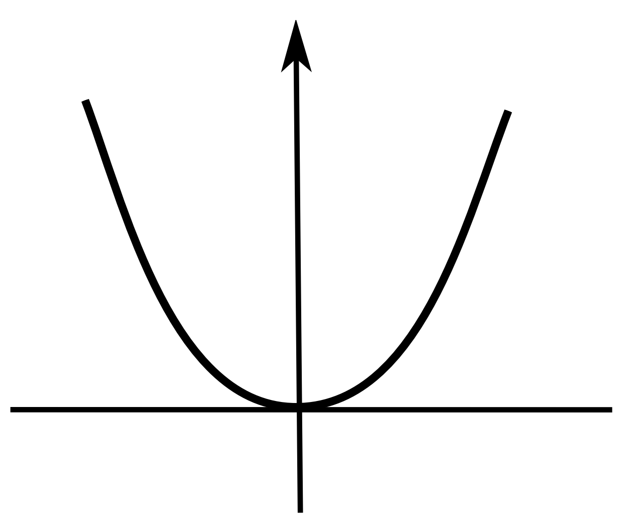


learn more here!

In practice, one can verify the absence of singularities and perform the resummation by using numerical techniques

Suppose that we are given a non-perturbative quantity in quantum theory, as well as its perturbative, asymptotic series. Can we reconstruct the non-perturbative object from the series?

When the perturbative series is Borel summable, its Borel resummation might agree with the original quantity. This is what famously happens in the quartic oscillator



$$\varphi(g) = \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \cdots$$

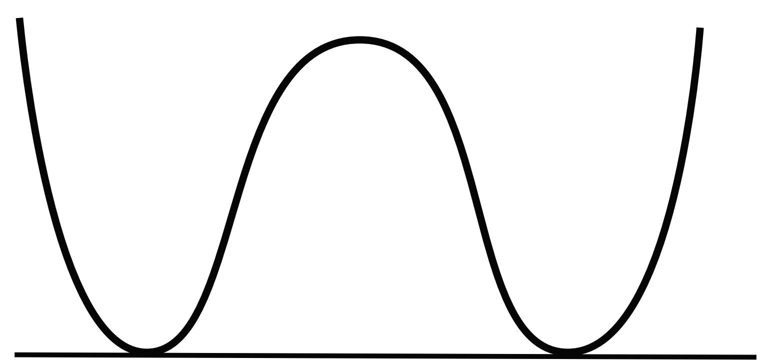
Borel summable

$$E_0(g) = s(\varphi)(g)$$

[Graffi-Grecchi-Simon]

In other examples, the series is still Borel summable, but its Borel resummation does *not* agree with the non-perturbative definition [Balian-Parisi-Voros, Grassi-M.M.-Zakany]

Finally, in many cases, the perturbative series is not even Borel summable, like in the double well potential



In all these cases, one needs to the very least additional information, on top of the perturbative series: *non-perturbative effects!*

Trans-series

Perturbative series are obtained by expanding around a trivial saddle of the path integral. If one expands around a non-trivial saddle or *instanton*, one finds a *trans-series*

$$\sum_{n\geq 0} a_n g^n + \underbrace{C}_{\text{parameter}}\mathrm{e}^{-\overbrace{A}^{\text{instanton action}}/g}\sum_{n\geq 0} a_n^{(1)} g^{n+b_1} + \cdots$$

Trans-series are made out of formal, asymptotic series. They can be resummed with Borel-Ecalle resummation, which takes into account possible singularities on the positive real axis. One obtains in this way *multi-parameter families* of non-perturbative completions.

We can now ask again whether one can reconstruct a non-perturbative quantity from its trans-series:

"Semiclassical decoding conjecture": non-perturbative functions in quantum theory can be written as the Borel-Ecalle resummation of a trans-series.

This seems to be true in many examples in Quantum Mechanics, and in some simple low-dimensional/topological/supersymmetric systems. It is probably not true in Yang-Mills theory (in infinite volume).

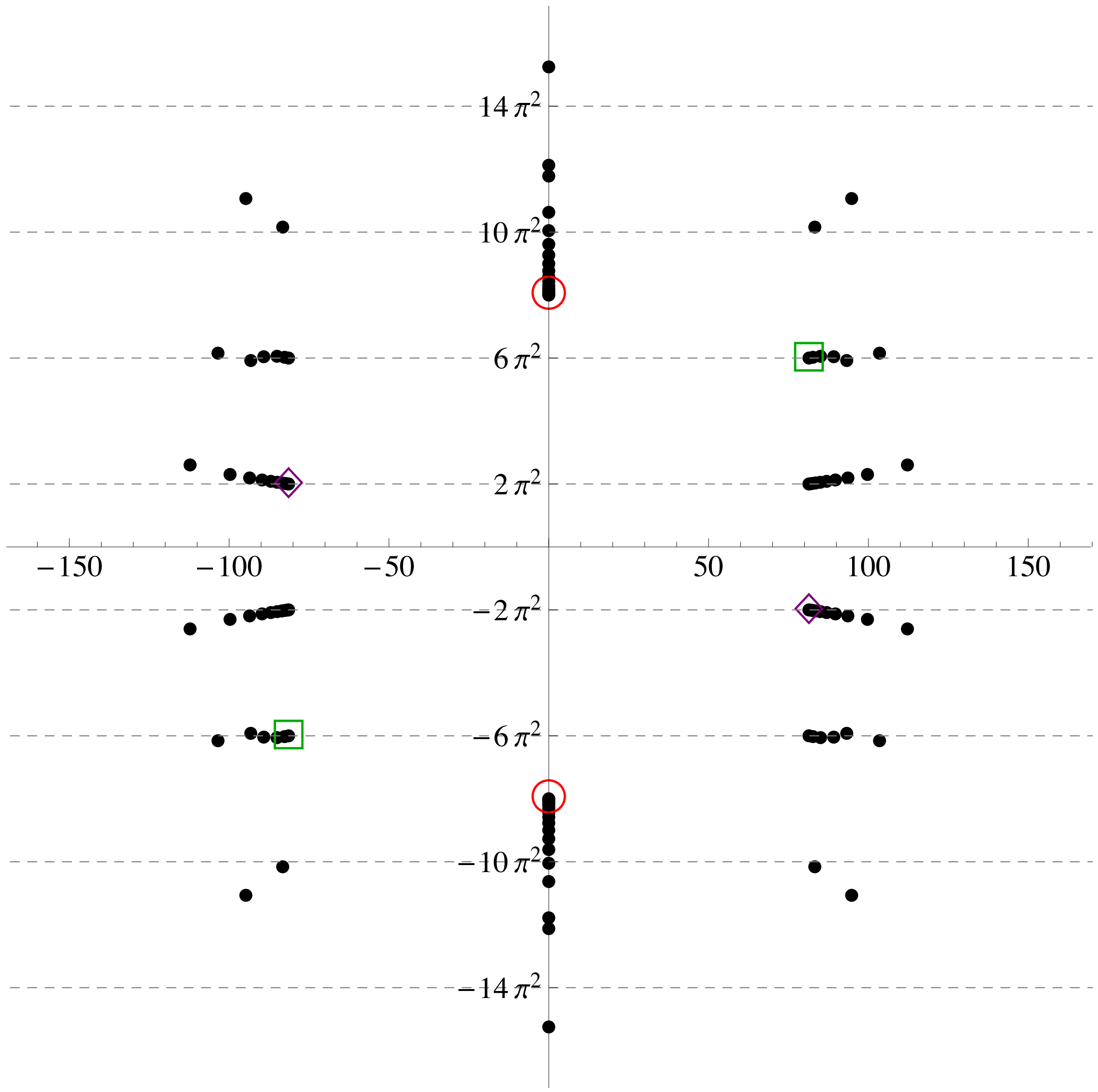
Let us then ask the following question:

Is the "semiclassical decoding conjecture" true for the non-perturbative definition I have just given for the topological string?

The first thing to do, in order to answer this question, is to study the Borel summability of the genus expansion, in the conifold frame

In the following, I will focus on local $\mathbb{P}^2$.

It turns out that the perturbative genus expansion is Borel summable for almost all real and positive λ



singularities in the Borel plane for

$$\lambda = \frac{1}{\pi}$$

We can then do standard Borel resummation of the perturbative series and get numerical answers.

For N=2 and $\hbar = 4\pi$ we obtain

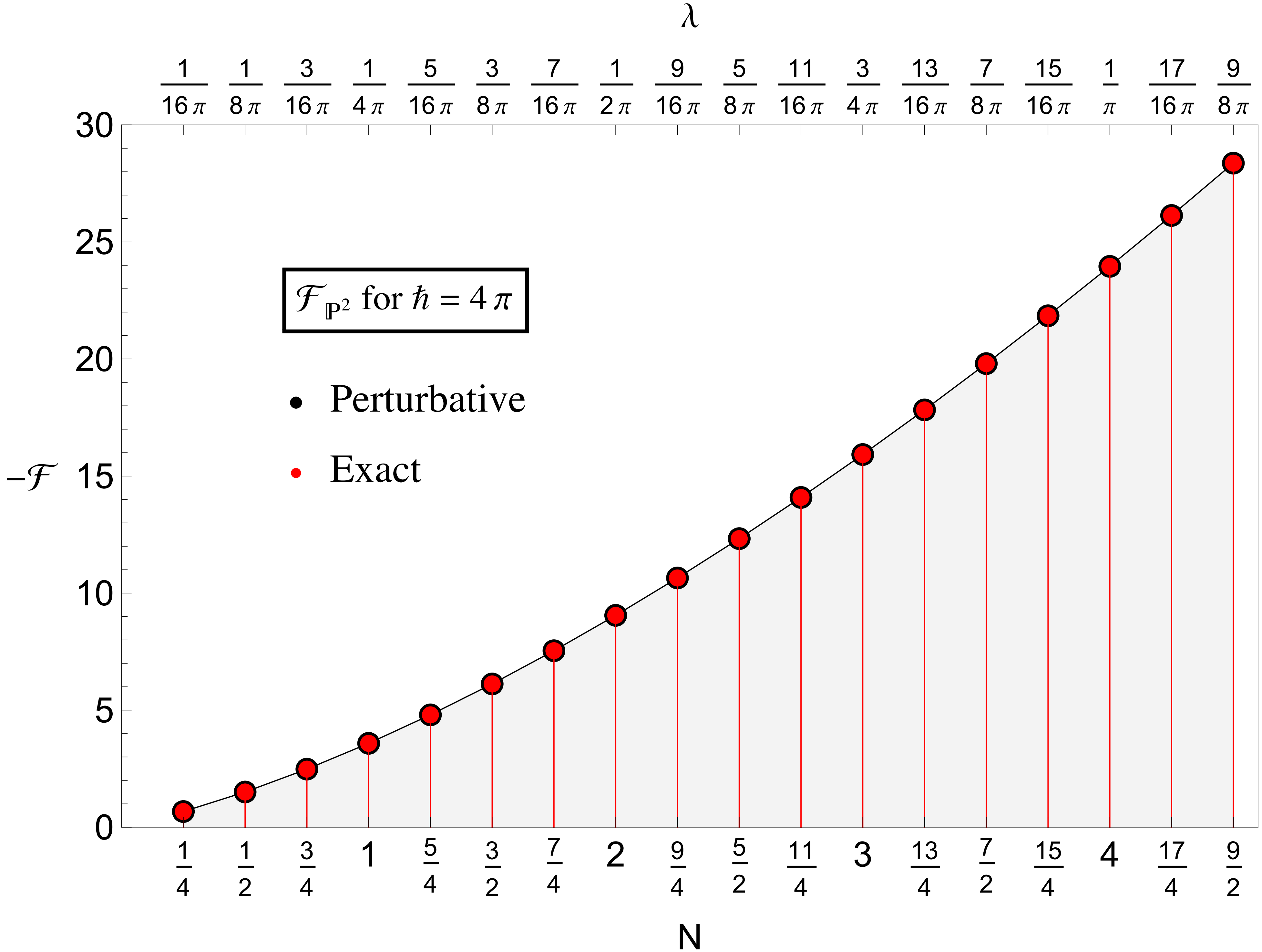
$$F_{\mathrm{Borel}}(N=2, \hbar = 4\pi) = -\underline{9.049\,862\,10}3\,051\,21\ldots$$

Our non-perturbative definition gives

$$F_{\mathbb{P}^2}(N=2, \hbar = 4\pi) = \log\left(\frac{5}{324} - \frac{1}{12\sqrt{3}\pi}\right)$$
$$= -\underline{9.049\,862\,10}2\,738\,02\ldots$$

This is *not* the same number, so the perturbative series is Borel summable but its resummation does not agree with our non-perturbative definition! Similar to the $1/N$ expansion of the ABJM matrix model [Grassi-M.M.-Zakany]

The numbers are however *incredibly close*, for many values of λ



The difference is not visible to the naked eye!

This is what we expect from our conjecture! The asymptotic 1/N expansion of $F_X(N,\hbar)$ should give the genus expansion of the topological string, so the difference between $F_X(N,\hbar)$ and the Borel resummation should be *exponentially small*, i.e. a non-perturbative effect

Can we compute this effect explicitly? Can we decode our non-perturbative definition in terms of a trans-series of the form

$$\sum_{g\geq 0} g_s^{2g-2} F_g(\lambda) + C e^{-A/g_s} \sum_{g\geq 0} g_s^{g-1} F_g^{(1)}(\lambda) + \cdots \quad ?$$

1-instanton correction

A general framework to calculate trans-series of this form for the topological string has been proposed by [Couso-Santamaria et al.]. They promote the trans-series to a non-holomorphic object and they require it to satisfy the holomorphic anomaly equation of BCOV

In this way they can solve for the non-perturbative corrections in terms of *perturbative* topological string data. The instanton action turns out to be a period of the CY, in agreement with previous proposals [Balian-Parisi-Voros, Drukker-M.M.-Putrov]

We can now consider the Borel resummation of the trans-series and compare it to our non-perturbative definition. We have included the one-instanton correction with a natural appropriate parameter C.

We get a remarkable agreement for a large range of values!

$$F_{\mathbb{P}^2}(N=2,\hbar=4\pi) = \log\left(\frac{5}{324} - \frac{1}{12\sqrt{3}\pi}\right)$$

$$= -\underline{9.049\,862\,102\,738\,02}042\ldots$$

$$F_{\text{Borel}} + C F^{(1)}_{\text{Borel}} = -\underline{9.049\,862\,102\,738\,02}\ldots$$

1-instanton correction

$$F_{\text{Borel}} = -\underline{9.049\,862\,10}3\,051\,21\ldots$$

This gives evidence that our non-perturbative completion can be "semiclassically decoded" in terms of the above trans-series

Conclusions

We have given a *rigorous* and *concrete* non-perturbative definition of topological string theory on toric CYs, in the spirit of large *N* dualities.

We have shown that this definition can be decoded in terms of a natural trans-series coming from the holomorphic anomaly equation. In particular, it is exponentially close to the Borel-resummed perturbative series, as required by a *bona fide* completion.

$Z_X(N, \hbar)$ can be written as a matrix model [M.M.-Zakany].
Can we compute the trans-series directly in this context?

Of course, they might be other non-perturbative definitions. Since in many cases the theory is Borel summable, there is always the "trivial" definition by Borel resummation.

The main interest of our non-perturbative definition is that it links topological strings to spectral theory in a highly non-trivial way (the TS/ST correspondence!), and it makes surprising predictions in both subjects

A particular case of this correspondence proved recently by [Bonelli-Grassi-Tanzini], see Grassi's talk. General proof?