# Monopoles and Vortices in 3d N=4 gauge theories

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Based on

arXiv :1609.xxxxx with M. Bullimore, T. Dimofte, D. Gaiotto, J. Hilburn

# *Introduction*

3d $\mathcal{N}=4$ supersymmetric gauge theories with gauge group $G$ .

- 8 real SUSY $Q_\alpha^{a\dot{a}}$ with R-symmetry $SU(2)_H \times SU(2)_C$
- Vector multiplet

  $\mathcal{N}=2$ vector multiplet $A_\mu,\ \sigma$ + Adjoint chiral multiplet $\varphi$

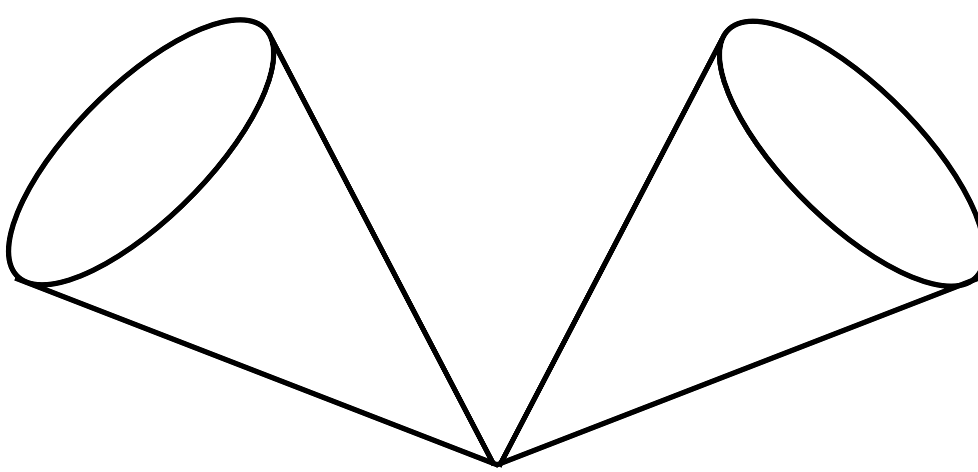
- Hypermultiplet

  Two $\mathcal{N}=2$ chiral multiplets $X, Y$ with $W = \mathrm{Tr}\varphi XY$

Rich structure of supersymmetric vacua parametrized by vacuum expectation value (VEV) of gauge invariant scalar operators.

Higgs branch

$X, Y$



Coulomb branch

$\sigma, \varphi, V_\pm$

## Higgs branch $\mathcal{M}_H$

- VEV of scalars $X, Y$ in hypermultiplets (and $\sigma = \varphi = 0$).
- Superpotential constraint (F-term)

$$\mu_{\mathbb{C}} \equiv \frac{\partial W}{\partial \varphi} = X \cdot Y = 0$$

- D-term constraint

$$\mu_{\mathbb{R}} \equiv X \cdot X^{\dagger} - Y^{\dagger} Y = 0$$

- Hyper-Kahler quotient

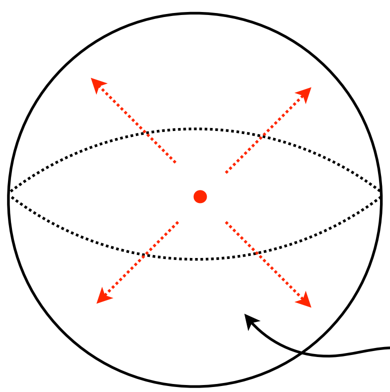
$$\mathcal{M}_M = \{X, Y | \mu_{\mathbb{C}} = \mu_{\mathbb{R}} = 0\} / G$$

- No quantum correction. So classical geometry is exact.

# Coulomb branch $\mathcal{M}_C$

- VEV of three real scalars $\sigma, \varphi$ in vectormultiplets together with monopole operators $V_m$.

- Monopole operator



$$A_\pm = \frac{m}{2}(\pm 1 - \cos\theta)d\phi + \cdots \qquad (m \in \Gamma_{G^\vee} \ \text{weight lattice of } G^\vee)$$

$$\int_{S^2} F = m$$

- Hyper-Kahler with $\dim_{\mathbb{C}} \mathcal{M}_C = 2 \ \text{rank}\ G$
- Quantum corrections on the classical geometry.

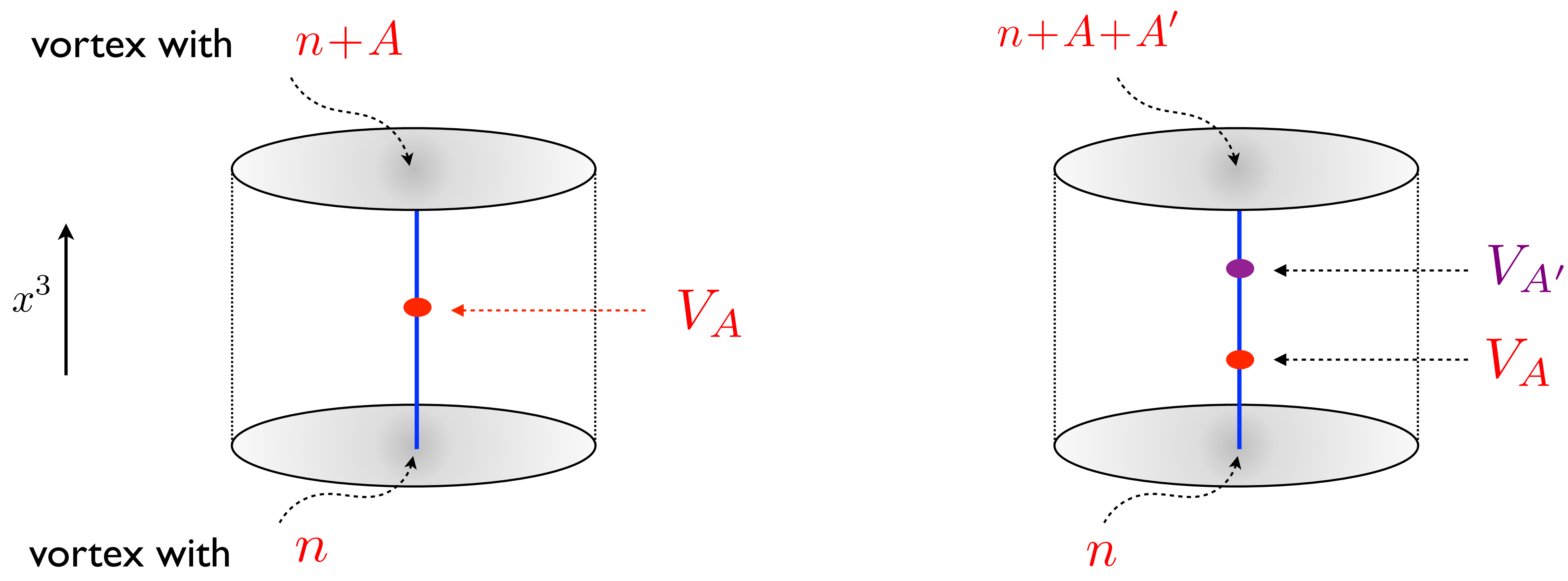
- Sometime we can study quantum-corrected Coulomb branch geometry indirectly using mirror symmetry.

# *Monopoles in Higgs branch*

In Higgs branch, there are vortex solitons carrying magnetic flux.

Monopoles become interfaces (or kinks, or domain walls) in 1d vortex quantum mechanics (QM) living on the vortex world line (blue line).

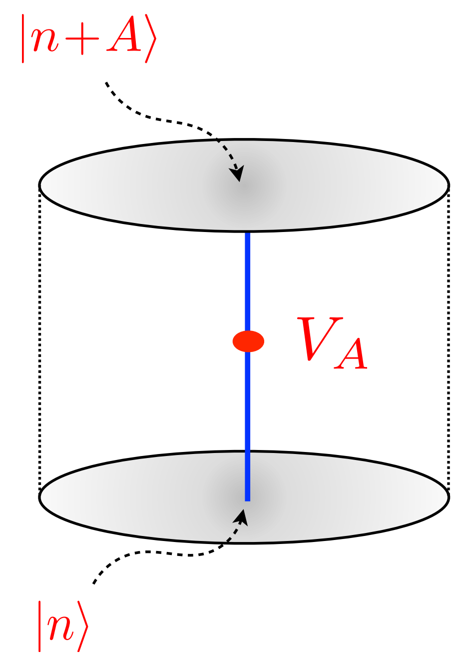
[Tong 03]




We shall find UV gauge theory description for the 1d vortex system with the monopole interface.

# *Monopole correlation functions*

Partition function of




$$= \langle n+A \,|\, V_A \,|\, n \rangle$$

We will compute correlation functions of monopole operators using supersymmetric localization computation in 1d vortex QM.

Main goal is to extract exact chiral rings and Coulomb branch algebra from the correlation functions.

I will also show that the Coulomb branch algebra is naturally quantized in our setting.

# Outline

- Introduction
- Moduli space of classical Coulomb branch.
- Vortex quantum mechanics with monopole operators.
- Correlation functions and Coulomb branch algebra.
- Conclusion

# *Symmetries and deformations*

We consider unitary gauge groups $G = \otimes_i U(N_i)$ and fundamental (and also bi-fundamental) hypermultiplets.

Global symmetry :

- Flavor symmetry acting on $\mathcal{M}_H$ : $G_H = \otimes_i SU(N_{f_i})$
- Topological symmetry acting on $\mathcal{M}_C$ : $G_C = (U(1)_T)^{\#\,\mathrm{of}\,U(1)'\mathrm{s\,in}\,G}$

Mass deformations for $G_H$

$$\sigma \to \sigma + m_{\mathbb{R}} \ , \quad \varphi \to \varphi + m_{\mathbb{C}}$$

Fayet-Iliopoulos (FI) deformations for $G_C$

$$\mu_{\mathbb{R}} = 0 \to \mu_{\mathbb{R}} = t_{\mathbb{R}} \ , \quad \mu_{\mathbb{C}} = 0 \to \mu_{\mathbb{C}} = t_{\mathbb{C}}$$

$\Omega$-deformation for $U(1)_z$ Lorentz rotation

$$\varphi \to \varphi - i\epsilon\mathcal{L}_z + \epsilon Q_H$$

# *Abelian Coulomb branch*

Abelian gauge field $A_\mu$ can be dualized into a periodic scalar field $a$, called dual photon.

$$da = *dA \quad \text{and} \quad a \sim a + 2\pi g^2$$

Monopole operators are defined as $V_\pm = e^{\pm\frac{1}{g^2}(\sigma + ia)}$

Classical Coulomb branch is $\mathcal{M}_C = \{\varphi, V_\pm | V_- = V_+^{-1}\} = \mathbb{R}^3 \times S^1$

Quantum corrections modify this classical geometry.

- For $N_f$ fundamental hypers, quantum-corrected Coulomb branch is

$$\mathcal{M}_C = \{\varphi, V_\pm\}/(*)\ , \quad (*) : V_+V_- = \prod_{i=1}^{N_f}(\varphi + m_{\mathbb{C}}^i)$$

[Borokhov, Kapustin, Wu 02]

$\longrightarrow$ $A_{N_f-1}$ singularity deformed by complex mass parameters

# *Non-Abelian Coulomb branch (classical analysis)*

At generic points on $\mathcal{M}_C$, scalar fields $\varphi$ take non-degenerate values in Cartan subalgebra of $\mathfrak{g}$ and the non-abelian gauge group will be broken to maximal abelian subgroup $U(1)^r \subset G$.

Monopole operators in IR abelian theory can be explicitly constructed as

$$V_m = e^{\frac{1}{g^2} m\cdot(\sigma + ia)}$$

Therefore, the classical Coulomb branch takes the form

$$\mathcal{M}_C^{clas} \sim \{(\mathbb{R}^3 \times S^1)^r\}/W_G$$

This description breaks down by quantum effects when W-bosons becomes massless and thus non-abelian gauge symmetry is restored.

We shall compute Coulomb branch chiral rings with full quantum corrections by studying monopole actions on vortices in Higgs branch.

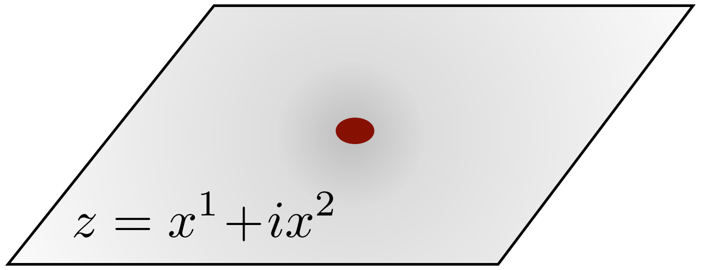
# Higgs branch and vortices

Focus on $U(N)$ gauge theory with $N_f$ fundamental hypermultiplets and a real FI parameter $t_{\mathbb{R}} > 0$ .

Higgs branch which admits vortex solution is Grassmannian $G(N, N_f)$ parametrized by $X$ (and $Y = \sigma = \varphi = 0$).

On a Higgs vacuum labelled by $\nu$, we find BPS vortices satisfying

$$F_{z\bar{z}} + X \cdot X^\dagger = t_{\mathbb{R}} \ , \quad D_{\bar{z}} X = 0$$
$$\int_{\mathbb{R}^2} F_{z\bar{z}} = n \ , \quad X \overset{z \to \infty}{\to} \langle X \rangle_\nu$$




Moduli space of vortex solutions $\mathcal{M}_\nu^n$ is a complex manifold with

$$\dim_{\mathbb{C}} \mathcal{M}_\nu^n = n N_f$$

which respects symmetry $U(1)_z \times S[U(N) \times U(N_f - N)]$ .

Complex masses and $\Omega$-deformation such as $\varphi \to \varphi + m_{\mathbb{C}} - i\epsilon\mathcal{L}_z + \epsilon Q_H$ leaves $\binom{N_f}{N}$ isolated massive vacua which are fixed points w.r.t equivariant gauge and flavor rotations.

Vortex moduli space on a massive vacuum $\nu$ will also be lifted and localized to isolated fixed points labelled by

$$\text{Vortex charge } n \text{ and } k = (k_1, \cdots, k_N)\,, \; \sum_{i=1}^{N} k_i = n$$

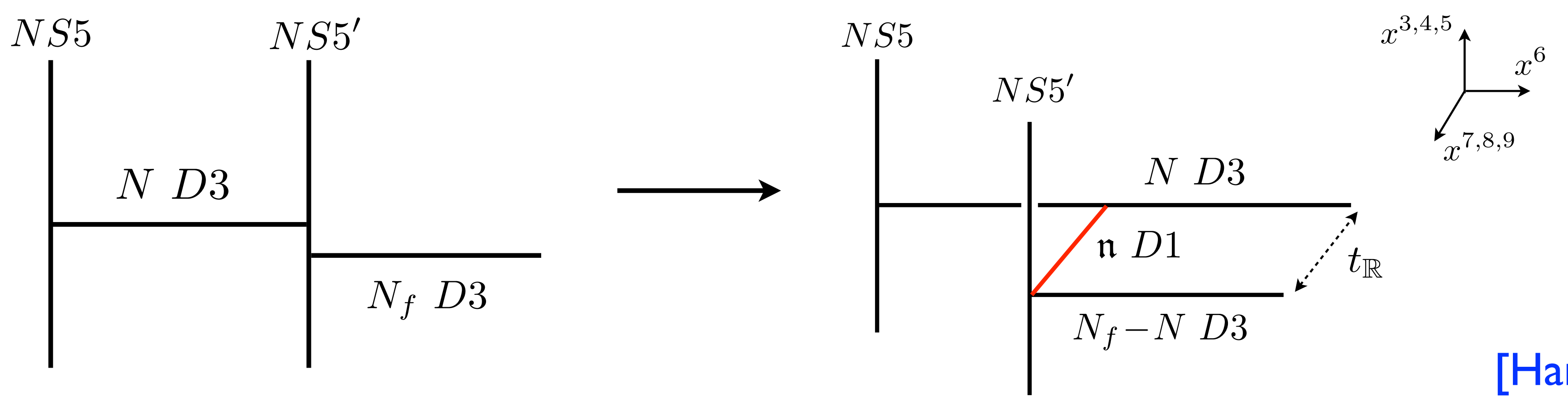
$\Omega$-deformation effectively compactifies 3d gauge theory to 1d quantum mechanics at $z = 0$.

Hilbert space of the 1d quantum mechanics is the equivariant cohomology of the moduli space of the 1/2 BPS vortices :

$$\mathcal{H}_\nu = \bigoplus_{\text{fixed } p} \mathbb{C}|p\rangle$$
$$= \bigoplus_{n,k} \mathbb{C}|n, k\rangle$$

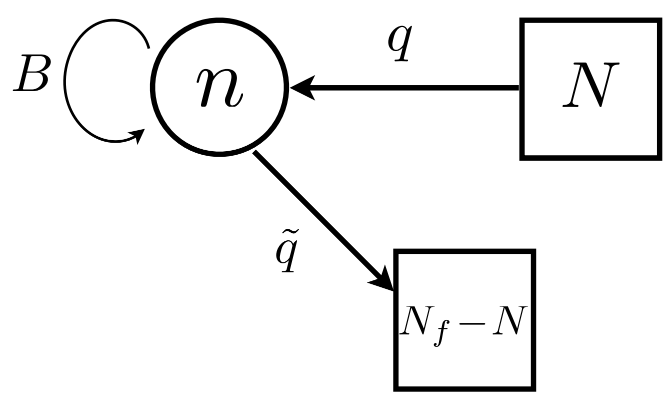
# *Vortex quantum mechanics*

Brane construction of vortices.




[Hanany, Tong 03]

D1-brane worldvolume theory is 1d $\mathcal{N}=(2,2)$ gauged quantum mechanics of

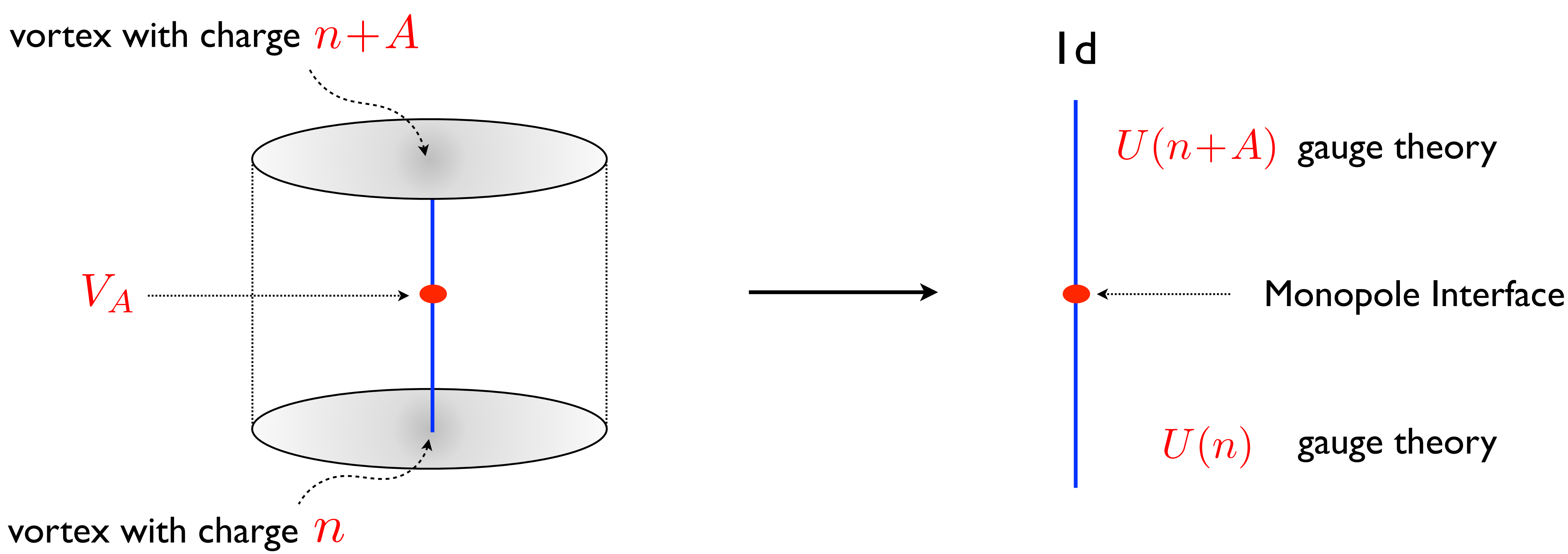



- Global symmetry : $U(1)_z \times S[U(N) \times U(N_f - N)]$
- Higgs branch agrees with vortex moduli space $\mathcal{M}_\nu^n$.

# *Monopole as Interface*

We propose

Monopole operator in 3d = 1/2 BPS interface in vortex QM



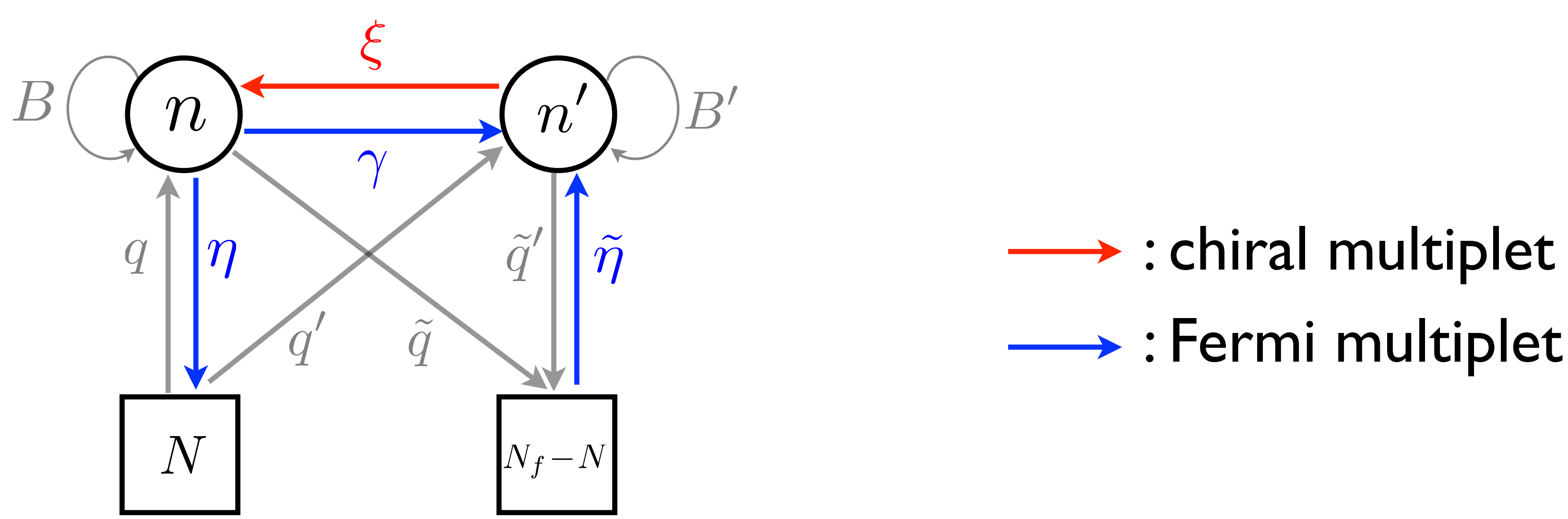

- 1d vortex system with the interface preserves $\mathcal{N}=(0,2)$ SUSY
- Interface connects vortex QM with U(n) gauge group and another vortex QM with U(n+A) gauge group.

Monopole interface between $n$ and $n'$ vortices is constructed as follows:

1. Neumann boundary condition on 1d vector and chiral multiplets.

   $\longrightarrow$ Induces $\mathcal{N}=(0,2)$ vector and chiral multiplets at the interface.

2. Additional 0-dimensional degrees of freedom $\xi$ and $\gamma, \eta, \tilde{\eta}$.




3. Superpotentials

$$J_\gamma = \xi B' - B\xi \ , \quad J_\eta = \xi q' - q \ , \quad J_{\tilde{\eta}} = \tilde{q}' - \tilde{q}\xi \quad (E_\gamma = E_\eta = E_{\tilde{\eta}} = 0)$$

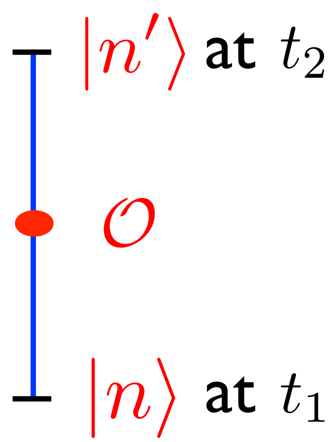
This interface provides a (surjective) map from $\mathcal{M}_\nu^{n'}$ to $\mathcal{M}_\nu^{n}$, when $n' \geq n$.
So, we claim that the interface realizes 3d monopole operator.

In mathematics, this map is known as Hecke correspondence.

[Nakajima 11], [Braverman, Feigin, Finkelberg, Rybnikov 11]

# *Correlation functions*

Partition function of the 1d vortex system with the interface on $t_2 > t > t_1$ .




$\longrightarrow \quad \langle n'|\mathcal{O}|n\rangle$ (for $N=1$)

- If $n' = n$ , $\mathcal{O} = 1$ and if $n' = n+1$, $\mathcal{O} = V_+$ .

Localization computation gives rise to the correlation function (at 1st vacuum)

$$\boxed{\langle n'|\mathcal{O}|n\rangle = \frac{\prod_{s=1}^{n}(n'+1-s)\epsilon \prod_{i=2}^{N_f}\prod_{s=1}^{n'}(m_1 - m_i + s\epsilon)}{\omega_n\, \omega_{n'}}}$$

where $\omega_n = \prod_{s=1}^{n}(n+1-s)\epsilon \prod_{i=2}^{N_f}\prod_{s=1}^{n}(m_1 - m_i + s\epsilon)$ is the equivariant weight at fixed point $|n\rangle$

Identity interface $\mathcal{O} = 1$ when $n' = n$ :

$$\langle n|\mathcal{O}|n\rangle = \langle n|n\rangle = \frac{1}{\omega_n}$$

- This gives the correct normalization of the vortex state $|n\rangle$ .
- 3d vortex partition function on interval with Neumann b.c. $|\mathcal{N}\rangle$ is

$$Z_{\text{vortex}} = \langle \mathcal{N}|\mathcal{N}\rangle = \sum_{n\geq 0} q^n \langle n|n\rangle = \sum_{n\geq 0} q^n \frac{1}{\omega_n}$$

Correlation functions of monopole operators $V_\pm$ when $n' = n+1$ :

$$\langle n{+}1|V_+|n\rangle = \langle n|V_-|n{+}1\rangle = \frac{1}{\prod_{s=1}^{n}\left[(n+1-s)\epsilon\prod_{i=2}^{N_f}(m_1 - m_i + s\epsilon)\right]}$$

# *Monopole action and Coulomb branch algebra*

We can compute monopole actions and Coulomb branch algebra from the monopole correlation functions.

Monopoles act on the state $|n\rangle$ as $V_+|n\rangle = C_+|n{+}1\rangle \ , \quad V_-|n\rangle = C_-|n{-}1\rangle$

where $C_+ = \dfrac{\langle n{+}1|V_+|n\rangle}{\langle n{+}1|n{+}1\rangle} \ , \quad C_- = \dfrac{\langle n{-}1|V_-|n\rangle}{\langle n{-}1|n{-}1\rangle}$

In addition, the 3d complex vectormultiplet scalar $\varphi$ acts as

$$\varphi|n\rangle = (-m_i - n\epsilon - \tfrac{\epsilon}{2})|n\rangle \qquad \text{(at i-th vacuum)}$$

**Monopole action :**

$$V_+|n\rangle = P(\varphi + \tfrac{\epsilon}{2})|n{+}1\rangle \ , \quad V_-|n\rangle = |n{-}1\rangle$$

with $P(x) = \prod_{a=1}^{N_f}(x + m_a)$

Hilbert space $\mathcal{H}_\nu$ is the highest-weight Verma module of the Coulomb branch algebra.

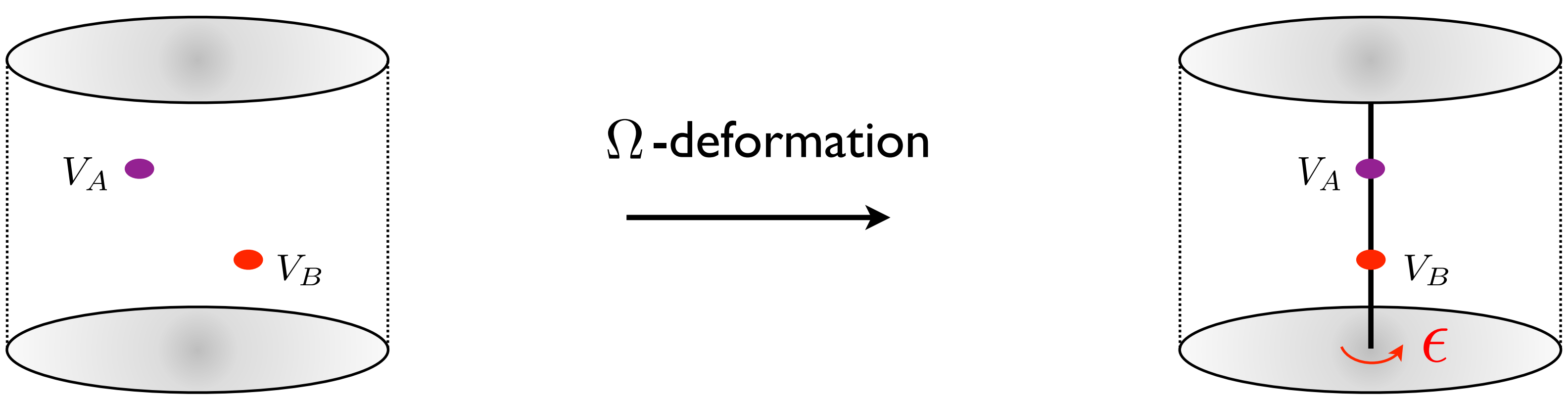
$$|n\rangle \sim (V_+)^n|0\rangle \quad \text{for} \quad \forall\ |n\rangle \in \mathcal{H}_\nu$$

The Coulomb branch algebra is

$$V_+V_- = P(\varphi + \tfrac{\epsilon}{2})\ , \quad V_-V_+ = P(\varphi - \tfrac{\epsilon}{2})\ , \quad [\varphi, V_\pm] = \mp\epsilon V_\pm \quad \text{for } N=1$$

"Spherical rational Cherednik algebra"

The algebra is quantized by $\Omega$-parameter $\epsilon$.




Commutative

Non-commutative

[Nekrasov, Shatashvili 09], [Yagi 14]

For general $N$, monopole operators acts on the vortex states as

$$V_a^+|n,k\rangle = \frac{P(\varphi_a + \frac{\epsilon}{2})}{\prod_{b\neq a}^N(\varphi_b - \varphi_a)}|n{+}1, k{+}\delta_a\rangle$$
$$V_a^-|n,k\rangle = \frac{1}{\prod_{b\neq a}^N(\varphi_b - \varphi_a)}|n{-}1, k{-}\delta_a\rangle$$
$$\varphi_a|n,k\rangle = (-m_a - k_a\epsilon - \tfrac{\epsilon}{2})|n,k\rangle$$

Quantized Coulomb branch algebra :

$$V_a^+V_a^- = \frac{P(\varphi_a + \frac{\epsilon}{2})}{\prod_{b\neq a}(\varphi_a - \varphi_b)(\varphi_b - \varphi_a - \epsilon)}\ , \quad V_a^-V_a^+ = \frac{P(\varphi_a - \frac{\epsilon}{2})}{\prod_{b\neq a}(\varphi_a - \varphi_b)(\varphi_b - \varphi_a + \epsilon)} \quad \cdots\cdots\cdots\cdots * $$

[Bullimore, Dimofte, Gaiotto 15]

Therefore, Coulomb branch chiral ring is

$$C[\mathcal{M}_C] = \{V_a^\pm, \varphi_a\}/\{*, W_G\}$$

# Conclusion

Exact Coulomb branch algebra can be computed using correlation functions of monopoles operators which are partition functions of 1d vortex QM with interfaces.

Generalization to "Triangular quiver gauge theories".

- Finite $W_N$ algebra and finite AGT correspondence

[Braverman, Feigin, Rybnikov, Finkelberg 11], [Nakajima 11]

Moduli matrix approach for monopole operators.

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Extensions :

- General unitary quiver theories [Bullimore, Dimofte, Gaiotto 15]
- Higher dimensional generalization
- N=3 or N=2 gauge theories
- Other gauge groups ?