Elliptic Algebras and Large-N Gauge Theories

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LET THERE BE LIGHT

Talk at workshop on Geometric Correspondences of Gauge Theories
Trieste, Italy September 16th 2016

In collaboration with

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Noppadol Mekareeya
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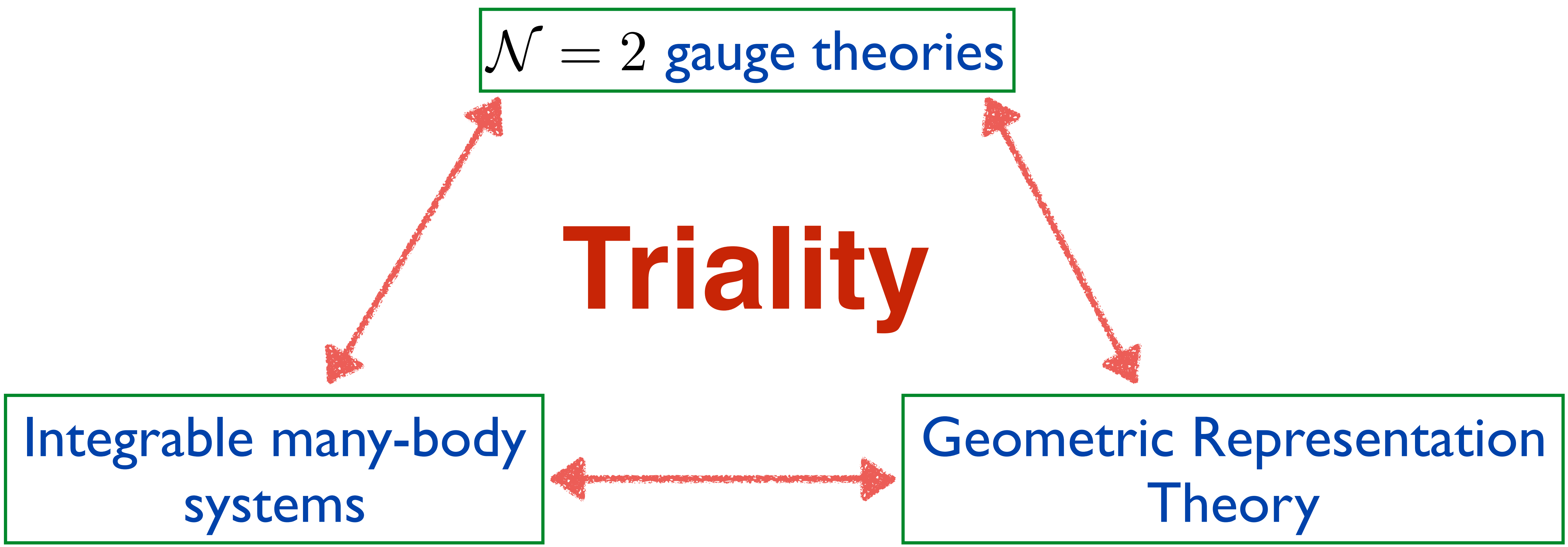
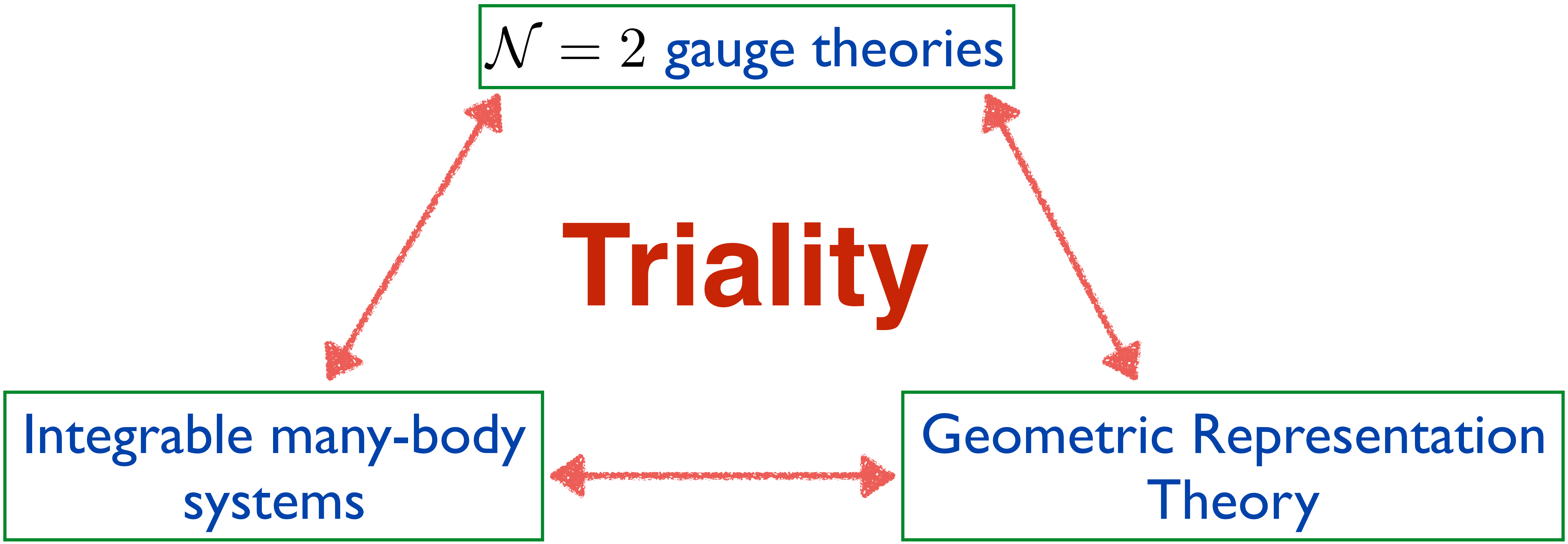
Large-N Gauge Theories

Gauge theories are known to have effective descriptions when the number of colors is large $U(N)$ $N \to \infty$

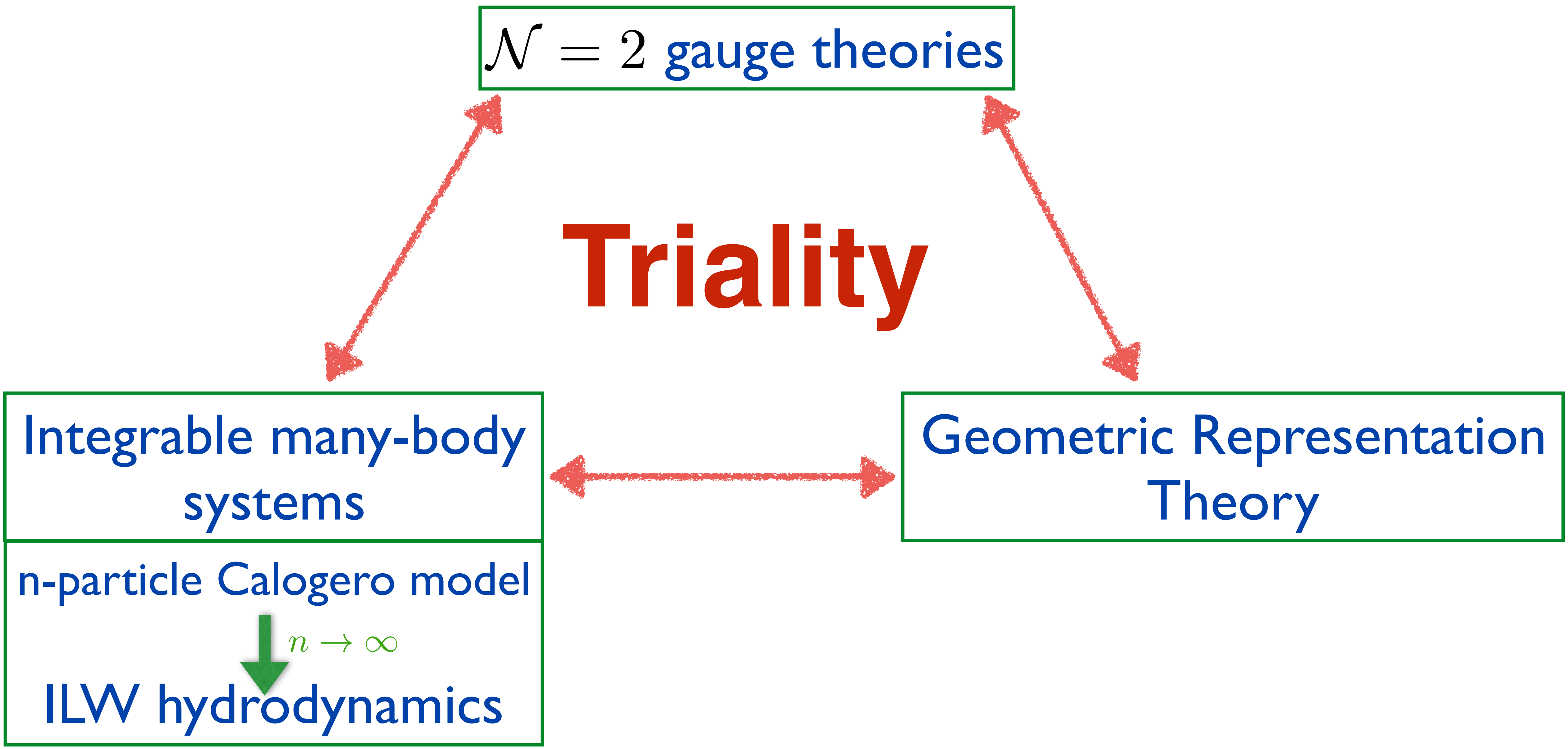
For supersymmetric gauge theories we expect to compute the effective large-N theory exactly

There are plenty of examples in the literature


$\mathcal{N} = 2$ gauge theories
Triality
Integrable many-body systems
Geometric Representation Theory



Large-n limits are manifest in each description!



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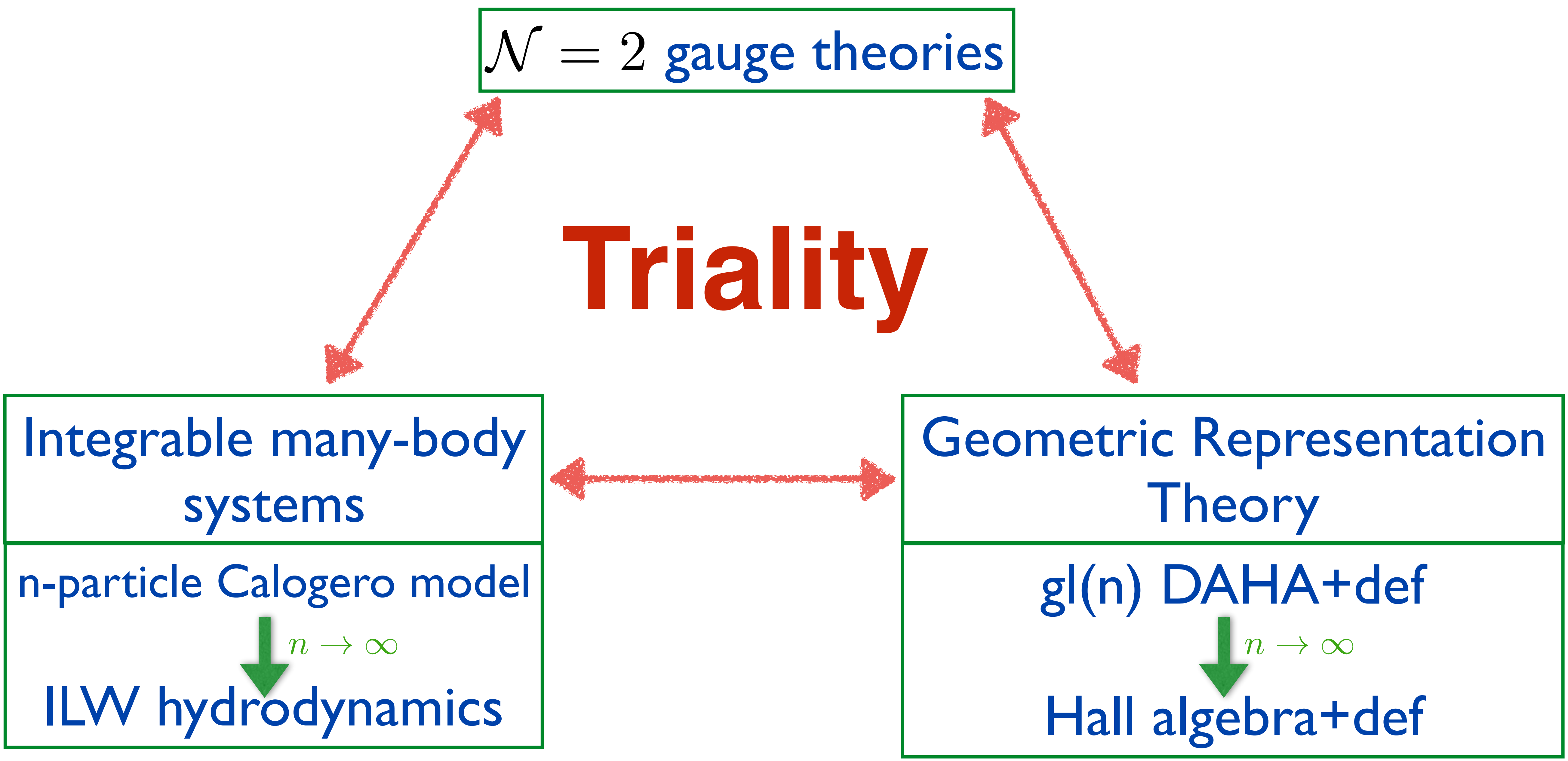

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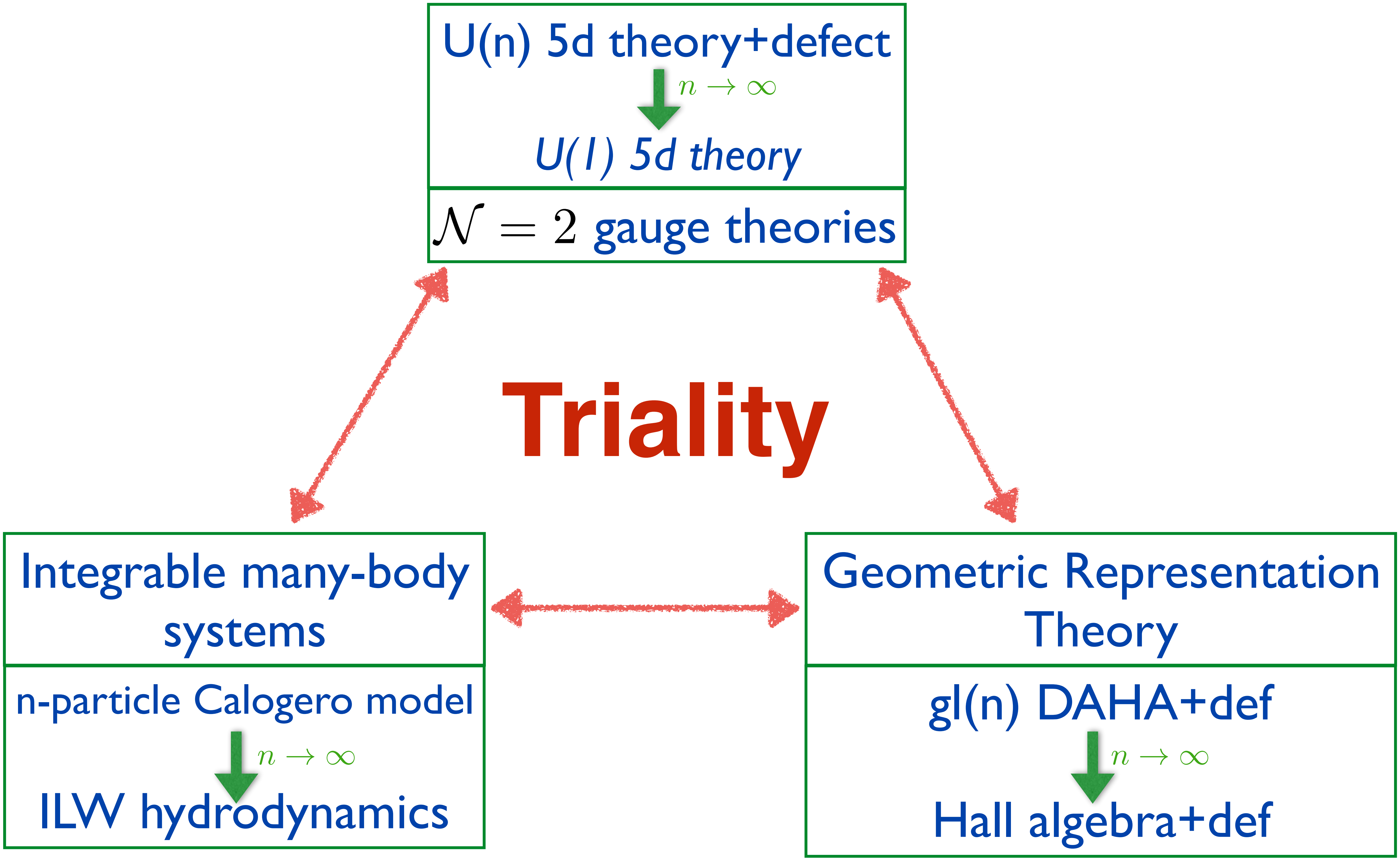

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N=2 Gauge Theories

We focus on N=2 gauge theories which have Seiberg-Witten description in IR

At the moment we have plethora of exact results for those theories thanks to Nekrasov's computation of instanton partition functions

Nekrasov's original work has been greatly extended in to:

- various supergravity backgrounds (e.g. spheres)
- quiver gauge theories
- five and six-dimensional theories on $X_D = \mathbb{R}^4 \times \Sigma$
- low dimensional theories

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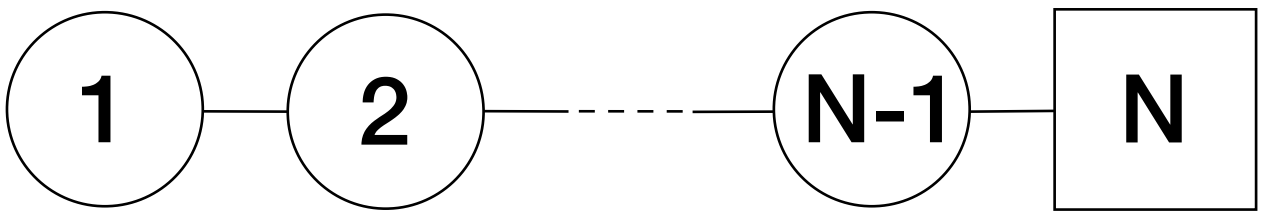
We shall study theories with adjoint matter on

$$X_3 = \mathbb{C}_{\epsilon_1} \times S^1_\gamma \qquad X_5 = \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times S^1_\gamma$$

3d Theory

$\mathcal{N}=2^*$ quiver gauge theory on $X_3 = \mathbb{C}_{\epsilon_1} \times S^1_\gamma$ T[U(N)]



$T^*\mathbb{F}_N$

Lagrangian depends on twisted masses μ_i and FI parameters τ_i

and $\mathcal{N}=2^*$ mass $t = e^m$

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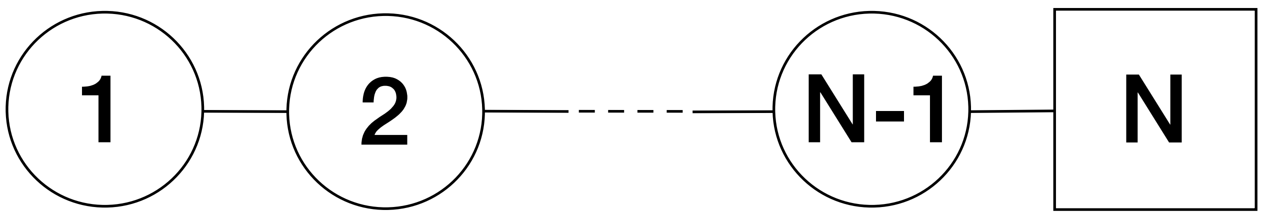

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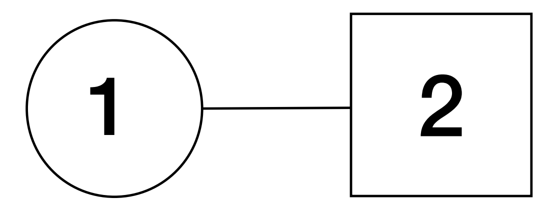
Partition function computed by localization for N=2



$$\mathcal{B}\sim\ {}_2\phi_1\left(t,t\frac{\mu_1}{\mu_2},q\frac{\mu_1}{\mu_2};q;\frac{\tau_1}{\tau_2}\right)$$

$q=e^{\epsilon_1}$

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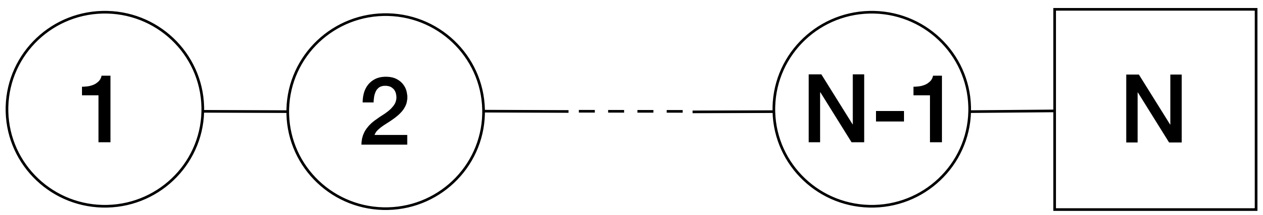

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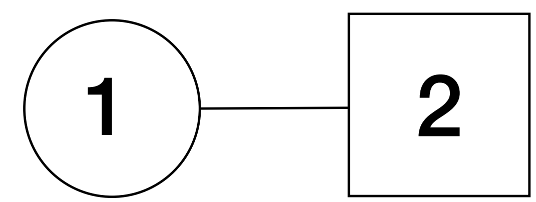

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$$q=e^{\epsilon_1}$$

is the eigenstate of the trigonometric Ruijsenaars-Schneider system!

$$D^{(1)}\mathcal{B}=(\mu_1+\mu_2)\mathcal{B} \qquad D^{(1)}\sim\sum_{i\neq j}\frac{t\tau_i-\tau_j}{\tau_i-\tau_j}e^{\hbar\partial_{\log\tau_i}}$$

3d A-type quiver

For T[U(N)] quiver

$$D^{(k)}\mathcal{B} = \left\langle W_k^{U(n)} \right\rangle \mathcal{B}$$

The eigenvalue of tRS Hamiltonian is a VEV of background Wilson loop around the compact circle

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Hamiltonians of n-particle tRS model form a commutative subalgebra inside spherical double affine Hecke algebra (DAHA) for gl(n)

[Cherednik]
[Oblomkov]
[PK Gukov Nawata in prog]

DAHA from line operators

Consider N=2* gauge theory on $\mathbb{R}^3 \times S^1$ with gauge group U(n)

Its moduli space of vacua $\mathcal{M}_n$ is described by VEVs of line operators wrapping the circle.

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$\mathcal{M}_n$ can be understood as the moduli space of GL(n;C) flat connections on punctured torus with a simple puncture

Its ***deformation quantization*** in complex structure **J** gives rise to *spherical* DAHA for gl(n) [Oblomkov]

sl(2) spherical DAHA

For sl(2) x,y,z are VEVs of Wilson, t'Hooft and dyonic loops

$$x = TrA \qquad y = TrB \qquad z = TrAB$$

y is tRS (symmetric Macdonald) operator

$\mathcal{M}_n$ $$x^2 + y^2 + z^2 + xyz \ = \ \mathrm{tr}V + 2$$

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quantization gives $[x, y]_q = (q - q^{-1})z$ +cyclic

'Casimir' $$\Omega = qx^2 + qy^2 + q^{-1}z^2 - q^{1/2}yzx$$

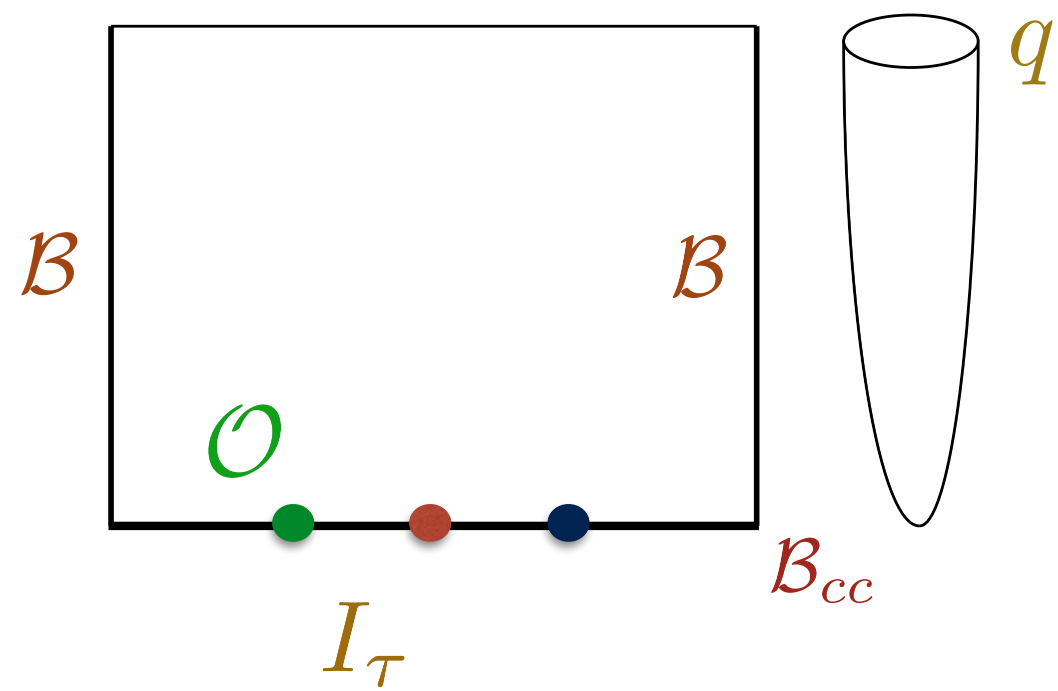
satisfying relation

$$\Omega = (q^{1/2}t^{-1} - q^{-1/2}t)^2 + (q^{1/2} + q^{-1/2})^2$$

DAHA cont'd

Physically we describe quantization by introducing Omega background to one of the spacetime 2-planes $\mathbb{R}^2_{\epsilon_1} \times \mathbb{R} \times S^1$

We can now reduce along the circle action which acts on this 2-plane



Line operators are forced to stay at the tip of the cigar and slide along the remaining line thus they do not commute anymore

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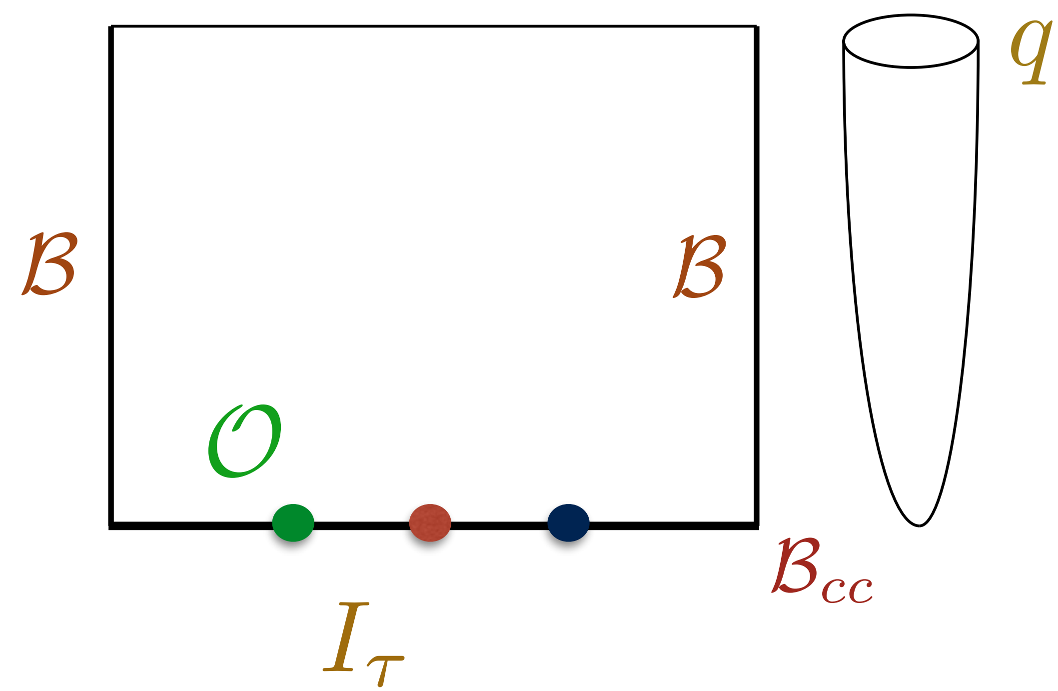

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Representations of DAHA can be understood by introducing boundaries

Elliptic Integrable System

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Couple 3d theory to 5d theory whose Seiberg-Witten solution gives elliptic Ruijsenaars model

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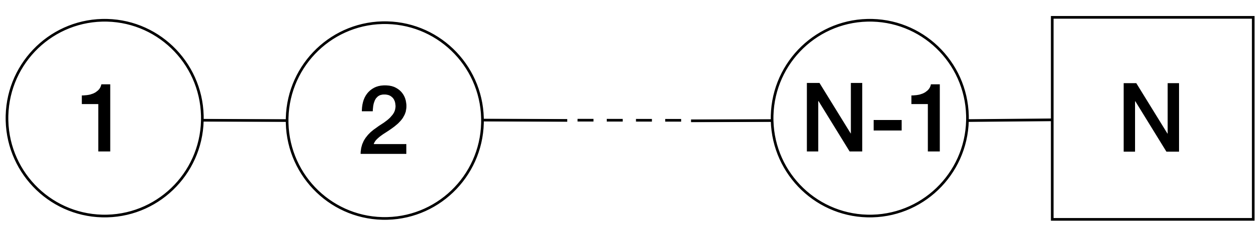
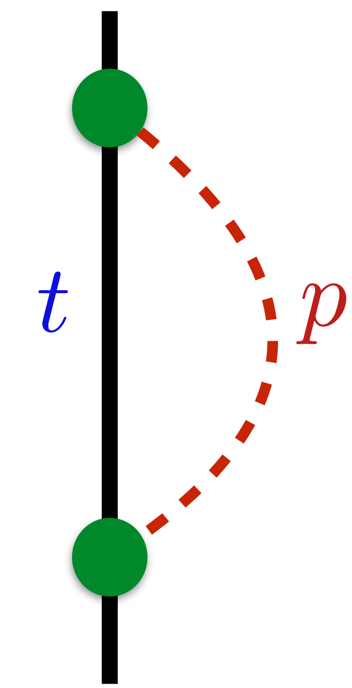
Gauging global symmetry of 3d theory
by gauge group of bulk 5d theory on $X_5 = \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times S^1_\gamma$



5d U(N)
N=1*



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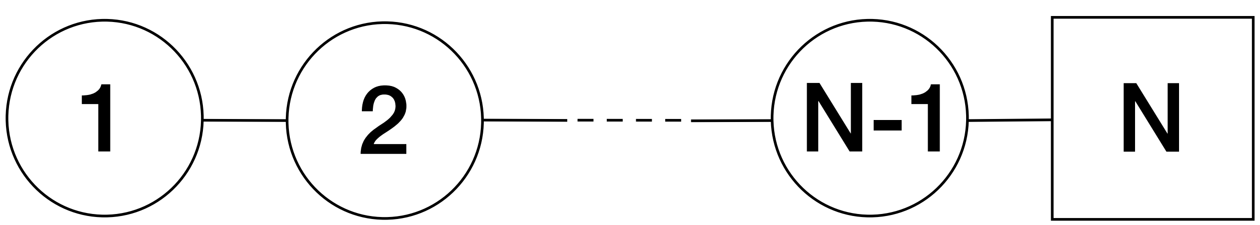

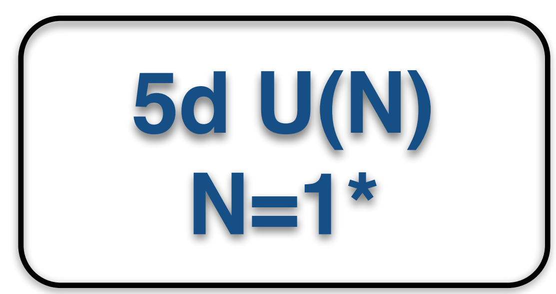

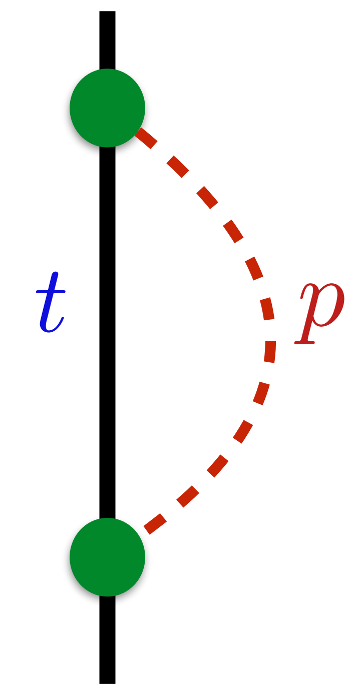

$$D^{(1)}_{p,q,t} \sim \sum \frac{\theta(t\frac{\tau_i}{\tau_j}|p)}{\theta(q\frac{\tau_i}{\tau_j}|p)} e^{\hbar \partial_{\log \tau_i}}$$

$$D^{(k)}_{p,q,t} \mathcal{Z}^{5d/3d} = \left\langle W^{U(n)}_{\Lambda^k} \right\rangle \mathcal{Z}^{5d/3d}$$

$\epsilon_2 \to 0$

Instanton partition functions

In the Nekrasov-Shatashvili limit

$$\mathcal{Z}^{5d/3d} = \sum_{k,l} R_{k,l}(t,q,\mu)\,\tau^k \left(\frac{p}{\tau}\right)^l$$

Wilson loop in fundamental representation

$$\langle W_{(1)} \rangle = \frac{\sum_{\vec{\lambda}} p^{|\vec{\lambda}|} \chi_{\vec{\lambda}}^{(\mathcal{E})} \prod_\alpha \left(2\sinh\left(\frac{w_\alpha}{2}\right)\right)^{-n_\alpha}}{\sum_{\vec{\lambda}} p^{|\vec{\lambda}|} \prod_\alpha \left(2\sinh\left(\frac{w_\alpha}{2}\right)\right)^{-n_\alpha}}$$

$$E_{(1)} = \lim_{\epsilon_2 \to 0} \langle W_{(1)} \rangle$$

$$E_{(1)}^{U(2)} = (\mu_1 + \mu_2)\left(1 + \sum_n F_n(q,t,\mu) p^n\right)$$

Gauge/Integrability duality

quantum eRS model	**5d/3d theory**
number of particles n	rank 3d flavor group / 5d gauge group
particle positions τ_j	3d Fayet-Iliopoulos parameters
interaction coupling t	3d $\mathcal{N}=2^*$ / 5d $\mathcal{N}=1^*$ deformation $e^{-i\gamma m}$
shift parameter q	Omega background $e^{i\gamma\epsilon_1}$
elliptic deformation p	5d instanton parameter $Q = e^{-8\pi^2\gamma/g_{YM}^2}$
eigenvalues	$\langle W_{\square}^{U(n)}\rangle$ for 5d $U(n)$ in NS limit
eigenfunctions	$Z_{\text{inst}}^{5d/3d}$ in NS limit at fixed μ_a

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Now we study large-n behavior of the operators (eigenvalues) and the eigenfunctions

Mapping States

Consider partition λ of $k < n$ (assume p=0)

Specify $\mu_a = q^{\lambda_a} t^{n-a} \quad , \quad a = 1, \ldots, n$ for T[U(n)] theory

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Partition function series truncates to Macdonald polynomials!

$$D^{(1)}_{n,\vec{\tau}}(q,t) P_\lambda(\vec{\tau}; q, t) = E^{(\lambda;n)}_{tRS} P_\lambda(\vec{\tau}; q, t)$$

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E.g. k=2

$$\mathcal{B}(\tau_1,\tau_2;t^{-1/2}q,t^{1/2}q) = P_{\square\square}(\tau_1,\tau_2;q,t)$$

$$\mathcal{B}(\tau_1,\tau_2;t^{-1/2},t^{-1/2}q^2) = P_{\substack{\square\\\square}}(\tau_1,\tau_2|q,t)\,.$$

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Their exact form depends on n

$$P_{(2,0)}(\tau_1,\tau_2;q,t) = \tau_1\tau_2 + \frac{1-qt}{(1+q)(1-t)}(\tau_1^2+\tau_2^2)$$

Change of Variables

However, after change of variables $$p_m = \sum_{l=1}^{n} \tau_l^m$$

Macdonald polynomials depend only on k and the partition

$$P_{\square\square} = \frac{1}{2}(p_1^2 - p_2)\,, \qquad P_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} = \frac{1}{2}(p_1^2 - p_2) + \frac{1-qt}{(1+q)(1-t)}p_2$$

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Starting with Fock vacuum $|0\rangle$

Construct Hilbert space $a_{-\lambda}|0\rangle \longleftrightarrow p_\lambda$

for each partition $a_{-\lambda}|0\rangle = a_{-\lambda_1}\cdots a_{-\lambda_l}|0\rangle$

Free boson realization

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}$$

(more involved with p)

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Vortex series encodes all states! Now need to describe eigenvalues

Free Boson Realization

Introduce vertex operators

[Ding Iohara]

$$\eta(z) =: \exp\left(-\sum_{k\neq 0}\frac{1-t^k}{k}a_k z^{-k}\right):$$

$$\phi(z) = \exp\left(\sum_{n>0}\frac{1-t^n}{1-q^n}a_{-n}\frac{z^n}{n}\right)$$

Define $\phi_n(\tau) = \prod_{i=1}^{n}\phi(\tau_i)$

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$$[\eta(z)]_1\phi_n(\tau)|0\rangle = \left[t^{-n} + t^{-n+1}(1-t^{-1})D^{(1)}_{n,\vec{\tau}}(q,t)\right]\phi_n(\tau)|0\rangle$$

Assuming |t|<1

$$\mathcal{E}_1^{(\lambda)} = \lim_{n\to\infty}\left[t^{-n+1}(1-t^{-1})E^{(\lambda;n)}_{tRS}\right]$$

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For elliptic model replace [Feigin Hashizume Hoshino Shiraishi Yanagida]

$$\eta(z;pq^{-1}t) = \exp\left(\sum_{n>0}\frac{1-t^{-n}}{n}\frac{1-(pq^{-1}t)^n}{1-p^n}a_{-n}z^n\right)\exp\left(-\sum_{n>0}\frac{1-t^n}{n}a_n z^{-n}\right)$$

Free Boson Realization

Assuming |t|<1

$$\mathcal{E}_1^{(\lambda)}(p) = \lim_{n\to\infty}\left[t^{-n+1}(1-t^{-1})\frac{(pt^{-1};p)_\infty(ptq^{-1};p)_\infty}{(p;p)_\infty(pq^{-1};p)_\infty}E_{eRS}^{(\lambda;n)}(p)\right]$$

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Free Boson Realization

From gauge theory we can compute

$$\frac{(pt^{-1};p)_\infty (ptq^{-1};p)_\infty}{(p;p)_\infty (pq^{-1};p)_\infty} E^{(\lambda;n)}_{eRS}(p) = \left\langle W^{U(1)}_{\Box} \right\rangle E^{(\lambda;n)}_{eRS}(p) = \left\langle W^{U(n)}_{\Box} \right\rangle \Big|_\lambda$$

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For elliptic model replace

[Feigin Hashizume Hoshino Shiraishi Yanagida]

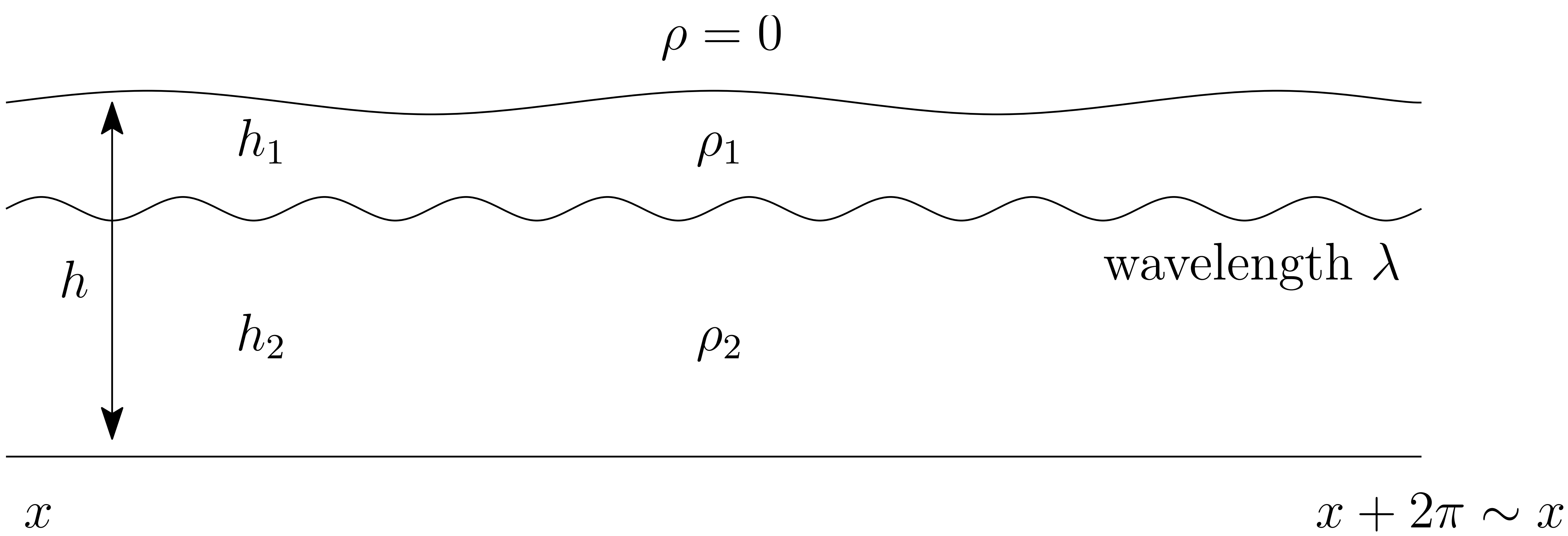
$$\eta(z;pq^{-1}t) = \exp\left(\sum_{n>0} \frac{1-t^{-n}}{n} \frac{1-(pq^{-1}t)^n}{1-p^n} a_{-n} z^n \right) \exp\left(-\sum_{n>0} \frac{1-t^n}{n} a_n z^{-n} \right)$$

Is there an effective large-n integrable system?

Which energies are we computing?

Intermediate Long Wave model

Describes propagation of waves along the interface of two 1d fluids



- $h \ll \lambda$, long wave: Korteweg-de Vries (KdV) regime for $\delta \to 0$
- $h \gg \lambda$, short wave: Benjamin-Ono (BO) regime for $\delta \to \infty$
- $h \sim \lambda$, intermediate wave: Intermediate Long Wave (ILW) regime for $\delta \sim 1$

Integrable ILW equation

$$u_t = 2u_{xx} - i\beta\partial_x^2 u^H \qquad u^H = \frac{1}{2\pi} P.V. \int_0^{2\pi} \zeta(y-x;\widetilde{p})u(y)dy$$

Kernel — Weierstrass zeta function, simplifies in Korteweg de-Vries and Benjamin-Ono limits

KdV equation $$u_t = 2uu_x + \frac{\beta}{3}u_{xxx}$$

Poisson bracket $$\{u(x), u(y)\} = \delta'(x-y)$$

Rewrite ILW as evolution equation $$u_t = \{u, I_2\}$$

Integrals of motion

$$I_1 = \int \left[\frac{1}{2}u^2\right] dx\,, \quad I_2 = \int \left[\frac{1}{3}u^3 + i\frac{\beta}{2}uu_x^H\right] dx\,,$$

$$\{I_l, I_m\} = 0$$

Soliton Solutions

n-Solitonic Ansatz

$$u(x,t) = \sum_{j=1}^{n} \left(\frac{i\beta}{x - a_j(t)} - \frac{i\beta}{x - a_j^*(t)} \right)$$

For non-periodic Benjamin-Ono we get equations of motion for Calogero-Moser-Sutherland (CMS) model

$$\ddot{a}_j = \sum_{l \neq j}^{n} \frac{2\beta^2}{(a_j - a_l)^3}$$

Poles describe propagation of solitons

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Poles describe propagation of solitons

Difference BO → Relativistic CMS

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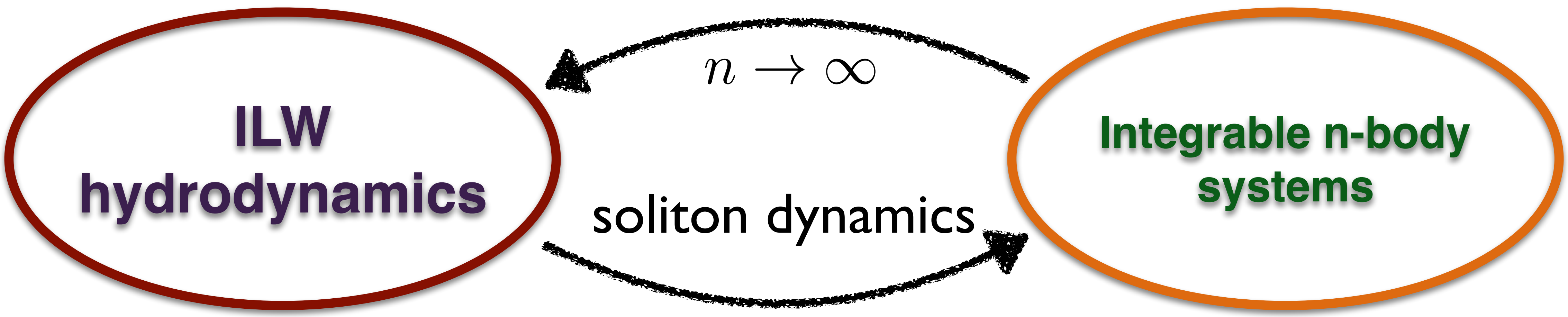
There exist a `hydro' version for most of known integrable many-body systems

Duality

Starting with an integrable many-body system we can take a thermodynamical limit by sending the number of its particles to infinity

EOM become hydrodynamical equations [Abanov Bettelheim Wiegmann]



Classically Large-n elliptic Calogero model turns into intermediate long wave (ILW) system

Elliptic Ruijsenaars-Schneider model becomes finite-difference ILW

Effective Large-n gauge theory

M-theory construction

Starting with M-theory on $S^1 \times \mathbb{C}_q \times \mathbb{C}_t \times T^*S^3$

With n M5 branes wrapping $S^1 \times \mathbb{C}_q \times S^3$ (with $S^3 \subset T^*S^3$)

This setup provides us with the U(n) theory on M5 branes with proper R-symmetry

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When n becomes large the background undergoes through the conifold transition and the resolved conifold becomes a deformed conifold

So we are left with M-theory on $S^1 \times \mathbb{C}_q \times \mathbb{C}_t \times Y$

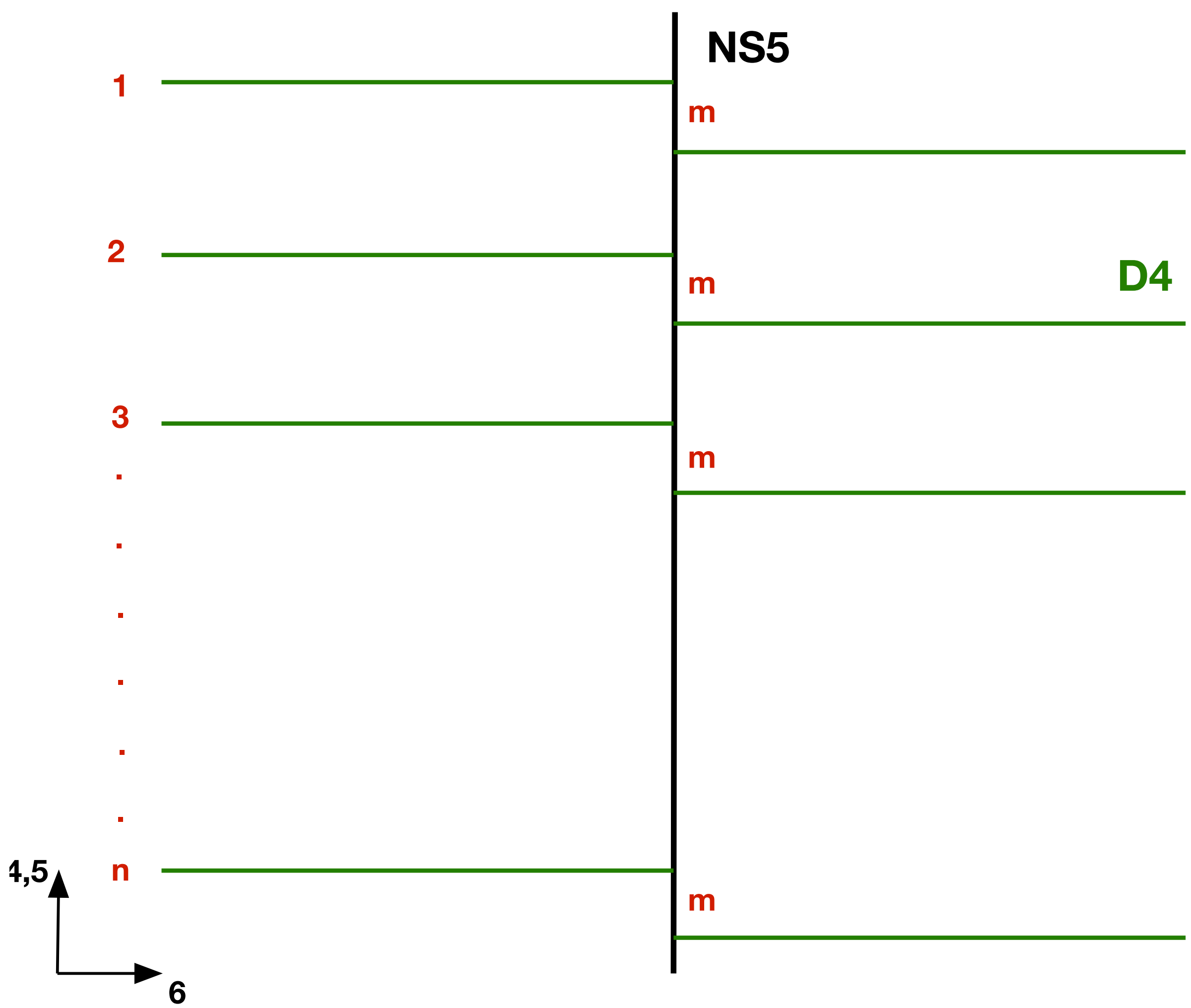
Reduction on Y leads us to a 5d U(1) theory with 8 supercharges

We shall now count instantons in this theory

Branes

N=1* theory on the Coulomb branch

5d theory on $\mathbb{R}^4 \times S^1_\gamma$.



NS5 012345

D4 0123(4 6)

Complex scalar

$$\mu_a = e^{-i\gamma a_a}$$

Adjoint hyper

$$t = e^{-i\gamma m}$$

Omega backgr
in 23-plane

$$q = e^{i\gamma \epsilon_1}$$

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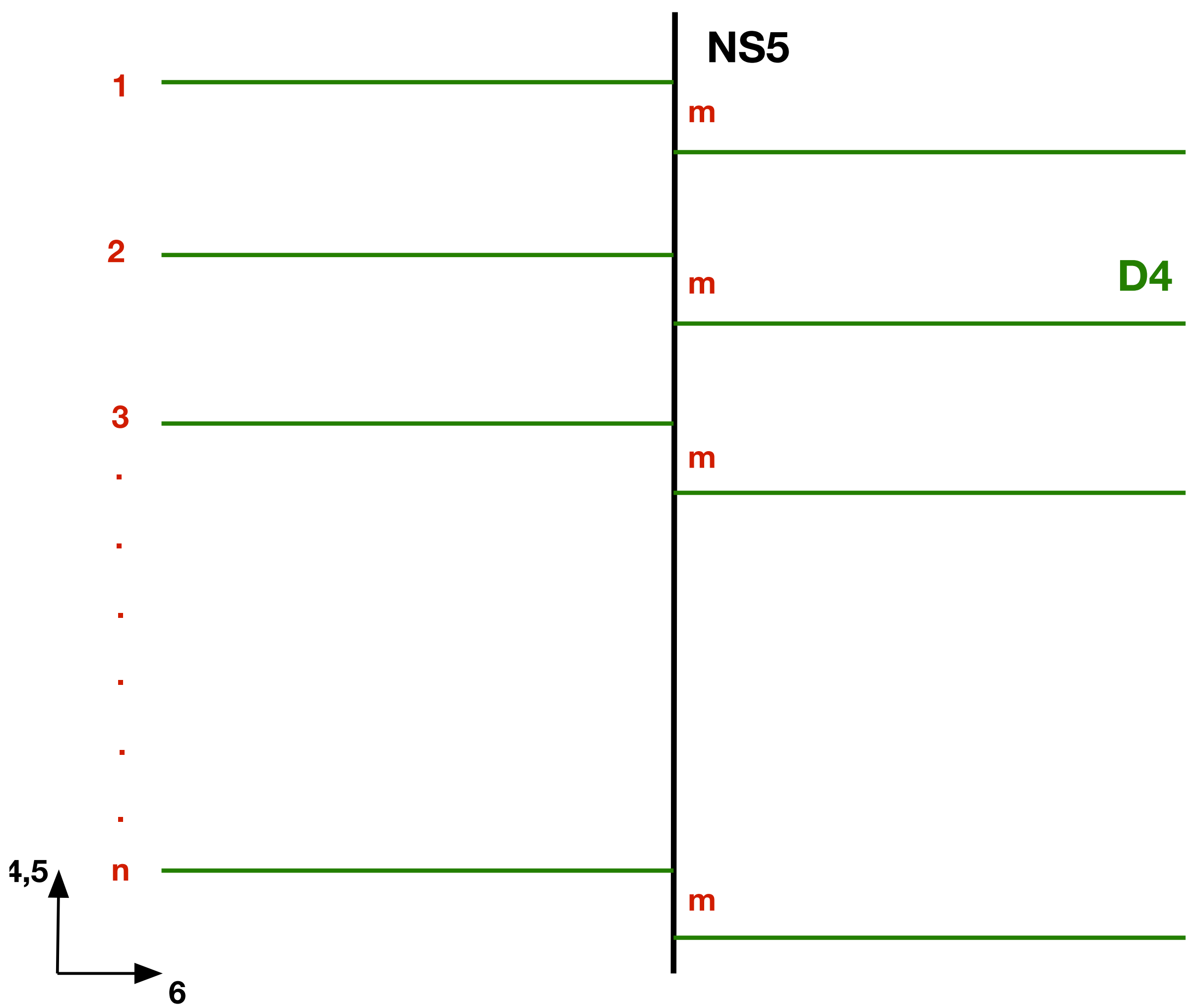

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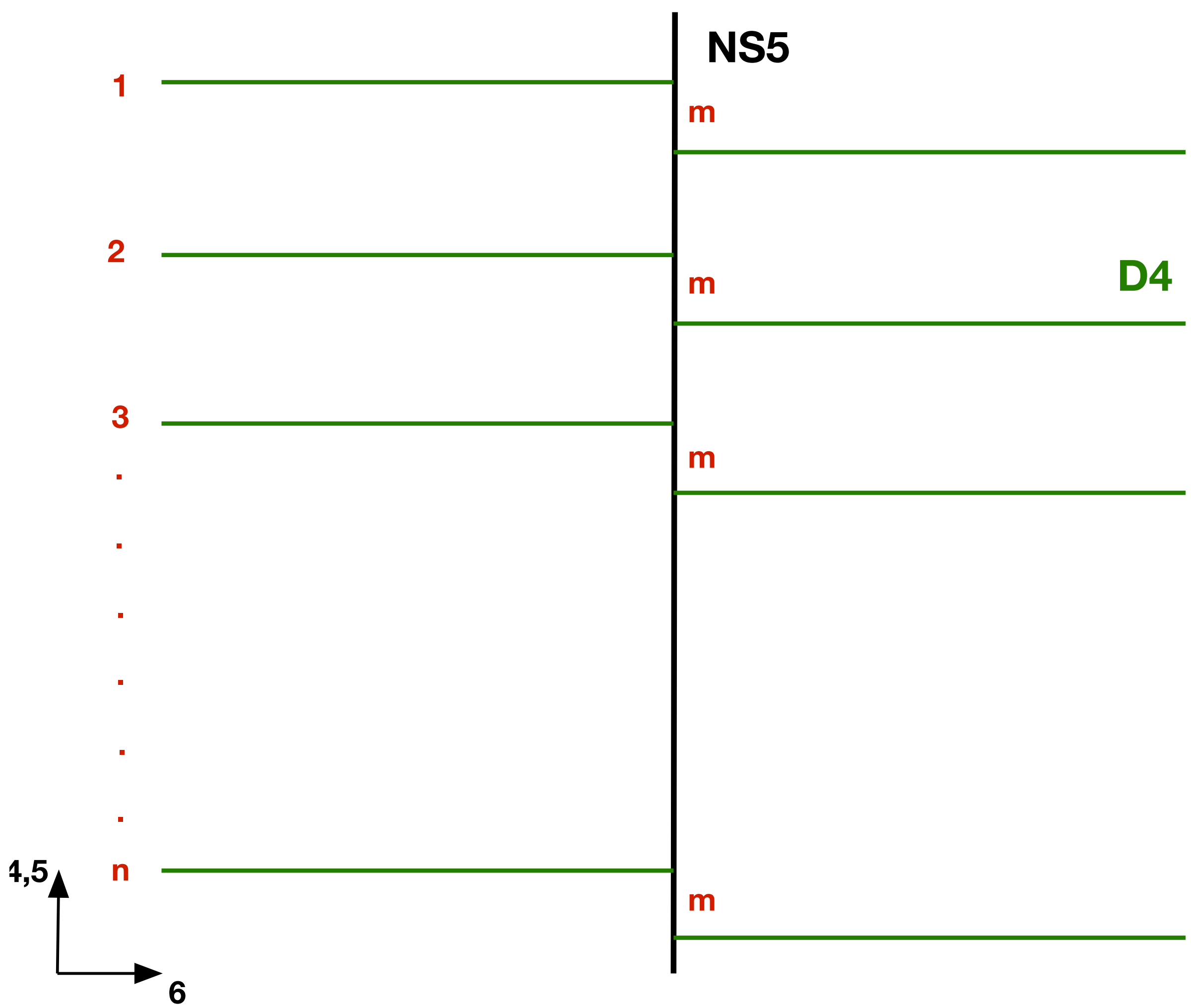

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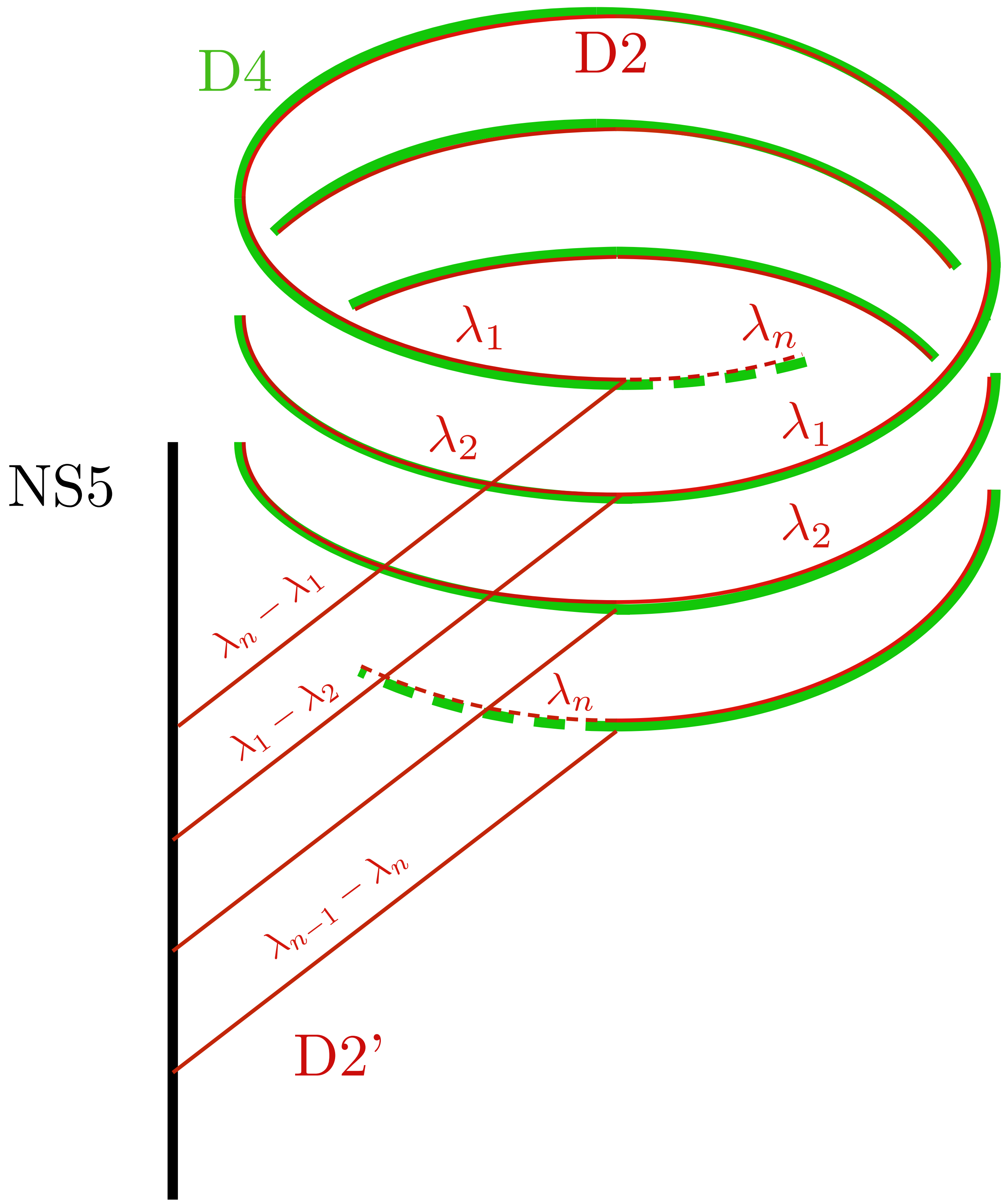
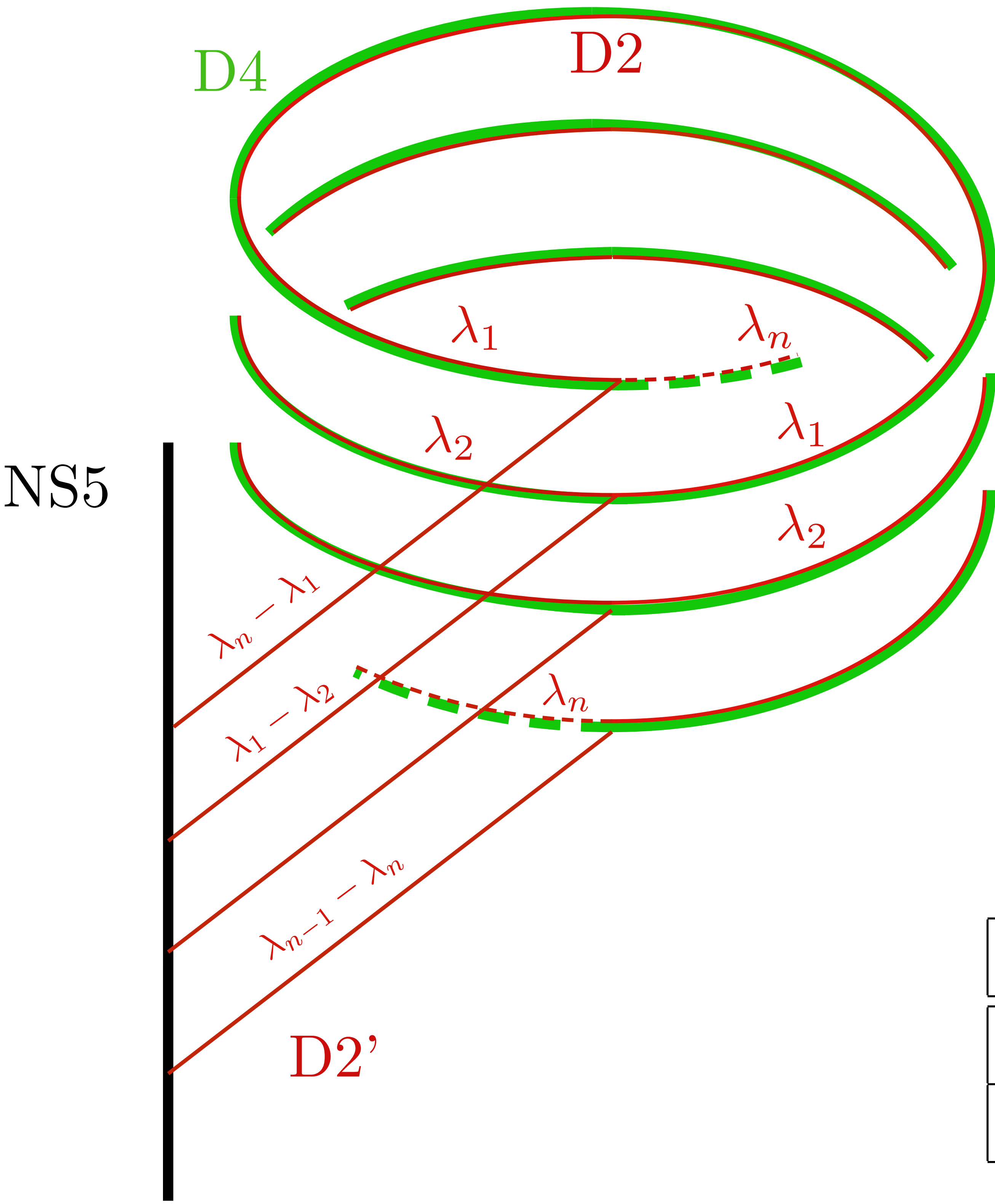
System undergoes geometric transition

ADHM from branes



	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D4	x	x	x	x	$\cos\delta$		$\sin\delta$			
D2'	x	x						x		
D2	x	x			$-\sin\delta$		$\cos\delta$			

ADHM from branes



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D2'	x	x						x		
D2	x	x			$-\sin\delta$		$\cos\delta$			

$$\delta \sim \tan^{-1} m/R$$

At large-n limit arrive at ADHM!!

	0	1	2	3	4	5	6	7	8	9
D6	x	x	x	x	x		x		x	
D2	x	x							x	

U(1) Instantons

Heisenberg algebra (and ***elliptic Hall algebra***) which we have seen earlier appears in the study of moduli space of U(1) (non-commutative) instantons

[Nakjima]
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Instanton - KK monopole propagating along the compact circle

KK modes yield different topological sectors

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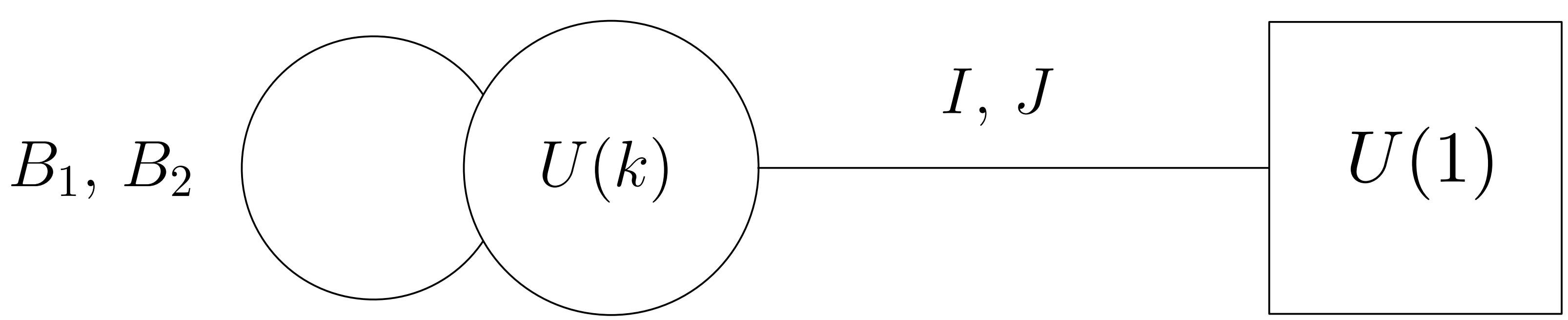
Higgs branch of the 3d N=2 ADHM quiver gauge theory on $\mathbb{C} \times S^1_\gamma$



$\mathcal{M}_{k,1}$

Quantum Cohomology

Using supersymmetry we can effectively describe quantum cohomology (K-theory) of the instanton moduli space $\mathcal{M}_{k,1}$

We need to find the twisted chiral ring of the ADHM gauge theory— Jacobian ring for effective twisted superpotential

$$H_T^\bullet(\mathcal{M}_{k,1}) \simeq \frac{\{\sigma_1, \ldots \sigma_s\}}{\{\partial\widetilde{\mathcal{W}}/\partial\sigma_s = 0\}}$$

[Nekrasov Shatashvili]

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$$(\sigma_s - 1)\prod_{\substack{t=1\\ t\neq s}}^{k} \frac{(\sigma_s - q\sigma_t)(\sigma_s - t^{-1}\sigma_t)}{(\sigma_s - \sigma_t)(\sigma_s - qt^{-1}\sigma_t)} = \frac{\widetilde{p}}{\sqrt{qt^{-1}}}(1 - qt^{-1}\sigma_s)\prod_{\substack{t=1\\ t\neq s}}^{k} \frac{(\sigma_s - q^{-1}\sigma_t)(\sigma_s - t\sigma_t)}{(\sigma_s - \sigma_t)(\sigma_s - q^{-1}t\sigma_t)}$$

where $\sigma_s = e^{i\gamma\Sigma_s},\ q = e^{i\gamma\epsilon_1},\ t = e^{-i\gamma\epsilon_2}$ $\quad \widetilde{p} = e^{-2\pi\xi}$ FI coupling

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Calogero ***Hamiltonian*** *contains the operator of* ***quantum multiplication*** *in small* ***quantum cohomology*** *ring of the instanton moduli space*

The Duality

Eigenvalues at large-n

[PK Sciarappa]

$$\left\langle W_{\square}^{U(n)} \right\rangle \Big|_{\lambda} \sim \mathcal{E}_1^{(\lambda)} = 1 - (1-q)(1-t^{-1}) \sum_s \sigma_s \Big|_{\lambda}$$

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elliptic RS	**3d ADHM theory**	**3d/5d coupled theory, $n \to \infty$**
coupling t	twisted mass $e^{-i\gamma\epsilon_2}$	5d $\mathcal{N} = 1^*$ mass deformation $e^{-i\gamma m}$
quantum shift q	twisted mass $e^{i\gamma\epsilon_1}$	Omega background $e^{i\gamma\widetilde{\epsilon}_1}$
elliptic parameter p	FI parameter $\widetilde{p} = -p/\sqrt{qt^{-1}}$	5d instanton parameter Q
eigenstates λ	ADHM Coulomb vacua	5d Coulomb branch parameters
eigenvalues	$\langle \mathrm{Tr}\,\sigma \rangle$	$\langle W_{\square}^{U(\infty)} \rangle$ in NS limit $\widetilde{\epsilon}_2 \to 0$

Mathematical Results

[Schiffmann Vasserot]

Hall algebra as large-n limit of DAHA

Trigonometric RS at large n $\lim_{n\to\infty} K_T(T^*\mathbb{F}_n) \simeq K^{\mathrm{cl}}_{q,t}\left(\widetilde{\mathcal{M}_1}\right)$

$\widetilde{\mathcal{M}_1} = \bigoplus_{k=0}^{\infty} \mathcal{M}_{1,k}$ Instanton moduli space

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No mathematical object is known to describe spectrum of elliptic RS

Our proposal

$$\mathcal{E}_T^Q(T^*\mathbb{F}_n) := \mathbb{C}[p_i^{\pm 1}, \tau_i^{\pm 1}, Q, t, \mu_i^{\pm 1}]/\mathcal{I}_{\mathrm{eRS}}$$

Large-n limit

$$\lim_{n\to\infty} \mathcal{E}_T^Q(T^*\mathbb{F}_n) \simeq K_{q,t}\left(\widetilde{\mathcal{M}_1}\right)$$

Open questions

Physics construction for elliptic cohomology

Knot homology

What happens for 6d theories at large n?   Holography?

Elliptic generalization of DAHA

Thanks to the
Organizers for
the
great Workshop!